ANNALES DE L'I. H. P., SECTION A

XAVIER GRÀCIA
JOSEP M. PONS

Noether transformations with vanishing conserved quantity

Annales de l'I. H. P., section A, tome 61, nº 3 (1994), p. 315-327
<http://www.numdam.org/item?id=AIHPA_1994__61_3_315_0>

Ann. Inst. Henri Poincaré,

Vol. 61, n° 3, 1994, p. 315-327. *Physique théorique*

Noether transformations with vanishing conserved quantity

by

Xavier GRÀCIA

Dep. de Matemàtica Aplicada i Telemàtica,
Universitat Politècnia de Catalunya,
Campus Nord edifici C3, C/Gran Capità s/n, 08071 Barcelona, Catalonia, Spain

and

Josep M. PONS

Dep. d'Estructura i Constituents de la Matèria,
Universitat de Barcelona,
Av. Diagonal 647, 08028 Barcelona, Catalonia, Spain

ABSTRACT. – Noether transformations whose conserved quantity is identically vanishing are studied. Such transformations occur only for singular lagrangians, and are related to the fact that the number of independent primary lagrangian constraints is less than the number of the hamiltonian ones, or, equivalently, the number of independant secondary hamiltonian constraints is less than the number of first-class primary hamiltonian constraints.

RÉSUMÉ. – On étudie les transformations de Noether dont la quantité conservée est identiquement nulle. Ces transformations apparaissent seulement avec les lagrangiennes singulières, et sont liées au fait que le nombre de contraintes lagrangiennes primaires indépendantes est plus petit que le nombre de hamiltoniennes, ou, de façon équivalente, le nombre des contraintes hamiltoniennes secondaires indépendantes est plus petit que le nombre de contraintes hamiltoniennes primaires de première classe.

1. INTRODUCTION

In a given lagrangian formalism, Noether's theorem establishes the relation between an infinitesimal transformation and a conserved quantity. These transformations are called Noether symmetries, and leave the action invariant–up to boundary terms. Moreover, when the lagrangian is regular, the canonical version of the conserved quantity becomes the generator–through Poisson bracket–of the Noether transformation.

In the singular case, however, there is room for different situations, either concerning the Noether transformation or the conserved quantity. As to the first, it can be a rigid or a gauge symmetry and, moreover, there is the possibility that the Noether transformation could not be expressed in phase space variables and therefore it makes no sense to consider a canonical generator for it. In [1] this case is considered to a full extent, and it is shown that in certain cases–when some of the usual regularity conditions are dropped–the relation between lagrangian and hamiltonian Noether transformations is by no means one to one.

With regard to the conserved quantity, it can exhibit special features when the lagrangian is singular; for instance, it can be a lagrangian constraint or even vanish identically. The purpose of this paper is to study which conditions allow this last possibility. In this case, the correspondence between Noether transformations and conserved quantities fails to be one-to-one; therefore, the usual statement of Noether's theorem is incorrect for some singular lagrangians. More precisely, we will show that this happens when the number of independent primary lagrangian constraints is less than the number of the hamiltonian ones; this also amounts to say that the number of independent secondary hamiltonian constraints is less than the number of first-class primary hamiltonian constraints.

The paper is organized as follows. In section 2 we introduce our notations and some useful results concerning projectability of functions, lagrangian and hamiltonian constraints, and evolution operators. In section 3 we state some basic results from [1] concerning Noether transformations. The core of the paper is section 4, where the construction of Noether transformations with vanishing conserved quantity is explored. Finally, section 5 considers briefly the specific case of gauge transformations, together with two examples: the reparametrization-invariant lagrangians and the bosonic string.

2. PREVIOUS RESULTS AND NOTATION

We consider a configuration space Q, with velocity space $V = \mathrm{T}(Q)$, and a lagrangian function $L(q, v)$ defined on it, that is to say, a function $L : \mathrm{T}(Q) \to \mathbf{R}$. From it we construct the Legendre's map, which is a function from velocity space to phase space $\mathrm{F}L : V \to P = \mathrm{T}(Q)^*$ locally defined by $\mathrm{F}L(q, v) = (q, \hat{p})$, where we have introduced the momenta $\hat{p} = \partial L/\partial v$.

