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## **On invariant measures for some infinite-dimensional dynamical systems**

by

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**ABSTRACT.** – We consider an abstract infinite-dimensional dynamical system recently introduced by M. Grillakis, J. Shatah and W. A. Strauss. A lot of nonlinear evolution equations of the mathematical physics may be represented in that form. The aim of the paper is the construction of an invariant measure for this system. Sufficient conditions for the boundedness of the constructed measure are presented. It allows us to apply the Poincaré recurrence theorem that explains the well-known Fermi-Pasta-Ulam phenomenon. The result is used to investigate concrete physical problems.

**RÉSUMÉ.** – Nous considérons un système dynamique abstrait de dimension infinie, du type de ceux récemment introduits par M. Grillakis, J. Shatah et W. A. Strauss. Beaucoup d'équations d'évolution non linéaires peuvent être présentées sous cette forme. L'objet de cet article est de construire une mesure invariante pour un tel système. Nous donnons des conditions suffisantes pour que cette mesure soit bornée. Celles-ci nous permettent d'appliquer le théorème de récurrence de Poincaré qui permet alors d'expliquer le phénomène de Fermi-Pasta-Ulam. Ce résultat est utilisé pour examiner quelques problèmes physiques concrets.

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## 1. INTRODUCTION

Recently several papers have been published on invariant measures for dynamical systems (DS) generated by nonlinear partial differential equations ([1]-[5]). In paper [1] that measure is constructed for the periodic problem for a nonlinear Klein-Gordon equation and in paper [2] a similar construction is made for a certain physical system. Unfortunately, in paper [1] some important steps of the proof are omitted. In the author's paper [3] the invariant measure is constructed for a nonlinear Schrödinger equation (NSE) under some severe constraints on the nonlinearity. Partially these difficulties are removed in paper [4] where the power nonlinearities are admissible. The next author's paper [5] contains a simpler approach to the same problem. A nonlinear wave equation is considered. However, as it is noted, one can easily apply this technique for the investigation of NSE.

Invariant measures play an important role in the theory of DS. It is well known that the whole ergodic theory is based on this concept. On the other hand, they are necessary in various physical considerations. In paper [6] they are used for constructing statistical mechanics corresponding to the NSE (however the proof of the invariance is not presented). Similar considerations are made in papers ([7]-[10]) where the Kubo-Martin-Schwinger states are constructed but without the proof of the invariance, too.

The first point which directed the author to this investigation was the so-called Fermi-Pasta-Ulam phenomenon. For an evolution equation with arbitrary initial data  $u_0$ , it implies the existence of a sequence of values of time  $t_n \rightarrow +\infty$  such that the corresponding values of the solution at these moments of time are close  $u_0$  (see [11], [12], for example). In the mathematical theory of dynamical systems a similar property of a trajectory is called the stability according to Poisson. Using a bounded invariant measure one can apply the Poincaré recurrence theorem which explains this phenomenon. It is essential to note that the results of the paper are in agreement with numerical simulations (see Remark 9).

In the present paper we consider an abstract Hamiltonian system introduced in paper [13] for the investigation of the soliton stability; a wide class of the "soliton" equations admits such a representation (see Section 5).

Finally, we note that the present paper contains complete proofs of the results of paper [3].

2. NOTATION. MAIN RESULTS

In what follows, we denote positive constants by  $C, C_1, C_2, C', C'', \dots$ . Let  $Y \subset X$  be real Hilbert spaces with the scalar products  $(\cdot, \cdot)_Y$  and  $(\cdot, \cdot)_X$  and the norms  $\|g\|_Y = (g, g)_Y^{\frac{1}{2}}$  and  $\|g\|_X = (g, g)_X^{\frac{1}{2}}$  respectively, satisfying the condition

$$\|g\|_X \leq C\|g\|_Y$$

with  $C > 0$  independent of  $g \in Y$ . Let  $Y$  be a dense set in  $X$ . Let  $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$  be a sequence of finite-dimensional subspaces of  $Y$ ,  $\dim X_n = d_n < \infty$ , and let  $\bigcup_n X_n$  be a dense set in  $Y$ . Let  $H$  be a  $C^1$ -functional on  $Y$  and a  $C^2$ -functional on  $X_n$  for any  $n$  with real values and let  $J : X^* \rightarrow X$  be a (generally unbounded) linear operator defined on a dense set  $D \subset X^*$  satisfying

$$g(Jh) = -h(Jg)$$

for any  $g, h \in D$  where  $g(h)$  is the value of  $g \in X^*$  of  $h \in X$ . It is clear that any  $g \in X^*$  belongs to  $Y^*$  (here  $X^*$  and  $Y^*$  are the dual spaces to  $X$  and  $Y$ , respectively).

Consider the problem

$$\dot{u}(t) = JH'(u(t)), \quad t \in R, \tag{1}$$

$$u(t_0) = \phi \in X. \tag{2}$$

Here  $t_0 \in R$ , the dot means the derivative with respect to  $t \in R$  and  $u(t)$  is the unknown function with values in  $X$ . In addition we consider the sequence of finite-dimensional problems

$$\dot{u}^n(t) = P_n J P_n^* H'(P_n u^n(t)), \quad t \in R, \tag{3}$$

$$u^n(t_0) = P_n \phi \tag{4}$$

where  $P_n$  is the orthogonal projector onto  $X_n$  in  $X$  and  $P_n^*$  is the adjoint operator to  $P_n$  in  $X$ .

It is obvious that  $X_n^* = P_n^* X^*$  is the dual space to  $X_n$ . We assume that  $J$  is defined on any  $X_n^*$ .

*Remark 1.* – As it is well known, the norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  are equivalent on any  $X_n$ .

We denote  $I = [t_0 - T, t_0 + T]$  for any  $T > 0, t_0 \in R$  and  $C(I; B)$  the space of continuous bounded functions from  $I$  into  $B$  with the norm  $\|g(t)\|_{C(I; B)} = \sup_{t \in I} \|g(t)\|_B$  where  $B$  is an arbitrary Banach space with

the norm  $\| \cdot \|_B$ . By the above assumptions, the operator from the right-hand side of (3) is of the class  $C^1$  as the map from  $X_n$  into  $X_n$ . Hence, for any  $\phi \in X$  there exists  $T > 0$  such that there exists the unique solution  $u^n(t)$  of the problem (3)-(4) of the class  $C(I; X_n)$ .

*Remark 2.* – In particular, the above solution  $u^n(t)$  belongs to  $C(I; X)$ .

*Assumption 1.* – Let for  $\phi \in X$  the solution  $u^n(t)$  be global in time.

