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## **Absolute spectral continuity for $N$ -body Stark Hamiltonians**

by

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ABSTRACT. – We prove that the spectrum for (a class of)  $N$ -body Stark Hamiltonians is purely absolutely continuous. This is done under a weak assumption on the decay of the pair potentials and for  $N > 3$  under an additional condition that holds for the “atomic” case.

RÉSUMÉ. – Nous prouvons que le spectre d’une classe de Hamiltoniens d’effet stark à  $N$ -corps est purement et absolument continu. Cette preuve requiert des hypothèses faibles sur la décroissance des potentiels de paires et, pour  $N > 3$ , une condition supplémentaire automatiquement vérifiée pour le cas d’un « atome ».

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### **1. INTRODUCTION**

This paper concerns spectral properties for  $N$ -body Stark Hamiltonians. We address ourselves to the problem of proving absence of pure point and singular continuous spectra for a large class of potentials. While it is well known (*see* [S] and references quoted there) that for  $N = 2$  and under the assumption of a vanishing (non-electric) force at infinity the pure point spectrum is empty, the case  $N > 2$  has received little attention and a complete answer is not known. As a matter of fact the only result in the literature the author is aware of is due to Sigal [S] who proved absence

of pure point spectrum for atoms in the infinite mass approximation model (or more generally for Born-Oppenheimer molecules). The condition of repulsion (between electrons) played a crucial role in the approach of [S]. In this paper no such assumption is imposed. We consider the following Hamiltonian  $H$  for a system of  $N$   $\nu$ -dimensional particles labelled  $1, \dots, N$  with masses, charges, positions and momenta denoted by  $m_i, q_i, x_i$ , and  $p_i$ , respectively, at the presence of an electric field  $\mathcal{E}$  and under the assumption

$$|v_{ij}(y)| + |\nabla v_{ij}(y)| = o(1) \quad \text{for } y \rightarrow \infty \quad (1.1)$$

on the pair potentials  $v_{ij}$ . It is given by the (standard) procedure of separating out the center of mass motion for the total Hamiltonian

$$\tilde{H} = \sum_{i=1}^N \left( \frac{p_i^2}{2m_i} - q_i \mathcal{E} \cdot x_i \right) + \sum_{1 \leq i < j \leq N} v_{ij}(x_i - x_j).$$

Under a geometric condition shown to be fulfilled for  $N = 3$  and in general (*i.e.* for  $N > 3$ ) for “atoms”:  $m_1 = M, q_1 = Q, m_i = m, q_i = -q; 2 \leq i \leq N$ , we prove

$$\sigma_{pp}(H) = \emptyset; \quad (1.2)$$

*i.e.* that  $H$  does not have bound states.

Although we don't include local singularities (opposed to [S]) we consider (1.1) as a very natural condition for the problem of proving (1.2). At least it is so from the point of view of our method which is a variant of the commutator method of Froese and Herbst for ordinary (without the electric field)  $N$ -body systems [FH]. Another source of inspiration is the construction of a certain vector field in recent papers [D] and [G] which there was applied successfully to the problem of asymptotic completeness (again for ordinary  $N$ -body systems). To explain its crucial property we notice that a basic feature of  $N$ -body Hamiltonians is the lack of decay of the potential (and its derivatives). Explicitly the pair potential  $v_{ij}(x_i - x_j)$  is constant along the collision plane:  $x_i = x_j$ , and hence it does not vanish at infinity. If one tries naively to imitate 2-body techniques this property usually will cause problems. However the point was circumvented in [D] and [G] by using a vector field that in an approximate sense is orthogonal to the gradient of the potential at the critical directions. In this paper we propose a similar construction which is applicable to the above problem (1.2). As indicated above we are only partially successful for  $N > 3$ . It should be remarked that the approach of [S] also involves a certain vector field although it is quite different.

Under an additional assumption on the second order derivatives we also prove

$$\sigma_{sc}(H) = \emptyset; \quad (1.3)$$

*i.e.* that the singular continuous spectrum of  $H$  is empty. This is accomplished by applying the commutator theory invented by Mourre [M], which in fact involves resolvent estimates of independent interest. Moreover we obtain uniform bounds for low energies.

Independently of this work Tamura [T] considered the same problems as here. His method does not involve a geometric condition, on the other hand it requires a certain short range condition on the potentials instead of the more general condition (1.1). It is interesting to point out the similarities of that method and the one of this paper. In both cases the vector field

$$\omega(x) = \left( 1 + \sum_{i=1}^N 2m_i |x_i|^2 \right)^{-\frac{1}{2}} (x_1, \dots, x_N)$$

plays an important role (*cf.* the discussion above) and the proof of the statement (1.2) is done by induction with respect to those subsystems for which the external (or intercluster) motion is purely kinetic (*i.e.* not determined by electric interaction). In order to reduce to such subsystems one needs a certain statement to be uniform with respect to translations of the energy but only to the left (*i.e.* a uniform statement for low energies). In case of this paper it is formulated as Lemma 3.4. A similar statement for high energies (put  $\lambda_0 = \infty$  in the lemma) was conceived by the author as being difficult to prove (*see* though the remark below). Roughly the main difference between the two methods seems to be that in order to deal with the weak condition (1.1) one needs the applied vector field to be conservative. This allows an approach more similar to the original method of [FH] giving a better result, but as explained the drawback is the appearance of a geometric condition that seems difficult to justify in general.

We present our results in the framework of generalized Schrödinger operators. In Section 2 we explain the necessary notations and state the geometric condition and our main results (Condition 2.2, Theorems 2.4 and 2.5 respectively). The next section is devoted to various preliminary results. The most important is Lemma 3.4. In Section 4 we prove some estimates by a variant of the method of [FH] and use them to prove Theorem 2.4 (that corresponds to (1.2)). In Section 5 we prove Theorem 2.5 (that contains the statement (1.3)). Finally in Appendix A we construct a vector field along the line of [D] and [G].

After the completion of the main bulk of this paper Ira Herbst and the author found out how to remove the geometric condition by

using instead the simple conservative vector field  $\kappa\omega(x) + E$ ;  $\kappa \gg 1$ ,  $E = \left( \frac{q_1}{2m_1}\mathcal{E}, \dots, \frac{q_N}{2m_N}\mathcal{E} \right)$ . This is accomplished by an elaborate analysis for high energies (*cf.* the discussion above). The details appear in [HMS] that also incorporates local singularities.

## 2. NOTATIONS, ASSUMPTIONS AND RESULTS

We consider a finite dimensional real vector space  $X$  with an inner product  $x \cdot y$ , and a finite family  $\{X_a \mid a \in \mathcal{A}\}$  of subspaces of  $X$ . It is assumed that this family is closed with respect to intersection and there exist  $a_{\min}, a_{\max} \in \mathcal{A}$  such that  $X_{a_{\min}} = X$  and  $X_{a_{\max}} = 0$ . The orthogonal complement of  $X_a$  in  $X$  is denoted by  $X^a$ . We order  $\mathcal{A}$  by writing  $a_1 \subset a_2$  if  $X^{a_1} \subset X^{a_2}$ . The uniquely determined  $a$  with  $X_a = X_{a_1} \cap X_{a_2}$  is denoted by  $a_1 \cup a_2$ . For given  $a$  we define  $\#a$  as the largest  $n \in \mathbb{N}$  for which there exists an increasing sequence  $a = a_n \subsetneq a_{n-1} \subsetneq \dots \subsetneq a_1 = a_{\max}$ . The largest such  $n$  is  $N := \#a_{\min}$ . The orthogonal projections onto  $X_a$  and  $X^a$  are denoted by  $\Pi_a$  and  $\Pi^a$ , respectively. The corresponding components for  $x \in X$  are denoted  $x_a$  and  $x^a$ . We use similar notations for the components of the momentum operator  $p = -i\nabla = (p_1, \dots, p_{\dim X})$ . We put  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ ;  $|x|^2 = x \cdot x$ . The notations  $\langle p \rangle$  and  $\langle \lambda \rangle$  for  $\lambda \in \mathbb{R}$  are (formally) identified by the same expression.

