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Propagation estimates for N-body Stark Hamiltonians

by

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ABSTRACT. – We prove some propagation estimates for N-body Stark Hamiltonians. These estimates imply that $p - Et = o(t)$ and $x - Et^2/2 = o(t^2)$ along the time evolution.

RÉSUMÉ. – Nous prouvons des estimations de propagation pour le Hamiltonians de Stark à N-corps.

Ces estimations entraînent que $p - Et = o(t)$ et $x - Et^2/2 = o(t^2)$ durant l'évolution temporelle.

1. INTRODUCTION

In this paper, we establish some propagation estimates for N-body Stark Hamiltonians.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^d$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. The N particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \le j < k \le N$. Then the total Hamiltonian for such a system is described by

$$\tilde{H} = \sum_{1\le j\le N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \mathcal{E} \cdot r_j \right\} + V,$$

where $\xi \cdot \eta = \sum_{j=1}^{d} \xi_j \eta_j$ for $\xi, \eta \in \mathbf{R}^d$ and the interaction V is given as the sum of the pair potentials

$$V = \sum_{1\le j<k\le N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center-of-mass frame. We introduce the metric $\langle r, \tilde{r}\rangle = \sum_{j=1}^{N} m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, ..., r_N)$ and $\tilde{r} = (\tilde{r}_1, ..., \tilde{r}_N) \in \mathbf{R}^{d\times N}$. We use the notation $|r| = \langle r, r\rangle^{1/2}$. Let X and X_{cm} be the configuration spaces equipped with the metric $\langle \cdot, \cdot \rangle$, which are defined by

$$X = \{r \in \mathbf{R}^{d\times N} : \sum_{1\le j\le N} m_j r_j = 0\},$$
$$X_{\mathrm{cm}} = \{r \in \mathbf{R}^{d\times N} : r_j = r_k \text{ for } 1 \le j < k \le N\}.$$

These two subspaces are mutually orthogonal. We denote by $\pi : \mathbf{R}^{d\times N} \to X$ and $\pi_{\mathrm{cm}} : \mathbf{R}^{d\times N} \to X_{\mathrm{cm}}$ the orthogonal projections onto X and X_{cm}, respectively. For $r \in \mathbf{R}^{d\times N}$, we write $x = \pi r$ and $x_{\mathrm{cm}} = \pi_{\mathrm{cm}} r$, respectively. Let $E \in X$ and $E_{\mathrm{cm}} \in X_{\mathrm{cm}}$ be defined by

$$E = \pi\left(\frac{e_1}{m_1}\mathcal{E}, ..., \frac{e_N}{m_N}\mathcal{E}\right), \qquad E_{\mathrm{cm}} = \pi_{\mathrm{cm}}\left(\frac{e_1}{m_1}\mathcal{E}, ..., \frac{e_N}{m_N}\mathcal{E}\right),$$

respectively. Then the total Hamiltonian $\tilde{H}$ is decomposed into $\tilde{H} = H \otimes \mathrm{Id} + \mathrm{Id} \otimes T_{\mathrm{cm}}$, where Id is the identity operator, H is defined by

$$H = -\frac{1}{2}\Delta - \langle E, x\rangle + V \quad \text{on } L^2(X),$$

T_{cm} denotes the free Hamiltonian $T_{\mathrm{cm}} = -\Delta_{\mathrm{cm}}/2 - \langle E_{\mathrm{cm}}, x_{\mathrm{cm}}\rangle$ acting on $L^2(X_{\mathrm{cm}})$, and Δ (resp. Δ_{cm}) is the Laplace-Beltrami operator on X (resp. X_{cm}). We assume that $|E| \ne 0$. This is equivalent to saying that $e_j/m_j \ne e_k/m_k$ for at least one pair (j, k). Then H is called an N-body Stark Hamiltonian in the center-of-mass frame.

A non-empty subset of the set $\{1, \ldots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1 \le j \le m} C_j = \{1, \ldots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \ldots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We denote by $\tilde{\mathcal{A}}$ the set of cluster decompositions and set $\mathcal{A} = \{a \in \tilde{\mathcal{A}} : \#(a) \ge 2\}$. We let $a, b \in \tilde{\mathcal{A}}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. We note that $a \subset a$ is regarded as a refinement of a itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \ne a$, this relation is denoted by $b \subsetneq a$. We denote by $\alpha = (j, k)$ the $(N-1)$-cluster decomposition $\{(j, k), (1), \ldots, (\hat{j}), \ldots, (\hat{k}), \ldots, (N)\}$.

Next we define the two subspaces X^a and X_a of X as

$$X^a = \{r \in X : \sum_{j \in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a\},$$
$$X_a = \{r \in X : r_j = r_k \text{ for each pair } \alpha = (j, k) \subset a\}.$$

We note that X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster interaction I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\frac{1}{2}\Delta - \langle E, x\rangle + V^a, \qquad V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes \mathrm{Id} + \mathrm{Id} \otimes T_a \quad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\frac{1}{2}\Delta^a - \langle E^a, x^a\rangle + V^a \quad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - \langle E_a, x_a \rangle \quad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a). By choosing the coordinates system of X, which is denoted by $x = (x^a, x_a)$, appropriately, we can write $\Delta^a = |\nabla^a|^2$ and $\Delta_a = |\nabla_a|^2$, where $\nabla^a = \partial_{x^a} = \partial/\partial x^a$ and $\nabla_a = \partial_{x_a} = \partial/\partial x_a$ and the gradients on X^a and X_a, respectively. We note that we denote by x^a (resp. x_a) a vector in X^a (resp. X_a) as well as the coordinates system of X^a (resp. X_a). We write $p = -i\nabla$, $p^a = -i\nabla^a$ and $p_a = -i\nabla_a$.

