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# Blow-up solutions and strong instability of standing waves for the generalized Davey-Stewartson system in $\mathbb{R}^2$

by

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ABSTRACT. – We study the instability of standing wave $e^{i\omega t}\varphi_\omega(x)$ for the equation

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)\,u = 0 \qquad (\star)$$

in $\mathbb{R}^2$, where $\varphi_\omega$ is a ground state. We prove that if $a(p-3) > 0$, then there exist blow-up solutions of $(\star)$ arbitrarily close to the standing wave.

RÉSUMÉ. – Nous étudions l'instabilité de l'onde stationnaire $e^{i\omega t}\varphi_\omega(x)$ pour l'équation

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)\,u = 0 \qquad (\star)$$

dans $\mathbb{R}^2$, où $\varphi_\omega$ est un état fondamental. Nous prouvons que si $a(p-3) > 0$, il existe solutions de $(\star)$ explosant en temps fini, arbitrairement voisine de l'onde stationnaire.

## 1. INTRODUCTION AND RESULT

We consider the instability of standing waves for the following equation:

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)\,u = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^n, \qquad (1.1)$$

where $a \in \mathbb{R}$, $1 < p < 2^* - 1$, $n = 2$ or 3, and $E_1$ is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Equation

(1.1), for $n = 2$ and $p = 3$, describes the evolution of weakly nonlinear water waves that travel predominantly in one direction (*see* [3], [4] and [2]). By a standing wave, we mean a solution of (1.1) with the form

$$u_\omega (t, x) = e^{i\omega t} \varphi_\omega (x),$$

where $\omega > 0$ and $\varphi_\omega$ is a ground state (least action solution) of the problem:

$$\left.\begin{array}{ll} -\Delta\psi + \omega\psi - a|\psi|^{p-1}\psi - E_1(|\psi|^2)\psi = 0, & x \in \mathbb{R}^n, \\ \psi \in H^1(\mathbb{R}^n), \qquad \psi \not\equiv 0. & \end{array}\right\} \qquad (1.2\,\omega)$$

Here the action $S_\omega$ of $(1.2\,\omega)$ is defined by

$$S_\omega (v) = \frac{1}{2}|\nabla v|_2^2 + \frac{\omega}{2}|v|_2^2 - \frac{a}{p+1}|v|_{p+1}^{p+1} - \frac{1}{4} B_1(|v|^2),$$

where $B_1(|v|^2) = \displaystyle\int |v|^2 E_1(|v|^2)\,dx$. We denote by $\mathcal{G}_\omega$ the set of all ground states for $(1.2\,\omega)$.

DEFINITION 1.1. – For $\Omega \subset H^1(\mathbb{R}^n)$, we say that the set $\Omega$ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^n)$ satisfies $\inf_{\varphi\in\Omega} \|u_0 - \varphi\|_{H^1} < \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies

$$\sup_{0\le t<\infty} \inf_{\varphi\in\Omega} \|u(t) - \varphi\|_{H^1} < \varepsilon.$$

Otherwise, $\Omega$ is said to be unstable. Moreover, for $\varphi_\omega \in \mathcal{G}_\omega$, we say that the standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is unstable if $\{ e^{i\theta}\varphi_\omega(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^n \}$ is unstable. Furthermore, we say that $u_\omega$ is strongly unstable if for any $\varepsilon > 0$ there exists $u_0 \in H^1(\mathbb{R}^n)$ such that $\|u_0 - \varphi_\omega\|_{H^1} < \varepsilon$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in a finite time.

For the standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$ with $\varphi_\omega \in \mathcal{G}_\omega$ of (1.1), Cipolatti [2] proved that if $a(p-3) \ge 0$, and $n = 2$ or 3, then $u_\omega$ is unstable for any $\omega \in (0, \infty)$, and that if $n = 2$, $p = 3$ and $a > -1$, then $u_\omega$ is strongly unstable for any $\omega \in (0, \infty)$. After that, the author [5] proved that if $a > 0$, $p \ge 1 + 4/n$, and $n = 2$ or 3, then $u_\omega$ is unstable for any $\omega \in (0, \infty)$, and that if $n = 3$, $a > 0$ and $1 < p < 7/3$, then there exists a positive constant $\omega_0 = \omega_0(a, p)$ such that $u_\omega$ is unstable for any $\omega \in (\omega_0, \infty)$. Moreover, the author [6] proved that if $n = 3$, $a > 0$ and $7/3 < p < 5$, or $a < 0$ and $1 < p < 3$, then $u_\omega$ is strongly unstable for any $\omega \in (0, \infty)$. On the other hand, when $n = 2$ and $a(p-3) < 0$, the author [6] showed the existence of stable standing waves of (1.1).