Let  $-\Delta = p^2$  and  $E \in X \setminus 0$ . Then the Hamiltonian

$$H = -\Delta - E \cdot x + V; \quad V = \sum_{a \in \mathcal{A}} V_a,$$

is essential selfadjoint on  $C_0^\infty(X) \subset L^2(X)$  under the following assumption on the potentials.

ASSUMPTION 2.1. – For all  $a \in \mathcal{A}$  the potential  $V_a(x^a) \in C^1(X^a)$  obeys

$$\partial^\beta V_a(x^a) = o(1) \quad \text{for } x^a \rightarrow \infty \quad \text{and } |\beta| \leq 1.$$

By convention  $\partial^\beta V_{a_{\min}} = 0$ .

Throughout the paper we impose this assumption and the following implicit

CONDITION 2.2. – There exists  $g \in C^\infty(X)$  such that

- (1)  $\partial^\beta g = O(|x|^{1-|\beta|})$  for  $x \rightarrow \infty$ ,
- (2)  $\exists \epsilon_0 > 0 : E \cdot \nabla g(x) \geq \epsilon_0$  for all  $|x| \geq 1$ ,
- (3) For all  $a \in \mathcal{A}$  with  $E_a \neq 0 : \nabla V_a(x^a) \cdot \nabla g(x) = o(1)$  for  $x \rightarrow \infty$ .

In Appendix A we verify Condition 2.2 under Assumption 2.1 and the additional Condition A, which holds for the 3-body problem  $N = 3$  and for general  $N$  under the assumption that  $E_a \neq 0$  for all  $a \neq a_{\max}$ .

To obtain resolvent estimates we shall need

ASSUMPTION 2.3. – For all  $a \in \mathcal{A}$  the potential  $V_a(x^a)$  admits bounded second order distributional derivatives.

Our main results are

THEOREM 2.4. – Suppose Assumption 2.1 and Condition 2.2. Then the pure point spectrum  $\sigma_{pp}(H)$  is empty.

THEOREM 2.5. – Suppose Assumptions 2.1, 2.3 and Condition 2.2. then

$$(1) \forall \delta > \frac{1}{2}, \lambda_0 \in \mathbf{R} : \sup_{\operatorname{Re} z \leq \lambda_0} \|\langle p \rangle^{-\delta} (H - z)^{-1} \langle p \rangle^{-\delta}\| < \infty,$$

$$(2) \forall \delta' > \frac{1}{4}, \lambda_0 \in \mathbf{R} :$$

$$\sup_{\operatorname{Re} z \leq \lambda_0} (\operatorname{Re} z)^\nu \|\langle x \rangle^{-\delta'} (H - z)^{-1} \langle x \rangle^{-\delta'}\| < \infty; \quad \nu < \frac{4\delta' - 1}{4\delta' + 1}.$$

In particular the singular continuous spectrum  $\sigma_{sc}(H)$  is empty. Moreover in both cases the (two) boundary values

$$\lim_{\operatorname{Im} z \downarrow (t)0} \langle p \rangle^{-\delta} (H - z)^{-1} \langle p \rangle^{-\delta}, \quad \lim_{\operatorname{Im} z \downarrow (t)0} \langle x \rangle^{-\delta'} (H - z)^{-1} \langle x \rangle^{-\delta'}$$

exists in  $\mathcal{B}(L^2(X))$  and are attained locally uniformly.

We complete Section 2 by some notations.

For  $R > 1$  the notation  $\chi(\cdot < R)$  stands for the characteristic function for the interval  $(-\infty, R)$ .

For  $\epsilon > 0$  the notation  $\eta_\epsilon$  stands for any function  $\eta \in C_0^\infty(\mathbf{R})$  obeying  $0 \leq \eta \leq 1$  and  $\eta(t) = 1$  for  $|t| \leq \frac{\epsilon}{2}$  and  $\eta(t) = 0$  for  $|t| > \epsilon$ .

For  $a \neq a_{\min}$  we introduce the sub-Hamiltonian

$$H^a = (p^a)^2 - E^a \cdot x^a + V^a; \quad V^a = \sum_{b \subset a} V_b,$$

and

$$H_a = p^2 - E \cdot x + V^a = (p_a)^2 - E_a \cdot x_a + H^a.$$

For  $\lambda \in \mathbf{R}$  we put  $H_\lambda = (H - \lambda)/\langle \lambda \rangle$ . A similar notation is used for  $H^a$  and  $H_a$ .

### 3. PRELIMINARIES

In this section we state various basic facts and in addition an induction hypothesis for Theorem 2.4. It is tacitly understood that Assumption 2.1 holds.

LEMMA 3.1. – *The domain  $\mathcal{D}(H) \subseteq H_{loc}^2$ , the local Sobolev space of order 2. Moreover*

(1)  $\exists C > 0 \forall 0 < \epsilon \leq 1, \lambda \leq 0$ :

$$\left\| p \langle x \rangle^{-\frac{1}{2}} \left( \epsilon H - \lambda - i \frac{C}{\epsilon} \right)^{-1} \right\| \leq 1,$$

(2)  $p^2 \langle x \rangle^{-1} (H - i)^{-1}$  is bounded.

*Proof of (1).* – By the well known formula

$$hp^2 h = \operatorname{Re}(h^2 p^2) + |\nabla h|^2$$

applied to  $h = \langle x \rangle^{-\frac{1}{2}}$  we obtain for  $\psi \in L^2(X)$  and  $\phi = \left( \epsilon H - \lambda - i \frac{C}{\epsilon} \right)^{-1} \psi$  that

$$\|ph\phi\|^2 = \langle \phi, \{ \operatorname{Re}(h^2 H) + h^2 (E \cdot x - V) + |\nabla h|^2 \} \psi \rangle.$$

Clearly

$$\begin{aligned} \langle \phi, \operatorname{Re}(h^2 H) \phi \rangle &\leq \frac{1}{\epsilon} \operatorname{Re} \left\langle \phi, h^2 \left( \epsilon H - \lambda - i \frac{C}{\epsilon} \right) \phi \right\rangle, \\ \left\| \left( \epsilon H - \lambda - i \frac{C}{\epsilon} \right)^{-1} \right\| &\leq \frac{\epsilon}{C}, \end{aligned}$$

and for some  $C' > 0$

$$|\langle \phi, \{ h^2 (E \cdot x - V) + |\nabla h|^2 \} \phi \rangle| \leq C' \|\phi\|^2.$$

Hence for  $\frac{1}{C} + \frac{C'}{C^2} \leq 1$

$$\|ph\phi\|^2 \leq \frac{1}{\epsilon} \|\phi\| \|\psi\| + C' \|\phi\|^2 \leq \|\psi\|^2. \quad \square$$

By a similar method one can prove

LEMMA 3.2. – Suppose  $0 \leq h \in C^\infty(X)$  and  $\psi \in \mathcal{D}(H)$  are given such that  $hH\psi, h\psi, E \cdot xh\psi, (\partial_j h)\psi \in L^2(X)$  for any component  $\partial_j h$  of  $\nabla h$ . Then

$$hp^2\psi, p_j h\psi, hp_j\psi \in L^2(X).$$

We shall prove Theorem 2.4 by showing by induction the following statement for  $1 \leq k \leq N - 1$ .