We now state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} : E^a = 0\}$ with respect to the relation $\subset$. As is easily seen, such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$$\text{(V.1)} \qquad \partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho'+|\beta|)}) \quad \alpha \subset c, \quad \rho' > 0,$$

$$\text{(V.2)} \qquad \partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|/2)}) \quad \alpha \not\subset c, \quad \rho > 0,$$

$$\text{(V.3)} \qquad \partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+\mu|\beta|)}) \quad \alpha \not\subset c, \quad \rho, \mu > 0$$

with $\rho + \mu > 1$.
Under this assumption, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. Throughout the whole exposition, the notations c, ρ', ρ and μ are used with the meanings described above. We make some remarks about potentials. For $\alpha \subset c$, if $\rho' > 1$ (resp. $0 < \rho' \leq 1$), V_α is called a short-range (resp. long-range) potential. For $\alpha \not\subset c$, if $\rho > 1/2$ (resp. $0 < \rho \leq 1/2$), V_α is called a short-range (resp. long-range) potential. If we consider the problem of the asymptotic completeness for long-range N-body Stark Hamiltonians, we should study the Dollard-type (resp. Graf-type) modified wave operators under the assumptions (V.1) and (V.2) [resp. (V.1) and (V.3)] (*cf.* [A], [AT1-2], [Gr2], [JO], [JY], [HMS2] and [W1-2]).

In this paper, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product on $L^2(X)$, respectively. Abusing notations, we also denote by $\|\cdot\|$ the norm of bounded operators on $L^2(X)$. We let $f \in C_0^\infty(\mathbf{R})$.

Propagation estimates are integral or pointwise estimates for large $|t|$ on $\|B_t\, e^{-itH} f(H)\,\psi\|$ for some time-dependent operator B_t. Integral estimates of the form

$$\int_1^\infty \frac{dt}{t}\, \|B_t\, e^{-itH} f(H)\,\psi\|^2 \leq C\,\|\psi\|^2 \tag{1.1}$$

play a basic role for the proof of the asymptotic completeness for N-body Hamiltonians (*see* [D], [Gr1], [SS] and [Z] in the case without constant electric fields, and [A] and [AT1-2] in the case with Stark effect). In these estimates, B_t can be a bounded pseudodifferential operator or a more general operator. We study here pointwise estimates of the form

$$\|B_t\, e^{-itH} f(H)\,\langle x\rangle^{-s}\| = O\left(t^k\right), \qquad t \geq 1 \tag{1.2}$$

for some $s > 0$ and $k \in \mathbf{R}$. For the case without constant electric fields, such estimates has been obtained by Skibsted [Sk] and Gérard [G]. In this paper, we obtain some pointwise propagation estimates for N-body Stark Hamiltonians.

We now formulate the results obtained in this paper. We use the following convention for smooth cut-off functions F with $0 \leq F \leq 1$, which is often used throughout the discussion below. For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \leq d) &= 1 \quad \text{for } s \leq d-\delta, && = 0 \quad \text{for } s \geq d,\\ F(s \geq d) &= 1 \quad \text{for } s \geq d+\delta, && = 0 \quad \text{for } s \leq d,\\ F(s = d) &= 1 \quad \text{for } |s-d| \leq \delta, && = 0 \quad \text{for } |s-d| \geq 2\delta \end{aligned}$$

and $F(d_1 \leq s \leq d_2) = F(s \geq d_1)\, F(s \leq d_2)$. The choice of $\delta > 0$ does not matter to the argument below, but we sometimes write F_δ for F when we want to clarify the dependence on $\delta > 0$.

THEOREM 1.1. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V.3). *Let $\varepsilon > 0$ and $s > s' > 0$. Then the following estimates hold for $t \geq 1$:*

$$\left\| F\left(\left|\frac{p}{t} - E\right| \geq \varepsilon\right) e^{-itH} f(H)\,\langle x\rangle^{-s/2} \right\| = O\left(t^{-s'}\right), \tag{1.3}$$

$$\left\| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \geq \varepsilon\right) e^{-itH} f(H)\,\langle x\rangle^{-s/2} \right\| = O\left(t^{-s'}\right). \tag{1.4}$$

This result implies that $|p-Et| = o(t)$ and $|x-Et^2/2| = o(t^2)$ along the time evolution. In particular, (1.4) implies that the particles asymptotically concentrate in any conical neighborhood of E, and this fact has played an important role for the proof of the asymptotic completeness for long-range N-body Stark Hamiltonians given by [A], [AT1-2] and [HMS2]. The following theorem is a refinement of the above properties. In particular, under the assumption that (V.2) with $0 < \rho \leq 1/2$ is satisfied, we obtain the difference between the growth order in t of the intercluster motion (x_c, p_c) and the total motion (x, p).

THEOREM 1.2. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V.3). *Let* $s_0(\rho) = \rho + 1/2$, $\chi_1(t) = t^{(1-2\rho)/2}$ *and* $\chi_2(t) = t^{(3-2\rho)/2}$ *if* (V.2) *with* $0 < \rho < 1/2$ *is satisfied,* $s_0(\rho) = 1$, $\chi_1(t) = (\log t)^{1/2}$ *and* $\chi_2(t) = t(\log t)^{1/2}$ *if* (V.2) *with* $\rho = 1/2$ *is satisfied, and* $s_0(\rho) = 1$, $\chi_1(t) = 1$ *and* $\chi_2(t) = t$ *if* (V.2) *with* $\rho > 1/2$ *or* (V.3) *is satisfied. Let* $0 < \varepsilon < \min_{\alpha \not\subset c} |E^\alpha|/2$. *Then the following estimates hold for* $t \geq 1$:

$$\left\| |p - Et| F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(\chi_1(t)), \quad s > s_0(\rho), \tag{1.5}$$

$$\left\| \left| x - \frac{E}{2}t^2 \right| F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(\chi_2(t)), \quad s > s_0(\rho). \tag{1.6}$$

Moreover, if V satisfies (V.1) *and* (V.2) *with* $0 < \rho \leq 1/2$, *the intercluster velocity* p_c *and position* x_c *satisfy the following estimates for* $t \geq 1$:

$$\left\| |p_c - Et| F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(1), \quad s > 1, \tag{1.7}$$

$$\left\| \left| x_c - \frac{E}{2}t^2 \right| F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t), \quad s > 1. \tag{1.8}$$

Comparing (1.6) with (1.8), we may conclude that $\| |x^c| F(|x/t^2 - E/2| \leq \varepsilon) e^{-itH} f(H) \langle x \rangle^{-s/2} \| = O(\chi_2(t))$. However, since the innercluster motion associated with the cluster decomposition c is not influenced by the constant electric field $\mathcal{E}$, we may expect that $\| |x^c| F(|x/t^2 - E/2| \leq \varepsilon) e^{-itH} f(H) \langle x \rangle^{-s/2} \| = O(t)$. If V satisfies (V.1) and (V.2) with $0 < \rho \leq 1/2$, (1.6) may be a weak estimate, but it seems to have the advantage that the form of $\chi_2(t)$ depends on ρ only and do not depend on ρ'. In this case, to obtain the stronger estimate than (1.6), we will need a more detailed analysis for the innercluster motion associated with the cluster decomposition c.

