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Curvature-induced resonances in a two-dimensional Dirichlet tube

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ABSTRACT. – Scattering problem is studied for the Dirichlet Laplacian in a curved planar strip which is assumed to fulfil some regularity and analyticity requirements, with the curvature decaying as $\mathcal{O}(|s|^{-1-\varepsilon})$ for $|s| \to \infty$. Asymptotic completeness of the wave operators is proven. If the strip width d is small enough we show that under the threshold of the j-th transverse mode, $j \geq 2$, there is a finite number of resonances, with the poles approaching the real axis as $d \to 0$. A perturbative expansion for the pole positions is found and the Fermi-rule contribution to the resonance widths is shown to be exponentially small as $d \to 0$.

RÉSUMÉ. – Nous étudions la diffusion quantique pour un laplacien de Dirichlet dans une bande plane qui satisfait à des propriétés d'analyticité et de régularité et dont la courbure décroît comme $\mathcal{O}(|s|^{-1-\varepsilon})$ à l'infini. Nous prouvons que l'opérateur d'onde est asymptotiquement complet. Si l'épaisseur « d » de cette bande est suffisamment petite, nous prouvons que sous le seuil du j-ième mode transverse et pour $j \geq 2$, il n'y a qu'un nombre fini de résonances dont les pôles s'approchent de l'axe réel si $d \to 0$. Nous donnons un développement perturbatif pour la position de ces pôles et nous montrons que la contribution de la règle d'or de Fermi à leur largeur devient exponentiellement petite si $d \to 0$.

1. INTRODUCTION

Spectral properties of Dirichlet Laplacians in curved tubes have attracted some attention recently (*cf.* [DE1], [EŠ] and references therein) in connection with the existence of bound states for a sufficiently small strip width. In this paper we are going to show that the scattering problem is equally interesting since it exhibits a resonance structure below the thresholds of the higher transverse modes.

Studies of scattering in tubes have been traditionally related to the problem of field propagation in waveguides. Recently an alternative motivation has appeared: Schrödinger equation in a tube represents a simple model of electron motion in quantum wires, *i.e.*, tiny strips of a highly pure semiconductor material prepared on a substrate – *cf.* [E] and references therein. This is why we formulate the scattering problem in consideration as a quantum-mechanical one, keeping in mind its closed relation, *e.g.*, to the TM-mode scattering in planar electromagnetic waveguides. In this connection, recall that resonances induced by time-periodic forces in infinitely stretched subsets of $\mathbb{R}^n$ or in stratified media have been discussed for the electro-magnetic as well as scalar wave equations – *cf.* [MW], [Wed] and references therein.

Even without an external force, however, there are various ways in which resonances can appear in thin curved tubes. If one cuts, for instance, the tube at a finite distance and couples both its ends to much wider tubes, then the infinite-tube bound states appear to embedded into the continuum and one expects them to turn into resonances – in a simplified model this effect was demonstrated in [E]. Here we are going to consider completely different resonances which appear in tubes of *infinite* length. The mechanism of their origin is the same as the one producing the above mentioned bound states: if the tube is "straightened" by passing to natural curvilinear coordinates, an effective curvature-induced potential appears which is for thin tubes dominated by a purely attractive term. In addition to this part, however, the Hamiltonian contains a transverse one with a purely discrete spectrum, and therefore eigenvalues associated with the j-th transverse mode, $j \geq 2$, appear to be embedded into the continuous spectrum of the lower modes. Finally, the Hamiltonian contains mode-coupling terms; for thin tubes they can be regarded as a perturbation turning the embedded eigenvalues into resonances. The aim of this paper is to show that this heuristic picture can be justified.

In addition to the physical motivation, the model discussed here represents at the same time a laboratory for studying the problems of non-adiabatic transitions. Recall that given a quantum system depending on a pair of variables of which one is slow and the other fast in a appropriate sense, one usually solves the spectral problem for each fixed value of the slow variable, replacing afterwards the corresponding part of the Hamiltonian by one of the obtained spectral parameters regarded as an effective potential for the slow variable; we say that we have projected our system on a particular mode of the fast Hamiltonian. This procedure is usually called the adiabatic or the Born-Oppenheimer approximation. The fast variable may be discrete as for systems of a finite number of coupled Schrödinger equations, or continuous as in the problem of a diatomic molecule. Our system is in a sense intermediate between the two cases, the fast variable being continuous but confined leading to an infinite system of coupled Schrödinger equations in the corresponding Fourier representation. The fast and the slow variables are in our case respectively the transverse and the longitudinal coordinate in the limit when the tube diameter d goes to zero. In addition, the slow kinetic part contains a metric factor [*see* (2)]; this case, up to our knowledge, has not yet been studied in the present context.

There is nowadays a great activity aimed at the problem of estimating the transitions due to a mode coupling. The case when the energies of two modes cross (for a real value of the slow variable) has been studied in [Kle, KR]. Here we address the question in the situation where the two energies do not cross. Under suitable analyticity assumptions on the slow part of the Hamiltonian one expects that these transitions probabilities should behave as $a_\varepsilon \, e^{-S/\varepsilon}$ where ε is the parameter characterizing the difference between the two time scales, S is the action of an "instanton" joining the two energy surfaces of the Hamiltonian corresponding to different modes in the phase space [LL], and a_ε is an amplitude which is polynomially bounded in ε^{-1} as $\varepsilon \longrightarrow 0$. In view of the obvious similarity to the usual tunneling in the configuration space, this effect has been named *dynamical tunneling* in [AD] and *microlocal tunneling* in [Ma1] (the last named paper contains references to earlier works). A behaviour of this type has been announced in [BD] for the Stark-Wannier resonances ladder problem, and proven rigorously in [Ma1] for a rather general two by two matrix of n-dimensional coupled Schrödinger operators; in the latter case, however, one does not always recover the full rate of exponential decay predicted by the heuristic argument (*see* [LL, Sec. 90] and Remark 4.2e below). The claim here is that we get the expected full decay rate but only for the Fermi-rule contribution to the transition probability; the same kind of result is obtained for a model of Stark-Wannier ladder resonances in [GMS]. Let us remark that while all these papers are concerned with resonances, other authors have treated recently the exponential decay rate of the transition probability in time-dependent adiabatic systems ([JK], [JKP], [Ne], [JS], [Ma2]).

The method is expected to work in any dimension greater or equal to two, however, in order not to burden the paper with technicalities we restrict ourselves to the simplest case of a tube in $\mathbb{R}^2$. Let us summarize briefly the contents. In the next section we formulate the problem and deduce transverse-mode decomposition of the Hamiltonian; we also collect here the used assumptions. Section 3 is devoted to proof of the wave operator asymptotic completeness which is performed by the smooth perturbations method. The key result is Theorem 4.1 which states existence of the resonances, *i.e.*, poles of the analytically continued resolvent, together with the perturbative expansion for the pole positions; the first two terms are computed explicitly. The proof is based on an appropriate modification of the complex scaling method [AC], [RS, Sec. XII.6] combined with the transverse-mode decomposition. Strictly speaking, we have not excluded the possibility that in some cases the imaginary part of the pole position remains zero after the perturbation; we conjecture, however, that with the used assumptions about the strip geometry this cannot happen. The proof shows at the same time that the Hamiltonian can have other resonances [*see* Remark 4.2(a)], however, since the corresponding poles do not approach the real axis when $d \longrightarrow 0$, the resonances coming from embedded eigenvalues are expected to dominate the scattering picture in thin tubes. Finally we illustrate Theorem 4.1 by a solvable model.