$q(\mathbf{k}) : \sigma_{pp}(H^a) = \emptyset$  for  $\#a \geq N - k$  and  $E_a = 0$ .

Clearly Theorem 2.4 follows from  $q(N - 1)$ .

Let  $\{j_a\}$ ,  $a \neq a_{\max}$ , be a family of functions on  $X$  each one being smooth and homogeneous of degree 0 outside a compact set. Moreover we assume that

$$\sum_a j_a(x) = 1 \quad \text{for } |x| \text{ large,} \tag{3.1}$$

and for any  $R > 1$  and  $b \not\subset a$

$$\chi(|x^b| < R)j_a = 0 \quad \text{for } |x| \text{ large.} \tag{3.2}$$

LEMMA 3.3. – Let  $\frac{1}{2} \geq \epsilon > 0$  and  $a \neq a_{\max}$ . Then

(1)  $\|\eta_\epsilon(H_\lambda)j_a - j_a\eta_\epsilon((H_a)_\lambda)\| = O(\langle \lambda \rangle^{-\frac{1}{2}})$  for  $\lambda \rightarrow -\infty$ ,

(2)  $\exists C > 0 \forall R > 1, \lambda \leq 0: \|\eta_\epsilon(H_\lambda)\chi(|x| < R)\| \leq CR^{\frac{1}{2}}\langle \lambda \rangle^{-\frac{1}{2}}$ .

*Proof.* – We pick an almost analytic extension  $\tilde{\eta} \in C_0^\infty(\mathbf{C})$ , that is

$$\tilde{\eta}(t) = \eta_\epsilon(t) \quad \text{for } t \in \mathbf{R},$$

$$\forall n \in \mathbf{N}, \quad z \in \mathbf{C} : |\partial_{\bar{z}} \tilde{\eta}(z)| \leq C_n |\text{Im } z|^n.$$

Then for any selfadjoint operator  $A$ , cf. [Hö, p. 63],

$$\eta_\epsilon(A) = \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\eta}(z)(A - z)^{-1} dt ds; \quad z = t + is. \tag{3.3}$$

Hence

$$\begin{aligned} \eta_\epsilon(H_\lambda)j_a - j_a\eta_\epsilon((H_a)_\lambda) &= -\frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\eta}(z)((H_\lambda - z)^{-1} \\ &\quad \times \{H_\lambda j_a - j_a(H_a)_\lambda\}((H_a)_\lambda - z^{-1}) dt ds. \end{aligned} \tag{3.4}$$



But since

$$\langle \lambda \rangle \{H_\lambda j_a - j_a (H_a)_\lambda\} = \sum_{b \neq a} V_b j_a + \Delta j_a - i 2 p \cdot \nabla j_a, \tag{3.5}$$

and, by Lemma 3.1 (1) with  $\epsilon = \langle \lambda \rangle^{-\frac{1}{2}}$  and  $\lambda \rightarrow \epsilon \lambda$ ,

$$\| (H_\lambda - z)^{-1} p \langle x \rangle^{-\frac{1}{2}} \| \leq C' \langle \lambda \rangle^{1/2} \frac{(1 + |z|)}{|\operatorname{Im} z|}, \tag{3.6}$$

the statement (1) follows.

As for the second estimate we can prove by a similar method that

$$\| E \cdot x \eta_\epsilon (H_\lambda) \chi(|x| < R) \| \leq C' R$$

uniformly in  $\lambda < 0$ . Hence (for  $\lambda$  sufficiently small negative)

$$\begin{aligned} \chi \eta^2 \chi &:= \chi(|x| < R) \eta_\epsilon (H_\lambda)^2 \chi(|x| < R) \\ &\leq \chi \eta \frac{-3}{\langle \lambda \rangle} H \eta \chi \\ &\leq \chi \eta \frac{-3}{\langle \lambda \rangle} E \cdot x \eta \chi - \chi \eta \frac{-3}{\langle \lambda \rangle} V \eta \chi \leq C^2 R \langle \lambda \rangle^{-1}. \quad \square \end{aligned}$$

LEMMA 3.4. – Suppose  $q(k)$ . Then for any  $\lambda_0 \in \mathbf{R}$  and  $R > 1$   $\forall b \subset a$  with  $E_b = 0$  and  $\#a \geq N - k$ :

$$\| \eta_\epsilon ((H^a)_\lambda) \chi(|x^b| < R) \| \rightarrow 0 \quad \text{for } \epsilon \rightarrow 0,$$

uniformly in  $\lambda \leq \lambda_0$ .

*Proof.* – The proof is by induction in  $k$ . For notational convenience we assume the statement for  $k < N - 1$  and look at  $k = N - 1$  in which case we only need to consider  $a = a_{\max}$ . (The start of induction  $k = 1$  follows immediately from the first step of the proof below.)

Suppose first that  $b = a_{\max}$ . We decompose

$$(-\infty, \lambda_0] = (-\infty, \lambda'_0] \cup (\lambda'_0, \lambda_0].$$

Then by applying Lemma 3.3 (2) with  $\epsilon = \epsilon' = \frac{1}{2}$  we obtain uniformly in  $\frac{1}{4} > \epsilon > 0$  and  $\lambda \in (-\infty, \lambda'_0]$

$$\| \eta_\epsilon (H_\lambda) \chi(|x| < R) \| \leq \| \eta_{\epsilon'} (H_\lambda) \chi(|x| < R) \| \leq C \langle \lambda'_0 \rangle^{-\frac{1}{2}}.$$

The right hand side goes to zero for  $\lambda'_0 \rightarrow -\infty$ . Thus it suffice to look at  $\lambda$  in a compact set  $[\lambda'_0, \lambda_0]$ . But the convergence statement of the lemma

holds uniformly with respect to such  $\lambda$  by a compactness argument since by assumption  $\sigma_{pp}(H) = \emptyset$ .

To treat a general  $b$  with  $E_b = 0$  we use the partition of unity introduced above. By the statement proved for  $b = a_{\max}$  and (3.1), (3.2) it suffices to prove that for any  $a \neq a_{\max}$  with  $b \subset a$

$$\|\eta_\epsilon(H_\lambda) j_a \chi(|x^b| < R)\| \rightarrow 0 \quad \text{for } \epsilon \rightarrow 0,$$

the statement being uniformly in  $\lambda \leq \lambda_0$ .