COROLLARY 1.3. – *Suppose that V satisfies* (V.1) *and* (V.2) *with* $0 < \rho \leq 1/2$. *Let $s_0(\rho)$, $\chi_1(t)$ and $\chi_2(t)$ be as in Theorem* 1.2. *Let* $0 < \tau < s_0(\rho)$. *Then the following estmates hold for $C > 0$ and $t \geq 1$:*

$$\| F(t^{-\tau} \chi_1(t)^{-1} |p - Et| \geq C) e^{-itH} f(H) \langle x \rangle^{-s/2} \| \\ = O(t^{-\tau}), \quad s > s_0(\rho), \tag{1.9}$$

$$\left\| F\left(t^{-\tau} \chi_2(t)^{-1} \left| x - \frac{E}{2} t^2 \right| \geq C\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| \\ = O(t^{-\tau}), \quad s > s_0(\rho). \tag{1.10}$$

We now consider only the case when c is N-cluster decomposition. The following theorem implies $pt - Et^2/2$ is a better approximation of x than $Et^2/2$ along the time evolution. The corresponding integral estimate has been obtained in [A], which may be used the (modified) wave operators.

THEOREM 1.4. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V3). *Let $s_0(\rho)$ and $\chi_1(t)$ be as in Theorem* 1.2. *Then the following estimates hold for any multi-index β, $s > 2|\beta| s_0(\rho)$ and $t \geq 1$:*

$$\left\| \left(x - pt + \frac{E}{2} t^2\right)^\beta F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| \\ = O(\chi_1(t)^{2|\beta|}). \tag{1.11}$$

The similar estimates may be obtained for the propagator $U_c(t)$ generated by the time-dependent Hamiltonian

$$H_c(t) = H_c + I_c(x) F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \leq \varepsilon\right).$$

Then we may apply these estimates, in particular (1.5) and (1.6), to prove the asymptotic completeness of N-body Stark Hamiltonians. We may also use these estimates to prove the existence of the time-delay operator for two-body Stark Hamiltonians, which was shown by Robert and Wang [RW].

Throughout this paper, we consider the case that V satisfies (V.1) and (V.2) with $0 < \rho < 1/2$ only for simplification. Other cases can be treated similarly.

The plan of this paper is as follows: In section 2, we collect the known results to be used in later sections. In sections 3 and 4, we prove Theorems 1.1-1.4.

2. KNOWN RESULTS

In this section, we collect the known results to be used in later sections. First, we recall the spectral properties of N-body Stark Hamiltonians, which has been studied by Herbst-Møller-Skibsted [HMS1]. We use the following notations throughout this paper. Let $\omega = E/|E|$ be the direction of E. We denote the coordinate $z \in \mathbf{R}$ by $z = \langle x, \omega \rangle$, so that H is written as $H = -\Delta/2 - |E|\, z + V$. Let $A = \langle \omega, p \rangle = -i\, \partial_z$. We should note that

$$\langle z \rangle^{-1/2} \nabla (H+i)^{-1}, \qquad \langle z \rangle^{-1} \nabla\nabla (H+i)^{-1} : L^2(X) \to L^2(X)$$

are bounded.

THEOREM 2.1. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V3). *Then*

(1) *H has no bound states.*

(2) *Let $R > 0$ be fixed and let $\Pi : X \to X$ be an orthogonal projection such that $\Pi E \neq 0$. Then*

$$\|F_\delta (H = \lambda)\, F(|\Pi\, x| \leq R)\| \to 0, \qquad \delta \to 0,$$

uniformly in $\lambda \in \mathbf{R}$. In particular, for $\alpha \not\subset c$,

$$\|F_\delta (H = \lambda)\, F(|x^\alpha| \leq R)\| \to 0, \qquad \delta \to 0.$$

(3) *Let* $0 < \sigma < |E| < \sigma'$. *Then one can take* $\delta > 0$ *so small (uniformly in* $\lambda \in \mathbf{R}$*) that*

$$F_\delta (H = \lambda)\, i\,[H,\, A]\, F_\delta (H = \lambda) \geq \sigma\, F_\delta (H = \lambda)^2, \tag{2.1}$$

$$F_\delta (H = \lambda)\, i\,[H,\, -A]\, F_\delta (H = \lambda) \geq -\sigma'\, F_\delta (H = \lambda)^2. \tag{2.2}$$

We should make some remarks. First, (2.1) has been obtained in [HMS1], and (2.2) may be obtained in the same way. Second, we should note that we can make σ and σ' very close to $|E|$. But we do not know whether this property holds or not in the case when the potentials have some singularities (also *see* [HMS1]).

Next we recall the almost analytic extension method due to Helffer and Sjöstrand [HeSj], which is useful in analyzing operators given by functions of self-adjoint operators. For two operators B_1 and B_2, we define

$$\mathrm{ad}^0_{B_1}(B_2) = B_2, \quad \mathrm{ad}^n_{B_1}(B_2) = [\mathrm{ad}^{n-1}_{B_1}(B_2),\, B_1], \qquad n \geq 1.$$

For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of functions $f \in C^\infty(\mathbf{R})$ such that

$$|f^{(k)}(s)| \leq C_k \langle s \rangle^{m-k}, \qquad k \geq 0.$$

If $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, then there exists $\tilde{f} \in C^\infty(\mathbf{C})$ such that $\tilde{f}(s) = f(s)$ for $s \in \mathbf{R}$, supp $\tilde{f}(\zeta) \subset \{\zeta \in \mathbf{C} : |\mathrm{Im}\,\zeta| \leq d(1 + |\mathrm{Re}\,\zeta|)\}$ for some $d > 0$ and

$$|\overline{\partial}_\zeta \tilde{f}(\zeta)| \leq C_M \langle \zeta \rangle^{m-1-M} |\mathrm{Im}\,\zeta|^M, \qquad M \geq 0.$$

