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On the asymptotic properties of quantum dynamics in the presence of a fractal spectrum

by

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ABSTRACT. – Asymptotic estimates for the dynamics induced in a separable Hilbert space by a discrete unitary group with a purely continuous spectrum are derived. They consist of upper and lower bounds for suitably defined exponents of growth. These bounds involve the capacity of the spectrum and the Hölder exponent of the spectral measure.

RÉSUMÉ. – On obtient une estimation asymptotique pour la dynamique induite par un groupe unitaire discret sur un espace de Hilbert séparable, ayant un spectre purement continu. Elle fournit une borne supérieure et une borne inférieure pour un exposant de croissance convenable. Ces bornes font intervenir la capacité du spectre et l'exposant de Hölder de la mesure spectrale.

1. INTRODUCTION

We consider a discrete-time evolution defined in a separable Hilbert space $\mathcal{H}$ by iterating the action of a given unitary operator U with a purely continuous spectrum; given an initial vector ψ, let $\psi(t) = U^t \psi$, with $t \in Z$.

The commonplace statement, that continuity of the spectrum of U enforces some sort of "unbounded spreading" of the "wavepacket" $\psi(t)$, is a qualitative summary for a number of well-known mathematical properties, the simplest of which is perhaps that $\forall t$ the subspace Λ_t spanned by

$\{\psi(s)\}_{0\leq s\leq t}$ has dimension $t+1$. This is just an example in the vast class of results known as RAGE (Ruelle-Amrein-Georgescu-Enss) [1] theorems, which relate the localization properties of the motion of a system in a configuration or phase space to the spectral properties of the Hamiltonian operator.

While RAGE theorems are usually concerned with properties valid in the infinite time limit, we will be concerned here with the description of the way the evolution attains this limit. This description involves the behaviour of the amplitudes $c_n(t)$ of the expansion of $\psi(t)$ over a given complete orthonormal basis $B = \{e_n\}_{n\in N}$. RAGE theorems imply that $\forall n \geq 1$:

$$\lim_{t\to\infty} p_n(t) \equiv \lim_{t\to\infty} \frac{1}{(1+t)} \sum_{s=0}^{t} |c_n(s)|^2 = 0$$

Since $\sum p_n(t) = const. = ||\psi||^2$, $\forall t$, the (non-normalized) distribution defined on the integer lattice by $p_n(t)$ spreads indefinitely as $t \to \infty$. If the moments of this distribution are finite, they must then diverge for $t \to \infty$. We seek asymptotic estimates of their growth in terms of generalized dimensions characterizing the fractal structure of the spectrum. Such estimates have a definite interest in various sectors of quantum physics [2], for instance in the study of low-temperature transport properties of particles in quasi-crystals or in disordered solids. Lower bounds on the spreading of the distribution $p_n(t)$ in terms of the Hölder exponent of the spectral measure of the vector ψ can be found in full generality ([3], [4], [5]); instead, upper bounds appear to depend on the specific choice of the base B.

We will define various quantities to gauge the spreading rate of quantum evolution. One of these will be an intrinsic exponent of growth, which does *not* depend on the basis chosen; we shall show that this exponent is bounded from below by the Hölder exponent of the spectral measure, and from above by the capacity of the support of the measure itself.

2. DEFINITIONS

The asymptotic algebraic growth in time of moments and of other quantities as a function of time can be characterized by a number of different parameters. Given a nonnegative sequence $g \equiv \{g_t\}$ tending to $+\infty$ in the limit $t \to +\infty$, obvious choices are:

$$\overline{\gamma}(g) = \lim \sup_{t\to+\infty} \frac{\log g_t}{\log t} \qquad \underline{\gamma}(g) = \lim \inf_{t\to+\infty} \frac{\log g_t}{\log t} \tag{1}$$

Another useful parameter can be introduced using a technical tool employed for different purpose in the study of singular measures [6]. This is the discrete Mellin transform of a sequence g, which is formally defined by:

$$\mathcal{M}(g, \beta) = \sum_{t=1}^{\infty} t^{-1-\beta} g_t$$

The aboves series is convergent for all complex β in a right half-plane $\{\Re e(\beta) > \overline{\beta}\}$. This defines a "convergence abscissa" $\overline{\beta} \equiv \overline{\beta}(g)$ which satisfies $\overline{\gamma}(g) \geq \overline{\beta}(g) \geq \underline{\gamma}(g)$. Though not exploited in the present context as yet, the usefulness of the Mellin transform lies with the investigation of its complex singularities.