**DEFINITION 1.** – Let  $\phi \in X$  be fixed and let there exist  $T > 0$  and  $u(t) \in C(I; X)$  such that there exists a sequence  $u^n(t)$  converging to  $u(t)$  in  $C(I; X)$ . Then, we call  $u(t)$  the solution of the problem (1)-(2).

*Assumption 2.* – Let for any  $\phi \in X$  there exists a unique global in  $t$  solution  $u(t)$  of the problem (1)-(2).

*Assumption 3.* – Let for any  $t_0 \in R$ ,  $\varepsilon > 0$ ,  $T > 0$  there exists  $\delta > 0$  such that

$$\|u_1^n(t) - u_2^n(t)\|_X < \varepsilon \quad (n = 1, 2, 3, \dots)$$

for any two solutions  $u_1^n$  and  $u_2^n$  of equation (3) such that

$$\|u_1^n(t_0) - u_2^n(t_0)\|_X < \delta$$

and for any  $t \in I$ .

**COROLLARY.** – For any  $t_0 \in R$ ,  $\varepsilon > 0$ ,  $T > 0$  there exists  $\delta > 0$  such that

$$\|u_1(t) - u_2(t)\|_X < \varepsilon$$

if

$$\|u_1(t_0) - u_2(t_0)\|_X < \delta$$

for all  $t \in I$  and for any two solutions  $u_1$  and  $u_2$  of the problem (1)-(2).

Now we briefly remind the general construction of a Gaussian measure on a Hilbert space (for details, see [14]-[16]). For a Hilbert space consider  $X$ . Let  $\{e_k\}$  be the orthonormal basis in  $X$  which consists of eigenvectors of some operator  $S = S^* > 0$  with corresponding eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ . We call a set  $M \subset X$  the cylindrical set iff

$$M = \{x \in X \mid [(x, e_{j_1})_X, \dots, (x, e_{j_m})_X] \in F\}$$

for some Borel set  $F \subset R^m$ , some positive integer  $m$  and  $j_i \neq j_l$  if  $i \neq l$ . We define the measure  $w$  as follows:

$$w(M) = (2\pi)^{-\frac{m}{2}} \prod_{l=1}^m \lambda_{j_l}^{-\frac{1}{2}} \int_F e^{-\frac{1}{2} \sum_{l=1}^m \lambda_{j_l} y_l^2} dy$$

where  $y = (y_1, \dots, y_m) \in R^m$  and  $dy$  is the Lebesgue measure in  $R^m$ . One can easily verify that the class  $\mathcal{A}$  of all cylindrical sets is an algebra on which the function  $w$  is additive. The function  $w$  is called the centered Gaussian measure on  $X$  with the correlation operator  $S^{-1}$ . The basic result is the following.

*Statement 1. – The measure  $w$  is countably additive on the algebra  $\mathcal{A}$  iff  $S^{-1}$  is an operator of trace class, i.e. iff  $\sum_{k=1}^{+\infty} \lambda_k^{-1} < +\infty$ .*

If the measure  $w$  is countably additive on  $\mathcal{A}$ , then it has a unique extension to the minimal sigma-algebra  $\mathcal{M}$  containing  $\mathcal{A}$ . In fact,  $\mathcal{M}$  is the Borel sigma-algebra of  $X$  (see [14]-[16]).

The following result is well-known (see, for example, [14]-[16]).

*Statement 2. – Let a centered Gaussian measure  $w$  be countably additive on  $X$ . Let  $B_r(a) = \{x \in X \mid \|x - a\|_X \leq r\}$ . Then,  $w(B_r(a)) > 0$  for any  $a \in X$ ,  $r > 0$ .*

We present the *proof* for convenience of the reader.

Obviously,  $B_r(a) = \bigcap_{m=1}^{+\infty} M_m$  where  $M_m = \{x \in X \mid [(x - a, e_1)_X]^2 + \dots + [(x - a, e_m)_X]^2 \leq r^2\}$ . Thus,  $w(B_r(a)) = \lim_{m \rightarrow \infty} w(M_m)$ . Fix  $m_0 > 0$  such that  $\sum_{k=m_0+1}^{+\infty} \lambda_k^{-1} < \frac{r^2}{16}$  and  $\left(\sum_{k=m_0+1}^{+\infty} a_k^2\right)^{\frac{1}{2}} < \frac{r}{4}$  where  $a_k = (a, e_k)_X$ . Taking  $m \geq m_0 + 1$  we obtain

$$\begin{aligned} w(M_m) &= (2\pi)^{-\frac{m}{2}} \prod_{k=1}^m \lambda_k^{\frac{1}{2}} \int_B e^{-\frac{1}{2} \sum_{k=1}^m \lambda_k y_k^2} dy_1 \dots dy_m \\ &\geq C(2\pi)^{-\frac{m-m_0}{2}} \prod_{k=m_0+1}^m \lambda_k^{\frac{1}{2}} \int_{B_1} e^{-\frac{1}{2} \sum_{k=m_0+1}^m \lambda_k z_k^2} dz_{m_0+1} \dots dz_m \end{aligned}$$

where  $C = \text{const} > 0$ ,

$$B = \{y = (y_1, \dots, y_m) \in R^m \mid (y_1 - a_1)^2 + \dots + (y_m - a_m)^2 \leq r^2\}$$

and

$$B_1 = \left\{ z = (z_{m_0+1}, \dots, z_m) \in R^{m-m_0} \mid (z_{m_0+1} - a_{m_0+1})^2 + \dots + (z_m - a_m)^2 \leq \frac{r^2}{4} \right\}$$

$$\left( \text{since } \left\{ y \in R^m \mid (y_1 - a_1)^2 + \dots + (y_{m_0} - a_{m_0})^2 \leq \frac{r^2}{4} \right\} \cap \left\{ y \in R^m \mid (y_{m_0+1} - a_{m_0+1})^2 + \dots + (y_m - a_m)^2 \leq \frac{r^2}{4} \right\} \subset B \right).$$

Further, since according to the choice of  $m_0$  the embedding

$$\left\{ z = (z_{m_0+1}, \dots, z_m) \mid z_{m_0+1}^2 + \dots + z_m^2 \leq \frac{r^2}{16} \right\} \subset B_1$$

is valid, we obtain:

$$w(M_m) \geq C(2\pi)^{-\frac{m-m_0}{2}} \prod_{k=m_0+1}^m \lambda_k^{\frac{1}{2}} \int_{z_{m_0+1}^2 + \dots + z_m^2 \leq \frac{r^2}{16}} e^{-\frac{1}{2} \sum_{k=m_0+1}^m \lambda_k z_k^2} dz_{m_0+1} \dots dz_m.$$

Now we use the following well-known inequality. Let  $T = T^* > 0$  be an operator on  $R^n$  for some  $n = 1, 2, 3, \dots$  and let  $w_n$  be a centered Gaussian measure on  $R^n$  with a correlation operator  $S^{-1} > 0$ . Then,

$$w_n((Tx, x) \geq 1) \leq \text{Tr}(TS^{-1})$$

(for the proof see [14], Lemma II.1.1). According to this result and the above inequality one has:

$$w(M_m) \geq C \left( 1 - \frac{16}{r^2} \sum_{k=m_0+1}^{+\infty} \lambda_k^{-1} \right) \geq C_1$$

where  $C_1$  is a positive constant. Thus, Statement 2 is proved.