For that we proceed somewhat similarly. We insert

$$\begin{aligned} \eta_\epsilon(H_\lambda) j_a &= \eta_\epsilon(H_\lambda) \{ \eta_{\epsilon'}(H_\lambda) j_a - j_a \eta_{\epsilon'}((H_a)_\lambda) \} \\ &\quad + \eta_\epsilon(H_\lambda) j_a \eta_{\epsilon'}((H_a)_\lambda), \end{aligned} \tag{3.7}$$

for  $\epsilon' \geq 2\epsilon$ . As for the contribution from the second term on the right hand side we use the fact that by a partial Fourier transform

$$\begin{aligned} \eta_{\epsilon'}((H_a)_\lambda) &= \int \oplus d\xi \eta_{\epsilon'}((H^a + |\xi|^2 - \lambda)/\langle \lambda \rangle) \\ &= \int \oplus d\xi \eta_{\epsilon'}((H^a + |\xi|^2 - \lambda)/\langle \lambda \rangle) \eta_{\tilde{\epsilon}}((H^a)_{\tilde{\lambda}}); \\ &\quad \tilde{\epsilon} \geq 2\langle \lambda_0 \rangle \epsilon', \quad \tilde{\lambda} = \lambda - |\xi|^2. \end{aligned} \tag{3.8}$$

(Notice that  $E_a = 0$ .) By the induction hypothesis

$$\|\eta_{\tilde{\epsilon}}((H^a)_{\tilde{\lambda}}) \chi(|x^b| < R)\| \rightarrow 0 \quad \text{for } \tilde{\epsilon} \rightarrow 0, \tag{3.9}$$

uniformly in  $\tilde{\lambda} \leq \lambda_0$ . We fix  $\tilde{\epsilon} = 2\langle \lambda_0 \rangle \epsilon'$  small to control the contribution from the second term on the right hand side of (3.7). The contribution from the first term can be estimated small, uniformly in  $2\epsilon \leq \epsilon'$  and  $\lambda \in (-\infty, \lambda'_0]$ , by choosing  $\lambda'_0$  sufficiently small negative. This follows from Lemma 3.3 (1). For  $\lambda$  in the remaining set  $[\lambda'_0, \lambda_0]$  it suffices by (3.4) to show that

$$\|\eta_\epsilon(H_\lambda) \{ H_\lambda j_a - j_a (H_a)_\lambda \}\| \rightarrow 0 \quad \text{for } \epsilon \rightarrow 0,$$

uniformly with respect to such  $\lambda$ . But this follows readily from (3.5) by using (3.2), Lemma 3.1 (1) and the statement of the first step of the proof above.  $\square$

## 4. ESTIMATES

In this section we complete the proof of Theorem 2.4. This is done by verifying the statement  $q(N-1)$  of Section 3 under the assumption of  $q(k)$  for  $k < N-1$ , a procedure justified by the fact that Condition 2.2 holds upon replacing  $H$  by  $H^a$  if  $E_a = 0$  (simply by restricting the function  $g$ ) and a change of notation. The start of induction  $k = 1$  can easily be extracted from the proof below.

We shall use a variant of the method of [FH] (see also [AHS] and [He]), which basically relies on the following computations.

For some smooth functions  $\theta : \mathbf{R}_+ \rightarrow \mathbf{R}$  and  $f : X \rightarrow \mathbf{R}_+$  to be specified below we put

$$A_f = (\nabla f \cdot p + p \cdot \nabla f)/2,$$

$$\theta = \theta(f), \quad \theta' = \left(\frac{d}{dt}\theta\right)(f), \quad \theta'' = \left(\frac{d^2}{dt^2}\theta\right)(f),$$

and compute the commutator

$$\left. \begin{aligned} i[H, e^\theta A_f e^\theta] &= e^\theta \{T_1 + \dots + T_4\} e^\theta; \\ T_1 &= 4 A_f \theta' A_f, \\ T_2 &= 2 p f^{(2)} p, \\ T_3 &= (E - \nabla V) \cdot \nabla f, \\ T_4 &= -\frac{1}{2} \Delta^2 f + \nabla f \cdot \nabla ((\theta'^2 - \theta'') |\nabla f|^2). \end{aligned} \right\} \quad (4.1)$$

Moreover

$$(H - \lambda - (\theta'^2 - \theta'') |\nabla f|^2 + 2i\theta' A_f) e^\theta = e^\theta (H - \lambda). \quad (4.2)$$

We shall use these formulas with

$$f(x) = f_\kappa(x) = \kappa \langle x \rangle + g(x),$$

where  $\kappa \gg 1$  and  $g$  is given by Condition 2.2.

As for  $\theta$  we have

$$\frac{d}{dt}\theta(t) \geq 0, \quad \left(\frac{d^n}{dt^n}\theta\right)(t) = O(t^{1-n}) \quad \text{for } t \rightarrow \infty; \quad n \geq 0.$$

Hence

$$T_1 \geq 0, \quad T_4 = O(\langle x \rangle^{-1}) \quad \text{for } x \rightarrow \infty. \quad (4.3)$$

To deal with the terms  $T_2$  and  $T_3$  we notice that

$$\nabla V \cdot \nabla \langle x \rangle = o(1) \quad \text{for } x \rightarrow \infty.$$

Hence

$$T_3 = E \cdot \kappa \frac{x}{\langle x \rangle} + \epsilon_0 - \sum_{a: E_a=0} \nabla V_a(x^a) \cdot \nabla g(x) + o(1).$$

On the other hand we can write

$$\begin{aligned} E \cdot \frac{x}{\langle x \rangle} &= \operatorname{Re} \left( \frac{1}{\langle x \rangle} \{p^2 + V - H\} \right) \\ &= p \cdot \frac{1}{\langle x \rangle} p + \left( \frac{V}{\langle x \rangle} - \left( \Delta \frac{1}{\langle 2x \rangle} \right) \right) - \operatorname{Re} \left( \frac{1}{\langle x \rangle} H \right). \end{aligned}$$

As for  $T_2$

$$pf^{(2)}p \geq pg^{(2)}p = pO(\langle x \rangle^{-1})p.$$

Putting together we obtain (for  $\kappa$  large enough)

$$T_2 + T_3 \geq \epsilon_0 - \sum_{a: E_a=0} \nabla V_a(x^a) \cdot \nabla g(x) - \kappa \operatorname{Re} \left( \frac{1}{\langle x \rangle} H \right) + o(1). \tag{4.4}$$

Finally, by (4.1), (4.3) and (4.4),

$$\left. \begin{aligned} &i[H, e^\theta A_f e^\theta] = e^\theta \{ \dots \} e^\theta; \\ &\{ \dots \} \geq T_1 + \epsilon_0 - \sum_{b: E_b=0} \nabla V_b(x^b) \cdot \nabla g(x) \\ &- \operatorname{Re} \left( \frac{\kappa}{\langle x \rangle} H \right) + o(1). \end{aligned} \right\} \tag{4.5}$$

In the following applications we consider some further specified functions  $\theta$  depending on parameters. The estimate (4.5) will be uniform with respect to these parameters. As above we suppress the dependence of  $\kappa$  in the notation for  $f = f_\kappa$  and  $\theta = \theta_\kappa(f_\kappa)$ .