2. PRELIMINARIES

We consider a *non-straight* strip $\Sigma \subset \mathbb{R}^2$ of a fixed width d; one of its boundaries will be called $\mathbf{\Gamma}$ and regarded as the reference curve. Up to Euclidean transformations, it is determined by its signed curvature $\gamma(s)$

where s denotes the arc length of $\mathbf{\Gamma}$. The object of our interest is the Dirichlet Laplacian,

$$\tilde{H} := -\Delta \quad \text{on } L^2(\Sigma). \tag{1}$$

A way to study it is to introduce the locally orthogonal coordinates $s,\ u$ on Σ. The unitary transformation $U\psi := (1+u\gamma)^{1/2}\,\psi \circ g$, $g(s,u) = \mathbf{\Gamma}(s) + u\,\mathbf{N}(s)$ with $\mathbf{N}$ being the normal to $\mathbf{\Gamma}$, maps then $\tilde{H}$ into the operator H on $L^2(\mathbb{R}\times(0,d),\, ds\,du) =: \mathcal{H}$ of the following form [EŠ]

$$H = -\partial_s (1+u\gamma)^{-2}\,\partial_s - \partial_u^2 + V, \tag{2}$$

$$V(s,u) := \frac{-\gamma^2}{4(1+u\gamma)^2} + \frac{u\gamma''}{2(1+u\gamma)^3} - \frac{5}{4}\,\frac{u^2\gamma'^2}{(1+u\gamma)^4}, \tag{3}$$

with Dirichlet b.c. at $u = 0,\ d$.

Spectral properties of H depends, of course, on the geometry of Σ. The assumptions we shall use can be divided into several groups. Let us list first some general regularity requirements:

($r1$) smoothness, $\gamma \in C^2$,

($r2$) smoothness at infinity, γ, γ' and γ'' bounded,

($r3$) regularity of the other boundary, $\inf \gamma > -d^{-1}$,

($r4$) injectivity of g which means that Σ is not self-interesting (this requirement can be avoided by considering Σ as a subset of a multisheeted Riemannian surface).

With these assumptions, the potential V as well as the functions $b := (1+u\gamma)^{-2}$ and b^{-1} are bounded so $D(H) = D(H_0) = \hat{\mathcal{H}}_0^1 \cap \hat{\mathcal{H}}^2(\mathbb{R}\times(0,d))$ (*see* the end of the section for the definitions), where $H_0 := -\partial_s^2 - \partial_u^2$. Next we shall need assumptions about the decay of curvature for $|s| \to \infty$, namely

(d) $\gamma(s)$, $(\gamma'(s))^2$ and $\gamma''(s)$ are $\mathcal{O}(|s|^{-1-\varepsilon})$ for some positive ε.

The borderline decay rate corresponds to logarithmic asymptotics of $\mathbf{\Gamma}$. Finally, the use of complex dilations requires

($a1$) γ can be continued analytically to the sector $\mathcal{M}_\alpha := \{z \in \mathbb{C} : |\arg(\pm z)| < \alpha\}$ for some $\alpha > 0$,

($a2$) $\lim_{|z|\to\infty} \gamma(z) = 0$ holds in $\mathcal{M}_\alpha$ and $\inf\{\operatorname{Re}\gamma(z) : z \in \mathcal{M}_\beta\} > -d^{-1}$ for any number $\beta \in [0,\alpha)$,

($a3$) γ can be continued analytically to the strip $\mathcal{S}_n := \{z \in \mathbb{C} : |\operatorname{Im} z| < \eta\}$ for some $\eta > 0$ and $\inf\{\operatorname{Re}\gamma(z) : z \in \mathcal{S}_{\eta'}\} > -d^{-1}$ for any η' in $[0,\eta)$.

Some of the arguments of the following sections rely on the transverse-mode decomposition. Denote by

$$\chi_j(u) := \sqrt{\frac{2}{d}}\,\sin\left(\frac{\pi j u}{d}\right) \tag{4}$$

the eigenfunctions of $-\partial_u^2$ with Dirichlet b.c. on $(0, d)$ corresponding to the eigenvalue $E_j := \left(\frac{\pi j}{d}\right)^2$, $j = 1, 2, \ldots$ Furthermore, denote by $\mathcal{J}_j$ the embedding $L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, ds) \otimes \chi_j \subset \mathcal{H}$, with $\mathcal{J}_j^*$ being the projection onto the j^{th}-mode. Then H can be expressed as an infinite matrix differential operator, $(H_{jk})_{j,k=1}^{\infty}$, with

$$H_{jk} := \mathcal{J}_j^*\, H\, \mathcal{J}_k = -\partial_s\, b_{jk}\, \partial_s + E_j\, \delta_{jk} + V_{jk}, \tag{5}$$

where $b_{jk} := \int_0^d (1 + u\gamma)^{-2}\, \chi_j\, \chi_k\, du$ and $V_{jk} := \int_0^d V\, \chi_j\, \chi_k\, du.$

We shall use the following standard Hilbert spaces. Let $w(s) := (1 + s^2)^{1/2}$ and n any integer, then $\mathcal{H}^n$ denotes $L^2(\mathbb{R}, w^{2n}\, ds)$ and $\hat{\mathcal{H}}^n$ its Fourier transform.

3. ASYMPTOTIC COMPLETENESS

The scattering problem for the curved waveguide means comparing H to the Hamiltonian of the straight strip, $H_0 = -\partial_s^2 - \partial_u^2$ with Dirichlet b.c. on $L^2(\mathbb{R} \times (0, d))$. This is possible even in the case when Γ has no asymptotes (which exist provided $\gamma(s) = \mathcal{O}(s^{-2-\varepsilon})$. To prove the asymptotic completeness, the smooth perturbations method may be used.

3.1. THEOREM. – *Assume* $(r1)$-$(r4)$ *and* (d), *then we wave operators* $\Omega_\pm(H, H_0)$ *exist, are complete and* $\sigma_{sc}(H) = \emptyset$.

Proof. – In the form sense, the difference of the two operators can be written as

$$H - H_0 = -\partial_s\,(b - 1)\,\partial_s + V = B^* A,$$

where $b := (1 + u\gamma)^{-2}$ and the operators $A, B : \mathcal{H} \to \mathbb{C}^2 \otimes \mathcal{H}$ are defined by

$$A := \begin{pmatrix} A_0 \\ A_1\,(-i\,\partial_s) \end{pmatrix}, \qquad B := \begin{pmatrix} B_0 \\ B_1\,(-i\,\partial_s) \end{pmatrix}$$

with $A_0 := |V|^{1/2}$, $B_0 := |V|^{1/2}\,\mathrm{sgn}\, V$, $A_1 := |b - 1|^{1/2}$ and $B_1 := |b - 1|^{1/2}\,\mathrm{sgn}\,(b - 1)$. Notice that the form domains of the operators

H_0 and H coincide, $D(H_0^{1/2}) = D(H^{1/2}) \subset D(A)$, and $D(H_0) \subset D(A)$, $D(H) \subset D(B)$.