Such a function $\tilde{f}(\zeta)$ is called an almost analytic extension of f. Let B be a self-adjoint operator. If $f \in \mathcal{F}^{-m}$ with $m > 0$, then $f(B)$ is represented by

$$f(B) = \frac{i}{2\pi} \int_{\mathbf{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)\, (B - \zeta)^{-1}\, d\zeta \wedge d\overline{\zeta}. \tag{2.3}$$

For $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, we have the following formulas of the asymptotic expansion of the commutator:

$$\begin{aligned}[B_1,\, f(B)] &= \sum_{n=1}^{M-1} \frac{(-1)^{n-1}}{n!}\, \mathrm{ad}^n_B(B_1)\, f^{(n)}(B) + R_M \\ &= \sum_{n=1}^{M-1} \frac{1}{n!}\, f^{(n)}(B)\, \mathrm{ad}^n_B(B_1) + R'_M,\end{aligned} \tag{2.4}$$

$$R_M = \frac{1}{2\pi i} \int_{\mathbf{C}} \overline{\partial}_\zeta \tilde{f}(\zeta)\, (B - \zeta)^{-1}\, \mathrm{ad}^M_B(B_1)\, (B - \zeta)^{-M}\, d\zeta \wedge d\overline{\zeta}, \tag{2.5}$$

$$R'_M = \frac{(-1)^{M+1}}{2\pi i} \int_{\mathbf{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(B-\zeta)^{-M}, \\ \times \operatorname{ad}_B^M(B_1)(B-\zeta)^{-1} d\zeta \wedge d\bar{\zeta}, \tag{2.6}$$

R_M is bounded if there exists k such that $m+k < M$ and $\operatorname{ad}_B^M(B_1)$ $(B+i)^{-k}$ is bounded. Similarly, R'_M is bounded if there exists k such that $m+k < M$ and $(B+i)^{-k} \operatorname{ad}_B^M(B_1)$ is bounded. For the proof, *see* [G]. We use the above formulas frequently.

Next, we state the maximal and minimal acceleration bounds, which has played a basic role in the proof of the asymptotic completeness for long-range N-body Stark Hamiltonians (*see* [A], [AT1-2] and [HMS2]). These estimates can be obtained by using Skibsted's abstract theory with slightly modification and Theorem 2.1 (3) (*see* [A]). To explain the results, we introduce some notations. We denote by D the Heisenberg derivative: $D\Phi(t) = \dfrac{\partial\Phi(t)}{\partial t} + i[H, \Phi(t)]$.

DEFINITION 2.2. – *For given $\beta, \alpha \geq 0$ and $\varepsilon > 0$, we take a function $\chi_{\alpha,\varepsilon}(y) = F(y \leq -\varepsilon)$ such that $\dfrac{d}{dy}\chi_{\alpha,\varepsilon}(y) \leq 0$ and $\alpha\chi_{\alpha,\varepsilon}(y) + y\dfrac{d}{dy}\chi_{\alpha,\varepsilon}(y) = \tilde{\chi}^2(y)$ for some $\tilde{\chi} \in C^\infty(\mathbf{R})$ with $\tilde{\chi}(y) \geq 0$, and define $g_{\beta,\alpha,\varepsilon}(y, t) = -t^{-\beta}(-y)^\alpha \chi_{\alpha,\varepsilon}(y/t)$ for $(y, t) \in \mathbf{R} \times \mathbf{R}^+$. We write $g^{(n)}_{\beta,\alpha,\varepsilon}(y, t) = \left(\dfrac{\partial}{\partial y}\right)^n g_{\beta,\alpha,\varepsilon}(y, t)$ for $n \in \mathbf{N}$.*

ASSUMPTION 2.3. – Let $n_0 \in \mathbf{N}$ with $n_0 \geq 2$, $t \geq 1$, $\beta_0 > 0$, $n_0 - 1/2 > \alpha_0 > 1$, f, $f_2 \in C_0^\infty(\mathbf{R})$, f_2 be real-valued with $f_2 f = f$, and $A(t)$, B be self-adjoint operators on $L^2(X)$. Assume that the operators $A(t)$ have a common domain $\mathcal{D}$, $\mathcal{D}(H) \cap \mathcal{D}$ is dense in $\mathcal{D}(H)$, $B \geq \mathrm{Id}$, and with $A = A(1)$ that $\langle A\rangle^{n_0/2} B^{-n_0/2} \in B(L^2(X))$. Assume moreover

(1) With $\operatorname{ad}^0_{A(t)}(H) = H$ and $1 \leq n \leq n_0$ the form $i^n \operatorname{ad}^n_{A(t)}(H) = i[i^{n-1}\operatorname{ad}^{n-1}_{A(t)}(H), A(t)]$ on $\mathcal{D}(H) \cap \mathcal{D}$ extends to a symmetric operator with domain $\mathcal{D}(H)$.

(2) If A is unbounded, $\sup_{|s|<1} \|H e^{iA(t)s}\psi\| < \infty$ for any $\psi \in \mathcal{D}(H)$ and $t \geq 1$.

(3) For any $t_1, t_2 \geq 1$, $A(t_1) - A(t_2)$ is bounded, and the derivative $\dfrac{d}{dt}A(t)$ exists in $B(L^2(X))$. For $n \leq n_0 - 1$ and $t \geq 1$, the form

$i^n \operatorname{ad}^n_{A(t)}\left(\frac{d}{dt} A(t)\right) = i\left[i^{n-1} \operatorname{ad}^{n-1}_{A(t)}\left(\frac{d}{dt} A(t)\right), A(t)\right]$ on $\mathcal{D}$ extends to a bounded self-adjoint operator on $L^2(X)$.

(4) For $n \leq n_0$, $\operatorname{ad}^n_{A(t)}(H)(H-i)^{-1}$ and $\operatorname{ad}^{n-1}_{A(t)}\left(\frac{d}{dt} A(t)\right)$ are continuous $B(L^2(X))$-valued functions of $t \geq 1$.

(5) (a) $\operatorname{ad}^{n_0-1}_{A(t)}\left(\frac{d}{dt} A(t)\right) = O(1)$ in $B(L^2(X))$ as $t \to \infty$. (b) For $n \leq n_0$, $\operatorname{ad}^{n-1}_{A(t)}\left(\frac{d}{dt} A(t)\right)(H-i)^{-1} = O(1)$ in $B(L^2(X))$ as $t \to \infty$. (c) For $n \leq n_0$, $\operatorname{ad}^n_{A(t)}(H)(H-i)^{-1} = O(1)$ in $B(L^2(X))$ as $t \to \infty$.