LEMMA 4.1. – *Let  $\lambda \in \mathbf{R}$ ,  $s > 0$  and*

$$\theta(t) = s \log \left( t \left( 1 + \frac{t}{\mu} \right)^{-1} \right); \quad \mu > 1.$$

Then for all sufficiently large  $\kappa$  there exists a constant  $C > 0$  and a compact operator  $K$  such that

$$\forall \mu > 1, \quad \phi \in C_0^\infty(X) : \langle \phi, i[(H - \lambda), e^\theta A_f e^\theta] \phi \rangle + C \|e^\theta (H - \lambda) \phi\|^2 \\ \geq \frac{1}{C} \|e^\theta \phi\|^2 - \|K e^\theta \phi\|^2.$$

LEMMA 4.2. – Let  $\lambda \in \mathbf{R}$ ,  $\alpha_0 \geq 0$  and

$$\theta(t) = \mu \log \left( 1 + \frac{\gamma t}{\kappa \mu} \right) + \frac{\alpha}{\kappa} t; \quad \mu > 1, \quad \gamma > 0, \quad \alpha \geq 0.$$

Then for a (small)  $\delta > 0$  and all sufficiently large  $\kappa$  there exists a constant  $C > 0$  and a compact operator  $K$  such that

$$\forall \mu > 1, \quad \gamma + |\alpha - \alpha_0| \leq \delta, \quad \phi \in C_0^\infty(X) : \\ \langle \phi, i[(H - \lambda), e^\theta A_f e^\theta] \phi \rangle + C \|e^\theta (H - \lambda) \phi\|^2 \\ \geq \frac{1}{C} \|e^\theta \phi\|^2 - \|K e^\theta \phi\|^2.$$

*Proof of Lemmas 4.1 and 4.2.* – We compute

$$\theta'(t) \leq \frac{s}{t}, \quad |\theta''(t)| \leq 2 \frac{s}{t^2}, \quad |\theta'''(t)| \leq \frac{8s}{t^3} \quad (4.6)$$

in case of Lemma 4.1, while

$$\left| \theta'(t) - \frac{\alpha}{\kappa} \right| \leq \frac{\gamma}{\kappa}, \quad |\theta''(t)| \leq \frac{\gamma}{\kappa t}, \quad |\theta'''(t)| \leq \frac{2\gamma}{\kappa t^2} \quad (4.7)$$

in case of Lemma 4.2.

Thus by inspection we obtain for  $h = T_4$  given by (4.1) that

$$|h(x)| \leq C \langle x \rangle^{-3}; \quad C = C_\kappa \text{ independent of } \mu, \quad (4.8)$$

under the condition (4.6), while

$$|h(x)| \leq C \langle x \rangle^{-1}; \quad C = C_\kappa \text{ independent of } \mu, \gamma, \alpha, \quad (4.9)$$

under the conditions (4.7) and  $\gamma + |\alpha - \alpha_0| \leq 1$ .

By (4.8) and (4.9) the fifth term on the right hand side of (4.5) is  $o(1)$  uniformly with respect to the parameters. Similarly all statements below will be uniform with respect to the parameter  $\mu$  in case of Lemma 4.1 and  $\mu, \gamma, \alpha$  in case of Lemma 4.2 (constrained as in the lemmas), and in

addition with respect to  $\phi \in C_0^\infty(X)$ . For notational convenience we put  $\alpha_0 = 0$  in case of Lemma 4.1, introduce  $\bar{\lambda} = \lambda + (\alpha_0)^2$  and abbreviate the expectation  $\langle e^\theta \phi, T e^\theta \phi \rangle = \langle T \rangle$  for any linear operator  $T$ .

As for the fourth term on the right hand site of (4.5) we obtain from (4.6) and (4.7) that

$$-\text{Re} \left\langle \frac{\kappa}{\langle x \rangle} H \right\rangle = -\text{Re} \left\langle \frac{\kappa}{\langle x \rangle} \{ H - \lambda - (\theta'^2 - \theta'') |\nabla f|^2 \} \right\rangle + O(\langle x \rangle^{-1}).$$

Hence by (4.2), the Cauchy Schwarz inequality and the same bounds again, it follows that for any  $\phi \in C_0^\infty(X)$

$$\begin{aligned} -\left\langle \text{Re} \left( \frac{\kappa}{\langle x \rangle} H \right) \right\rangle &= -\text{Re} \left\langle e^\theta \phi, \frac{\kappa}{\langle x \rangle} e^\theta (H - \lambda) \phi \right\rangle \\ &\quad + \text{Re} \left\langle \frac{2\kappa i}{\langle x \rangle} \theta' A_f \right\rangle + \langle O(\langle x \rangle^{-1}) \rangle \\ &\geq -\epsilon \| e^\theta \phi \|^2 - \frac{\kappa^2}{\epsilon} \| e^\theta (H - \lambda) \phi \|^2 \\ &\quad - \left\langle \frac{T_1}{3} \right\rangle - \left\langle \frac{\kappa^2}{\langle x \rangle^2} \theta' \right\rangle + \langle O(\langle x \rangle^{-1}) \rangle \\ &\geq -\epsilon \| e^\theta \phi \|^2 - \frac{\kappa^2}{\epsilon} \| e^\theta (H - \lambda) \phi \|^2 \\ &\quad - \left\langle \frac{T_1}{3} \right\rangle + \langle O(\langle x \rangle^{-1}) \rangle; \quad \epsilon = \frac{\epsilon_0}{7}. \end{aligned} \tag{4.10}$$

As for the third term on the right hand side of (4.5) we obtain by choosing  $R$  large enough that it is estimated by

$$\left. \begin{aligned} - \sum_{b: E_b=0} h_b(x) &\geq - \sum_{b: E_b=0} \chi(|x^b| < R) h_b(x) - \frac{\epsilon_0}{7}; \\ h_b(x) &= \nabla V_b(x^b) \cdot \nabla g(x). \end{aligned} \right\} \tag{4.11}$$

We look more closely at the first term on the right hand side of (4.11). The idea is to write

$$\left. \begin{aligned} \chi(|x^b| < R) &= \eta_\epsilon(H_{\bar{\lambda}}) \chi(|x^b| < R) + \mathcal{E}_b; \\ \mathcal{E}_b &= H_{\bar{\lambda}} \zeta_\epsilon(H_{\bar{\lambda}}) \chi(|x^b| < R), \quad \zeta_\epsilon(t) = t^{-1}(1 - \eta_\epsilon(t)), \end{aligned} \right\} \tag{4.12}$$

and choose  $\epsilon$  small to treat the first term by Lemma 3.4 and its proof. The appearance of functions  $h_b$  as factors is a minor complication since these are bounded. By mimicking the second step of the proof of Lemma 3.4

(cf. (3.1)-(3.8)) we can write

$$\begin{aligned} \eta_\epsilon(H_{\tilde{\lambda}})\chi(|x^b| < R) &= \sum_{b \subset a} j_a \eta_\epsilon((H_a)_{\tilde{\lambda}})\chi(|x^b| < R) + K \\ &= \sum_{b \subset a} j_a \eta_\epsilon((H_a)_{\tilde{\lambda}}) \int \oplus d\xi \eta_{\tilde{\epsilon}}((H_a)_{\tilde{\lambda}})\chi(|x^b| < R) + K; \\ &K \text{ compact, } \quad \tilde{\epsilon} = 2\langle \tilde{\lambda} \rangle \epsilon, \quad \tilde{\lambda} = \bar{\lambda} - |\xi|^2. \end{aligned}$$

The first term goes to zero as  $\epsilon \rightarrow 0$  since (3.9) holds as guaranteed by Lemma 3.4 and our induction hypothesis. In conclusion for  $\epsilon$  small enough

$$-\text{Re} \sum_{b: E_b=0} \eta_\epsilon(H_{\tilde{\lambda}})\chi(|x^b| < R) h_b \geq -\frac{\epsilon_0}{7}I - K; \quad K \text{ compact.} \quad (4.13)$$