Given a function $g(q, p)$ in phase space, its pull-back (through the Legendre's map $\mathrm{F}L$) is the function $\mathrm{F}L^*(g)$ in velocity space obtained by substituting the momenta by their lagrangian expression: $\mathrm{F}L^*(g)(q, v) = q(q, \hat{p})$. A function $f(q, v)$ in velocity space is called ($\mathrm{F}L$-)projectable if it is the pull-back of a certain function $g(q, p)$.

We shall always assume that the hessian matrix $W = \partial^2 L/\partial v\, \partial v$ has constant rank, which amounts to say that $\mathrm{F}L$ has constant rank. Let $\gamma_\mu\,(\mu = 1, \ldots, p_0)$ be a basis for the null vectors of W; then *the necessary and sufficient condition for a function $f(q, v)$ in $\mathrm{T}Q$ to be (locally) projectable to $\mathrm{T}^* Q$* is

$$\Gamma_\mu \cdot f = 0 \tag{2.1}$$

for each μ, where $\Gamma_\mu := \gamma_\mu \dfrac{\partial}{\partial v}$.

Under the same assumption on the hessian matrix W, the image P_0 of the Legendre's map can be locally taken as the submanifold of the phase space described by the vanishing of p_0 primary hamiltonian constraints ϕ_μ, linearly independent at each point of p_0. Then the basis γ_μ can be taken as [2]

$$\gamma_\mu := \mathrm{F}L^* \left(\frac{\partial \phi_\mu}{\partial p}\right). \tag{2.2}$$

The time-derivative operator acting on, a function $f(q, v; t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial q} + a\frac{\partial}{\partial v},$$

with the acceleration a as a independent variable. Then the Euler-Lagrange equations can be written $[L]_{(q, \dot{q}, \ddot{q})} = 0$, where we have defined

$$[L] := \frac{\partial L}{\partial q} - \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \alpha - a\,W, \tag{2.3}$$

with $\alpha = \dfrac{\partial L}{\partial q} - v\,\dfrac{\partial^2 L}{\partial q\, \partial v}$. The primary lagrangian constraints arise from it,

$$\chi_\mu := \alpha \gamma_\mu = [L]\,\gamma_\mu, \tag{2.4}$$

though they are not necessarily independent; their vanishing defines a subset $V_1 \subset V$.

As a matter of notation, we write $f \underset{M}{\simeq} 0$ to mean that $f(x) = 0$ for all $x \in M$ (Dirac's weak equality), and $f \underset{M}{\cong} 0$ to mean $f \underset{M}{\simeq} 0$ and $\mathrm{d}f \underset{M}{\simeq} 0$ (Dirac's strong equality); for instance $\phi_\mu \underset{P_0}{\simeq} 0$ and $\chi_\mu \underset{V_1}{\simeq} 0$.

Now we will introduce a useful differential operator K connecting lagrangian and hamiltonian formalisms. It takes a function $g(q, p; t)$ in phase space and gives its time-derivative $(K \cdot g)(q, v; t)$ in velocity space, without any ambiguity:

$$K \cdot g := v \cdot \mathrm{F}L^* \left(\frac{\partial g}{\partial q}\right) + \frac{\partial L}{\partial q} \cdot \mathrm{F}L^* \left(\frac{\partial g}{\partial p}\right) + \mathrm{F}L^* \left(\frac{\partial g}{\partial t}\right). \tag{2.5}$$

This operator was introduced in [2], and is especially worth in the study of singular lagrangians. For instance, all the lagrangian constraints are obtained by applying it to the hamiltonian constraints [3]; on the other hand the geometric formulation of K allows to write the Euler-Lagrange equations in an intrinsic way [4], even for higher-order lagrangians [5].