*Assumption 4.* – Let  $H(u) = \frac{1}{2}(Su, u)_X + g(u)$  where  $S^* = S > 0$  is an (unbounded) operator on  $X$  mapping  $X_n$  into  $X_n$  ( $n = 1, 2, 3, \dots$ ) and  $g(u)$  is a continuous real functional on  $X$  bounded on any bounded

set  $\Omega \subset X$ . Let  $S^{-1}$  be an operator of trace class on  $X$ . Let  $H(u)$  be a  $C^2$ -functional on  $X_n$  for any  $n = 1, 2, 3, \dots$

We use the concept of DS. There are various definitions for it. We introduce the following.

DEFINITION 2. – Let  $M$  be a metric space and let  $f(x, t)$  be a homeomorphism from  $M$  into  $M$  for any fixed  $t \in R$  such that

- (a)  $f(x, 0) = x$ ;
- (b)  $f(f(x, \tau), t) = f(x, t + \tau)$  for any  $x \in M$  and  $t, \tau \in R$ .

The function  $f$  is called DS on the phase space  $M$ . Let  $\mu$  be a measure defined on the Borel sigma-algebra of  $M$ . It is called the invariant measure if  $\mu(f(\Omega, t)) = \mu(f(\Omega, \tau))$  for any Borel set  $\Omega \subset M$  and for any  $t, \tau \in R$ .

DEFINITION 3. – By  $f(\phi, t)$  we denote the function from  $X$  into  $X$  mapping  $\phi$  into  $u(t + t_0)$  where  $u(t)$  is the solution of the problem (1)-(2). By analogy, let  $f_n(\phi, t)$  be the function from  $X$  into  $X_n$  mapping  $\phi \in X$  into  $u^n(t + t_0)$  where  $u^n(t)$  is the solution of the problem (3)-(4). It is clear that  $f$  and  $f_n$  are DS on the phase spaces  $X$  and  $X_n$ , respectively.

The first main result of the present paper is the following:

THEOREM 1. – Let Assumptions 1-4 be valid and let  $\mu$  be a Borel measure on  $X$  defined for any Borel set  $\Omega \subset X$  by the rule

$$\mu(\Omega) = \int_{\Omega} e^{-g(u)} w(du)$$

where  $w$  is the centered Gaussian measure corresponding to the correlation operator  $S^{-1}$ . Then,  $\mu$  is an invariant measure for DS  $f$ .

DEFINITION 4. – We call the measure  $\mu$  bounded if  $\mu(X) < +\infty$ .

Remark 3. – Since we do not claim the boundedness of the functional  $g$ , generally the measure  $\mu$  is not bounded. It is not difficult to formulate the conditions for the boundedness of  $\mu$ . For example, the measure  $\mu$  is bounded if  $g$  is bounded from below in addition to the above assumptions.

As we will see in Section 5, Assumption 3 highly reduces the class of nonlinearities of the admissible partial differential equations. So, we present one more result which allows us to prove the invariance of the measure  $\mu$  for a wider class of nonlinearities.

Let  $H_N(u) = \frac{1}{2}(Su, u)_X + g_N(u)$  ( $N = 1, 2, 3, \dots$ ). Consider the sequence of the problems

$$\dot{u}_N(t) = JH'_N(u(t)), \quad t \in R, \tag{5}$$



$$u(t_0) = \phi \in X \quad (6)$$

in place of the problem (1)-(2). Let for any  $N$  Assumptions 1-4 be valid for the problem (5)-(6) with  $H = H_N$ . We denote solutions of this problem by  $u_N(t)$ .

*Assumption 5.* – Let  $G(u)$  be a real functional on  $X$  such that  $e^{-g_N(u)}$  converges to  $G(u)$  as  $N \rightarrow \infty$  non-increasingly or non-decreasingly simultaneously for almost all points  $u \in X$  in the sense of the measure  $w$ . Let  $u_N(t + t_0)$  tend to some  $f(\phi, t) \in X$  as  $N \rightarrow +\infty$  for any  $\phi \in X$  and  $t \in R$ . We suppose that the function  $f(u, t)$  is continuous on  $X$  for any fixed  $t$ . Thus, we have DS  $f$  on the phase space  $X$ , again.

Let  $\Omega \subset X$  be a Borel set. We set

$$\nu(\Omega) = \int_{\Omega} G(u) w(du).$$

**THEOREM 2.** – Under Assumption 5 the measure  $\nu$  is invariant for DS  $f$ .

*Remark 4.* – In each situation, one should verify that the measure  $\nu$  is non-trivial, i.e. that there exist Borel sets  $A$  satisfying  $\nu(A) \neq 0$ . In particular, in the case of NSE that proof was made in paper [4].

### 3. PROOF OF THEOREM 1

Since  $S$  maps  $X_n$  into  $X_n$ , there exists the orthonormal basis  $\{e_k\}$  of the space  $X$  consisting of the eigenvectors of the operator  $S$  with corresponding eigenvalues  $\{\lambda_k\}$  ( $k = 1, 2, 3, \dots$ ) such that  $e_1, \dots, e_{d_n}$  is the basis of  $X_n$  for any  $n$ . Let  $u^n(t) = \sum_{k=1}^{d_n} a_k(t) e_k$ ,  $h(\mathbf{a}) = H\left(\sum_{k=1}^{d_n} a_k e_k\right)$  where  $\mathbf{a} = (a_1, \dots, a_{d_n}) \in R^{d_n}$  and let  $J_n$  be the matrix of the operator  $P_n J P_n^*$  from  $X_n^*$  into  $X_n$  in the bases  $\{e_k^*\}$  and  $\{e_k\}$  where  $\{e_k^*\}$  is the dual basis to  $\{e_k\}$ . Then the matrix  $J_n$  is skew-symmetric and the problem (3)-(4) takes the form

$$\dot{\mathbf{a}}(t) = J_n \nabla_{\mathbf{a}} h, \quad (7)$$

$$a_k(t_0) = (\phi, e_k) \quad (k = 1, 2, \dots, d_n). \quad (8)$$

The phase space of this problem is  $R^{d_n}$ .