Our goal is to show that the task can be reducted to a one-dimensional problem, allowing therefore a straightforward application of the standard smooth perturbations method [RS, Sch]. Doing so, we have to pay attention to the fact that contrary to the usual situation, both A and B are first-order differential operators.

The method can be used for energies outside the so-called set of critical points. In the one-dimensional case it is just the origin; in the strip it is replaced by the family of threshold values E_j, $j = 1, 2, \ldots$ We set $\Xi := \mathbb{R}\backslash\{E_1, E_2, \ldots\}$.

Introducing the weight factor $\varrho(s) := w(s)^{-(1+\varepsilon)/2}$, we have, owing to the assumption (d), $\| A_l\, \varrho^{-1} \|_\infty < \infty$ and $\| B_l\, \varrho^{-1} \|_\infty < \infty$ for $l = 0, 1$. Since the free resolvent expresses as $R_0(z) = \sum_j (-\partial_s^2 + E_j - z)^{-1} \mathcal{J}_j \mathcal{J}_j^*$, we have the estimate

$$\| A_l (-i\,\partial_s)^l R_0(z) \| \le \| A_l\, \varrho^{-1} \|_\infty \sup_j \| \varrho(s) (-i\,\partial_s)^l (-\partial_s^2 + E_j - z)^{-1} \|$$

for $l = 0, 1$, and the analogous relation for B_l. A simple argument based on the first resolvent identity then yields the existence of the operator-norm limit of $I - A[B\,R_0(x \pm ia)]^*$ as $a \to 0+$, whenever $x \in \Xi$. In a similar way, we get the inequality

$$\| A\,R_0(x \pm ia) \|^2 + \| B\,R_0(x \pm ia) \|^2 \le C_I\, a^{-1}, \qquad x \in I, \qquad a > 0,$$

for any compact interval $I \subset \Xi$. Furthermore one can check easily that the operator $AR_0(z)\,[BR_0(z_1)]^*$ is compact (in fact, trace class) for any $z, z_1 \in \mathbb{C}\backslash\mathbb{R}$, since the functions $A, B \in L^2$ due to the condition (d).

In order to use an abstract smoothness result [Sch, Thm. 10.2.2], it remains to check several technical conditions which allow us to handle the kernels of the operators $I - A[BR_0(x \pm i\,0)]^*$. This can be done again by an easy modification of the argument used in the one-dimensional case; the reason is that all the expressions in question contain the spectral projection $E_I(H_0)$ to a compact interval I, and therefore only a finite number of modes are involved. In analogy with [Sch, Thm. 10.5.1] we arrive at the requirement

$$\int_{\mathbb{R}} \sup_{0 \le u \le d} \{ | V(s, u) |^2 + | b(s, u) - 1 |^2 \} (1 + | s |)^\delta \, ds < \infty$$

for some $\delta > 1$ which is fulfilled in view of (d). ■

4. RESONANCES

For a thin strip, the main contribution to (5) comes from the diagonal operator in the transverse-mode decomposition

$$H^0 := T - \partial_u^2, \qquad T := -\partial_s^2 - \frac{1}{4}\gamma^2 \tag{6}$$

(not to be confused with H_0 of the previous section).

As it is well known, the eigenvalues of T, $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ are simple ([Wei, Thm. 8.29]) and due to the decay assumption (d), their number N is non-zero and finite; we denote by φ_n, $1 \le n \le N$, the corresponding eigenfunctions. If d is small enough, H^0 has therefore N embedded eigenvalues

$$E_{j,n} := E_j + \lambda_n, \qquad 2 \le j, \quad 1 \le n \le N \tag{7}$$

below the threshold of the j-th transverse mode for any $j \ge 2$. The aim of the present section is to show that they turn into resonances of the full Hamiltonian H, *i.e.*, poles of its resolvent (*see* Definition 4.3). To state the main theorem of this section we introduce the following notations for the perturbation $H - H^0$:

$$W := H - H^0 = -\partial_s (b-1)\,\partial_s + V + \frac{1}{4}\gamma^2,$$

$$W_{j,k} := \mathcal{J}_j^*\, W\, \mathcal{J}_k = -\partial_s\,(b_{j,k} - \delta_{j,k})\,\partial_s + V_{j,k} + \frac{1}{4}\gamma^2\,\delta_{j,k},$$

$$W_j := W_{j,j}$$

We shall use the formal expansion of W in powers of u,

$$W = \sum_{l=1}^{\infty} \frac{u^l}{l!}\, W^{(l)}, \tag{8}$$

which is actually convergent for u small enough (*see* Section 4.5). Notice that $W_{j,k}$ as well as $W^{(l)}$ are second-order differential operators in the variable s only; in particular one has

$$W^{(1)} := 2\,\partial_s\,\gamma\,\partial_s + \frac{1}{2}\,(\gamma^3 + \gamma''). \tag{9}$$

4.1. THEOREM. – *Assume* $(r1-4)$, (d) *and* $(a1,\,2)$. *For all sufficiently small* d *one has:*

(i) *each eigenvalue* $E^0 := E_j + \lambda_n$ *(for* $j \ge 2$*) of the diagonal operator* H^0 *gives rise to a resonance* E *of the operator* H, *with multiplicity one, whose position is given by a convergent series*

$$E = E^0 + \sum_{m=1}^{\infty} e_m,$$

where $e_m = \mathcal{O}(d^m)$ *as* $d \to 0$;

(ii) *the coefficient* e_1 *is analytic in* d *around* $d = 0$ *and*

$$\left.\begin{aligned} e_1 &= (W_j\,\varphi_n,\,\varphi_n) = \sum_{l=1}^{\infty} e_{1;\,l}\,d^l, \\ e_{1;\,l} &:= \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k\,(2\,\pi\,j)^{-2\,k}}{(l-2\,k+1)!}\,(W^{(l)}\,\varphi_n,\,\varphi_n); \end{aligned}\right\} \tag{10}$$

as usually $[x]$ *denotes the integer part of* x. *Furthermore, the second coefficient is given by*

$$e_2 = -\frac{d^2}{4}\,([(T-\lambda_n)^{-1}]^\wedge\,W^{(1)}\,\varphi_n,\,W^{(1)}\,\varphi_n) + \mathcal{O}(d^3), \tag{11}$$

where the hat denotes the reduced resolvent in the sense of [Ka].