The operator K can be given several interesting expressions. Some of them are

$$K \cdot g = [L] \cdot \mathrm{F}L^* \left(\frac{\partial g}{\partial p}\right) + \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{F}L^* (g)), \tag{2.6}$$

and

$$K \cdot g = \mathrm{F}L^* \{g, H\} + \sum_\mu \mathrm{F}L^* \{g, \phi_\mu\} \lambda^\mu + \mathrm{F}L^* \left(\frac{\partial g}{\partial t}\right). \tag{2.7}$$

In the last expression H is any hamiltonian function and the $\lambda^\mu(q, v)$ are functions uniquely determined by this equality; these functions are not $\mathrm{F}L$-projectable, and indeed

$$\Gamma_\nu \cdot \lambda^\mu = \delta^\mu_\nu, \tag{2.8}$$

so in a certain sense they correspond to the velocities lost through the Legendre's map. On the other hand, from (2.6) and (2.4) we obtain another expression for the primary lagrangian constraints:

$$\chi_\mu = K \cdot \phi_\mu. \tag{2.9}$$

3. NOETHER TRANSFORMATIONS

In general, we call dynamical symmetry transformations those transformations that map solutions into solutions of an evolution equation.

A careful study of these transformations in the hamiltonian formalism is in [6]. Let us only point up that an infinitesimal function $G_H(q, p; t)$ generates–through Poisson bracket, $\delta f = \{f, G_H\}$–a hamiltonian dynamical symmetry transformation if and only if

$$K \cdot G_H \underset{V_f}{\cong} 0, \tag{3.1}$$

where $V_f \subset V$ is the final lagrangian constraint submanifold.

Now we are mainly interested in the lagrangian formalism. A particular type of lagrangian dynamical symmetry transformations are Noether transformations: they are infinitesimal transformations $\delta q(q, v; t)$ in velocity space $\mathrm{T}Q$ such that δL is a total derivative. In this case, Noether's theorem [7]-[14] guarantees that δq maps solutions into solutions, and that a conserved quantity $G(q, v; t)$ arises from it, as expressed by

$$[L] \cdot \delta q + \frac{\mathrm{d}G}{\mathrm{d}t} = 0. \tag{3.2}$$

A careful analysis of this relation has been carried out in [1], where the fact that G is projectable is used to prove the following:

THEOREM 1. – *Let L be a lagrangian whose hessian matrix W has constant rank. Then every infinitesimal lagrangian Noether transformation $\delta q(q, v; t)$* (3.2) *can be obtained as*

$$\delta q = \mathrm{F}L^* \left(\frac{\delta G_H}{\partial p}\right) + \sum_\mu \tau^\mu \gamma_\mu, \tag{3.3}$$

where the infinitesimal function $G_{\mathrm{H}}(q, p; t)$ satisfies

$$\mathrm{F}L^*(G_{\mathrm{H}}) = G \tag{3.4}$$

and

$$K \cdot G_{\mathrm{H}} \underset{V_1}{\simeq} 0, \tag{3.5}$$

and the τ^μ satisfy

$$K \cdot G_{\mathrm{H}} + \sum_\mu \tau^\mu \chi_\mu = 0. \tag{3.6}$$

Conversely, and infinitesimal function $G_{\mathrm{H}}(q, p; t)$ satisfying (3.5) *generates through* (3.3) *a lagrangian Noether transformation.* ■

The function $G_{\rm H}$ is a kind of "hamiltonian generator" [in the sense of (3.3)] for the lagrangian transformation δq. Notice, however, that such $G_{\rm H}$ is not necessarily a hamiltonian gange generator, since in general it does not satisfy (3.1). Both conditions coincide only when $K \cdot G_{\rm H} = 0$. On the other hand, notice that $G_{\rm H}$ is determined from G up to primary hamiltonian contraints; this indetermination does not change the Noether transformation, because it amounts to a change of the τ^μ.