We use the following result. Consider DS with the phase space  $R^r$  generated by the following system of ordinary differential equations

$$\dot{z} = f(z) \quad (9)$$

where  $z = z(t) \in R^r$  with some positive integer  $r$  and  $f = (f_1, \dots, f_r)$  is a smooth function. Let for any Borel set  $C \subset R^r$

$$\rho(C) = \int_C \lambda(z) dz$$

where  $\lambda(z) > 0$  is a smooth function and  $dz$  is the Lebesgue measure in  $R^r$ . Then, the measure  $\rho$  is invariant for the above  $DS$  generated by the system (9) iff

$$\sum_{i=1}^r \frac{\partial}{\partial z_i} (\lambda f_i) = 0$$

for all  $z$  (for the proof, see [17]).

Using this result one can easily verify that the Borel measure

$$\mu'_n(A_n) = (2\pi)^{-\frac{d_n}{2}} \prod_{k=1}^{d_n} \lambda_k^{\frac{1}{2}} \int_A e^{-\frac{1}{2} \sum_{k=1}^{d_n} \lambda_k a_k^2 - g \left( \sum_{k=1}^{d_n} a_k e_k(x) \right)} da$$

is invariant for the problem (7)-(8). Also, we introduce the measures

$$w'_n(A) = (2\pi)^{-\frac{d_n}{2}} \prod_{k=1}^{d_n} \lambda_k^{\frac{1}{2}} \int_A e^{-\frac{1}{2} \sum_{k=1}^{d_n} \lambda_k a_k^2} da.$$

Let  $\Omega_n \subset X_n$  and  $\Omega_n = \left\{ u \in X \mid u = \sum_{k=1}^{d_n} a_k e_k, \mathbf{a} \in A \right\}$  where  $A \subset R^{d_n}$  is a Borel set. We set  $\mu_n(\Omega_n) = \mu'_n(A)$ ; by analogy,  $w_n(\Omega_n) = w'_n(A)$ . Since the measure  $\mu'_n$  is invariant for (7)-(8), the measure  $\mu_n$  is invariant for the problem (3)-(4).

Although  $w_n$  and  $\mu_n$  are the measures on  $X_n$ , we can define them on the Borel sigma-algebra of  $X$  by the rule:  $w_n(\Omega) = w_n(\Omega \cap X_n)$  and  $\mu_n(\Omega) = \mu_n(\Omega \cap X_n)$ . Since the set  $\Omega \cap X_n$  is open as a set in  $X_n$  for any open set  $\Omega \subset X$ , this procedure is correct.

LEMMA 1. – *The sequence  $\{w_n\}$  weakly converges to  $w$  in  $X$ .*

*Proof.* – Since  $S^{-1}$  is an operator of trace class, the trace  $\text{Tr}(S^{-1}) = \sum_k \lambda_k^{-1} < \infty$ . It is clear that there exists a continuous positive function  $p(x)$  defined on  $(0, \infty)$  with the property  $\lim_{x \rightarrow \infty} p(x) = +\infty$  such that  $\sum_k \lambda_k^{-1} p(\lambda_k) < +\infty$ . We define an (unbounded) operator  $T = p(S)$  and let  $Q = S^{-1}T$ . According to the definition  $0 < \text{Tr} Q < \infty$ . Consider

$R > 0$ . Let  $B_R = \{u \in X | T^{\frac{1}{2}} u \in X, \|T^{\frac{1}{2}} u\|_X \leq R\}$  and let  $B$  be the closure of  $B_R$  in  $X$ . It is clear that  $B$  is compact for any  $R > 0$ . By the well-known inequality (see [14], Lemma II.1.1)

$$w_n(X \setminus B) = w_n(u : (Tu, u)_X > R^2) \leq \frac{\text{Tr } Q}{R^2}.$$

Therefore, by the Prokhorov theorem  $\{w_n\}$  is weakly compact on  $X$ .

In view of the definition  $w_n(M) \rightarrow w(M)$  for any cylindrical set  $M \subset X$  (because  $w_n(M) = w(M)$  for all sufficiently large  $n$ ). Hence, since the extension of a measure from an algebra to a minimal sigma-algebra is unique, we have proved that the sequence  $w_n$  converges to  $w$  weakly, and Lemma 1 is proved.

LEMMA 2.

$$\liminf_{n \rightarrow \infty} \mu_n(\Omega) \geq \mu(\Omega) \text{ for any open set } \Omega \subset X.$$

$$\limsup_{n \rightarrow \infty} \mu_n(K) \leq \mu(K) \text{ for any closed bounded set } K \subset X.$$

*Proof is usual.* – It is obvious that any bounded set has a finite measure  $\mu$ . Let  $\Omega \subset X$  be open and let  $B_R = \{u \in X | \|u\|_X < R\}$  for some  $R > 0$ . Consider  $\Omega_R = \Omega \cap B_R$ . For any  $\varepsilon > 0$  there exists a continuous function  $\phi(u) : 0 \leq \phi(u) \leq 1$  with the support belonging to  $\Omega_R$  such that

$$\int_X \phi(u) e^{-g(u)} w(du) \geq \mu(\Omega_R) - \varepsilon.$$

Then,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mu_n(\Omega_R) &\geq \liminf_{n \rightarrow \infty} \int_X \phi(u) e^{-g(u)} w_n(du) \\ &= \int_X \phi(u) e^{-g(u)} w(du) \geq \mu(\Omega_R) - \varepsilon. \end{aligned}$$

Therefore, due to the arbitrariness of  $\varepsilon > 0$  one has:

$$\liminf_{n \rightarrow \infty} \mu_n(\Omega) \geq \limsup_{n \rightarrow \infty} \mu_n(\Omega_R) \geq \mu(\Omega_R).$$

Taking  $R \rightarrow +\infty$  in this inequality, we obtain the first statement of Lemma 2.

Let  $K$  be a closed bounded set. Fix  $\varepsilon > 0$ . We take a continuous function  $\phi \in [0, 1]$  such that  $\phi(u) = 1$  for any  $u \in K$ ,  $\phi(u) = 0$  if  $\text{dist}(u, K) > \varepsilon$  and  $\int_X \phi(u) e^{-g(u)} w(du) < \mu(K) + \varepsilon$ . Then,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu_n(K) &\leq \limsup_{n \rightarrow \infty} \int_X \phi(u) e^{-g(u)} w_n(du) \\ &= \int_X \phi(u) e^{-g(u)} w(du) \leq \mu(K) + \varepsilon \end{aligned}$$

and due to the arbitrariness of  $\varepsilon > 0$  Lemma 2 is proved.