Our result in this paper is the following.

THEOREM 1.2. – *Assume that* $n = 2$ *and* $a\,(p-3) > 0$, *or* $n = 3$, $a > 0$ *and* $p = 7/3$.. *Then,* for any $\omega \in (0, \infty)$, *the standing wave* $u_\omega\,(t) = e^{i\omega t}\,\varphi_\omega$ *with* $\varphi_\omega \in \mathcal{G}_\omega$ *is strongly unstable in the sense of Definition* 1.1.

*Remark* 1.3. – As stated above, we showed in [6] that if $n = 3$, $a > 0$ *and* $7/3 < p < 5$ or $a < 0$ and $1 < p < 3$, then $u_\omega$ is strongly unstable for any $\omega \in (0, \infty)$, by extending the method of Berestycki and Cazenave [1] to an anisotropic case [(1.1) contains an anisotropic nonlinearity $E_1\,(|u|^2)\,u$]. Following Berestycki and Cazenave [1], we consider the same minimization problem as in [6] (*see* Proposition 2.1 below). In the case of Theorem 1.2, we need some devices to obtain that its minimizing sequence is bounded in $H^1\,(\mathbb{R}^n)$, and is not vanishing in $L^q\,(\mathbb{R}^n)$ for some $2 < q < 2^*$, although it is easy in the case of [6] (*see* Proposition 2.2 below, and Lemma 4.2 in [6]). In particular, in order to show that the minimizing sequence is not vanishing in $L^{p+1}\,(\mathbb{R}^2)$ when $n = 2$, $a > 0$ and $p > 3$, we need an estimate for the critical value of minimization problem (*see* Lemma 2.3 below).

In what follows, we omit the integral variables with respect to the spatial variable $x$, and we omit the integral region when it is the whole space $\mathbb{R}^n$. We denote the norms of $L^q\,(\mathbb{R}^n)$ and $H^1\,(\mathbb{R}^n)$ by $|\cdot|_q$ and $\|\cdot\|_{H^1}$, respectively. We put $v^\lambda\,(x) = \lambda^{n/2}\,v\,(\lambda\,x)$, $\lambda > 0$.

## 2. PROOF OF THEOREM 1.2

In this section, we give the proof of Theorem 1.2. We prove the case when $n = 2$ and $a\,(p-3) > 0$ only. The case when $n = 3$, $a > 0$ and $p = 7/3$ can be proved analogously to the case when $n = 2$, $a > 0$ and $p > 3$. Thus, we assume that $n = 2$ and $a\,(p-3) > 0$ throughout this section. Moreover, since we fix the parameter $\omega$, we drop the subscript $\omega$. Thus, we write $\varphi$ for $\varphi_\omega$, $S$ for $S_\omega$, and so on. We put

$$P\,(v) = |\nabla\,v|_2^2 - \frac{p-1}{p+1}\,a|v|_{p+1}^{p+1} - \frac{1}{2}\,B_1\,(|v|^2). \tag{2.1}$$

We note that $P(v) = \partial_\lambda\,S\,(v^\lambda)|_{\lambda=1}$. We first prove a key proposition to obtain Theorem 1.2.

PROPOSITION 2.1. – *Assume that* $n = 2$ *and* $a\,(p-3) > 0$. *Then,* $\varphi$ *is a ground state of* (1.2) *if and only if* $\varphi \in M$ *and* $m = S\,(\varphi)$, *where*

$$\begin{aligned} m &= \inf\{\,S\,(v)\;:\;v \in M\,\},\\ M &= \{\,v \in H^1\,(\mathbb{R}^2)\;:\;v \neq 0,\;P\,(v) = 0\,\}. \end{aligned} \tag{2.2}$$

In order to obtain a minimizer for (2.2), we consider the following minimization problem (2.3), instead of (2.2):

$$m_1 = \inf \{ S^1 (v) \ : \ v \in H^1 (\mathbb{R}^2), \ v \neq 0, \ P (v) \leq 0 \}, \qquad (2.3)$$

where

$$S^1 (v) = S (v) - \frac{1}{2} P (v) = \frac{\omega}{2} |v|_2^2 + \gamma |v|_{p+1}^{p+1}, \qquad \gamma = \frac{a (p-3)}{2 (p+1)} > 0.$$