(iii) *if an addition* $(a3)$ *is valid, then for any* μ *in* $(0,\,\eta)$ *there exists* C_μ *such that:*

$$|\,\mathrm{Im}\,e_2\,| \le C_\mu\,d^3\,e^{-2\,\mu\,\pi\,\sqrt{2\,j-1}\,d^{-1}}. \tag{12}$$

The proof (i) and (ii) of this theorem is the contents of Sections 4.1-4.5; (iii) is proven in Section 4.6.

4.2. R*emarks.* – (a) The resonances considered in the theorem describe transitions between different transverse modes. In contrast to that, the Hamiltonian may have other resonances (perturbations of the poles $E_j + \tau_m$ – *see* Sec. 4.2 below) which we call inherent since they are related to one mode only. However, the latter do not approach the real axis as $d \to 0$, and therefore the resonances of the theorem are expected to dominate for a sufficiently thin strip.

(b) We shall not consider in this paper the imaginary part of the higher coefficients e_m. They are also of interest, however, since there are indications they might have the same exponential asymptotic behaviour as e_2 contributing therefore to the leading behaviour of the full imaginary part of the resonance E.

(c) The term d^3 on the *rhs* (12) comes from the estimate; the power need not be preserved when we choose for η the limiting value which produces the optimal exponential decay – *cf.* the example of Section 4.7.

(d) The exact expression for the coefficient e_m in terms of the unperturbed quantities is given in the relation (17) below.

(e) The rate S of exponential decay for $\mathrm{Im}\,e_2$ in (12) as $d \to 0$ is in agreement with the heuristic semiclassical prediction [LL],

$$\begin{aligned}\frac{S}{d} &= 2\,\mathrm{Im}\int_0^{s_0} \left(\sqrt{E - V_j^0(s)} - \sqrt{E - V_{j-1}^0(s)}\right) ds \\ &= \frac{2\,\mu\pi}{d}\sqrt{2j-1} + \mathcal{O}\left(\frac{d^2}{(2j-1)\,\mu}\right),\end{aligned} \tag{13}$$

where $V_k^0 := -\frac{1}{4}\gamma^2 + E_k^0$, $E = E_j + \lambda_n + \mathcal{O}(d)$, and $s_0 = i\,\mu$ with $\mu < \eta$ where η is the imaginary part of the nearest singularity (pole, branching point, ...) of the integrated function at the lhs (13). Without specifying the structure of these singularities, the best result one can expect is an upper bound on $\mathrm{Im}\, e_2$, *i.e.*, a lower bound on the rate of exponential decay. On the other hand, in Example 4.7 below where the structure of singularities is known we are able to show that (13) is valid with $s_0 = i\,\eta$.

4.1. Resonances and complex scaling

We adopt the standard mathematical definition of resonances (*cf.* [AC], [RS, Sect. XII.6]:

4.3. DEFINITION. – Let $F_\psi(z) := ((H-z)^{-1}\psi, \psi)$ be defined for every ψ in $\mathcal{H}$ and $\mathrm{Im}\, z$ strictly positive. Suppose that for all ψ in a dense subset $\mathcal{A}$ of $\mathcal{H}$, the function F_ψ admits a meromorphic continuation from the open upper half plane to a domain of the lower half plane. Then any E in this domain which is a pole for F_ψ for some ψ in $\mathcal{A}$ is called a resonance *of* H.

4.4. R*emarks*. – (a) The usual definition of resonances requires to check whether the pole of F_ψ is not simultaneously a pole of $F_\psi^{(0)}(z) := ((H_0 - z)^{-1}\psi, \psi)$ where H_0 is the comparison operator for the scattering theory under consideration. In our case, however, H_0 defined in Section 2 produces no poles.

(b) Although we are convinced that the resonances of theorem 4.1 have indeed a strictly negative imaginary part, at least generically, this is not proven in the present article. Thus for the sake of rigour the "lower-half-plane" in the above definition must be understood as "closed lower half-plane".

The analytic continuation of F_ψ is constructed with help of an extra parameter θ. Let U_θ be the following scaling transformation on $\mathcal{H}$:

$$(U_\theta\,\psi)(s, u) := e^{\theta/2}\,\psi(e^\theta s, u), \qquad \theta \in \mathbb{R}.$$

The operators U_θ are unitary and generate the family:

$$H_\theta := U_\theta\, H U_\theta^{-1} = -e^{-2\theta}\,\partial_s\, b_\theta\,\partial_s + V_\theta - \partial_u^2, \qquad \theta \in \mathbb{R},$$

where $b_\theta(s, u) := b(e^\theta s, u)$ and $V_\theta(s, u) := V(e^\theta s, u)$. We shall consider it as a perturbation to

$$H_\theta^0 := U_\theta H^0 U_\theta^{-1} = T_\theta - \partial_u^2, \quad T_\theta := -e^{-2\theta} \partial_s b_\theta^0 \partial_s + V_\theta^0, \quad \theta \in \mathbb{R},$$

where $b_\theta^0(s, u) := b_\theta(s, 0) = 1$, $V_\theta^0(s, u) := V_\theta(s, 0) = -\frac{1}{4}\gamma_\theta^2$ and $\gamma_\theta(s) := \gamma(e^\theta s)$.

Both of these two families extend to the sector $\mathcal{M}_\alpha$ as selfadjoint holomorphic families of type A of m-sectorial operators (*cf.* [Ka, Secs. VII.2 and V.3] for these definitions). This is easily seen from the fact that b_θ is sectorial and V_θ is bounded for all θ in $\mathcal{M}_\alpha$ due to $(a1, 2)$. Notice that $D(H_\theta^0) = D(H_\theta) = D(H)$ as defined in the preliminaries. Finally one can show (as, *e.g.*, in [Hu]) that there is a one-to-one correspondence between the discrete eigenvalues of $\{H_\theta : 0 \leq \operatorname{Im}\theta < \alpha\}$ and the resonances of Definition 4.3 provided $\mathcal{A}$ is chosen as the set of analytic vectors of the generator of the dilation group $\{U_\theta : \theta \in \mathbb{R}\}$.

Finally let us recall that the algebraic multiplicity of an eigenvalue of H_θ is usually called the *multiplicity* of the corresponding resonance.

Thus we shall look in the sequel for eigenvalues of H_θ with $0 < \operatorname{Im}\theta < \alpha$. This will be done perturbatively starting form discrete eigenvalue of H_θ^0, and therefore it requires a detailed analysis of

4.2. The structure of the spectrum of H_θ^0

Since T_θ and $-\partial_u^2$ commute, one has

$$\sigma(H_\theta^0) = \bigcup_{j=1}^{\infty} \{E_j + \sigma(T_\theta)\}; \tag{14}$$

due to our decay assumption $(a2)$ on γ_θ, V_θ^0 is relatively compact with respect to $-\partial_s^2$ in $L^2(\mathbb{R})$ (*cf.* [RS, Sect. XIII.14] and therefore falls into the well known class of dilation analytic potentials ([AC], [RS]). Thus we may refer to standard arguments when discussing such operators. In particular, $\sigma(T_\theta)$ has the following structure:

$$\left.\begin{aligned} \sigma(T_\theta) = e^{-2\theta}\,\mathbb{R}_+ \cup \{\lambda_1, \dots, \lambda_N\} \cup \{\nu_1, \dots, \nu_M\}, \\ N < \infty, \quad M \leq \infty, \end{aligned}\right\} \tag{15}$$

where the ν_m are possible resonances of T (with a non-zero imaginary part since T cannot have embedded eigenvalues in the continuum due to our decay assumption (d) on γ – *cf.* [RS, Sect. XIII.13]). The essential spectrum of T_θ is simply $e^{-2\theta}\,\mathbb{R}_+$.