LEMMA 3. – Let  $\Omega \subset X$  be open,  $t \in R$ . Then  $\mu(\Omega) = \mu(\Omega_1)$  where  $\Omega_1 = f(\Omega, t)$ .

*Proof.* – Using Assumption 2 and the Corollary one has that  $\Omega_1$  is open, too. First, let us suppose that  $\mu(\Omega) < \infty$ ,  $\mu(\Omega_1) < \infty$ .

Fix  $\varepsilon > 0$ . Then, there exists a compact set  $K \subset \Omega$  such that  $\mu(\Omega \setminus K) < \varepsilon$ . Let  $K_1 = f(K, t)$ . Then  $K_1 \subset \Omega_1$  is compact. Let  $\alpha = \min \{\text{dist}(K, \partial\Omega); \text{dist}(K_1, \partial\Omega_1)\}$  where  $\text{dist}(A, B) = \inf_{x \in A, y \in B} \|x - y\|_X$  and  $\partial A$  is the boundary of a set  $A \subset X$ . One obviously has  $\alpha > 0$ . By Assumption 3 for any  $u \in K$  there exists a ball  $B(u) \subset \Omega$  with the center in  $u$  such that  $\text{dist}(f_n(u, t); f_n(g, t)) < \frac{\alpha}{3}$  for all  $g \in B(u)$  and for all  $n$ . Let  $\Omega_\beta = \{g \in \Omega_1 | \text{dist}(g, \partial\Omega_1) \geq \beta\}$  for any  $\beta > 0$  and let  $B(u_1), \dots, B(u_l)$  be a finite covering of  $K$  by the above balls,  $D = \bigcup_{i=1}^l B(u_i)$ . Since  $f_n(u_i, t) \rightarrow f(u_i, t)$  ( $n \rightarrow \infty$ ) for any  $i$ , using Assumption 2 one obtains that  $\text{dist}(f_n(u, t), K_1) < \frac{\alpha}{3}$  for large  $n$ . Thus,  $f_n(D, t)$  belongs to a closed bounded subset of  $\Omega_{\frac{\alpha}{4}}$  for all sufficiently large  $n$ . Hence, by Lemma 2

$$\begin{aligned} \mu(\Omega) &\leq \mu(D) + \varepsilon \leq \liminf_{n \rightarrow \infty} \mu_n(D) + \varepsilon \\ &= \liminf_{n \rightarrow \infty} \mu_n(f_n(D \cap X_n, t)) + \varepsilon \leq \mu(\Omega_1) + \varepsilon. \end{aligned}$$

Due to the arbitrariness of  $\varepsilon > 0$  one obtains the inequality

$$\mu(\Omega) \leq \mu(\Omega_1).$$

Since  $\Omega = f(\Omega_1, -t)$ , the opposite inequality is valid, too:

$$\mu(\Omega) \geq \mu(\Omega_1).$$

Thus, we have proved the equality

$$\mu(\Omega) = \mu(\Omega_1)$$

for any two open sets with finite measures. If  $\Omega$  is open and  $\mu(\Omega) = +\infty$ , then we take the sequence  $\Omega^k = \Omega \cap \{u \in X \mid |g(u)| + |g(f(u, t))| < k\}$  ( $k = 1, 2, 3, \dots$ ) and set  $\Omega_1^k = f(\Omega^k, t)$ . One has  $\mu(\Omega^k) = \mu(\Omega_1^k) < \infty$ . Taking  $k \rightarrow +\infty$ , we obtain the statement of the lemma. Lemma 3 is proved.

For any Borel set  $\Omega \subset X$  we obtain the equality  $\mu(\Omega) = \mu(\Omega_1)$  approximating  $\Omega$  and  $\Omega_1$  by open sets from outside and by closed sets from inside. Thus, Theorem 1 is proved.

#### 4. PROOF OF THEOREM 2

By  $f_N(u, t)$  we denote  $DS$  generated by the problem (1)-(2) with  $H = H_N$ . Let  $\mu_N$  be the corresponding invariant measure from Theorem 1. Since  $G(u)$  is a pointwise limit of continuous functionals, it is measurable. Then, since  $G(u) \geq 0$ , the measure  $\nu$  is well-defined. By the classical result

$$\lim_{N \rightarrow \infty} \mu_N(\Omega) = \nu(\Omega) \leq +\infty$$

for any measurable  $\Omega \subset X$ .

Let us fix  $t \in R$  and a measurable  $\Omega \subset X$ . Let  $\Omega_N = f_N(\Omega, t)$ ,  $A_k = \bigcap_{N \geq k} \Omega_N$ ,  $A = \bigcup_{k \geq 1} A_k$ . It is clear that  $A_1 \subset A_2 \subset A_3 \subset \dots \subset A_k \subset \dots$

LEMMA 4. – Let  $\Omega$  be open. Then,  $\Omega_1 = f(\Omega, t) \subset A$ .

*Proof.* – Let  $u \in \Omega_1$ . In accordance with Assumption 5  $f_N(u, -t) \in \Omega$  for all sufficiently large numbers  $N$ . Hence,  $u \in A_k$  for sufficiently large  $k$ , and Lemma 4 is proved.

Let  $\Omega$  and  $\Omega_1$  be open. Then, we have

$$\mu_N(\Omega) = \mu_N(\Omega_N) \geq \mu_N(A_k)$$

for  $N \geq k$ . Taking  $N \rightarrow \infty$ , we obtain:  $\nu(\Omega) \geq \nu(A_k)$ . Hence, by Lemma 4

$$\nu(\Omega) \geq \nu(A) \geq \nu(\Omega_1).$$

The opposite inequality may be proved by analogy. For an arbitrary measurable set  $\Omega \subset X$  we obtain the same equality as at the end of Theorem 1. Thus, Theorem 2 is proved.

#### 5. APPLICATIONS

As it is noted in Section 1, one of the well-known applications of invariant measures in the theory of dynamical systems is the Poincaré recurrence theorem (*see* [17]).

THEOREM (Poincaré). – Let  $f$  be  $DS$  on a separable phase space  $X$  with a bounded invariant measure  $\mu$ :  $\mu(X) < \infty$ . Then, almost all points of  $X$  lie on the trajectories stable according to Poisson.