Having introduced these various characterizations of growth exponent, we point out that the results we are about to prove will not make specific reference to any of them, but apply to all. In fact by "growth exponent" of a sequence g we shall mean a real number $\beta(g)$ such that (i) $\beta(g) = \alpha$ if $g_t \sim const. \cdot t^{\alpha}$, and (ii) $\beta(g) \geq \beta(g')$ if $g_t \geq g'_t$ eventually, equality holding if $g_t = g'_t$ eventually.

Now let B be an orthonormal system (OS) of vectors in $\mathcal{H}$, and let $\psi \in \mathcal{H}$. We shall always assume that B and ψ have the following property: there exists $\Delta > 0$ such that for all $t \geq 0$, $\|P_B \psi(t)\| > \Delta$, P_B being the projection onto the closed subspace spanned by B; this is clearly the case when B is complete. If B and ψ are such that the moments

$$M_m(\psi, t, B) = \sum_{n=1}^{\infty} n^m p_n(t) \tag{2}$$

are finite for all t, then the above assumption along with continuity of the spectrum of U imply that these moments diverge in the limit $t \to +\infty$. In fact for any positive integer N a time t_0 can be found, such that $p_n(t) < \dfrac{\Delta}{2N}$ for $\forall n < N$ and for $\forall t > t_0$. Then, for such t, $M_m(\psi, t, B) > N^m \Delta/2$. Thus there is a (possibly infinite) growth exponent $\beta_m(\psi, B)$ associated with the m-th moment, and $\beta_m(\psi, B) \geq \beta_n(\psi, B)$ for $m \geq n > 0$.

An additional exponent β_0 was introduced in [4], which has the advantage of being defined $\forall \varphi \in \mathcal{H}$. For $\varepsilon \in (0, 1)$ and $\psi \neq 0$ we define

$$\overline{n}(\varepsilon, \psi, t, B) = \min \left\{ n : \sum_{k \geq n}^{\infty} p_k(t) < \varepsilon \|\psi\|^2 \right\} \tag{3}$$

This definition entails some immediate consequences that are summarized in the following Lemma:

LEMMA 1. – (i) $\varepsilon < \varepsilon' \Rightarrow \overline{n}(\varepsilon, \psi, t, B) \geq \overline{n}(\varepsilon', \psi, t, B)$;

(ii) $\lim_{t\to\infty} \overline{n}(\varepsilon, \psi, t, B) = +\infty$;

(iii) *if the m-th moment* (2) *is finite then*

$$\overline{n}(\varepsilon, \psi, t, B) \leq \left\{ \frac{M_m(\psi, t, B)}{\varepsilon \|\psi\|^2} \right\}^{\frac{1}{m}} \tag{4}$$

For given ε, ψ let us denote the growth exponent of the sequence $\overline{n}(\varepsilon, \psi, t, B)$ by $\overline{\beta}(\psi, \varepsilon, B)$. Point (i) in the Lemma entails that $\overline{\beta}(\psi, \varepsilon, B)$ is non decreasing for $\varepsilon \searrow 0$; therefore we define

$$\beta_0(\psi, B) = \lim_{\varepsilon\to 0+} \overline{\beta}(\psi, \varepsilon, B) = \sup_{0<\varepsilon<1} \overline{\beta}(\psi, \varepsilon, B) \tag{5}$$

From this definition and from (iii) of lemma 1 one easily deduces that

$$\beta_0(\psi, B) \leq m^{-1} \beta_m(\psi, B) \tag{6}$$

Whereas all the above defined growth exponents make reference to a given orthornormal system, it is possible to give an intrinsic measure for the growth of wavepackets, that does not rely on such choice. This can be done as follows.