If $P (v) < 0$, then we have

$$P (\lambda v) = \lambda^2 \, |\nabla v|_2^2 - \frac{p-1}{p+1} \, a \, \lambda^{p+1} |v|_{p+1}^{p+1} - \frac{1}{2} \lambda^4 B_1 (|v|^2) > 0$$

for sufficiently small $\lambda > 0$, so there exists a $\lambda_0 \in (0, 1)$ such that $P (\lambda_0 v) = 0$. Moreover, since we get

$$S^1 (\lambda_0 v) = \frac{\omega}{2} \lambda_0^2 |v|_2^2 + \gamma \, \lambda_0^{p+1} |v|_{p+1}^{p+1} < S^1 (v),$$

we obtain that

$$m_1 = \inf \{ S^1 (v) \ : \ v \in H^1 (\mathbb{R}^2), \ v \neq 0, \ P (v) = 0 \} = m. \qquad (2.4)$$

PROPOSITION 2.2. – *The minimization problem* (2.3) *is attained at some* $w \in M$.

Before giving the proof of Proposition 2.2, we prepare one lemma. We use Lemma 2.3 to show that a minimizing sequence for (2.3) is not vanishing in $L^{p+1} (\mathbb{R}^2)$ when $a > 0$ and $p > 3$.

LEMMA 2.3. – *Let* $a > 0$ *and* $p > 3$. *Then, we have* $m_1 < \omega \mu_0 / 2$, *where*

$$\mu_0 = \inf \left\{ |v|_2^2 \ : \ v \in H^1 (\mathbb{R}^2), v \neq 0, \mathcal{E}_0 (v) \equiv \frac{1}{2} |\nabla v|_2^2 - \frac{1}{4} B_1 (|v|^2) \leq 0 \right\}.$$

*Proof.* – From Proposition 2.1 in [6], there exists a function $Q \in H^1 (\mathbb{R}^2)$ such that $Q \not\equiv 0$, $|Q|_2^2 = \mu_0$ and $\mathcal{E}_0 (Q) = 0$. For $0 < \delta < 1$ and $\lambda > 0$, we have by $\mathcal{E}_0 (Q) = 0$

$$\begin{aligned} P (\delta \, Q^\lambda) &= \delta^2 \, \lambda^2 |\nabla Q|_2^2 - \frac{p-1}{p+1} \, a \, \delta^{p+1} \, \lambda^{p-1} |Q|_{p+1}^{p+1} - \frac{1}{2} \, \delta^4 \, \lambda^2 \, B_1 (|Q|^2) \\ &= \delta^2 \, \lambda^2 \, (1 - \delta^2) |\nabla Q|_2^2 - \frac{p-1}{p+1} \, a \, \delta^{p+1} \, \lambda^{p-1} |Q|_{p+1}^{p+1}. \end{aligned}$$

If we take $0 < \delta < 1$ and $\lambda > 0$ such that $P(\delta Q^\lambda) = 0$, then we have

$$\lambda = C(a, p, Q)\, \delta^{(1-p)/(p-3)} (1-\delta)^{1/(p-3)}$$

and

$$S^1(\delta Q^\lambda) = \frac{\omega}{2}\delta^2 |Q|_2^2 + \gamma \delta^{p+1} \lambda^{p-1} |Q|_{p+1}^{p+1}$$
$$= \frac{\omega}{2}\delta^2 |Q|_2^2 + \frac{p-3}{2(p-1)}\delta^2 \lambda^2 (1-\delta^2) |\nabla Q|_2^2.$$

Thus, if we take $\delta$ sufficiently close to 1, then we have $S^1(\delta Q^\lambda) < \omega |Q|_2^2/2$. Hence, from the definition of $m_1$, we obtain that $m_1 < \omega |Q|_2^2/2 = \omega \mu_0/2$. □

REMARK 2.4. – It is important to note that $m_1$ is strictly less than $\omega \mu_0/2$ in Lemma 2.3. This fact plays an essential role in the proof of Proposition 2.2.

*Proof of Proposition* 2.2. – Let $\{ v_j \}$ be a minimizing sequence for (2.3). Since $\gamma > 0$, $\{ v_j \}$ is bounded in $L^2(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2)$.

First, we show that $\{ v_j \}$ is bounded in $H^1(\mathbb{R}^2)$. When $a > 0$ and $p > 3$, we see that $\{ v_j \}$ is bounded in $L^4(\mathbb{R}^2)$, $B_1(|v_j|) \le |v_j|_4^4$ and $P(v_j) \le 0$, so that we have $\sup_j |\nabla v_j|_2^2 < \infty$. When $a < 0$ and $1 < p < 3$, we have from $P(v_j) \le 0$

$$|\nabla v_j|_2^2 \le |\nabla v_j|_2^2 + \frac{p-1}{p+1}|a|\, |v_j|_{p+1}^{p+1} \le \frac{1}{2} B_1(|v_j|^2)$$
$$\le \frac{1}{2}|v_j|_4^4 \le C_1 |v_j|_{p+1}^{p+1} |\nabla v_j|_2^{3-p}$$

for some $C_1 > 0$. Here we have used the Gagliardo-Nirenberg inequality. Since $\{ v_j \}$ is bounded in $L^{p+1}(\mathbb{R}^2)$, we have $|\nabla v_j|_2^2 \le C_2 |\nabla v_j|_2^{3-p}$ for some $C_2 > 0$, so that we have $|\nabla v_j|_2^{p-1} \le C_2$.