The substantial feature we need to known on H_θ^0 is that its possible resonances $E_k + \nu_m$ are not too close to its eigenvalues $E_{j,n} := E_j + \lambda_n$ in the limit d going to zero, so that regular perturbation theory can be applied to compute the perturbation of $E_{j,n}$ by $H_\theta - H_\theta^0$. This is ensured by the following result.

4.5. LEMMA. – *For any θ, $0 < \operatorname{Im}\theta < \alpha$, there are two constants d_0 and C such that*

$$\inf\{\operatorname{dist}(E_k + \nu_m,\, E_j + \lambda_n) : k,\, j \geq 1\} \geq C > 0$$

if d is in $(0,\, d_0)$.

Proof. – By standard arguments [RS, Sect. XII.6] the resonances ν_m cannot belong to $\{z \in \mathbb{C} : 0 \leq \arg z < 2\pi - 2\operatorname{Im}\theta\}$. Using an elementary result on the location of resonances ([CE, D]) one also knows that all ν_m do not belong to $\{z \in \mathbb{C} : \| V_{i\operatorname{Im}\theta}^0 \| < \operatorname{dist}(z,\, e^{-2i\operatorname{Im}\theta}\mathbb{R}_+)\}$. These two results together with the fact that E_j behaves like d^{-2} as d goes to zero imply easily the assertion. ■

Given any unperturbed eigenvalue $E_{j,n}$ in the j-th transverse mode, Lemma 4.5 allows us to draw a contour in the resolvent set of H_θ^0 enclosing $E_{j,n}$, only, whose distance from the rest of the spectrum of H_θ^0 remains bounded below for all d small enough. Denote therefore $W_\theta := H_\theta - H_\theta^0$; obviously W_θ is H_θ^0-bounded by the closed graph theorem since W_θ is defined on $D(H_\theta^0)$ and the resolvent set of H_θ^0 is not empty (*cf.* Sections 4.1 and 4.2). The symbol for W_θ must not be confused with the one for W_j. This qualitative information is not yet sufficient, however, to ensure convergence of the perturbation expansions; we shall need also

4.3. Estimates on the perturbation of H_θ^0 by $H_\theta - H_\theta^0$

Suppose that $E^0 := E_{j,n}$ is an eigenvalue of H_θ^0, furthermore, let $R_\theta^0(z) := (H_\theta^0 - z)^{-1}$ and $\hat{R}_\theta^0(E^0)$ be the reduced resolvent of H_θ^0 at E^0, respectively. Then we have

4.6. LEMMA. – *Assume* $(a1,\, 2)$ *and let* Γ *be a contour enclosing a single eigenvalue* $E^0 := E_{j,n}$ *of* H_θ^0. *Then for any* $0 < \operatorname{Im} < \alpha$, *there exist* $d_{\Gamma,\theta}$, $C_\theta^{(1)}$ *and* $C_{\Gamma,\theta}^{(2)}$ *which do not depend on j and n, such that for any d in* $(0,\, d_{\Gamma,\theta})$ *one has*

(i) $\| W_\theta(-\partial_s^2 + 1)^{-1} \| \leq C_\theta^{(1)} d$

(ii) $\sup\{ \| (-\partial_s^2 + 1) R_\theta^0(z) \| : z \in \Gamma\}$
$\leq C_{\Gamma,\theta}^{(2)}$ *and* $\| (-\partial_s^2 + 1) \hat{R}_\theta^0(E^0) \| \leq C_{\Gamma,\theta}^{(2)}$.

Proof. – To simplify the notation we drop the index θ when it is not necessary. To prove (i) we observe that in the form sense on $D(-\partial_s^2 \otimes 1)$ one has

$$\begin{aligned}
|W|^2 &= |-\partial_s (b-b^0)\,\partial_s + V - V^0|^2 \\
&\le 2\,|\partial_s (b-b^0)\,\partial_s|^2 + 2\,|V-V^0|^2 \\
&\le 2\,|(b-b^0)\,\partial_s^2 + (b-b^0)'\,\partial_s|^2 + 2\,|V-V^0|^2 \\
&\le 4\,|(b-b^0)\,\partial_s^2|^2 + 4\,|b'\,\partial_s|^2 + 2\,|V-V^0|^2 \\
&\le 4\,\|b-b^0\|_\infty^2\,\partial_s^4 - 4\,\|b'\|_\infty^2\,\partial_s^2 + 2\,\|V-V^0\|_\infty^2
\end{aligned}$$

which is a quadratic form with the argument $(-\partial_s^2,\,1)$ and the matrix

$$\begin{pmatrix} 4\,\|b-b^0\|_\infty^2 & 2\,\|b'\|_\infty^2 \\ 2\,\|b'\|_\infty^2 & 2\,\|V-V^0\|_\infty^2 \end{pmatrix}$$

the modulus of its maximal eigenvalue is then estimated by $\max\{4\,\|b-b^0\|_\infty^2,\ 2\,\|V-V^0\|_\infty^2\} + 2\,\|b'\|_\infty^2$ which is for a given θ bounded by $(C_\theta^{(1)}\,d)^2$. Hence we have proven (i).

To estimate the operator $(-\partial_s^2+1)\,R_\theta^0(z)$ we use its transverse-mode decomposition $\bigoplus_{k=1}^{\infty}(-\partial_s^2+1)\,(T_\theta+E_k-z)^{-1}$. If $k=j$ we denote for a moment $C_{\Gamma,\theta}^{(2)} := \sup\{\|-(\partial_s^2+1)\,(T_\theta+E_j-z)^{-1}\| : z\in\Gamma\}$ which is finite since $z\mapsto(-\partial_s^2+1)\,(T_\theta+E_j-z)^{-1}$ is bounded and continuous on Γ and d-independent because the position of Γ with respect to $E_{j,n}$ is d-independent. If $k\ne j$ we use a perturbative argument; one has

$$\begin{aligned}
(-\partial_s^2+1)\,(T_\theta+E_k-z)^{-1} &= (-\partial_s^2+1)\,(-e^{-2\theta}\,\partial_s^2+E_k-z)^{-1} \\
&\quad\times(1+V_\theta^0\,(-e^{-2\theta}\,\partial_s^2+E_k-z)^{-1})^{-1}.
\end{aligned}$$