The expression (3.3) can also be written

$$\delta q = \mathrm{F}L^* \{q,\, G_{\rm H}\} + \sum_\mu \tau^\mu \, \mathrm{F}L^* \{q,\, \phi_\mu\}. \tag{3.7}$$

Let us also say that the transformations of the momenta can be shown to be

$$\delta \hat{p} = \mathrm{F}L^* \{p,\, G_{\rm H}\} + \sum_\mu \tau^\mu \, \mathrm{F}L^* \{p,\, \phi_\mu\} - [L]\, \frac{\partial \delta q}{\partial v}, \tag{3.8}$$

in agreement with (3.7) when the equations of motion are taken into account.

Observe that the determination of the functions τ^μ from (3.5) suffers of two types of ambiguities:

1) The change $\tau^\mu \to \tau^\mu + \sum_\nu A^{\mu\nu} \chi_\nu$, with $A^{\mu\nu}$ antisymmetric, leaves (3.6) invariant, and the corresponding transformation changes as $\delta q \to \delta q + \sum_{\mu,\,\nu} \gamma_\mu \, A^{\mu\nu} \chi_\nu = \delta q + B\,[L]$, where B is an antisymmetric matrix. This corresponds to adding a trivial gauge transformation (antisymmetric combination of the equations of motion) [15]. The presence of such terms in the r.h.s. of the commutator of two gauge transformations gives rise to open gauge algebras [16].

2) Suppose that the primary lagrangian constraints are not all them independent, that is to say, there exist functions A^μ such that $\sum_\mu \chi_\mu \, A^\mu = 0$. Then additional indetermination appears, since under the change $\tau^\mu \to \tau^\mu + A^\mu$ relation (3.6) still holds. These are the ambiguities we are interested in, and the rest of the paper is devoted to their exploration.

Finally, one can consider the particular case of *gauge* Noether transformations, that is to say, transformations depending on arbitrary functions of time. The general form for the (infinitesimal) "generators" can be taken as [1], [6], [17]-[20]

$$G_{\rm H}(q,\, p;\, t) = \sum_{k \geq 0} \varepsilon^{(-k)}(t)\, G_k(q,\, p), \tag{3.9}$$

where $\varepsilon^{(-k)}$ is a k-th primitive of an arbitrary function of time ε. Since G_{H} has to satisfy (3.5) and ε is arbitrary, a recursive algorithm is obtained to complete the G_k [1]:

$$\mathrm{F}L^* \,(G_0) \underset{V_1}{\simeq} 0, \tag{3.10 a}$$

$$\mathrm{F}L^* \,(G_{k+1}) \underset{V_1}{\simeq} -K \cdot G_k. \tag{3.10 b}$$

This also shows that G_{H} is made up of hamiltonian constraints.

4. NOETHER TRANSFORMATIONS WITH VANISHING CONSERVED QUANTITY

We will study the multiplicity of Noether transformations deduced from a fixed lagrangian "generator" G. This amounts to study the Noether transformations δq which admit as a lagrangian "generator" the function $G = 0$, that is to say, the transformations with vanishing conserved quantity.

Consider (3.2) with $G = 0$:

$$[L]\,\delta q = \alpha\delta q - a\,W\,\delta q = 0.$$

The coefficient of the acceleration a yields $W\,\delta q = 0$, so there exist infinitesimal functions τ^μ, uniquely determined by δq, such that

$$\delta q = \sum_\mu \gamma_\mu\,\tau^\mu. \tag{4.1}$$

The other term is $\alpha\,\delta q = 0$; with the preceding expression of δq, and using the expression (2.4) for the primary lagrangian constraints χ_μ, it becomes

$$\sum_\mu \chi_\mu\,\tau^\mu = 0. \tag{4.2}$$

If the transformation δq of the solutions does not vanish, then not all the functions τ^μ are constraints. Therefore the primary lagrangian constraints χ_μ are not all them independent (on the lagrangian final constraint submanifold V_f). This reasoning can be reversed: if the primary lagrangian constraints satisfy a relation (4.2) then we can construct the corresponding $\delta q \neq 0$ which satisfies Noether's theorem with $G = 0$.