On the other hand for fixed  $\epsilon$  we can estimate

$$\begin{aligned} |\langle \mathcal{E}_b h_b \rangle| &\leq \|H_{\tilde{\lambda}} e^\theta \phi\| C \|e^\theta \phi\| \\ &\leq C' \|H_{\tilde{\lambda}} e^\theta \phi\|^2 + \epsilon' \|e^\theta \phi\|^2 \\ &\leq 2C' \|(H - \lambda - (\theta'^2 - \theta'')|\nabla f|^2 + 2i\theta' A_f)\epsilon^\theta \phi\|^2 \\ &\quad + 2C' \|( (\alpha_0)^2 - (\theta'^2 - \theta'')|\nabla f|^2 \} + 2i\theta' A_f)\epsilon^\theta \phi\|^2 + \epsilon' \|e^\theta \phi\|^2; \\ &\epsilon' > 0, \quad C' = C'_{\epsilon, \epsilon'}. \end{aligned}$$

As for the first term we use (4.2). To estimate the second term we notice the bound

$$\theta' \leq C \kappa^{-1}.$$

Moreover

$$(\alpha_0)^2 - (\theta'^2 - \theta'')|\nabla f|^2 = O(\langle x \rangle^{-1})$$

under the condition (4.6), while

$$|(\alpha_0)^2 - (\theta'^2 - \theta'')|\nabla f|^2| \leq C(\langle x \rangle^{-1} + \kappa^{-1} + \delta)$$

under the condition (4.7). In both cases  $C$  is independent of  $\kappa$  and the parameters (constrained to  $\gamma + |\alpha - \alpha_0| \leq \delta$  in case of Lemma 4.2). We apply the above estimates for  $\kappa$  large, and in case of Lemma 4.2 for  $\delta$  small. As a result

$$\begin{aligned} -\left\langle \text{Re} \sum_{b: E_b=0} \mathcal{E}_b h_b \right\rangle &\geq -\epsilon' \|e^\theta \phi\|^2 - C \|e^\theta (H - \lambda)\phi\|^2 \\ &\quad - \left\langle \frac{T_1}{3} \right\rangle + \langle O(\langle x \rangle^{-2}) \rangle; \quad \epsilon' = \frac{\epsilon_0}{7}. \quad (4.14) \end{aligned}$$

By combining (4.5), (4.10), (4.11), (4.13) and (4.14) we finally obtain that for a constant  $C > 0$  and compact  $K$  (both independent of  $\mu$ , and in case of Lemma 4.2 also of  $\gamma$  and  $\alpha$ )

$$\begin{aligned} & \langle \phi, i[(H - \lambda), e^\theta A_f e^\theta] \phi \rangle + C \| e^\theta (H - \lambda) \phi \|^2 \\ & \geq \frac{3}{7} \epsilon_0 \| e^\theta \phi \|^2 + \left\langle \frac{T_1}{3} \right\rangle - \| K e^\theta \phi \|^2 + \langle h \rangle; \quad h = o(1). \end{aligned} \quad (4.15)$$

It remains to treat the last term on the right hand side of (4.15). For that we pick  $C', R > 1$  independent of the parameters such that

$$h \geq -C' \chi(|x| < R) - \frac{\epsilon_0}{7}; \quad (4.16)$$

cf. (4.8) and (4.9). Since for any  $\sigma > 0$  the operator  $\eta_\sigma(H) \chi(|x| < R)$  is compact we can estimate

$$-C' \chi(|x| < R) \geq -K - C' \operatorname{Re}(H \zeta_\sigma(H) \chi(|x| < R)), \quad (4.17)$$

where  $K$  is compact and  $\zeta_\sigma$  is given by (4.12).

By an argument now familiar there exists a constant  $\tilde{C} > 0$  such that

$$\tilde{C}^{-1} \| H e^\theta \phi \|^2 \leq \| e^\theta \phi \|^2 + \| e^\theta (H - \lambda) \phi \|^2 + \langle T_1 \rangle. \quad (4.18)$$

By choosing  $\sigma$  large enough we obtain from the bound  $\| \zeta_\sigma(H) \| \leq \frac{2}{\sigma}$  and (4.18) that

$$\begin{aligned} & -C' \operatorname{Re} \langle H \zeta_\sigma(H) \chi(|x| < R) \rangle \\ & \geq -\frac{\epsilon_0}{7} \| e^\theta \phi \|^2 - \left\langle \frac{T_1}{3} \right\rangle - \| e^\theta (H - \lambda) \phi \|^2. \end{aligned} \quad (4.19)$$

Putting together the statements (4.15), (4.16), (4.17) and (4.19) it follows that

$$\begin{aligned} & \langle \phi, i[(H - \lambda), e^\theta A_f e^\theta] \phi \rangle + C \| e^\theta (H - \lambda) \phi \|^2 \\ & \geq \frac{\epsilon_0}{7} \| e^\theta \phi \|^2 - \| K e^\theta \phi \|^2. \end{aligned}$$

Hence we have proved the lemmas.  $\square$

The previous estimates lead to

LEMMA 4.3. – Suppose  $(H - \lambda) \psi = 0$ ;  $\lambda \in \mathbf{R}$ ,  $\psi \in \mathcal{D}(H)$ . Then

$$\forall \alpha \geq 0 : \quad e^{\alpha(x)} \psi(x) \in L^2(X).$$

*Proof.* – First we prove

$$\forall s \geq 0 : \quad \langle x \rangle^s \psi(x) \in L^2(X). \quad (4.20)$$



For that we use Lemmas 3.1 and 4.1 to prove by an approximation argument that

$$C \| K e^\theta \psi \|^2 \geq \| e^\theta \psi \|^2 \tag{4.21}$$

with notation of Lemma 4.1. If (4.20) does not hold we obtain a contradiction by letting  $\mu \rightarrow \infty$  in (4.21).

Now let

$$\alpha_0 = \sup \{ \alpha \geq 0 \mid e^{\alpha(x)} \psi(x) \in L^2(X) \}.$$

We need to show that  $\alpha_0 = \infty$ . Suppose this statement is false.

If  $\alpha_0 = 0$  we come to a contradiction by using (4.20), Lemma 3.2, and Lemma 4.2 with  $\alpha = \alpha_0$ . It is obtained by letting  $\mu \rightarrow \infty$  in an estimate similar to (4.21).

If  $\alpha_0 > 0$  the contradiction arises from Lemma 4.2 by first fixing  $\alpha < \alpha_0$  such that  $\alpha_0 < \alpha + \frac{\delta}{2}$ , letting  $\mu = \frac{\delta}{2}$  and then choosing  $\kappa$  large. As above we apply Lemma 3.2 to show that the left hand side of the estimate of Lemma 4.2 (with the described inputs) vanishes and let  $\mu \rightarrow \infty$ .  $\square$

*Proof of  $q(N-1)$ .* – By using Lemmas 3.2 and 4.3 the statement follows readily from the following formula valid for any  $\phi \in C_0^\infty(X)$ :

$$\begin{aligned} & \| e^{\alpha E \cdot x} \{ (H - \lambda) - V \} \phi \|^2 \\ &= \| \{ p^2 - E \cdot x - (\lambda + \alpha^2 |E|^2) \} e^{\alpha E \cdot x} \phi \|^2 \\ & \quad + 4 \alpha^2 \| E \cdot p e^{\alpha E \cdot x} \phi \|^2 + 2 \alpha |E|^2 \| e^{\alpha E \cdot x} \phi \|^2; \\ & \quad \lambda \in \mathbf{R}, \quad \alpha \geq 0. \end{aligned}$$