The norm of the product of the first two terms of the *rhs* of the above formula is explicitely computable by the spectral theorem and tends to zero uniformly with respect to k and z in Γ as d tends to zero for every fixed θ. Since V_θ^0 is bounded the same holds for $V_\theta^0\,(-e^{-2\theta}\,\partial_s^2+E_k-z)^{-1}$ and therefore also for $(-\partial_s^2+1)\,(T_\theta+E_k-z)^{-1}$. So when d tends to zero, then only the component $k=j$ of $(-\partial_s^2+1)\,R_\theta^0(z)$ plays a role; this yields the first assertion of (ii). The second one can be proved in a similar way; $C_{\Gamma,\theta}^{(2)}$ is finally the bigger one of the two bounds. The above analysis shows also that $C_\theta^{(1)}$, $C_{\Gamma,\theta}^{(2)}$ and $d_{\Gamma,\theta}$ can be chosen independently of j and n. ■

We are now ready to perform the

4.4. Perturbation expansions of the resonances of H_θ

Let $e^{-2\theta}$ be in $\mathcal{M}_\alpha$ with $\operatorname{Im}\theta$ positive and Γ a contour in the resolvent set of H_θ^0 enclosing a single eigenvalue $E^0 := E_{j,n}$, $j \geq 2$ and $1 \leq n \leq N$; we denote by P_θ^0 the corresponding eigenprojection. In view of Lemma 4.6, $\| W_\theta R_\theta^0(z) \|$ can be made smaller than one uniformly on Γ for all d small enough; so for such d, Γ belongs also to the resolvent set of H_θ. Thus one can construct the eigenprojection of H_θ associated with Γ:

$$P_\theta := \frac{i}{2\pi} \int_\Gamma (H_\theta - z)^{-1}\, dz$$

and compute perturbatively the difference

$$P_\theta - P_\theta^0 = -\frac{i}{2\pi} \int_\Gamma R_\theta^0(z)\, W_\theta\, R_\theta^0(z)\,(1 + W_\theta\, R_\theta^0(z))^{-1}\, dz.$$

Using Lemma 4.6 again we obtain

$$\| P_\theta - P_\theta^0 \| \leq \frac{|\Gamma|}{2\pi} \frac{C_\theta^{(1)}\,(C_{\Gamma,\theta}^{(2)})^2\, d}{1 - C_\theta^{(1)}\, C_{\Gamma,\theta}^{(2)}\, d}$$

which shows that for d small enough P_θ and P_θ^0 have same dimension (notice that $C_{\Gamma,\theta}^{(2)}$ is *a fortiori* a bound on $R_\theta^0(z)$). Since $\dim P_\theta^0$ is one we have shown that H_θ posseses in the interior of Γ an eigenvalue E of multiplicity one. Standard arguments show that $\operatorname{Im} E$ cannot be positive, so E is a resonance of H by Definition 4.3. To compute $E - E^0$ perturbatively we use the formula

$$\left.\begin{aligned} E - E^0 &= \sum_{m=1}^{\infty} e_m, \\ e_m &:= \frac{i}{2\pi} \operatorname{tr} \int_\Gamma (z - E^0)\, R_\theta^0(z)\, (-W_\theta\, R_\theta^0(z))^m\, dz. \end{aligned}\right\} \tag{16}$$

Performing the contour integral in (16) leads to (*cf.* [Ka, Sect. II.2])

$$e_m = \sum_{p_1+p_2+\ldots+p_m=m-1} \frac{(-1)^m}{m} \operatorname{tr} \prod_{i=1}^{m} W_\theta\, S^{(p_i)} \tag{17}$$

where $S^{(p_i)}$ denotes $-P_\theta^0$ if $p_i = 0$ and $(\hat{R}_\theta^0(E^0))^{p_i}$ otherwise. Since each term of the product of the rhs of (17) contains at least one projection P_θ^0 the trace can be estimated by the norm. We have

$$\| W_\theta\, P_\theta^0 \| \leq \frac{1}{2\pi} \int_\Gamma \| W_\theta\, R_\theta^0(z) \|\, dz \leq \frac{|\Gamma|}{2\pi}\, C_\theta^{(1)}\, C_{\Gamma,\theta}^{(2)}\, d,$$

$$\begin{aligned} \| W_\theta\, \hat{R}_\theta^0(E^0)^{p_i} \| &\leq \| W_\theta\, (-\partial_s^2 + 1)^{-1} \|\ \| (-\partial_s^2 + 1)\, \hat{R}_\theta^0(E^0) \|^{p_i} \\ &\leq C_\theta^{(1)}\, (C_{\Gamma,\theta}^{(2)})^{p_i}\, d \end{aligned}$$

for $p_i = 0$ and $p_i \neq 0$, respectively. Since there can be at most $m - 1$ projections among the $S^{(p_i)}$ we get

$$\left\| \prod_{i=1}^{m} W\, S^{(p_i)} \right\| \leq (C_\theta^{(1)})^m\, d^m\, (\tilde{C}_{\Gamma,\theta}^{(2)})^{2m-2}$$

where we have set $\tilde{C}_{\Gamma,\theta}^{(2)} := \max\left\{1,\, C_{\Gamma,\theta}^{(2)},\, \frac{|\Gamma|}{2\pi}\, C_{\Gamma,\theta}^{(2)}\right\}$. Finally the number of terms in the sum of the rhs of (17) is $\binom{2m-2}{m-1}$ so we get

$$|e_m| \leq \left(\sqrt{C_\theta^{(1)}}\, \tilde{C}_{\Gamma,\theta}^{(2)}\right)^{2m} \binom{2m-2}{m-1} \frac{d^m}{m}. \tag{18}$$

The rhs of (18) is easily seen to define a convergent series; this shows that the rhs of (16) defining $E - E^0$ is absolutely convergent. We conclude this section by analysing more precisely

4.5. The behaviour of the first two coefficients with respect to d

According to (17) one has $e_1 = \operatorname{tr} W_\theta\, P_\theta^0$. Standard arguments show that the rhs of this equation is constant with respect to θ and thus:

$$e_1 = \operatorname{tr} W\, P^0 = (W\, \varphi_n \otimes \chi_j,\, \varphi_n \otimes \chi_j) = (W_j\, \varphi_n,\, \varphi_n). \tag{19}$$

To find the behaviour of e_1 with respect to d we remark that for d small enough W can be regarded as a bounded multiplication operator from $L^2((0, d), \hat{\mathcal{H}}^2)$ into $L^2((0, d), L^2(\mathbb{R}))$ by an analytic operator-valued function in u. So we may expand W around $u = 0$; this will give the expansion (8) where the $W^{(l)}$ are bounded operators from $\hat{\mathcal{H}}^2$ into $L^2(\mathbb{R})$. From this expansion we deduce immediately that

$$W_j = \sum_{l=1}^{\infty} \frac{\langle u^l \rangle_j}{l!}\, W^{(l)}$$

where we have denoted $(u^l\, \chi_j,\, \chi_j)$ by $\langle u^l \rangle_j$. Next we observe that W_j is an analytic (operator-valued) function in the variable d around $d = 0$ and thus the same holds true for e_1; this shows that the expansion in (10) is convergent.

