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Resonances of relativistic Schrödinger operators with homogeneous electric fields*

by

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ABSTRACT. – In this paper, we apply the analytic distortion technique and the calculus of pseudodifferential operators to study the resonances of the relativistic Schrödinger operators with homogeneous electric fields. By constructing a suitable approximate operator, we give precise locations for resonances generated by eigenvalues below the bottom of the essential spectra and as a consequence, we obtain an upper bound on the widths of resonances.

RÉSUMÉ. – Dans cet article, nous utilisons la technique de déformation analytique et le calcul pseudodifférentiel pour étudier les résonances d'un opérateur de Schrödinger relativiste en présence d'un champ électrique homogène. En construisant une approximation convenable de cet opérateur, nous fournissons une localisation précise des résonances engendrées par les valeurs propres situées au-dessous du spectre essentiel, ce qui fournit en conséquence une borne supérieure sur la largeur des résonances.

1. INTRODUCTION

The study of resonances in Schrödinger operator theory has been an object of growing interest in last years. There are several mathematical

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concepts, but there is not a general theory which can describe all physical phenomena considered as resonances. A very powerful method is the complex scaling which was introduced by Aguilar, Balslev and Combes in [AC], [BC]. It was extended in several directions, *see* [CFKS] for a survey. In [HS], using the calculus of pseudodifferential operators on I-Lagrange manifolds, Fourier integral operators and the microlocal analysis, Helffer and Sjöstrand founded the microlocal theory of resonances for a wide class of h-pseudodifferential operators.

The purpose of this work is to discuss the resonances of relativistic Schrödinger operators with homogeneous electric fields, $H(\beta) = \sqrt{-\Delta + m^2} - m + v(x) + \beta x_1$, where β is a small parameter. In this paper, we assume $v(x)$ satisfies the following conditions.

(H1). There are two constants C, $R_0 > 0$, such that $v(x)$ is an analytic function in variable x_1 in complex region: $\{x_1, |Im x_1| \leq C |Re x_1|, Re x_1 < -R_0\}$.

(H2). $v(x) \to 0$, as $|x| \to \infty$, and $|\partial^\alpha v(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}$, for any $\alpha \geq 0$, and x in the above analyticity region.

One can find many authors who have studied the spectral properties of $H(0)$ in literature. *See* [CMS] and its references there. In general the spectra of $H(0)$ include eigenvalues and the essential spectrum $(0, +\infty)$ (*cf.* [CMS], [Fa2] and [We]). In [HP], Helffer and Parisse studied the decay properties of eigenfunctions and tunneling effect of $H(0)$ in semiclassical limit, and they showed the decays of the eigenfunctions of $H(0)$ are more rapid than that of the Dirac operators. For the operator $H(\beta)$, $\beta \neq 0$, if $v(x)$ is real valued, one can prove $H(\beta)$ is essentially self-adjoint on $C_0^\infty(R^n)$ (*cf.* [Fa1]). Furthermore, in [Fa3], we proved that the essential spectrum of $H(\beta)\,(\beta \neq 0)$ is $(-\infty, +\infty)$. In the following we will study the properties of the discrete eigenvalues of $H(0)$ under the perturbation βx_1, for β small enough. We will prove that the discrete eigenvalues of $H(0)$ become the resonances of $H(\beta)$, when β is sufficiently small.

The main ideas of this paper are similar to those for usual Schrödinger operators with Stark effect (*cf.* [Wa1], [Wa2] and [Wa3]), but many technical points are quite different. As in [Wa1], we used analytic distortion to define resonances for $H(0)$ and to construct an appropriate Grushin problem to prove the existence and the locations of resonances. The latter is in the spirit of the work of Helffer-Sjöstrand on resonances in semiclassical limit (*cf.* [HS]). The differences between relativistic and non relativistic Schödinger operators arise from the fact that the former is a pseudodifferential operator (non local operator) while the latter is a differential operator (local operator).

The arrangement of this paper as following. In section 2, we introduce the complex dilation and give the definition of resonances. In section 3, we will study the behavior of eigenfunctions with discrete eigenvalues. In section 4, we will study the location of resonances.

2. COMPLEX DILATION AND THE RESONANCES

In this section, we assume $v(x)$ satisfies the condition (H1), (H2). We will introduce the complex dilation as in [CDKS], [Hu], [Wa1]. Let $\lambda \in R$, $\lambda < 0$, and $\delta > 0$ sufficiently small. One can choose $\chi \in C^\infty(R)$, such that $\chi(t) = 1$, $t < (\lambda + \delta)/\beta$, $\chi(t) = 0$, $t > (\lambda + 2\delta)/\beta$, and $|\chi(t)| \leq 1$, $\chi^{(k)}(t) \leq C_k \left(\frac{\beta}{\delta}\right)^k$.