Our problem is therefore the analysis of relation (4.2) in generality. First let us quote the following trivial case:

PROPOSITION. – *Given a primary hamiltonian constraint ϕ such that*

$$K \cdot \phi = 0, \tag{4.3}$$

then we have a projectable Noether transformation with $G = 0$:

$$\delta q = \mathrm{F}L^* \{q, \varepsilon \phi\}, \tag{4.4}$$

where ε *is an infinitesimal parameter.* ■

Now let us assume for a while that (4.2) is satisfied with the functions τ^μ projectable through $\mathrm{F}L$, $\tau^\mu = \mathrm{F}L^* (\tau^\mu_{\mathrm{H}})$. That is to say, we are assuming δq projectable through $\mathrm{F}L$. Then

$$K \cdot (\sum_\mu \phi_\mu \tau^\mu_{\mathrm{H}}) = \sum_\mu \chi_\mu \tau^\mu = 0.$$

Therefore there exists a (non-vanishing) primary hamiltonian constraint $\phi = \sum_\mu \phi_\mu \tau^\mu_{\mathrm{H}}$ which is cancelled out by the differential operator K. According to the preceding proposition, we obtain a Noether transformation with vanishing conserved quantity. We claim that this is the usual case, as stated by the following theorem:

THEOREM 2. – *Consider a singular lagrangian* L *such that the following conditions are satisfied:*

1) *The hessian matrix* W *has constant corank* p_0; *let* ϕ_μ *be* p_0 *primary hamiltonian constraints defining* P_0, *and whose differentials are linearly independent on it.*

2) *The matrix* $\{\phi_\mu, \phi_\nu\}$ *has constant rank* $p''_0 = p_0 - p'_0$ on P_0.

3) *The primary lagrangian subset* $V_1 \subset V$ *is a submanifold of codimension* $n_1 \leqq p_0$, *and the differentials of the primary lagrangian constraints* $\chi_\mu = K \cdot \phi_\mu$ *have constant rank* n_1 *on* V_1.

Then there exist $p_0 - n_1$ *independent primary hamiltonian constraints* ϕ_d *such that* $K \cdot \phi_d = 0$.

Proof. – The constancy of the rank of the matrix $\{\phi_\mu, \phi_\nu\}$ implies that the primary hamiltonian constraints can be redefined into two subsets $\phi_{\mu'}$ and $\phi_{\mu''}$ in the following way:

1) The p'_0 constraints $\phi_{\mu'}$ are first-class on P_0 : $\{\phi_{u'}, \phi_\nu\} \underset{P_0}{\simeq} 0$.

2) The p''_0 constraints $\phi_{\mu''}$ are second-class on P_0; that is to say, the matrix $C_{\mu''\nu''} = \{\phi_{\mu''}, \phi_{\nu''}\}$ is invertible on P_0.

The secondary hamiltonian submanifold $P_1 \subset P$ is thus defined by the secondary constraints $\psi_{\mu'} := \{\phi_{\mu'}, H\}$, though they are not necessarily independent. Let us also say that, by redefining the hamiltonian, one can enforce the constraints $\phi_{\mu''}$ to be stable under H on P_0, $\{\phi_{\mu''}, H\} \underset{P_0}{\simeq} 0$, though this will not be needed.

The primary lagrangian constraints can be expressed in the following way, using (2.7):

$$\chi_\mu = K \cdot \phi_\mu = \mathrm{F}L^* \{\phi_\mu, H\} + \sum_\nu \mathrm{F}L^* \{\phi_\mu, \phi_\nu\} \lambda^\nu. \tag{4.5}$$

Therefore the splitting of the primary hamiltonian constraints into first- and second-class classifies these lagrangian constraints in two types:

$$\chi_{\mu'} = K \cdot \phi_{\mu'} = \mathrm{F}L^* \{\phi_{\mu'}, H\}, \tag{4.6 a}$$

$$\chi_{\mu''} = K \cdot \phi_{\mu''} = \mathrm{F}L^* \{\phi_{\mu''}, H\} + \sum_{\nu''} \mathrm{F}L^* (C_{\mu''\nu''}) \lambda^{\nu''}, \tag{4.6 b}$$

which are respectively $\mathrm{F}L$-projectable and non $\mathrm{F}L$-projectable, since the functions λ^ν are not projectable (2.8) and the matrix C is invertible on P_0:

$$\Gamma_{\nu''} \cdot (K \cdot \phi_{\mu''}) = \mathrm{F}L^* (C_{\mu''\nu''});$$

indeed the constraints $\chi_{\mu''}$ describe the same submanifold as the

$$\lambda^{\mu''} + \mathrm{F}L^* (\sum_{\nu''} (C^{-1})^{\mu''\nu''} \{\phi_{\mu''}, H\}),$$

which shows moreover that these p_0'' non-projectable constraints $\chi_{\mu''}$ are independent.