(6) $q(\beta_0, \alpha_0, \delta)$ for some $\delta > 0$: There exists a bounded operator $B_1(t)$ on $L^2(X)$ such that

$$f_2(H)\, DA(t)\, f_2(H) \geq B_1(t) + O(t^{-\delta}),$$

and, with $\alpha_0' = \max\{m \in \mathbf{N} | m < \alpha_0\}$, the following estimate holds for $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0)$: For $g_{\beta, \alpha, \varepsilon}(y, t)$, there exists $C > 0$ such that with $\zeta(t) = (g^{(1)}_{\beta, \alpha, \varepsilon}(A(t), t))^{1/2} e^{-itH} f(H) B^{-\alpha/2} \phi$,

$$\int_1^\infty |(\zeta(t), B_1(t)\zeta(t))| dt \leq C \|\phi\|^2, \qquad \phi \in L^2(X).$$

(7) $\alpha_0 + 3/2 < \beta_0 + n_0$ and $\alpha_0' + 3/2 < n_0$.

(8) For any $g \in C_0^\infty(\mathbf{R})$ and $1 \leq n \leq n_0$, the form $\operatorname{ad}^n_{A(t)}(g(H)) = [\operatorname{ad}^{n-1}_{A(t)}(g(H)), A(t)]$ on $\mathcal{D}$ extends to a bounded operator on $L^2(X)$. Moreover $(H+i)\operatorname{ad}^n_{A(t)}(g(H))$ and $\operatorname{ad}^n_{A(t)}(g(H))(H+i)$ are continuous $B(L^2(X))$-valued functions of $t \geq 1$, and $O(1)$ as $t \to \infty$.

(9) For $1 \leq n \leq n_0$, $\operatorname{ad}^n_A(e^{-itH} f(H))$ is a continuous $B(L^2(X))$-valued function of $t \in \mathbf{R}$.

(10) For any real s with $0 \leq s \leq n_0$, $\langle A\rangle^s e^{-itH} f(H) \langle A\rangle^{-s}$ is a continuous $B(L^2(X))$-valued function of $t \in \mathbf{R}$.

The following lemma is an extension of Corollary 2.5 of [Sk] for the N-body Stark Hamiltonian H. We recall that H is not bounded from below. For the proof, *see* [A].

LEMMA 2.4. – *Suppose that Assumption* 2.3 *is satisfied. Then for* $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0)$, *any* $\sigma > 0$ *and* $0 \leq \theta' \leq 1$,

$$\|(-g_{0, \alpha(1-\theta'), \varepsilon}(A(t), t))^{1/2} e^{-itH} f(H) B^{-\alpha/2}\| = O(t^{(\beta - \alpha\theta')/2}).$$

PROPOSITION 2.5. – *Suppose that* V *satisfies* (V.1), *and* (V.2) *or* (V.3). *Let* $f \in C_0^\infty(\mathbf{R})$ *and* $s \geq l \geq 0$. *Then there exists* $M > 0$ *such that the following estimate holds for* $t \geq 1$:

$$\left\| \langle x \rangle^{l/2} F\left(\frac{\langle x \rangle}{t^2} \geq M\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{l-s}). \tag{2.7}$$

Proof. – We follow the proof of Theorem 3.4 of [Sk]. We take any $n_0 \geq 4$ and set $\alpha_0 = n_0 - 2$, $\beta_0 = 1$, $A(t) = vt - \langle x \rangle^{1/2}$ and $B = \langle x \rangle^{1/2}$. It can be verified that Assumption 2.3 (1)-(5) and (7) are satisfied. We also see that (6) holds, using the fact that for any $f_2 \in C_0^\infty(\mathbf{R})$ there exists $v_0 > 0$ such that for $v \geq v_0$

$$f_2(H) DA(t) f_2(H) = f_2(H)(v - b_{-1/2}p - b_{-3/2}) f_2(H) \geq 0.$$

We now verify that Assumption 2.3 (8)-(10) are satisfied. We prove (8) only. (9) and (10) can be proved similarly. Integrating the derivatives, we have by induction

$$\begin{aligned} &\mathrm{ad}_{A(t)}^n (e^{ikH})(H+i)^{-n} \\ &= i\int_0^k dl \sum_{\substack{n_1+n_2+n_3=n-1 \\ n_j \geq 0}} \frac{(n-1)!}{n_1!\, n_2!\, n_3!} \mathrm{ad}_{A(t)}^{n_1}(e^{i(k-l)H}) \\ &\quad \times \mathrm{ad}_{A(t)}^{n_2+1}(H)\, \mathrm{ad}_{A(t)}^{n_3}(e^{ilH})(H+i)^{-n}. \end{aligned}$$

Using this formula and noting $[e^{ilH}, A(t)](H+i)^{-1} = O(\langle l \rangle) O(1)$, we have for $n \in \mathbf{N}$

$$\|\mathrm{ad}_{A(t)}^n (e^{ikH})(H+i)^{-n}\| \leq C\langle k \rangle^n, \qquad k \in \mathbf{R}.$$

Assumption 2.3 (8) follows from this estimate. Therefore, we have the proposition. □

LEMMA 2.6. – *Suppose that* V *satisfies* (V.1), *and* (V.2) *or* (V.3). *Let* $f \in C_0^\infty(\mathbf{R})$. *Let* σ *and* σ' *be as in Theorem* 2.1. *Then the following estimate holds for* $t \geq 1$:

$$\left\| F\left(\frac{A}{t} \leq \sigma\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s}), \qquad s > 0, \tag{2.8}$$

$$\begin{aligned} &\left\| \left(\sigma - \frac{A}{t}\right)^{1/2} F\left(\frac{A}{t} \leq \sigma\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| \\ &= O(t^{-s}), \qquad s > \frac{1}{2}, \end{aligned} \tag{2.9}$$

$$\left\| F\left(\frac{A}{t} \geq \sigma'\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s}), \qquad s > 0, \tag{2.10}$$

$$\left\| \left(\frac{A}{t} - \sigma'\right)^{1/2} F\left(\frac{A}{t} \geq \sigma'\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s}), \qquad s > \frac{1}{2}. \tag{2.11}$$