For $\varepsilon \in (0, 1)$ let us consider a sequence $\{\sigma_t\}_{1\leq t\leq\infty}$ of spheres of radius ε, the t-th sphere having its centre in $\psi(t)$. For given t let us consider finite-dimensional subspaces spanned by t-ples of vectors $\xi_s \in \sigma_s$, $1 \leq s \leq t$; let $d_t(\varepsilon, \psi)$ be the minimum dimension of subspaces in the class. At fixed ε, $d_t(\varepsilon, \psi)$ is a non-decreasing sequence and we can associate to it a growth exponent $\nu(\varepsilon, \psi)$. Moreover, since at fixed t $d_t(\varepsilon, \psi)$ is non-decreasing as $\varepsilon \searrow 0$, $\nu(\varepsilon, \psi)$ is itself a non-decreasing function of ε. This preparatory work leads us to the definition of a new quantity, $\theta(\psi)$:

$$\theta(\psi) = \sup_{0<\varepsilon<1} \nu(\varepsilon, \psi) \tag{7}$$

We shall now look for bounds on the quantities β_0, β_m, $\theta(\psi)$.

3. LOWER BOUNDS

Estimates of this sort can be established under rather general assumptions on the scaling properties of the spectral measure μ_ψ of the vector ψ. Let us assume that the local scaling (Hölder) exponent:

$$\lambda(x, \psi) = \lim_{\delta\to 0+} \frac{\log \mu_\psi(I_\delta(x))}{\log \delta} \tag{8}$$

exists for μ_ψ-almost all points x in the spectrum S, where $I_\delta(x)$ is an interval of width δ centred at x. The value λ is a sort of local dimension, since the mass in a sphere of radius δ centred at x scales as δ^λ.

We will suppose that the spectral measure is such that

$$\lambda(x, \psi) = const. = \overline{\lambda}(\psi) \qquad \mu_\psi - a.e. \tag{9}$$

Then the following result holds:

THEOREM 1. – *Let B be an orthonormal system as specified in the above, and let the spectral measure μ_ψ satisfy* (8), (9). *Then, for $\varepsilon < \frac{\Delta}{2}$, $\overline{\beta}(\varepsilon, \psi, B) \geq \overline{\lambda}(\psi)$. Therefore also $\beta_0(\psi, B) \geq \overline{\lambda}(\psi)$.*

This result was proven in [4]. A proof is also presented in Appendix 1, because both the formulation and the definitions used here slightly differ from those of [4].

We can collect the results obtained so far:

$$\frac{\beta_m}{m} \geq \beta_0 \geq \overline{\beta}(\varepsilon) \geq \overline{\lambda}$$

Theorem 1 also provides a lower bound for $\theta(\psi)$:

THEOREM 2. – *If the spectral measure μ_ψ satisfy* (8), (9) *then $\theta(\psi) \geq \overline{\lambda}(\psi)$.*

Proof. – In the following we shall assume $\|\psi\| = 1$. Given $d(\varepsilon, \psi, t) = \delta$, let $\Gamma_{\varepsilon, \psi, t}$ be a finite set of vectors such that (i) the subspace spanned by $\Gamma_{\varepsilon, \psi, t}$ has dimension δ and (ii) $\forall s \leq t$ a vector $\xi_s^{(\varepsilon, t)} \in \Gamma_{\varepsilon, \psi, t}$ can be found such that $\|\psi(s) - \xi_s^{(\varepsilon, t)}\| < \varepsilon$. Let P_t denote projection onto the subspace Λ_t spanned by $\{\psi(s)\}_{0 \leq s \leq t}$ and for given s consider the sequence $\{u_t^{(s)}\}_{t \geq s}$ defined by $u_t^{(s)} = P_s \xi_s^{(\varepsilon, t)}$. This sequence belongs in the finite-dimensional subspace Λ_s, and $\forall t \geq s$ $u_t^{(s)}$ is ε-close to $\psi(s)$. We can now find a sequence of integers $\{t_k\}$ such that $\{u_{t_k}^{(1)}\}$ converges to a limit $z_1^{(\varepsilon)}$; from $\{t_k\}$ we can extract another subsequence $\{t_{k_l}\}$ such that $\{u_{t_{k_l}}^{(2)}\}$ converges to $z_2^{(\varepsilon)}$, and so on. In this way we generate a sequence $\{z_s^{(\varepsilon)}\}_{s \geq 1}$ with the following properties: (i) $\forall t$, $\|\psi(t) - z_t^{(\varepsilon)}\| < \varepsilon$; (ii) the (not necessarily distinct) vectors $\{z_s^{(\varepsilon)}\}_{s \leq t}$ span a subspace of dimension $d(\varepsilon, \psi, t)$.