The stability of the trajectory  $f(\phi, t)$  according to Poisson means in particular that there exists a sequence of points  $t_n \rightarrow +\infty$  such that

$f(\phi, t_n) \rightarrow \phi$  as  $n \rightarrow \infty$ . In view of Statement 2 elements  $\phi$  of  $X$  satisfying this property form a dense set in  $X$ .

According to Theorems 1 and 2 we have constructed the invariant measure for our  $DS$ . As we will see further, it is not difficult to formulate conditions for the measure being bounded. Unfortunately, we have to remark that Assumptions 1-5 are rigorously proved for concrete partial differential equations only in several special cases. Assumptions 1, 2 and 4 seem to be sufficiently natural but Assumption 3 is very strong (now it is proved in some simple situations). We find Assumption 5 to be natural, too. Despite the mentioned difficulties we are able to present the invariant measures in several cases for concrete nonlinear partial differential equations.

By  $L_2(0, A)$  we denote the usual (real or complex) Lebesgue space of functions of the argument  $x \in (0, A)$  with the scalar product  $(u, v)_{L_2(0, A)} = \int_0^A u(x) \overline{v(x)} dx$ . Let also  $H_0^1(0, A)$  be the Sobolev space of absolutely continuous real functions  $u(x)$  ( $x \in (0, A)$ ) satisfying the conditions  $u(0) = u(A) = 0$  with the finite norm  $\|u\|_{H_0^1(0, A)} = \left\{ \int_0^A |u'(x)|^2 dx \right\}^{\frac{1}{2}}$ .

**5.1. A nonlinear Schrödinger equation**

Consider the problem

$$iu_t + u_{xx} + f(x, |u|^2)u = 0, \quad x \in (0, A), \quad t \in R, \quad (10)$$

$$u(0, t) = u(A, t) = 0, \quad (11)$$

$$u(x, t_0) = u_0(x). \quad (12)$$

Our main hypothesis is the following:

(f1) Let  $f$  be a smooth real function and let there exist  $C > 0$  such that

$$|f(x, s)| + \left| (1 + s) \frac{\partial}{\partial s} f(x, s) \right| < C$$

for all  $x, s$ .

We remark that the hypothesis (f1) is weaker than in paper [3].

We rewrite the problem (10)-(12) for the functions  $u^1 = \text{Re } u, u^2 = \text{Im } u, \phi_1 = \text{Re } \phi, \phi_2 = \text{Im } \phi$ :

$$u_t^1 + u_{xx}^2 + f(x, (u^1)^2 + (u^2)^2)u^2 = 0, \quad x \in (0, A), \quad t \in R, \quad (13)$$

$$u_t^2 - u_{xx}^1 - f(x, (u^1)^2 + (u^2)^2) u^1 = 0, \quad (14)$$

$$u^i(0, t) = u^i(A, t) = 0, \quad i = 1, 2, \quad (15)$$

$$u^i(x, t_0) = \phi_i(x). \quad (16)$$

We introduce the following definitions. Let  $X = L_2(0, A) \otimes L_2(0, A)$ ,  $Y = H_0^1(0, A) \otimes H_0^1(0, A)$  where  $L_2(0, A)$  and  $H_0^1(0, A)$  are the real spaces. Let  $Q$  be the operator mapping  $u^* \in (L_2(0, A))^*$  into  $u \in L_2(0, A)$  such that  $u^*(g) = (u, g)_{L_2(0, A)}$  for any  $g \in L_2(0, A)$  and let  $J = \begin{pmatrix} 0 & -Q \\ Q & 0 \end{pmatrix}$ . It is clear that the operator  $J$  maps the whole space  $X^*$  into  $X$  with the property  $h^*(Jg^*) = -g^*(Jh^*)$  where  $h^*, g^* \in X^*$ . Then, let  $\Delta$  be the closure of the operator  $-\frac{d^2}{dx^2}$  in  $L_2(0, A)$  defined first on  $C_0^\infty(0, A)$  and let  $S = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$ . It is clear that  $S = S^* > 0$ . Let  $F(x, s) = \frac{1}{2} \int_0^s f(x, p) dp$  and

$$\begin{aligned} H(u^1, u^2) &= \int_0^A \left\{ \frac{1}{2} ((u_x^1(x))^2 + (u_x^2(x))^2) \right. \\ &\quad \left. - F(x, (u^1(x))^2 + (u^2(x))^2) \right\} dx \\ &= \frac{1}{2} (Su, u)_X - \int_0^A F(x, (u^1(x))^2 + (u^2(x))^2) dx \end{aligned}$$

where  $u = (u^1, u^2) \in X$ . In this notation one obtains the representation of the system (13)-(16) in the form (1)-(2).

Further, let  $\{e_n\}$  be the orthonormal basis of eigenvectors of the operators  $\Delta$  with corresponding eigenvalues  $\{\lambda_n\}$ . We set  $X_n = \text{span}\{e_1, \dots, e_n\} \otimes \text{span}\{e_1, \dots, e_n\}$  and let  $P_n$  be the orthogonal projector onto  $\text{span}\{e_1, \dots, e_n\}$  in  $L_2(0, A)$ . Then, the approximate problem (3)-(4) takes the following form:

$$\begin{aligned} u_t^{1n} + u_{xx}^{2n} + P_n [f(x, (u^{1n})^2 + (u^{2n})^2) u^{2n}] &= 0, \\ u_t^{2n} - u_{xx}^{1n} - P_n [f(x, (u^{1n})^2 + (u^{2n})^2) u^{1n}] &= 0, \quad (17) \\ (u^{in})(x, t_0) &= P_n \phi_i(x). \end{aligned}$$

We can now present

**THEOREM 3.** – *Let the hypothesis (f1) be valid. Then, NSE (13)-(16) satisfies Assumptions 1-4. Hence, the Borel measure*

$$\mu(\Omega) = \int_{\Omega} e^{\int_0^A F(x, (u^1)^2 + (u^2)^2) dx} w(du^1 du^2)$$

*is invariant for DS generated on the phase space X by this problem (here w is the centered Gaussian measure on X with the correlation operator S<sup>-1</sup>).*

*Proof.* – It is convenient to consider the complex problem (10)-(12) again, because this is the usual approach to NSE. So, L<sub>2</sub>(0, A) is the complex space in this proof. Of course, the results presented below for the complex problem imply the corresponding results for the real problem (13)-(16).