With notation of Lemma 4.3 we can apply it to  $\phi = \psi$ . This gives the condition

$$\| e^{\alpha E \cdot x} V \psi \|^2 \geq 2 \alpha |E|^2 \| e^{\alpha E \cdot x} \psi \|^2,$$

which clearly holds for all  $\alpha \geq 0$  only if  $\psi = 0$ .  $\square$

### 5. RESOLVENT ESTIMATES

This section is devoted to a proof (or sketch of proof) of Theorem 2.5. We apply the commutator technique due to Mourre [M]. With the notations of Section 4 we choose  $\kappa$  so large that (4.4) holds. Under this condition we claim that  $A = A_f$  is a conjugate operator in the sense of Mourre. In fact we claim that given  $\lambda_0 \in \mathbf{R}$  then for  $\epsilon_1 > 0$  sufficiently small

$$\eta_{\epsilon_1}(H_\lambda) i [H, A] \eta_{\epsilon_1}(H_\lambda) \geq \eta_{\epsilon_1}(H_\lambda) \left. \frac{1}{2} \left( \epsilon_0 + \kappa \frac{\langle \lambda \rangle}{\langle x \rangle} \right) \eta_{\epsilon_1}(H_\lambda); \right\} \tag{5.1}$$

$$\lambda \leq \lambda_0.$$

To indicate a proof of this it is noticed that the commutator  $i[H, A]$  extends to a symmetric operator on  $\mathcal{D}(H)$  (cf. Lemma 3.1) and that this operator, denoted by the same notation, has the lower bound given by the right hand side of (4.4). To proceed from there we substitute

$$-\operatorname{Re}\left(\frac{\kappa}{\langle x \rangle} H\right) = -\langle \lambda \rangle \operatorname{Re}\left(\frac{\kappa}{\langle x \rangle} H_\lambda\right) - \frac{\kappa \lambda}{\langle x \rangle} \tag{5.2}$$

and estimate each term (when localized in energy) separately under use of Theorem 2.4 and Lemma 3.4. As for the contribution from (5.2) the second term on the right hand side dominates the first for small negative  $\lambda$  and small  $\epsilon_1 > 0$  as can easily be shown by using the bound

$$\|\langle x \rangle^{\frac{1}{2}} H_\lambda \eta_{\epsilon'}(H_\lambda) \langle x \rangle^{-\frac{1}{2}}\| \leq \epsilon' + C_{\epsilon'} \langle \lambda \rangle^{-\frac{1}{2}}; \quad \epsilon' > 0, \quad \lambda < 0,$$

which in turn follows from (3.3) and (3.6).

We shall not use the full strength of (5.1) but rather the following result which includes an upper bound that follows in a similar way.

LEMMA 5.1. – *Let  $\lambda_0 \in \mathbf{R}$ . Then there exists  $C, \epsilon_1 > 0$  such that*

$$C \eta_{\epsilon_1}(H_\lambda)^2 \geq \eta_{\epsilon_1}(H_\lambda) i[H, A] \eta_{\epsilon_1}(H_\lambda) \geq \frac{\epsilon_0}{2} \eta_{\epsilon_1}(H_\lambda)^2; \quad \lambda \leq \lambda_0.$$

To show the part of Theorem 2.5 that involves the weight  $\langle p \rangle^{-\delta}$  it suffices to show the statements with this weight replaced by  $\langle A \rangle^{-\delta}$  due to the fact that  $\langle A \rangle^\delta \langle p \rangle^{-\delta}$  is bounded for any  $\delta > \frac{1}{2}$ . This can be done by mimicking [M] and [PSS] under use of the uniform bounds of Lemma 5.1. With notation of [M] we define  $B^* B$  to be the localized commutator in Lemma 5.1 with  $\lambda = \operatorname{Re} z$  where  $z$  is the resolvent parameter, and use it to establish a certain differential inequality from which the statements follow. To control the double commutator in the Mourre theory we need Assumption 2.3. Some other terms can be dealt with by use of (3.3) and (3.6). We omit the details.

As for the remaining part of Theorem 2.5 that involves the weight  $\langle x \rangle^{-\delta'}$  we basically use Lemma 3.1 (1) to replace  $\langle x \rangle^{-\delta'}$  by  $\langle p \rangle^{-2\delta'}$  and thus reduce to the first part. Here we shall give a detailed proof of the low energy bound (2) omitting the (easy) proof of the result on existence of boundary values. Let  $\delta' > \frac{1}{4}$  be given. Then for a constant  $C > 0$  and any  $\frac{1}{4} < \delta'' \leq \min\left(\delta', \frac{1}{2}\right)$

$$\sup_{\lambda < 0} \|B_1(\lambda)\| < \infty; \quad B_1(\lambda) = \langle p \rangle^{2\delta''} (H - \lambda - iC)^{-1} \langle x \rangle^{-\delta''}, \tag{5.3}$$

as it follows by interpolating the result for  $\delta'' = \frac{1}{2}$  which in turn is valid by Lemma 3.1 (1) and a commutation. With  $\epsilon = \frac{1}{2}$  we decompose

$$\left. \begin{aligned} B_2(\lambda) &:= \langle p \rangle^{2\delta''} (H - \lambda - iC)^{-1} \eta_\epsilon(H_\lambda) \langle x \rangle^{-\delta'} = B_1(\lambda) B_3(\lambda); \\ B_3(\lambda) &= \langle x \rangle^{\delta''} \eta_\epsilon(H_\lambda) \langle x \rangle^{-\delta'}. \end{aligned} \right\} \quad (5.4)$$

Now to obtain (2) it suffices to show boundedness of the expression

$$\begin{aligned} &\langle \lambda \rangle^\nu \|\langle x \rangle^{-\delta'} \eta_\epsilon(H_\lambda) (H - z)^{-1} \eta_\epsilon(H_\lambda) \langle x \rangle^{-\delta'}\|; \\ &\lambda = \operatorname{Re} z, \quad \nu < \frac{4\delta' - 1}{4\delta' + 1}, \end{aligned}$$

uniformly in  $\lambda < 0$ . By using the first resolvent equation and the bound

$$\|\langle x \rangle^{-\delta'} \eta_\epsilon(H_\lambda)\|^2 = O(\langle \lambda \rangle^{-\frac{2\delta'}{1+2\delta'}}) \quad (5.5)$$

which is valid by Lemma 3.3 (2) (applied with  $R = \langle \lambda \rangle^{\frac{1}{1+2\delta'}}$ ), it is enough to bound

$$\langle \lambda \rangle^\nu \|B_2(\lambda)^* \langle p \rangle^{-2\delta''} (H - z)^{-1} \langle p \rangle^{-2\delta''} B_2(\lambda)\|$$

uniformly in  $\lambda = \operatorname{Re} z < 0$  and for some conveniently chosen  $\delta''$  as above. By (5.3), (5.4) and (1) we need to show that

$$\sup_{\lambda < 0} \langle \lambda \rangle^\nu \|B_3(\lambda)\|^2 < \infty. \quad (5.6)$$

But (5.6) holds for  $\delta'' - \frac{1}{4}$  small enough since (5.5) applies for  $\delta'$  replaced by  $\delta' - \delta''$  and since

$$\sup_{\lambda < 0} \langle \lambda \rangle^\nu \|\langle x \rangle^{\delta''}, \eta_\epsilon(H_\lambda)\|^2 < \infty.$$

The latter follows from (3.3) and (3.6). We have proved (2).  $\square$

## APPENDIX A

We shall verify Condition 2.2 under Assumption 2.1 and the following condition on  $E$  and the family  $\{X_a \mid a \in \mathcal{A}\}$ .