By orthonormalizing the sequence $\{z_t^{(\varepsilon)}\}$ we obtain an orthonormal set $B \equiv \{u_n\}$ to which our previous results can applied, for indeed $\forall t$ the projection of $\psi(t)$ onto the subspace spanned by B cannot be less than $1 - \varepsilon$ in norm. On the other hand, the projection of $\psi(s)$, $(s \leq t)$ on the subspace spanned by $\{u_n\}_{n > d(\varepsilon, \psi, t)}$ has norm less than ε; we therefore have $\overline{n}(\varepsilon^2, \psi, t, B) \leq d(\varepsilon, \psi, t)$ whence it follows that $\overline{\beta}(\varepsilon^2, \psi, \overline{B}) \leq \nu(\varepsilon, \psi)$ and finally $\beta_0(\psi, \overline{B}) \leq \theta(\psi)$. To complete the proof we have to use thm.1. ■

4. UPPER BOUNDS

It is fairly obvious that further assumptions about the orthonormal system B are needed in order to find upper estimates for the growth exponents associated with the moments. In fact the type of growth of the moments can be changed from algebraic to exponential just by reordering the base vectors. We shall here introduce a special basis, intrinsically associated with the spectral measure μ_ψ, for which upper estimates are easily obtained. This result will be used to obtain an upper bound for the (basis-independent) index θ.

First of all, we restrict to the cyclic subspace $\mathcal{H}_\psi$ generated in the Hilbert space $\mathcal{H}$ by $\{U^t\psi,\ t \in Z\}$. Then we use the Spectral Theorem to identify $\mathcal{H}_\psi$ with $L^2(S,\ \mu_\psi)$, ψ with the constant function $=1$, and U^t with multiplication by e^{ixt}. Finally we define an orthornormal base as follows: we consider partitions of $[0,\ 2\pi]$ in dyadic intervals $I_{N,\,k}$ of width 2^{-N}, with $0 \leq k \leq 2^N - 1$. For any given N let us consider those integers j, $(0 \leq j \leq 2^{N-1} - 1)$ such that both $I_{N,\,2j}$ and $I_{N,\,2j+1}$ have nonzero measure. For such N, j let us define functions

$$\phi_{N_j} = a_{Nj}\chi_{I_{N,\,2j}} - b_{Nj}\chi_{I_{N,\,2j+1}}$$

where the χ's are characteristic functions, and $a_{N,\,j}$, b_{Nj} are chosen such that

$$\int_S \phi_{Nj}^2\, d\mu_\psi = 1, \qquad \int_S \phi_{Nj}\, d\mu_\psi = 0$$

Explicit computation yields:

$$\begin{aligned} a_{Nj} &= \left\{ \frac{\mu_\psi(I_{N,\,2j+1})}{\mu_\psi(I_{N,\,2j})\,\mu_\psi(I_{N-1,\,j})} \right\}^{\frac{1}{2}} \\ b_{Nj} &= \left\{ \frac{\mu_\psi(I_{N,\,2j})}{\mu_\psi(I_{N,\,2j+1})\,\mu_\psi(I_{N-1,\,j})} \right\}^{\frac{1}{2}} \end{aligned} \tag{10}$$

Upon ordering the functions thus defined according to increasing N (and to increasing j at fixed N) we obtain an orthonormal sequence $\{e_n\}_{n\geq 1}$ where $\phi_{Nj} = e_{n\,(N,\,j)}$. To this sequence we add $e_0 \equiv 1$ and thus obtain a complete set B_0 of vectors in $L^2(S,\ \mu_\psi)$ (See Appendix). The n-th function has support in a dyadic interval that will be denoted I_n, the length of which is $2^{-N(n)}$.