From these considerations we conclude that, if the differentials of the χ_μ are not linearly independent on V_1, but still they have constant rank $n_1 < p_0$, the dependence is among the p_0' projectable ones, $\chi_{\mu'}$, and $p_0 - n_1$ of these ones, say χ_d, can be (locally) isolated as function of the other $p_0' - n_1 - p_0$ ones, let us call them χ_i:

$$\chi_d = \sum_i \chi_i \hat{f}_d^i. \tag{4.7}$$

Moreover, using appropriate coordinates if needed, it is easily seen that the coefficients $\hat{f}_d^i$ in this last expression are projectable, so $\hat{f}_d^i = \mathrm{F}L^* (f_d^i)$.

Finally we obtain

$$K \cdot (\phi_d - \sum_i \phi_i f_d^i) = \chi_d - \sum_i \chi_i \hat{f}_d^i = 0, \tag{4.8}$$

and the expressions in parentheses are the primary hamiltonian constraints we looked for. ■

Bearing in mind (4.6 *a*), which relates the secondary hamiltonian constraints with the projectable primary lagrangian constraints, we can restate Theorem 2 in the following way:

THEOREM 3. – *We keep the preceding notations as well as the hypotheses in Theorem 2, but changing the third one to*

3) *The secondary hamiltonian subset $P_1 \subset P_0$ is a submanifold of codimension $p_1 \leqq p'_0$ and the differentials of the secondary hamiltonian constraints $\psi_{\mu'} = \{\phi_{\mu'}, H\}$ have constant rank p_1 on P_0.*

Then there exist $p'_0 - p_1$ independent primary hamiltonian constraints ϕ_d such that $K \cdot \phi_d = 0$. ■

Combining these theorems with the preceding proposition we have proved the following result:

COROLLARY. – *Assuming the hypotheses and notations in the preceding theorems, there exist $p_0 - n_1 = p'_0 - p_1$ independent Noether transformations with vanishing conserved quantity.* ■

Of course, a relation (4.2) can arrive with non-projectable coefficients, but our theorems assert the existence of a certain number of independent relations with projectable coefficients, which can be obtained from the primary lagrangian constraints as in the proof of Theorem 2. If our mild regularity conditions are not satisfied then one can not exclude the possibility of a linear dependence (4.2) with non-projectable coefficients τ^μ, and which can not be obtained from Noether transformations constructed from the theorems; we do not know if such pathologies can occur.

5. GAUGING THE TRANSFORMATIONS WITH VANISHING CONSERVED QUANTITY

Consider a Noether transformation constructed from a relation $\sum_\mu \chi_\mu \tau^\mu = 0$ between the primary lagrangian constraints. This relation still holds when multiplied by any function; in particular, by an arbitrary function of time $f(t)$. Then the corresponding Noether transformation (4.1) is also multiplied, so it is a lagrangian *gauge* transformation

$$\delta q = f(t) \sum_\mu \gamma_\mu \tau^\mu. \tag{5.1}$$

This is the simplest form that can have a gauge transformation, when there does not appear any derivative of the arbitrary functions. Let us have a look at two examples of it.

Example 1. – Reparametrizations.