Using the well-known methods (*see*, for example, [18], [19]) one can prove the existence and uniqueness of the solution of the problem (10)-(12) for any u<sub>0</sub> ∈ L<sub>2</sub>(0, A) belonging to C(R; L<sub>2</sub>(0, A)) with the property  $\frac{d}{dt} \|u(\cdot, t)\|_{L_2(0, A)} = 0$ . The solution is understood here as a function from the above space satisfying the equation

$$u(\cdot, t) = e^{-i(t-t_0)\Delta} \phi + i \int_{t_0}^t e^{-i(t-s)\Delta} [f(x, |u(x, s)|^2) u(x, s)] ds. \tag{18}$$

Further, the finite-dimensional problem (17) is equivalent to the following complex one:

$$iu_t^n + P_n u_{xx}^n + P_n [f(x, |P_n u^n|^2) P_n u^n] = 0, \\ u^n(x, t_0) = P_n \phi$$

where  $u^n = u^{1n} + iu^{2n}$  and  $P_n g = \sum_{k=1}^n e_k \int_0^A (g^1(x) + ig^2(x)) e_k(x) dx$  for any  $g \in L_2(0, A)$ . Obviously, this problem has a unique local solution for any  $\phi \in L_2(0, A)$  which satisfies the following equation:

$$u^n(x, t) = e^{-i(t-t_0)\Delta} P_n \phi + i \int_{t_0}^t e^{-i(t-s)\Delta} P_n \\ \times [f(x, |P_n u^n(x, s)|^2) P_n u^n(x, s)] ds \tag{19}$$

In the following three estimates we suppose that  $t > t_0$ . Of course, similar inequalities take place for  $t < t_0$ . First, according to the estimate following from (19)

$$\|u^n(x, t)\|_{L_2(0, A)} \leq C_1 \|\phi\|_{L_2(0, A)} + C_2 \int_{t_0}^t \|u^n(x, s)\|_{L_2(0, A)} ds$$



solutions  $u^n$  exist for all  $t \in R$ . Thus, Assumption 1 is valid. Then, in view of the inequality ( $t > t_0$ )

$$\begin{aligned} \|u_1^n(x, t) - u_2^n(x, t)\|_{L_2(0, A)} &\leq C_3 \|u_1^n(x, t_0) - u_2^n(x, t_0)\|_{L_2(0, A)} \\ &+ C_4 \int_{t_0}^t \|u_1^n(x, s) - u_2^n(x, s)\|_{L_2(0, A)} ds \end{aligned}$$

where  $u_1^n$  and  $u_2^n$  are any two solutions of equation (19), Assumption 3 holds, too.

Let us verify Assumption 2. One has:

$$\begin{aligned} &\|u(x, t) - u^n(x, t)\|_{L_2(0, A)} \\ &\leq C' \|P_n \phi - \phi\|_{L_2(0, A)} + C'' \int_{t_0}^t \|u^n(x, s) - u(x, s)\|_{L_2(0, A)} ds \\ &+ C''' \int_{t_0}^t \|P_n [f(x, |u(x, s)|^2) u(x, s)] \\ &- f(x, |u(x, s)|^2) u(x, s)\|_{L_2(0, A)} ds. \end{aligned}$$

Since the last term in the right-hand side tends to  $+0$  as  $n \rightarrow \infty$ , Assumption 3 follows from this inequality, too.

The validity of Assumption 4 is obvious and Theorem 3 is proved.

*Remark 5.* – For the system (13)-(16) a result similar to Theorem 2 is presented in paper [4] for the power nonlinearity  $f(x, |u|^2)u = \lambda |u|^p u$  where  $p \in (0, 4)$  if  $\lambda < 0$  and  $p \in (0, 2)$  if  $\lambda > 0$ . In this paper the non-triviality and boundedness of the constructed measure  $\nu$  on any ball in  $X$  are proved, *i.e.* it is proved that  $0 < \nu(B) < +\infty$  for any ball  $B \subset X$ .

*Remark 6.* – The described approach is applicable also to the problem periodic in  $x$  for NSE without any essential modifications.

*Remark 7.* – The hypothesis (f1) is valid for two physical nonlinearities:  $f(x, s) = \frac{\alpha s}{1+s}$  and  $f(x, s) = e^{-\alpha s}$  with  $\alpha \geq 0$  in the second case (for physical applications of the equation with nonlinearities of these kinds *see* [20], for example). According to Theorem 3 any ball  $B_R = \{u \in X \mid \|u\|_X \leq R\}$  is an invariant set for our DS. So, a ball  $B_R$  may be taken for the new phase space. It is clear that  $\mu$  is bounded on any such ball for each of nonlinearities presented. Thus, the Poincaré recurrence theorem is applicable in this case and therefore the points from a dense subset of  $X$  are stable in the sense of Poisson.

**5.2. A nonlinear wave equation**

In this section all variables are real and  $L_2(0, A)$  is the real space. Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + f(x, u) = 0, \quad x \in (0, A), \quad t \in R, \quad (20)$$

$$u(0, t) = u(A, t) = 0, \quad (21)$$

$$u(x, t_0) = \phi(x), \quad u'_t(x, t_0) = \psi(x). \quad (22)$$

The hypothesis for the function  $f$  is the following.

(f2) Let the function  $f$  be continuously differentiable and let there exist  $C > 0$  such that

$$|f(x, u)| + \left| \frac{\partial}{\partial u} f(x, u) \right| < C$$

for all  $x, u$ .

Let  $\Delta$  be the operator from Section 3.1 and let  $H^{-1}(0, A)$  be the completion of the space  $L_2(0, A)$  with respect to the norm  $\|u\|_{H^{-1}(0, A)} = \|\Delta^{-\frac{1}{2}} u\|_{L_2(0, A)}$ . Then, the following result is valid (for the proof, see [5]).

**THEOREM 4.** - Let the hypothesis (f2) be valid. Then for any  $u(x, t_0) = \phi \in L_2(0, A)$ ,  $u'_t(x, t_0) = \psi \in H^{-1}(0, A)$  there exists a unique solution  $u(x, t)$  to the problem (20)-(22) satisfying  $(u(x, t), u'_t(x, t)) \in C(I; L_2(0, A) \otimes H^{-1}(0, A))$  for any finite interval  $I \in R$ .