The capacity $\sigma(\psi)$ of the support of the measure μ_ψ is defined by

$$\sigma(\psi) = \lim\sup_{N\to\infty} \frac{\log_2 \sharp_N}{N}$$

where $\sharp_N$ is the minimum number of intervals $I_{N,\,k}$ needed to cover the support of μ_ψ. We can now prove:

THEOREM 3. – $\beta_0\,(B_0,\,\psi) \le \sigma\,(\psi)$, *the capacity of the support of the measure* μ_ψ.

Proof. – Let $f\,(x)$ be a Lipschitz function with $|f\,(x) - f\,(x')| \le c|x - x'|$; let us estimate its amplitudes over the base B:

$$|f_{Nj}| = \left| a_{Nj} \int_{I_{N,\,2j}} f d\mu_\psi - b_{Nj} \int_{I_{N,\,2j+1}} f d\mu_\psi \right|$$
$$\le 2^{-N}\, c\,[a_{Nj}\mu_\psi\,(I_{N,\,2j}) + b_{Nj}\mu_\psi\,(I_{N,\,2j+1})]$$

whence, on account of formula (10):

$$|f_{Nj}| \le 2^{-N}\, c\,[\mu_\psi\,(I_{N-1,\,j})]^{\frac{1}{2}} \tag{11}$$

From this, taking $f\,(x) = U^t\psi = e^{itx}$, we get:

$$|c_n\,(t)|^2 \le t^2 2^{-2N\,(n)}\,\mu_\psi\,(I_n) \tag{12}$$

If follows that:

$$\sum_{n \ge n_0} p_n\,(t) \le t^2 \sum_{N \ge N\,(n_0)} 2^{-2N} \le 2t^2 \cdot 2^{-2N\,(n_0)} \tag{13}$$

The lhs will be smaller than a given ε, if n_0 is taken so large that

$$2^{N\,(n_0)} > \left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}} t \tag{14}$$

In order to estimate the minimum such n_0 we first take the least integer N satisfying (14) and then determine n_0 as the total number of base functions supported by dyadic intervals not smaller than 2^{-N}. If the number of dyadic intervals of size 2^{-L} needed to cover the support of μ_ψ is denoted by $\sharp_L$, then certainly $n_0 \le \sum_{L \le N} \sharp_L$. On the other hand, if $\sigma \equiv \sigma\,(\psi)$ is the capacity of the support of μ_ψ, then for arbitrary $\varepsilon_1 > 0$ a L_{ε_1} exists, such that $\sharp_L < 2^{(\sigma+\varepsilon_1)\,L}$ for $L > L_{\varepsilon_1}$; therefore, if $N > L_{\varepsilon_1} + 1$, then $n_0 \le \sum_{L \le N} \sharp_L < c'_{\varepsilon_1} + c''_{\varepsilon_1}\, 2^{(\sigma+\varepsilon_1)\,N}$ with appropriate constants c'_{ε_1}, c''_{ε_1}. Putting this estimate and (14) together we get

$$\overline{n}\,(\varepsilon,\,\psi,\,t,\,B_0) \sim n_0 < c'_{\varepsilon_1} + c''_{\varepsilon_1} \left(\frac{2}{\varepsilon}\right)^{\frac{\sigma+\varepsilon_1}{2}} t^{\sigma+\varepsilon_1}$$

which shows that $\beta_0\,(\psi,\,B_0) \le \sigma + \varepsilon_1$, $\forall \varepsilon_1 > 0$. ■

The proof just given yields as a corollary an upper bound for the index $\theta\,(\psi)$. In fact from (12) it follows that $\forall t' \le t$ the squared norm of the projection of $\psi(t')$ on the subspace spanned by $\{e_n\}_{n>n_0}$ is less than ε, where n_0 has an exponent of growth not larger than $\sigma(\psi)$. This means that the subspace spanned by the first n_0 vectors of B_0 is sufficient to approximate $\psi(1), \ldots, \psi(t)$ within an approximation $\varepsilon^{\frac{1}{2}}$; therefore, $d(\varepsilon^{\frac{1}{2}}, \psi, t) \leq n_0$, and also $\theta(\psi) \leq \sigma(\psi)$.