Let us consider a reparametrization-invariant lagrangian. This is a lagrangian which is homogeneous of degree 1 in the velocity. Then it is easily seen that $W v = 0$, so we know at least one of the null vectors of the hessian matrix. From it we obtain the lagrangian constraint $\chi = \alpha(q, v) \cdot v = 0$, which vanishes identically due to homogeneity of L. Accordingly we have an infinitesimal Noether transformation with vanishing conserved quantity, $\delta q = \varepsilon\, v$. This can be turned into a gauge transformation:

$$\delta q = \varepsilon(t)\, v,$$

which is the reparametrization. Notice that this transformation is not projectable through FL, since $\Gamma \cdot \delta q = v \dfrac{\partial \delta q}{\partial v} = \varepsilon \neq 0$, but Theorem 2 guarantees that it is equivalent to projectable transformations.

For instance, for the free relativistic particle, with $L = \sqrt{v^2}$, there is only one null vector; then the transformation

$$\delta q = \varepsilon(t) \frac{v}{\sqrt{v^2}}$$

is the covariant projectable reparametrization. Both transformations are related through a change of the arbitrary function.

Example 2. – The bosonic string.

A different example is provided by Polyakov's string, whose lagrangian density is $\mathcal{L} = \sqrt{g} g^{\alpha\beta}\, \partial_\alpha\, x^\mu\, \partial_\beta\, x_\mu$. Here we shall follow notations in [1], [21]. The Weyl transformation

$$\delta g_{\alpha\beta} = \Lambda\, g_{\alpha\beta},$$

with Λ an arbitrary function, satisfies

$$\sum_{\alpha \leqq \beta} [L]_{g_{\alpha\beta}}\, \delta g_{\alpha\beta} = 0,$$

and therefore its associated conserved quantity vanishes.

We can trace back this occurrence to the fact that there are three primary hamiltonian constraints $\pi^{\alpha\beta}$ (the momenta of $g_{\alpha\beta}$), whereas their corresponding primary lagrangian constraints $K \cdot \pi^{\alpha\beta}$ are not independent; indeed,

$$\sum_{\alpha \leqq \beta} g_{\alpha\beta}\, (K \cdot \pi^{\alpha\beta}) = 0.$$

This relation implies that there are only two independent primary lagrangian constraints, and since its coefficients $g_{\alpha\beta}$ are projectable we obtain at once a primary hamiltonian constraint

$$\phi_{\mathrm{W}} = \sum_{\alpha \leqq \beta} g_{\alpha\beta}\, \pi^{\alpha\beta}$$

that satisfies $K \cdot \phi_{\mathrm{W}} = 0$. This yields a Noether transformation

$$\delta g_{\alpha\beta}(\sigma) = \Lambda \int \mathrm{d}\sigma'\, g_{\alpha\beta}(\sigma')\, \delta(\sigma' - \sigma) = \Lambda\, g_{\alpha\beta}(\sigma),$$

which is the Weyl symmetry when Λ is considered as an arbitrary function.

6. CONCLUSIONS

In this paper we have exhibited the necessary and sufficient conditions (under certain regularity assumptions) for a lagrangian system to present Noether transformations with vanishing conserved quantity.

These transformations are identified as those generated by the primary hamiltonian constraints whose lagrangian stability (2.9) is identically zero; in order words, the primary hamiltonian constraints that do not yield secondary constraints.

Due to the general form (3.9) of a hamiltonian gauge generator we can identify gauge Noether transformations with vanishing conserved quantity as those transformations whose hamiltonian gauge generator has the simplest form, $G = \varepsilon(t)\, \phi$. Since such G generate gauge transformations, the fact that the correspondence between Noether transformations and conserved quantities is not one-to-one in the lagrangian formalism has no special implication on the physical states, which remain unchanged under these transformations.

REFERENCES

[1] X. GRÀCIA and J.-M. PONS, A Hamiltonian Approach to Lagrangian Noether Transformations, *J. Phys. A: Math. Gen.*, Vol. **25**, 1992, pp. 6357-6369.

[2] C. BATLLE, J. GOMIS, J.-M. PONS and N. ROMÁN-ROY, Equivalence between the Lagrangian and Hamiltonian Formalisms for Constrained Systems, *J. Math. Phys.*, Vol. **27**, 1986, pp. 2953-2962.