Consider now the second coefficient e_2; according to (17) one has for non-real θ: $e_2 = \operatorname{tr} P_\theta^0\, W_\theta\, \hat{R}_\theta^0(E^0)\, W_\theta\, P_\theta^0$. Standard arguments used to derive the Fermi golden rule (*cf.* [RS, Sec. XII.6]) apply here also and thus

$$e_2 = -\operatorname{tr} P^0\, W\, \hat{R}^0(E^0 + i\,0)\, W\, P^0. \tag{20}$$

To find the leading behaviour of e_2 with respect to d we expand it with respect to the transverse modes of the waveguide:

$$e_2 = -\sum_{k=1}^{\infty} ([(T + E_k - E_j - \lambda_n - i0)^{-1}]^\wedge W_{k,j}\,\varphi_n, W_{k,j}\,\varphi_n).$$

We split this sum in three parts $e_2 = e_2^{k<j} + e_2^{k=j} + e_2^{k>j}$ and analyse then separately. The coefficient $e_2^{k=j}$ which depends on d through W_j only, is therefore analytic in d around $d = 0$ with the leading behaviour as in (11). As for $e_2^{k>j}$, all $E_k^0 := E_j + \lambda_n - E_k$ fall into the resolvent set of T so we estimate it easily as

$$0 \leq -e_2^{k>j} \leq \sum_{k=j+1}^{\infty} \operatorname{dist}(E_k^0, \sigma(T))^{-1} \| W_{j,k}\,\varphi_n \|^2 = \mathcal{O}(d^4);$$

notice that $e_2^{k\geq j}$ is real.

To study $e_2^{k<j}$ it is convenient to introduce the following operator on $\mathcal{H}^1$:

$$\omega(E + i0) := (1 + V^0(-\partial_s^2 - E - i0)^{-1})^{-1}, \qquad E > 0$$

which is easily seen to be bounded for E large enough due to the assumption (d); moreover, $\omega(E + i0)$ tends to one as $E \to \infty$. Then each term of $e_2^{k<j}$ may be rewritten as

$$e_2^k = -((-\partial_s^2 - E_k^0 - i0)^{-1}\,\omega(E_k^0 + i0)\,W_{k,j}\,\varphi_n, W_{k,j}\,\varphi_n) \tag{21}$$

where the above scalar product should be understood as the duality between $\mathcal{H}^1$ and $\mathcal{H}^{-1}$. Due the assumption (d) one can show easily that the bound obtained in the proof of Lemma 4.6, $W(-\partial_s^2 + 1)^{-1} = \mathcal{O}(d)$, holds true for $w\,W(-\partial_s^2 + 1)^{-1}$ [*see* also the proof of Lemma 4.7 (iii)]. We recall that $w(s) := (1 + s^2)^{1/2}$. Using finally the fact that φ_n is in $\hat{\mathcal{H}}^2$ and the well known bound (*cf.* [RS, XIII.8])

$$\| w^{-1}(-\partial_s^2 - E - i0)^{-1} w^{-1} \| \leq \frac{\pi}{2\sqrt{E}}. \tag{22}$$

we get that $e_2^k = \mathcal{O}(d^3)$. The same is true for $e_2^{k<j}$ since it contains only finitely many such terms; therefore gathering all the above results we obtain that the leading behaviour of e_2 is given by the one of $e_2^{k=j}$.

4.6. Exponential smallness of $\operatorname{Im} e_2(d)$

The analysis of Section 4.5 shows that only $e_2^{k<j}$ can contribute to the imaginary part of e_2. Let us consider each term of this sum. We shall use

the following two formulas. First

$$\operatorname{Im}(T-E-i0))^{-1} = \omega(E+i0)^* \operatorname{Im}(-\partial_s^2-E-i0)^{-1}\,\omega(E+i0), \quad E>0 \tag{23}$$

which follows from simple algebra and the fact that V^0 is real, and second

$$\operatorname{Im}(-\partial_s^2 - E - i0)^{-1} = \frac{\pi}{2\sqrt{E}} \sum_{\varepsilon=\pm} \tau_E^\varepsilon, \qquad E>0, \quad \varepsilon = \pm 1 \tag{24}$$

where τ_E^ε denotes the trace operator on $\hat{\mathcal{H}}^1$ defined by:

$$\tau_E^\varepsilon\,\varphi := \hat{\varphi}(\varepsilon\sqrt{E}), \qquad \varepsilon = \pm 1, \qquad E>0, \quad \varphi \in \hat{\mathcal{H}}^1,$$

$\hat{\varphi}$ being the Fourier transform of φ. Using these relations we get

$$-\operatorname{Im} e_2^k = \frac{\pi}{2\sqrt{E_k^0}} \sum_{\varepsilon=\pm} |\,\tau_{E_k^0}^\varepsilon\,\omega(E_k^0+i0)\,W_{k,j}\,\varphi_n\,|^2 \tag{25}$$

Further we employ the unitary group of translation, $\{e^{i\sigma\rho},\ \sigma \in R\}$ where $\rho := -i\,\partial_s$, to define several families of operators:

$$T_\sigma := e^{i\sigma\rho}\,T\,e^{-i\sigma\rho} = -\partial_s^2 + V_\sigma^0, \qquad V_\sigma^0(s) := V^0(s-\sigma)$$

and similarly, W_σ, $\omega_\sigma(E+i0)$ denote respectively the image of W and $\omega(E+i0)$ by $e^{i\sigma\rho}$ (the subscript σ will always denote the image by $e^{i\sigma\rho}$).

4.7. Lemma. – *Under the assumptions* (a3), (r4) *and* (d) *one has for all σ in S_η:*

(i) *T_σ extends to a type-A selfadjoint holomorphic family,*

(ii) $\sigma_{\text{ess}}(T_\sigma) = \mathbb{R}_+$,

(iii) *W_σ is bounded by $c_\sigma\, d$ as an operator from $\hat{\mathcal{H}}^2$ into $\mathcal{H}^1$,*

(iv) *$\omega_\sigma(E+i0)$ is bounded for E large enough as an operator on $\mathcal{H}^1$, more precisely*

$$\|\,w\,\omega_\sigma(E+i0)\,w^{-1}\,\| \le \left(1 - \frac{\pi}{2\sqrt{E}}\,\|\,w^2\,V_\sigma^0\,\|\right)^{-1}.$$

We postpone the proof of this lemma to the end of this section. Since $\tau_E^\varepsilon\,e^{-i\sigma\rho} = e^{-i\varepsilon\sigma\sqrt{E}}\,\tau_E^\varepsilon$, one gets

$$\tau_{E_k^0}^\varepsilon\,\omega(E_k^0+i0)\,W_{k,j}\,\varphi_n = e^{-i\varepsilon\sigma\sqrt{E_k^0}}\,\tau_{E_k^0}^\varepsilon\,\omega_\sigma(E_k^0+i0)\,W_{k,j,\sigma}\,\varphi_{n,\sigma}. \tag{26}$$

From the previous lemma we see that the rhs of (26) is analytic in the strip S_η. Since (26) shows that this rhs is also constant with respect to σ we may choose σ in S_η, at our will. Taking $\sigma := -i\,\varepsilon\mu$ with $0 < \mu < \eta$, we obtain

$$|\,lhs \text{ of } (26)\,|^2 \leq \frac{e^{-2\mu\sqrt{E_k^0}}\, c_{-i\,\varepsilon\mu}^2\, d^2\, \|\,(-\partial_s^2 + 1)\,\varphi_n\,\|^2}{\left(1 - \frac{\pi}{2\sqrt{E_k^0}}\,\|\,w^2\, V_{-i\,\varepsilon\mu}^0\,\|\right)^2}$$
$$\leq C_\mu^{(k)}\, d^2\, e^{-2\mu\sqrt{E_k^0}}$$

for d small enough where the $C_\mu^{(k)}$ is a constant which depends only on μ and k. Summing up for all values of k and ε and multiplying by the prefactor $\frac{\pi}{2\sqrt{E_k^0}}$ will give (12) of theorem 4.1; notice that the smallest exponential decay rate comes from the term $k = j - 1$.