As a concluding summary we explicitly write the bounds:

$$\overline{\lambda}(\psi) \leq \lim_{\varepsilon \searrow 0} \lim \inf_{t \to +\infty} \frac{\log d(\varepsilon, \psi, t)}{\log t}$$

$$\lim_{\varepsilon \searrow 0} \lim \sup_{t \to +\infty} \frac{\log d(\varepsilon, \psi, t)}{\log t} \leq \sigma(\psi)$$

We recall that the former of these is valid under assumption (8) (9) for the spectral measure.

The above bounds yield a sharp asymptotic estimate in the case $\overline{\lambda}(\psi) = \sigma(\psi)$. This is the case in particular when the spectral measure is homogeneous, in the sense that associated with it there is only one scaling exponent.

5. APPENDIX

A Proof of Theorem 1

Without limitation of generality we can assume $\|\psi\| = 1$. Following Strichartz [7] we shall say that the measure μ_ψ is *locally uniformly* α *– dimensional* if $\mu_\psi(I_\delta(x)) \leq c\delta^\alpha$ for all x in the spectrum and for all $\delta \leq 1$. For this class of spectral measures, the following result is straightforward:

LEMMA A1. – *If* μ_ψ *is locally uniformly* α*-dimensional then* $\overline{\beta}(\varepsilon, \psi, B) \geq \alpha$ $\forall \varepsilon < \Delta$. *Therefore,* $\beta_0(\psi, B) \geq \alpha$.

Proof. – If the spectral measure has the stated property, then $\forall t$, $\forall n$:

$$p_n(t) \leq ct^{-\alpha} \tag{15}$$

As remarked by Combes [5] this estimate follows from general results proven by Strichartz [7]. A weaker form of (15) including a logarithmic factor was proven by elementary methods in [3]. Recalling that $\sum_1^\infty p_n(t) > \Delta$ we get:

$$\sum_{n=N+1}^{\infty} p_n(t) \geq \Delta - cNt^{-\alpha}$$

which immediately entails that $\overline{n}(\varepsilon, \psi, t, B) \geq (\Delta - \varepsilon)\, c^{-1}\, t^{\alpha}$. The result now follows from the properties of the growth exponent. ■

LEMMA A2. – *If the spectral measure* μ_ψ *satisfies* (8), (9) *then* $\forall \eta \in (0, 1)$ *and* $\forall \lambda \in (0, \overline{\lambda}(\psi))$ *a vector* $\psi_{\eta,\lambda}$ *can be found, such that* (i) $\|\psi - \psi_{\eta,\lambda}\| < \eta$, (ii) *the spectral measure of* $\psi_{\eta,\lambda}$ *is locally uniformly* λ*-dimensional*

Proof. – By Egorov's theorem [8] we can select a subset J_η of the spectrum, of measure $\mu(J_\eta) > 1 - \eta^2$, in which the limit (8) is uniform. If P_{J_η} is the corresponding spectral projection, then $\psi_{\eta,\lambda} = P_{J_\eta}\psi$ has the required properties. ■

Finally we show that if η is suitably small then the spreading of $\psi(t)$ over the base B cannot be slower than that of $\psi_{\eta,\lambda}$. Dropping for simplicity the suffix λ let us not that $\phi_\eta = \psi - \psi_\eta$ is orthogonal to ψ_η, that $\|\phi_\eta\|^2 + \|\psi_\eta\|^2 = 1$ and that $\psi_\eta(t) \equiv U^t\,\psi_\eta = P_{J_\eta}\,\psi(t)$. Let us denote by $p_n^{(\eta)}(t)$ the average distribution defined on the base B by the orbit of ψ_η. The following estimate is a straightforward consequence of the Schwarz inequality:

$$\sum_{n>n_0} p_n(t) \geq \sum_{n>n_0} p_n^{(\eta)}(t) - 2\|\psi_\eta\|\,\|\phi_\eta\| \tag{16}$$