We replace the problem (20)-(22) by the following:

$$u_t = v, \quad (23)$$

$$v_t = u_{xx} - f(u), \quad x \in (0, A), \quad t \in R, \quad (24)$$

$$u(0, t) = u(A, t) = 0, \quad t \in R, \quad (25)$$

$$u(x, t_0) = \phi(x), \quad v(x, t_0) = \psi(x). \quad (26)$$

Since this problem is considered in paper [5] and the result of the present paper is identical to the above result, we only demonstrate the applicability of our abstract scheme to this problem. Let

$$(u, v) \in X = L_2(0, A) \otimes H^{-1}(0, A),$$

$$Y = H_0^1(0, A) \otimes L_2(0, A), \quad F(x, u) = \int_0^u f(x, s) ds,$$

$$H(u, v) = \int_0^A \left\{ \frac{1}{2} (v^2 + u_x^2) + F(x, u) \right\} dx, \quad J = \begin{pmatrix} 0 & Q_1 \\ -Q_1 & 0 \end{pmatrix}.$$

Here  $Q_1$  is defined as follows. Let  $Q_1$  map elements  $g^* \in \{H^{-1}(0, A)\}^*$  into  $g \in L_2(0, A)$  with the property  $g^*(w) = (g, w)_{L_2(0, A)}$  for any  $w \in L_2(0, A)$ . Obviously,  $Q_1$  is defined on the whole space  $(H^{-1}(0, A))^*$  which is the dense set in  $(L_2(0, A))^*$ . Hence,  $J$  is defined on a dense subset  $D \subset X^*$ . Finally, the property  $h^*(Jg^*) = -g^*(Jh^*)$  where  $g^*, h^* \in D$  is valid, too.

Further, we take the closure of  $\begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix}$  on  $X$  defined first on

$C_0^\infty \otimes C_0^\infty$  for  $S$  and the spaces  $X_n$  from Section 3.1. In this notation, one obtains the problem (23)-(26) again in the form (1)-(2).

As in paper [5], one can verify that Assumptions 1-4 are valid. So, the measure

$$\mu(\Omega) = \int_{\Omega} e^{-\int_0^A F(x, u) dx} w(du dv),$$

where  $w$  is the centered Gaussian measure on  $X$  with the correlation operator  $S^{-1}$ , is invariant.

*Remark 8.* – Unfortunately, the author does not know any results verifying Assumptions 1-4 or 5 on the space  $X$  for a wider class of nonlinearities to make Theorems 1 and 2 applicable.

*Remark 9.* – To use the Poincaré recurrence theorem ensuring trajectories being stable according to Poisson, the measure constructed is to be bounded. Consider, in particular, two physical nonlinearities  $f(x, u) = \frac{\alpha u^2}{1 + u^2} u$  and  $f(x, u) = ue^{\alpha u^2}$ ,  $\alpha \leq 0$ , satisfying the hypothesis (f2). Since the integral  $\int_X e^{-\alpha \|u\|_X^2} w(du)$  is finite for small  $\alpha_0 > 0$  and  $\alpha > -\alpha_0$  (see [16]), our measure  $\mu$  is finite for the same values of  $\alpha$  for the first function. In the second case, the measure  $\mu$  is finite of all  $\alpha \leq 0$ . As the author is informed by Professor V. G. Makhankov, this fact for the function  $f$  of the first type is confirmed by the results of his numerical investigations which show that the Fermi-Pasta-Ulam phenomenon takes place for the same values of  $\alpha$ .

### 5.3. A generalized Korteweg-de Vries equation

Consider the problem

$$u_t + (a(x)u)_x + u_{xxx} = 0, \quad x, t \in R, \tag{27}$$

$$u(x, t_0) = \phi(x). \tag{28}$$

We assume that the functions  $a, u, \phi$  are periodic in  $x$  with a period  $A > 0$ . Using the method of paper [21] one can prove

**THEOREM 5.** – *For any periodic  $a, \phi \in C^\infty$  with a period  $A > 0$  there exists a unique solution of the problem  $u(x, t)$  of the class  $C^\infty$  defined for all  $x, t$  which is periodic in  $x$  with the same period.*

We take the spaces of periodic real functions from  $L_2(0, A)$  and  $H^1(0, A)$  for  $X$  and  $Y$ , respectively, with the norms  $\|g\|_X = \int_0^A g^2(x) dx$  and  $\|g\|_Y = \|g'\|_X + \|g\|_X$ . Let  $J = \frac{\partial}{\partial x} Q$  where the operator  $Q$  maps  $v^* \in X^*$  into  $v \in X$  such that  $v^*(g) = (v, g)_X$  for any  $g \in X$ . Finally, let  $S = -\Delta + E$  where  $\Delta$  is the closure of the operator  $\frac{d^2}{dx^2}$  in  $X$  defined first on periodic functions from  $C^\infty$  and  $E$  is the identical operator. We set  $H(u) = \frac{1}{2} \left\{ (Su, u) - \int_R (1+a(x)) u^2(x) dx \right\}$ . Using the trivial estimate ( $t > t_0$ )

$$\begin{aligned} & \int_0^A (u(x, t) - v(x, t))^2 dx \\ & \leq C \int_{t_0}^t \int_0^A (u(x, s) - v(x, s))^2 dx ds \\ & \quad + \int_0^A (u(x, t_0) - v(x, t_0))^2 dx \end{aligned}$$

where  $C = \text{const} > 0$  and  $u$  and  $v$  are arbitrary  $C^\infty$ -solutions of the problem (27)-(28), one proves the existence and uniqueness of the solution

in the space  $C(I; X)$ . Then, we take  $e_{2n-1}(x) = \left(\frac{2}{A}\right)^{\frac{1}{2}} \sin\left(\frac{\pi n x}{A}\right)$ ,

$e_{2n}(x) = \left(\frac{2}{A}\right)^{\frac{1}{2}} \cos\left(\frac{\pi n x}{A}\right)$  ( $n = 1, 2, 3, \dots$ ),  $e_0 = \left(\frac{1}{A}\right)^{\frac{1}{2}}$  and let

$X_n = \text{span}\{e_0, e_1, \dots, e_{2n}\}$ . In addition to the above inequality one can prove the following:

$$\begin{aligned} & \int_0^A (u(x, t) - u^n(x, t))^2 dx \\ & \leq C \int_{t_0}^t \int_0^A (u(x, s) - u^n(x, s))^2 dx ds \\ & \quad + \int_0^A (u(x, t_0) - u^n(x, t_0))^2 dx. \end{aligned}$$

where  $u^n$  is the approximate solution introduced in Section 2. It is easy to verify that Assumptions 1-4 follow from these two inequalities and one more inequality similar to the first of them written for  $u^n$  and  $v^n$ . Then, the Borel measure

$$\mu(\Omega) = \int_{\Omega} e^{\int_0^A \frac{1}{2} \{u^2(x) + a(x)u^2(x)\} dx} w(du)$$

is invariant for DS generated on the phase space  $X$  by the problem (27)-(28). Here  $w$  is the centered Gaussian measure on  $X$  with the correlation operator  $S^{-1}$ .

When the present paper was already written the author learned about results for the nonlinear cubic Schrödinger equation similar to ours which are obtained by H. McKean and K. Vaninsky [22].

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