We conclude this section with the

Proof of Lemma 4.7. – (i) is obvious since by $(a3)$, V_σ^0 is bounded and analytic for all σ in S_η. To prove (ii) we notice that V_σ^0 is relatively compact with respect to $-\partial_s^2$; this is a consequence of $(a3)$ and [RS, Thm. XI.20]. This fact and the simple form of $\sigma_{\text{ess}}(-\partial_s^2)$ ensures that $\sigma_{\text{ess}}(T_\sigma)$ is $\mathbb{R}_+$. To prove (iii) one has to mimick the estimate of $W_\theta\,(-\partial_s^2 + 1)^{-1}$ in the proof of Lemma 4.6 with a slight modification due to the presence of the weight function w. It yields

$$|\,W_\sigma\,|^2 \leq 4\,\|\,w\,(b_\sigma - b^0)\,\|_\infty^2\,\partial_s^4 - 4\,\|\,w b_\sigma'\,\|^2\,\partial_s^2 + 2\,\|\,w\,(V_\sigma - V_\sigma^0)\,\|_\infty^2$$

in form sense on $\hat{\mathcal{H}}^2 \otimes L^2(0,\ d)$. Since $b_\sigma - b^0$, b_σ', $V_\sigma - V_\sigma^0$ decay sufficiently fast at infinity to control w, the rest of the estimate is obvious. The proof of (iv) is also obvious once one knows the formula (22).

4.7. Example

Consider the curve Γ given parametrically by the formulae

$$x(s) = \rho \int_0^{\arctan \sinh (s/\rho)} \frac{\cos \alpha y}{\cos y}\, dy,$$

$$y(s) = \rho \int_0^{\arctan \sinh (s/\rho)} \frac{\sin \alpha y}{\sin y}\, dy,$$

which can be integrated for particular values of α. The corresponding curvature is $\gamma(s) = \frac{\alpha}{\rho} \cos\text{h}^{-1}\left(\frac{s}{\rho}\right)$ so Γ has the minimum curvature radius ρ/α and the total bending angle $\pi\alpha$. The condition $(r4)$ requires $0 < \alpha \leq 1$, otherwise we suppose the strip is supported by a multisheeted surface. The eigenvalue problem for T is explicitly solvable ([LL], Sec. 23): as long as $\alpha \leq 2\sqrt{2}$, there is just one eigenvalue, $\lambda_1 = -(\sigma/\rho)^2$, where

$$\sigma := \frac{1}{2}(-1 + \sqrt{1+\alpha^2}),$$

corresponding to $\varphi_1(s) = \left(\cosh\frac{s}{\rho}\right)^{-\sigma}$. We find easily $\|\varphi_1\|^2 = \sqrt{\pi}\,\rho\,\Gamma(\sigma)/\Gamma\left(\sigma+\frac{1}{2}\right)$ ([GR], 3.985, 8.335) and

$$W^{(1)}\varphi_1 = \frac{-\alpha\,(2\,\sigma+1)}{\rho^3}\left(\cosh\frac{s}{\rho}\right)^{-\sigma-3} + \frac{\alpha\,(\alpha^2+1)}{2\,\rho^3}\left(\cosh\frac{s}{\rho}\right)^{-\sigma-1}$$

so the first perturbation-series term (10) is given by

$$e_1 = \frac{\alpha^2\sqrt{\pi}}{2\,\rho^3}\,\frac{\Gamma\left(\frac{\sigma+3}{2}\right)}{\Gamma(\sigma+2)}\left[2\,\sigma+1-\frac{\sigma+1}{(\sigma+2)\,(\sigma+3)}\right]; \qquad (27)$$

it is positive for any values of α, ρ. On the other hand, the scattering solutions are

$$\begin{aligned}\psi_{\pm\kappa}(s) &= \text{const}\,(1-\tanh^2(s/\rho))^{\mp\kappa\rho/2}\\ &\times F\left(\mp i\,\kappa\rho-\sigma,\ \mp i\,\kappa\rho+\sigma+1,\ \mp i\,\kappa\rho+1;\ \frac{1-\tanh(s/\rho)}{2}\right); \end{aligned}\qquad (28)$$

they determine the Fermi-rule contribution to the resonance width by

$$\operatorname{Im} e_2 = -\pi\sum_{k=1}^{j-1}\sum_{\nu=\pm}\left|\sum_{n=1}^{\infty}\frac{\langle u^n\rangle_{k,j}}{n!}\,(W^{(n)}\varphi_1,\,\psi_{\nu,\kappa})\right|^2 \qquad (29)$$

with $\kappa := \sqrt{\frac{\pi^2\,(j^2-k^2)}{d^2}+\lambda_1}$ and $\langle u^n\rangle_{k,j} := (u^l\,\chi_k,\,\chi_j) := b_n^{(k,j)}\,d^n$ where the coefficients $b_n^{(k,j)}$ are easily computed. For simplicity, we restrict our attention to the case $\alpha = 2\sqrt{2}$ when the "background scattering" is reflectionless [MT] and (28) reduces to

$$\psi_{\pm\kappa}(s) = \frac{1}{2\sqrt{\pi\kappa}}\,\frac{1\mp i\,(\kappa\rho)^{-1}\tanh(s/\rho)}{\sqrt{1+(\kappa\rho)^{-2}}}\,e^{\mp i\kappa\rho\,s}.$$

Using the relation

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi s} \left(\cosh \frac{s}{\rho}\right)^{-\beta} ds$$
$$= \sqrt{2\pi} \frac{\rho^{\beta} |\xi|^{\beta-1}}{\Gamma(\beta)} e^{-\pi\rho|\xi|/2} (1 + \mathcal{O}(|\xi|^{-1})) \quad \text{as } |\xi| \to \infty$$

([GR], 3.985, 8.328) we find that the leading term corresponds to $k = j - 1$ and equals

$$-\mathrm{Im}\, e_2 = C(j) \rho d^{-3} e^{-\pi^2 \rho \sqrt{2j-1}/d} (1 + \mathcal{O}(d))$$

where $C(j)$ is a numerical constant depending only on j.

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