[3] J.-M. PONS, New Relations between Hamiltonian and Lagrangian Constraints, *J. Phys. A: Math. Gen.*, Vol. **21**, 1988, pp. 2705-2715.

[4] X. GRÀCIA and J.-M. PONS, On an Evolution Operator Connecting Lagrangian and Hamiltonian Formalisms, *Lett. Math. Phys.*, Vol. **17**, 1989, pp. 175-180.

[5] X. GRÀCIA, J.-M. PONS and N. ROMÁN-ROY, Higher-Order Lagrangian Systems: Geometric Structures, Dynamics, and Constraints, *J. Math. Phys.*, Vol. **32**, 1991, pp. 2744-2763.
[6] X. GRÀCIA and J.-M. PONS, Gauge Transformations, Dirac's Conjecture and Degrees of Freedom for Constrained System, *Ann. Phys.*, (N.Y.), Vol. **187**, 1988, pp. 355-368.
[7] A. E. NOETHER, Invariante Variationsprobleme, *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.*, 1918, pp. 235-257.
[8] E. BESSEL-HAGEN, Über die Erhaltungssätze der Elektrodynamik, *Math. Ann.*, Vol. **84**, 1921, pp. 258-276.
[9] E. L. HILL, Hamilton's Principle and the Conservation Theorems of Mathematical Physics, *Rev. Mod. Phys.*, Vol. **23**, 1951, pp. 253-260.
[10] J. ROSEN, Generalized Noether's theorem. I. Theory, *Ann. Phys.* (N.Y.), Vol. **82**, 1974, pp. 54-69.
[11] P. J. OLVER, *Applications of Lie Groups to Differential Equations*, GTM 107, Springer, New York, 1986.
[12] J.-F. CARIÑENA, C. LÓPEZ and E. MARTÍNEZ, A New Approach to the Converse of Noether's theorem, *J. Phys. A: Math. Gen.*, Vol. **22**, 1989, pp. 4777-4786.
[13] C. FERRARIO and A. PASSERINI, Symmetries and Constants of Motion for Constrained Lagrangian Systems: a Presymplectic Version of the Noether Theorem, *J. Phys. A: Math. Gen.*, Vol. **23**, 1990, pp. 5061-5081.
[14] L. LUSANNA, The second Noether Theorem as the Basis of the Theory of Singular Lagrangians and Hamiltonian Constraints, *Riv. Nuovo Cim.*, Vol. **14** (3), 1991
[15] M. HENNEAUX, Lectures on the antifield-BRST Formalism for Gauge Theories, *Nucl. Phys. B (Proc. Suppl.)*, Vol. **18A**, 1990, pp. 47-105.
[16] I. A. BATALIN and G. A. VILKOVISKY, Closure of the Gauge Algebra, Generalized Lie Equations and Feynman Rules, *Nucl. Phys. B*, Vol. **234**, 1984, pp. 106-124.
[17] J.-L. ANDERSON and P. G. BERGMANN, Constraints in Covariant Field Theory, *Phys. Rev.*, Vol. **83**, 1951, pp. 1018-1025.
[18] L. CASTELLANI, Symmetries in Constrained Hamiltonian Dynamics, *Ann. Phys. (N.Y.)*, Vol. **143**, 1982, pp. 357-371.
[19] J. GOMIS, K. KAMIMURA and J.-M. PONS, To Construct Gauge Transformations from Singular Lagrangians, *Europhys. Lett.*, Vol. **2**, 1986, pp. 187-194.
[20] J. GOMIS, M. HENNEAUX and J.-M. PONS, Existence Theorem for Gauge Symmetries in Hamiltonian Constrained Systems, *Class. Quantum Grav.*, Vol. **7**, 1990, pp. 1089-1096.
[21] C. BATLLE, J. GOMIS, X. GRÀCIA and J.-M. PONS, Noether's Theorem and Gauge Transformations: Application to the Bosonic String and CP_2^{n-1} Model, *J. Math. Phys.*, Vol. **30**, 1989, pp. 1345-1350.

(Manuscript received June 21, 1993; revised October 5, 1993.)