Proof. – The lemma can be proved as in the proof of Lemma 5.5 of [A] (also *see* Example 1 in section 3 of [Sk]). □

PROPOSITION 2.7. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V.3). *Let* $f \in C_0^\infty(\mathbf{R})$. *Let* σ *and* σ' *be as in Theorem* 2.1, *and* $0 < s' < s$. *Then the following estimate holds for* $t \geq 1$:

$$\left\| F\left(\frac{z}{t^2} \leq \frac{\sigma}{2}\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s'}), \tag{2.12}$$

$$\left\| F\left(\frac{z}{t^2} \leq \frac{\sigma'}{2}\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s'}), \tag{2.13}$$

Proof. – The proof is similar to that of Theorem 5.4 of [A] (also *see* Example 2 in section 3 of [Sk]). We prove (2.12) only. (2.13) can be proved similarly. Fix $0 < \sigma'' < \sigma$. For $\varepsilon'' > 0$, take $g_{0,1,\varepsilon''}(y, t)$ and let $A(t)$ be a multiplication by $g_{0,1,\varepsilon''}(-tM, t) = -tMF(-M \leq -\varepsilon'')$, where $M = M(x, t) = (\sigma''/2 - z/t^2)^{1/2}$. Let $A'(t) = A - \sigma t$. $B = \langle x \rangle^{(1+\kappa)/2}$ for $\kappa > 0$. We take any $n_0 \in \mathbf{N}$ and set $\alpha_0 = n_0 - 2$ and $\beta_0 = 1$. Assumption 2.3 (7) holds automatically.

It can be verified that Assumption 2.3 (1)-(5) are satisfied. As for (6), we have only to verify the condition $q(\beta_0, \alpha_0, 1)$ as follows: We compute

$$\begin{aligned} DA(t) = &\left(\frac{\partial}{\partial t} g_{0,1,\varepsilon''}\right)(-tM, t) + \frac{1}{2}(g_{0,1,\varepsilon''}^{(1)}(-tM, t))^{1/2} M^{-1/2} \\ &\times \left\{\frac{A'(t)}{t} + (\sigma - \sigma'') + O(t^{-1})\right\} M^{-1/2} (g_{0,1,\varepsilon''}^{(1)}(-tM, t))^{1/2}. \end{aligned}$$

The first term is non-negative. As for the second, we observe that

$$\frac{A'(t)}{t} + (\sigma - \sigma'') \geq \frac{A'(t)}{t} F^2\left(\frac{A'(t)}{t} \leq -\varepsilon'\right) \quad \text{with } \varepsilon' = \sigma - \sigma''.$$

Hence it suffices to show that for f, $f_2 \in C_0^\infty(\mathbf{R})$ such that $f f_2 = f$,

$$\left(\frac{-A'(t)}{t}\right)^{1/2} F\left(\frac{A'(t)}{t} \leq -\varepsilon'\right) M^{-1/2} (g_{0,1,\varepsilon''}^{(1)}(-tM, t))^{1/2}$$
$$\times f_2(H) (g_{\beta,\alpha,\varepsilon}^{(1)}(A(t), t))^{1/2} e^{-itH} f(H) B^{-\alpha/2} = O(t^{-(1+\alpha\kappa)/2}),$$

since $-(1+\alpha\kappa) < -1$ for $\alpha > 0$. This can be shown by commuting $G_1 = (-A'(t)/t)^{1/2} F(A'(t)/t \leq -\varepsilon')$ with $G_2 = M^{-1/2}(g_{0,1,\varepsilon''}^{(1)}(-tM, t))^{1/2} f_2(H)\,(g_{\beta,\alpha,\varepsilon}^{(1)}(A(t), t))^{1/2}$ as follows: We note that by virtue of Proposition 2.5, there exists $C > 0$ such that

$$F\left(\frac{A(t)}{t} \leq -C\right) e^{-itH} f(H) B^{-\alpha/2} = O(t^{-(1+\kappa)\alpha/2}).$$

Now we set $G_3 = G_2 F(A(t)/t \leq -C)$ and $G_4 = G_2 - G_3$. Then we have

$$G_1 G_3 e^{-itH} f(H) B^{-\alpha/2} = O(t^{-(1+\alpha\kappa)/2}). \tag{2.14}$$

Using the estimates $[A'(t)/t, M^{-1/2}(g_{0,1,\varepsilon''}^{(1)}(-tM, t))^{1/2}] = O(t^{-3})$, $[A'(t)/t, f_2(H)] = O(t^{-1})$ and $[A'(t)/t, (g_{\beta,\alpha,\varepsilon}^{(1)}(A(t), t))^{1/2}(1 - F(A(t)/t \leq -C))] = O(t^{(\alpha-7)/2})$, we have, by almost analytic extension method, that $[G_1, G_4] = O(t^{(\alpha-3)/2}) F(A'(t)/t \leq -\varepsilon''') + O(t^{-\infty})$ for some $0 < \varepsilon''' < \varepsilon'$. Hence, we have

$$[G_1, G_4] e^{-itH} f(H) B^{-\alpha/2} = O(t^{-1-(1+\alpha\kappa)/2}). \tag{2.15}$$

By Lemma 2.6, we have

$$G_4 G_1 e^{-itH} f(H) B^{-\alpha/2} = O(t^{-(1+\alpha\kappa)}). \tag{2.16}$$

By (2.14)-(2.16), we see that Assumption 2.3 (6) holds.

Assumption 2.3 (8)-(10) can be verified as in the proof of Proposition 2.5. Now we apply Lemma 2.4. Since $g_{\beta,\alpha,\varepsilon}(A(t), t) = g_{\beta,\alpha,\varepsilon}(-tM, t)$ for any $\varepsilon > 2\varepsilon''$, and $\varepsilon'' > 0$ is arbitrary, (2.12) is a consequence of Lemma 2.4 with $\theta' = 1$. □

In this paper, we call (2.7) and (2.12) the maximal and minimal acceleration bounds, respectively.