CONDITION A

$$\forall a_{\min} \neq b \subset a \neq a_{\max} : E_a = 0 \Rightarrow E_b = 0.$$

We shall construct  $g = \nabla R \cdot E$  outside the unit-ball by cutting off a certain smooth function  $R$  defined outside zero. The latter will be convex and homogeneous of degree two, which in particular means that the property (1) holds. Its construction is inspired by [D] and [G].

Let us denote the set of all  $a \in \mathcal{A}$  such that  $E_a \neq 0$  by  $\mathcal{A}(E)$  and label its elements  $a_1, \dots, a_J$ . For  $j = 1, \dots, J$ ,  $\rho = (\rho_1, \dots, \rho_J) \in \mathbf{R}^J$  and  $0 < \epsilon < 1$  we put

$$T_{j,\rho} = (1 + \rho_j \epsilon^{\#a_j}) \Pi_{a_j}.$$

For  $\rho$  and  $\epsilon$  as above let

$$R_\rho(x) = \frac{1}{2} \max_{j \leq J} x \cdot T_{j,\rho} x; \quad x \in X \setminus 0.$$

Finally with  $\phi$  given as a smooth non-negative function on  $\mathbf{R}^J$  supported in  $[1 - \epsilon^N, 1]^J$  and with  $\int \phi(\rho) d\rho = 1$  we define the function  $R$  by

$$R(x) = \int \phi(\rho) R_\rho(x) d\rho; \quad x \in X \setminus 0.$$

As can easily be seen  $R$  has the properties mentioned above. (As for the smoothness we change variables.) To show the property (3) we shall need  $0 < \epsilon \ll 1$ . At first we prove (2). Let for  $j \leq J$  and  $\rho \in \mathbf{R}^J$

$$Q_{j,\rho} = \{x \in X \mid x \cdot T_{j,\rho} x > x \cdot T_{k,\rho} x; k \neq j\}$$

and  $\chi_{j,\rho}$  be the corresponding characteristic function. Then

$$R(x) = \sum_{j \leq J} \int \phi(\rho) \frac{1}{2} x \cdot T_{j,\rho} x \chi_{j,\rho}(x) d\rho, \tag{A.1}$$

$$\nabla R(x) = \sum_{j \leq J} \int \phi(\rho) T_{j,\rho} x \chi_{j,\rho}(x) d\rho, \tag{A.2}$$

$$R^{(2)}(x) \geq \sum_{j \leq J} \int \phi(\rho) T_{j,\rho} \chi_{j,\rho}(x) d\rho. \tag{A.3}$$

The statement (A.1) follows from a change of variables. As for (A.2) and (A.3) these can be verified in the sense of distribution using for (A.3)

that  $R$  is convex. Then the pointwise statements follow since all functions are continuous.

By (A.3)

$$E \cdot R^{(2)}(x) E \geq \epsilon_0; \quad \epsilon_0 = \min_{a \in \mathcal{A}(E)} |E_a|^2,$$

so (2) holds.

To show (3) let  $a \in \mathcal{A}(E)$  be given. We claim that with  $Q = \{x \in X \mid |x^a|^2 < \epsilon^N |x|^2\}$  and for  $\epsilon$  small enough

$$\Pi^a \nabla R(x) = 0 \quad \text{for } x \in Q. \tag{A.4}$$

Postponing the proof of (A.4) for a moment we prove (3) as follows. Since

$$\nabla V_a(x^a) \cdot R^{(2)}(x) E \rightarrow 0 \quad \text{for } x^a \rightarrow \infty,$$

it suffices to show that this function vanishes in  $Q$ . For that we notice that it is given by  $\nabla V_a(x^a) \cdot \Pi^a R^{(2)}(x) E$ , and that  $\Pi^a R^{(2)}(x) E = E \cdot \nabla \Pi^a(\nabla R)(x)$ . Hence the result follows from (A.4).

It remains to show (A.4). By (A.2) the statement will follow if we can show that  $\int \phi(\rho) T_{j,\rho} x \chi_{j,\rho}(x) d\rho = 0$  for all  $x \in Q$  and all  $j \leq J$  given such that  $a \not\subset a_j$ , which in turn will follow from the statement

$$|x^a|^2 \geq \epsilon^{\#a_j} |x|^2 \tag{A.5}$$

that we claim is valid for all  $x \in Q_{j,\rho}$  where  $\rho \in [1 - \epsilon^N, 1]^J$  and under the above assumptions  $a \in \mathcal{A}(E)$ ,  $a \not\subset a_j$  (and  $\epsilon$  small enough).

To prove (A.5) we first show that

$$|x^{a_j}|^2 \leq \epsilon^{\#a_j} |x|^2 \quad \text{for } x \in Q_{j,\rho}. \tag{A.6}$$

The statement is trivial if  $a_j = a_{\min}$ . Otherwise we have for such  $x$  that  $x \cdot T_{j,\rho} x > x \cdot T_{k,\rho} x$ ;  $a_k = a_{\min}$ . This means that

$$|x^{a_j}|^2 < \frac{\rho_j \epsilon^{\#a_j} - \rho_k \epsilon^{\#a_k}}{1 + \rho_j \epsilon^{\#a_j}} |x|^2 \leq \epsilon^{\#a_j} |x|^2.$$

Moreover we claim that

$$|x^{a \cup a_j}|^2 > \frac{1}{2} \epsilon^{\#a \cup a_j} |x|^2 \quad \text{for } x \in Q_{j,\rho}. \tag{A.7}$$

To see (A.7) we notice that the statement is trivial for  $a \cup a_j = a_{\max}$ . Otherwise  $a \cup a_j =: a_k \in \mathcal{A}(E)$  by Condition A, and thus  $x \cdot T_{j, \rho} x > x \cdot T_{k, \rho} x$ . This means that

$$(1 + \rho_k \epsilon^{\#a_k}) |x^{a_k}|^2 > (\rho_k \epsilon^{\#a_k} - \rho_j \epsilon^{\#a_j}) |x|^2 + (1 + \rho_j \epsilon^{\#a_j}) |x^{a_j}|^2.$$

In particular (since  $\#a_k < \#a_j$ )

$$|x^{a_k}|^2 > \frac{1}{2} \epsilon^{\#a_k} |x|^2,$$

which prove (A.7).

We can assume that

$$4\epsilon |x^{a \cup a_j}|^2 \leq |x^a|^2 + |x^{a_j}|^2. \tag{A.8}$$

Now suppose (A.5) is false. Then by (A.6)-(A.8)

$$\begin{aligned} \frac{1}{2} \epsilon^{\#a \cup a_j} |x|^2 &< |x^{\#a \cup a_j}|^2 \leq \frac{1}{4\epsilon} (|x^a|^2 + |x^{a_j}|^2) \\ &\leq \frac{1}{4\epsilon} 2\epsilon^{\#a_j} |x|^2 \leq \frac{1}{2} \epsilon^{\#a \cup a_j} |x|^2; \quad x \in Q_{j, \rho}, \end{aligned}$$

which obviously is a contradiction.  $\square$

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