Next, we show that $\liminf_{j\to\infty} |v_j|_{p+1}^{p+1} > 0$ when $a > 0$ and $p > 3$. In fact, suppose that $|v_j|_{p+1}^{p+1} \to 0$. Then, since we have

$$B_1(|v_j|^2) \le |v_j|_4^4 \le |v_j|_2^{2(p-3)/(p-1)} |v_j|_{p+1}^{2(p+1)/(p-1)}$$

and $\{ v_j \}$ is bounded in $L^2(\mathbb{R}^2)$, we have $B_1(|v_j|^2) \to 0$, and from $P(v_j) \le 0$ we have $|\nabla v_j|_2 \to 0$. From the fact that $P(v_j) \le 0$, Proposition 2.1 in [6] and the Gagliardo-Nirenberg inequality, we have

$$|\nabla v_j|_2^2 \le \frac{p-1}{p+1} a |v_j|_{p+1}^{p+1} + \frac{1}{2} B_1(|v_j|^2)$$
$$\le \frac{p-1}{p+1} a\, C_3 |v_j|_2^2\, |\nabla v_j|_2^{p-1} + \frac{1}{\mu_0} |v_j|_2^2 |\nabla v_j|_2^2$$
$$\le C_4 |\nabla v_j|_2^{p-1} + \frac{1}{\mu_0} |v_j|_2^2 |\nabla v_j|_2^2$$

for some positive constants $C_3$ and $C_4$, so that we have

$$1 \leq C_4 |\nabla v_j|_2^{p-3} + \frac{1}{\mu_0} |v_j|_2^2.$$

It follows from $|\nabla v_j|_2 \to 0$ that $\mu_0 \leq \liminf_{j\to\infty} |v_j|_2^2$. Since $S^1(v_j) \to m_1$, we have $\omega\mu_0/2 \leq m_1$. However, this contradicts Lemma 2.3. Therefore, we obtain that $\liminf_{j\to\infty} |v_j|_{p+1}^{p+1} > 0$ in the case of $a > 0$ and $p > 3$.

Next, we show that $\liminf_{j\to\infty} |v_j|_4^4 > 0$ when $a < 0$ and $1 < p < 3$. In fact, suppose that $|v_j|_4^4 \to 0$. Then, from $P(v_j) \leq 0$ we have $|\nabla v_j|_2 \to 0$. Again from $P(v_j) \leq 0$, we have

$$|\nabla v_j|_2^2 + \frac{p-1}{p+1} |a|\, |v_j|_{p+1}^{p+1} \leq \frac{1}{2} B_1(|v_j|^2)$$
$$\leq \frac{1}{2} |v_j|_4^4 \leq \frac{p-1}{p+1} |a|\, |v_j|_{p+1}^{p+1} + C_5 |v_j|_5^5,$$

so that we have

$$|\nabla v_j|_2^2 \leq C_5 |v_j|_5^5 \leq C_6 |v_j|_2^2 |\nabla v_j|_2^3 \leq C_7 |\nabla v_j|_2^3$$

for some positive constants $C_5$, $C_6$ and $C_7$. However, this contradicts $|\nabla v_j|_2 \to 0$. Therefore, we obtain that $\liminf_{j\to\infty} |v_j|_4^4 > 0$ in the case of $a < 0$ and $1 < p < 3$.

From the above results, we can prove Proposition 2.2 in the same way as the proof of Lemma 4.2 in [6]. □

From (2.4) and Proposition 2.2, we obtain a minimizer of (2.2), that is, there exists a $w \in M$ such that $m = S(w)$.

LEMMA 2.5. – *If $w \in M$ satisfies $m = S(w)$, then we have $S'(w) = 0$.*

We can prove Lemma 2.5 similarly to the proof of Lemma 4.3 in [6]. Moreover, since we have $P(\psi) = 0$ for any solution $\psi$ of (1.2), Proposition 2.1 follows from Proposition 2.2 and Lemma 2.5. Finally, we can prove Theorem 1.2 from Proposition 2.1 in the same way as the proof of Theorem 1.2 in [6].

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