Now let us choose $n_0 < \overline{n}(\varepsilon, \psi_\eta, t, B)$: then the sum on the rhs of (16) will not be smaller than $\varepsilon\|\psi_\eta\|^2$:

$$\sum_{n>n_0} p_n(t) \geq \varepsilon\|\psi_\eta\|^2 - 2\|\psi_\eta\|\sqrt{1 - \|\psi_\eta\|^2} \tag{17}$$

Since the rhs of (17) tends to ε as $\eta \to 0+$, it can be made greater than $\frac{\varepsilon}{2}$ by choosing $\eta = \eta_\varepsilon$ conveniently small. Consequently, with such a choice of η we obtain $\overline{n}\left(\frac{\varepsilon}{2}, \psi, t, B\right) > n_0$ if $n_0 < \overline{n}(\varepsilon, \psi_{\eta_\varepsilon}, t, B)$, hence $\overline{n}\left(\frac{\varepsilon}{2}, \psi, t, B\right) \geq \overline{n}(\varepsilon, \psi_{\eta_\varepsilon}, t, B)$. Since the spectral measure of ψ_{η_ε} is locally uniformly λ-dimensional, from Lemma A1 we finally get $\overline{\beta}(\varepsilon, \psi, B) \geq \lambda$ if $\varepsilon < \frac{\Delta}{2}$, $\forall \lambda < \overline{\lambda}(\psi)$ and the proof of thm. 1 is concluded. ■

6. APPENDIX

Completeness of the Base B_0.

We shall here prove that the base B_0 is complete in $L^2(\mathcal{S}, \mu_\psi)$. Let $f \in \mathcal{L}^2(\mathcal{S}, \mu_\psi)$ be such that $\int f e_n d\mu_\psi = 0\ \forall n$. Let x be any point in the spectrum which is not an extreme of any dyadic interval. There is an infinite sequence $\{e_{n_k}\}$ with $x \in J_{n_k}$ (the support of e_{n_k}), for otherwise we could find a sequence of dyadic intervals of constant nonzero measure, shrinking to $\{x\}$, in contradiction with the continuous character of the measure itself. Then let L_k, R_k be the two halves of J_{n_k} and χ'_k, χ''_k their characteristic functions. In this way we can write $e_{n_k} = a_k\chi'_k - b_k\chi''_k$. From orthonormality of B_0 and from $\int f e_{n_k} d\mu_\psi = 0$ we get:

$$a_k\,\mu_\psi(L_k) - b_k\,\mu_\psi(R_k) = 0, \qquad a_k \int_{L_k} f d\mu_\psi = b_k \int_{R_k} f d\mu_\psi$$

which together imply that:

$$\frac{1}{\mu_\psi(L_k)} \int_{L_k} f d\mu_\psi = \frac{1}{\mu_\psi(R_k)} \int_{R_k} f d\mu_\psi \tag{18}$$

Let us then consider the next basis function $e_{n_{k+1}}$ the support $J_{n_{k+1}}$ of which still contains x. Supposing to fix ideas that $x \in L_k$ we have

$$\mu_\psi(L_{k+1}) + \mu_\psi(R_{k+1}) = \mu_\psi(L_k)$$

and

$$\left(\int_{L_{k+1}} + \int_{R_{k+1}}\right) f d\mu_\psi = \int_{L_k} f d\mu_\psi.$$

From these equations and from (18) we obtain:

$$\frac{1}{\mu_\psi(L_{k+1})} \int_{L_{k+1}} f d\mu_\psi = \frac{1}{\mu_\psi(R_{k+1})} \int_{R_{k+1}} f d\mu_\psi = \frac{1}{\mu_\psi(L_k)} \int_{L_k} f d\mu$$

Continuing in this way we see that $\frac{1}{\mu_\psi(I)} \int_I f d\mu_\psi$, evaluted over a sequence of dyadic intervals I shrinking to $\{x\}$, has a constant value. Now on one hand this value must coincide with $f(x)$ for μ_ψ-almost all x, and on the other it must be zero, because $J_0 \equiv [0, 2\pi]$ is one of these intervals, and $\int_{J_0} f d\mu_\psi = 0$. We have thus proven that $f = 0$, $\mu_\psi - a.e.$; therefore B_0 is a complete base.

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