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. First we show that (1.3) holds. We write $\zeta(t) = F(|p/t - E| \geq \varepsilon) F(\langle x\rangle/t^2 \leq M)\, e^{-itH} f(H) \langle x\rangle^{-s/2} \psi$, where $M > 0$ is as in Proposition 2.5, and $\psi \in L^2(X)$. We compute for $k \in \mathbf{N}$

$$
\begin{aligned}
&\| |p - Et|^k \zeta(t) \|^2 \\
&\quad = (\zeta(t), \{(p - Et)^2\}^k \zeta(t)) \\
&\quad = (\zeta(t), \{2(H - V) + 2|E| z - 2|E| A t + |E|^2 t^2\}^k \zeta(t)) \\
&\quad = \Bigg(\zeta(t), \Bigg[2(H - V) + 2|E| \\
&\qquad \times \left\{ \left(z - \frac{|E|}{2} t^2 \right) - t(A - |E| t) \right\} \Bigg]^k \zeta(t) \Bigg). \qquad (3.1)
\end{aligned}
$$

Here we should note that (2.8), (2.10), (2.12) and (2.13) imply that for any $\varepsilon' > 0$ and $s > s' > 0$,

$$
\left\| F\left(\left| \frac{A}{t} - |E| \right| \geq \varepsilon' \right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s}), \qquad (3.2)
$$

$$
\left\| F\left(\left| \frac{z}{t^2} - \frac{|E|}{2} \right| \geq \varepsilon' \right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s'}). \qquad (3.3)
$$

We also note that $(H - V)(H + i)^{-1}$ is bounded. Using (3.2) and (3.3), and using the almost analytic extension method to control the commutators, we estimate the both sides of (3.1) as follows:

$$
\varepsilon^{2k} t^{2k} \|\zeta(t)\|^2 \leq C t^{2k - 2s'} \|\psi\|^2 + C' \varepsilon'^k t^{2k} \|\zeta(t)\|^2, \quad 0 < s' < s \leq k.
$$

Hence, taking $\varepsilon' > 0$ so small that $C' \varepsilon'^k < \varepsilon^{2k}$, we have

$$
\|\zeta(t)\| \leq C t^{-s'} \|\psi\|, \qquad 0 < s' < s \leq k.
$$

Combining this estimate with the maximal acceleration bound (2.7), we obtain (1.3).

Next we prove (1.4). We need the following proposition.

PROPOSITION 3.1. – *Let $\varepsilon > 0$ and $s > s' > 0$. Then there exists $\varepsilon' > 0$ such that the following estimate holds for $t \geq 1$:*

$$
\left\| F\left(\left| \frac{x}{t^2} - \frac{E}{2} \right| \geq \varepsilon \right) F\left(\left| \frac{p}{t} - E \right| \leq \varepsilon' \right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\| = O(t^{-s'}).
$$

Proof. – We should note that by virtue of the maximal acceleration bound (2.7), there exists $M > 0$ such that the following estimate holds:

$$\left\| F\left(\left|\frac{x}{t^2}-\frac{E}{2}\right| \geq M\right) e^{-itH} f(H) \langle x\rangle^{-s/2} \right\| = O(t^{-s'}). \tag{3.4}$$

Here we write $\zeta(t) = F(\varepsilon \leq |x/t^2 - E/2| \leq M)\, F(|p/t - E| \leq \varepsilon')\, e^{-itH} f(H) \langle x\rangle^{-s/2} \psi$ with $\psi \in L^2(X)$, where $F'(\varepsilon \leq s \leq M) \geq 0$ for $\varepsilon \leq s \leq \varepsilon + \delta$. We compute for $k \in \mathbf{N}$

$$\left\| \left| x - \frac{E}{2} t^2 \right|^k \zeta(t) \right\|^2 = \left(\zeta(t), \left\{\left(x - \frac{E}{2}t^2\right)^2\right\}^k \zeta(t)\right).$$

We note that the Heisenberg derivatives of $x - Et^2/2$, $F(\varepsilon \leq |x/t^2 - E/2| \leq M)$ and $F(|p/t - E| \leq \varepsilon')$ are as follows:

$$D\left(x - \frac{E}{2}t^2\right) = p - Et, \tag{3.5}$$

$$\begin{aligned} D\left(F\left(\varepsilon \leq \left|\frac{x}{t^2}-\frac{E}{2}\right| \leq M\right)\right) &= F'\left(\varepsilon \leq \left|\frac{x}{t^2}-\frac{E}{2}\right| \leq M\right) \left|\frac{x}{t^2}-\frac{E}{2}\right|^{-1} \\ &\quad \times \left\langle \left(\frac{x}{t^2}-\frac{E}{2}\right), \left\{-2t^{-3}\left(x-\frac{E}{2}t^2\right)\right.\right. \\ &\qquad \left.\left. + t^{-2}(p - Et)\right\}\right\rangle + O(t^{-4}), \end{aligned} \tag{3.6}$$

$$D\left(F\left(\left|\frac{p}{t} - E\right| \leq \varepsilon'\right)\right) = O(t^{-1})\, F\left(\left|\frac{p}{t} - E\right| \geq \varepsilon''\right) + O(t^{-\infty}), \tag{3.7}$$

for some $0 < \varepsilon'' < \varepsilon'$. Here we used the almost analytic extension method and pseudodifferential calculus to obtain (3.7). Noting that $F'(\varepsilon \leq s \leq M) \geq 0$ for $\varepsilon \leq s \leq \varepsilon + \delta$, and using (1.3) and (3.4)-(3.7), $\dfrac{d}{dt}(\||x - Et^2/2|^k \zeta(t)\|^2)$ can be estimated as follows: For $0 < s' < s \leq 2(k-1)$,

$$\frac{d}{dt}\left(\left\| \left| x - \frac{E}{2}t^2 \right|^k \zeta(t) \right\|^2\right) \leq O(t^{4k-1-2s'}) \|\psi\|^2 + C\varepsilon' t^{4k-1} \|\zeta(t)\|^2.$$

Hence we have for $0 < s' < s \leq 2(k-1)$,

$$\varepsilon^{2k} t^{4k} \|\zeta(t)\|^2 \leq \left\| \left| x - \frac{E}{2} t^2 \right|^k \zeta(t) \right\|^2$$
$$\leq O(t^{4k-2s'}) \|\psi\|^2 + \int_1^t C \varepsilon' \tau^{4k-1} \|\zeta(\tau)\|^2 d\tau.$$

Taking $\varepsilon' > 0$ so small that $C\varepsilon' \varepsilon^{-2k} \leq 1$, we obtain by virtue of Gronwall's lemma,

$$t^{4k} \|\zeta(t)\|^2 \leq O(t^{4k-2s'}) \|\psi\|^2, \qquad 0 < s' < s \leq 2(k-1).$$

This implies the proposition. □

Combining (1.3) and this proposition, we have (1.4) immediately. Therefore the proof of Theorem 1.1 is completed.

4. PROOF OF THEOREMS 1.2-2.4

First, we prove Theorem 1.2. We begin with showing (1.5). Recalling

$$|p - Et|^2 = 2(H - V) + 2|E| \left(z - At + \frac{|E|}{2} t^2 \right),$$

we need the following proposition.

PROPOSITION 4.1. – *Set*

$$\zeta(t) = F(|x/t^2 - E/2| \leq \varepsilon) e^{-itH} f(H) \langle x \rangle^{-s/2} \psi \qquad \textit{for} \quad \psi \in L^2(X).$$

Then the following estimate holds $t \geq 1$:

$$\left| \left(\zeta(t), \left(z - At + \frac{|E|}{2} t^2 \right) \zeta(t) \right) \right| = O(t^{1-2\rho}) \|\psi\|^2, \quad s > \rho + \frac{1}{2}. \quad (4.1)$$

Proof. – The Heisenberg derivative of $z - At + |E| t^2/2$ is as follows:

$$D \left(z - At + \frac{|E|}{2} t^2 \right) = t \partial_z V.$$

Hence, by assumption, we see that $D(z - At + |E|\,t^2/2) = O(t^{-2\rho})$ on $\operatorname{supp} F(|x/t^2 - E/2| \le \varepsilon)$. We note that, by using the almost analytic extension method, we have for some $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned} D\left(F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \le \varepsilon\right)\right)(H+i)^{-1} \\ = O(t^{-1})\,F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \ge \varepsilon'\right) + O(t^{-\infty}). \end{aligned} \tag{4.2}$$

Noting these facts, by straightforward computation, we have

$$\left(\zeta(t), \left(z - At + \frac{|E|}{2}t^2\right)\zeta(t)\right) = \{O(t^{1-2\rho}) + O(t^{\max(2-2s',0)})\}\,\|\psi\|^2.$$

This implies the proposition. □

Now we prove (1.5). Noting that $(H - V)(H + i)^{-1}$ is bounded, we have by virtue of Proposition 4.1

$$\begin{aligned} \||p - Et|\zeta(t)\|^2 &= \left(\zeta(t), 2(H - V) + 2|E|\left(z - At + \frac{|E|}{2}t^2\right)\zeta(t)\right) \\ &\le O(t^{1-2\rho})\,\|\psi\|^2. \end{aligned}$$

This implies that (1.5) holds.

Next we prove (1.6). Noting (3.5) and (4.2), we estimate

$$\left\| \frac{d}{dt}(e^{itH}(x - Et^2/2)\,F(|x/t^2 - E/2| \le \varepsilon)\,e^{-itH} f(H)\langle x\rangle^{-s/2}) \right\|$$

as follows:

$$\begin{aligned} &\left\| \frac{d}{dt}\left(e^{itH}\left(x - \frac{E}{2}t^2\right)F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \le \varepsilon\right)e^{-itH} f(H)\langle x\rangle^{-s/2}\right) \right\| \\ &\le O(t^{(1-2\rho)/2}) + O(t^{\max(1-s',0)}). \end{aligned}$$

By integration in t, we have (1.6).

Next we prove (1.7) and (1.8). First we note that the Heisenberg derivatives of $p_c - Et$ and $x_c - Et^2/2$ are $-\nabla_c V$ and $p_c - Et$, respectively, and that, by virtue of the definition of the cluster decomposition c, on $\operatorname{supp} F(|x/t^2 - E/2| \le \varepsilon)$,

$$D(p_c - Et) = O(t^{-(1+2\rho)}). \tag{4.3}$$

Then, in the way similar to that in the proof of (1.6), we have

$$\left\| |p_c - Et| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\|$$
$$= O(1) + O\left(t^{\max(1-s', 0)}\right).$$

This implies that (1.7) holds. Similarly, we also have by (1.7)

$$\left\| \left| x_c - \frac{E}{2} t^2 \right| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\|$$
$$= O(t) + O\left(t^{\max(2-s', 0)}\right).$$

This implies that (1.8) holds. Hence, the proof of Theorem 1.2 is completed. Noting that (1.4)-(1-6) hold, Corollary 1.3 is an immediate consequence of Theorem 1.2.

Finally, we prove Theorem 1.4. First, we should note that the Heisenberg derivative of $x - pt + Et^2/2$ is $t \nabla V$, and that, since the cluster decomposition c is N-cluster decomposition, that is $X_c = X$, we have on supp $F(|x/t^2 - E/2| \leq \varepsilon)$

$$D\left(x - pt + \frac{E}{2} t^2\right) = O\left(t^{-2\rho}\right). \tag{4.4}$$

Using the way similar to that in the proof of Theorem 1.2, we prove the theorem by induction in $|\beta|$. When $|\beta| = 1$, noting (4.2) and (4.4), we have

$$\left\| \left(x - pt + \frac{E}{2} t^2\right)^\beta F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\|$$
$$= O\left(t^{1-2\rho}\right) + O\left(t^{\max(2-s', 0)}\right).$$

This implies that (1.11) holds for $|\beta| = 1$. Next we assume that (1.11) holds for any β with $|\beta| \leq k$. Here we should note that for $l \in \mathbf{N}$

$$i\left[t^l (\nabla)^l V, \left(x - pt + \frac{E}{2} t^2\right)\right] = t^{l+1} (\nabla)^{l+1} V,$$

which is $O\left(t^{-2\rho}\right)$ on supp $F(|x/t^2 - E/2| \leq \varepsilon)$. From this fact and (4.2), by the assumption of induction, we have for any β with $|\beta| = k + 1$

$$\left\| \left(x - pt + \frac{E}{2} t^2\right)^\beta F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s/2} \right\|$$
$$= O\left(t^{|\beta|(1-2\rho)}\right) + O\left(t^{\max(2|\beta|-s', 0)}\right).$$

This implies that (1.11) holds for any β with $|\beta| = k + 1$. Hence, we finish the proof of Theorem 1.4.

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