For $\theta \in R$, put $\phi_\theta(x) = (e^{\theta\chi(x_1)} x_1, x')$, $x = (x_1, x')$. Since $\det \phi'_\theta(x) = e^{\theta\chi(x_1)}(1 + \theta\chi'(x_1)x_1)$, one knows that ϕ_θ is invertible when $|\theta|$ is sufficiently small, say $|\theta| < \theta_0$. In this case, let $\psi_\theta = \phi_\theta^{-1}$. In the following, we always assume $|\theta| < \theta_0$. Let $U(\theta)$ be the operator defined by

$$U(\theta) f(x) = f(\phi_\theta(x)) \, |\phi'_\theta(x)|^{1/2}.$$

In this paper, we will use the Weyl corresponding of symbol $\sigma(x, \xi)$, that is

$$\sigma(x, D) f(x) = (2\pi)^{-n} \int_{R^{2n}} e^{i(x-y)\xi} \, \sigma\left(\frac{x+y}{2}, \xi\right) f(y) \, dy d\xi.$$

The symbol of $H(\beta)$ is given by

$$p(x, \xi) = \sqrt{m^2 + \xi^2} - m + v(x) + \beta x_1$$

The symbol $p_\theta(x, \xi)$, of $U(\theta) p(x, D) U(\theta)^{-1}$, is given by

$$p_\theta(x, \xi) = (2\pi)^{-n} \int\int p\Big(\frac{1}{2}\Big(\phi_\theta\Big(x + \frac{t}{2}\Big) + \phi_\theta\Big(x - \frac{t}{2}\Big)\Big), \; {}^T\Psi_\theta(x, t)^{-1}(\xi + \eta)\Big) \,.\, J_\theta(x, t) \, e^{i(\eta, t)} \, d\eta dt, \qquad (2.1)$$

where $J_\theta(x, t) = \left|\phi'_\theta\left(x + \frac{t}{2}\right)\right|^{1/2} \left|\phi'_\theta\left(x - \frac{t}{2}\right)\right|^{1/2} |\det \Psi_\theta(x, t)|^{-1}$, and $\Psi_\theta(x, t)$ satisfies: $\phi_\theta\left(x + \frac{t}{2}\right) - \phi_\theta\left(x - \frac{t}{2}\right) = {}^T\Psi_\theta(x, t)\, t$ and $\Psi_\theta(x, 0) = \phi'_\theta(x)$.

Let S^k be the symbol classes of Hörmander type $S^k_{1,0}$, that is

$$S^k = \{ a(x,\xi) \in C^\infty(R^{2n}),\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{k-|\beta|},\ \alpha,\ \beta \ge 0 \}.$$

From (2.1) and Taylor expansion in variable t, one can get

$$p_\theta(x,\ \xi) = p_0(x,\ \xi) + R_\theta(x,\ \xi), \tag{2.2}$$

where $p_0(x,\ \xi) = p(\phi_\theta(x),\ \psi'_\theta(x)\,\xi)$, $R_\theta(x,\ \xi) \in S^0$, and satisfy

$$|\partial_\xi^k\, \partial_x^\alpha\, R_\theta(x,\ \xi)| \le O(\beta/\delta)\, C_{k\alpha} \langle \xi \rangle^{-|k|}, \qquad \alpha,\ \beta \ge 0. \tag{2.3}$$

Therefore

$$\| R_\theta(x,\ \xi) \| \le C\frac{\beta}{\delta}.$$

Let $\Omega = \{ \theta \in C,\ |\theta| < \rho,\ \mathrm{Im}\,\theta > 0 \}$. We should notice that in the following, $\rho > 0$ will always be sufficiently small. For $\theta \in \Omega$, we define

$$E(\theta,\ \delta) = \{ e^\theta\, r + s,\ s > -4\,\delta,\ r < 4\,\delta \}.$$

For $\beta > 0$ sufficiently small, $p_\theta(x, D)$ defined for $\theta \in R$ and $|\theta| < \rho$ has a natural extension in θ into Ω. In fact, by the analyticity assumption (H1) of $v(x)$, one knows that, for $\beta > 0$ sufficiently small, $U(\theta)\, v(x)\, U(\theta)^{-1}$ is analytic in $\theta \in \Omega$. In order to prove that $p_\theta(x,\ D)$ has an analytic extension in $\theta \in \Omega$, one note that

$$U(\theta)\,\sqrt{m^2-\Delta}\,U(\theta)^{-1}\, f(x) = \frac{1}{(2\,\pi)^n} \int\int e^{i\,(\phi_\theta(x)-\phi_\theta(y)).\xi} \times \sqrt{m^2+\xi^2}\, |\phi'_\theta(x)|^{1/2}\, |\phi'_\theta(y)|^{1/2}\, f(y)\, d\xi\, dy,$$

for $f \in S(R^n)$. $\phi_\theta(x) - \phi_\theta(y) = ((x_1 - y_1)\,\tilde{\psi}_\theta(x_1,\ y_1),\ x' - y')$, where $\tilde{\psi}_\theta(x_1,\ y_1) = \int_0^1 \phi'_{\theta,\,1}(x_1 + \tau\,(y_1 - x_1))\, d\tau$ and $\phi_{\theta,\,1}$ is the first component of ϕ_θ. Since $\phi'_{\theta,\,1} = e^{\theta\chi(x_1)}(1 + \theta\chi'(x_1)\, x_1)$, for $\theta \in \Omega$ small enough, one can check that $\tilde{\psi}_\theta(x_1,\ y_1) \neq 0$, and $|\mathrm{Im}\,\tilde{\psi}_\theta(x_1,\ y_1)| \le C(\theta)\,|\mathrm{Re}\,\tilde{\psi}_\theta(x_1,\ y_1)|$, for $x_1,\ y_1 \in R$, where $|C(\theta)| < C\,|\theta|$ and C is a constant. Since $\sqrt{m^2+\xi^2}$ is an analytic function in variable ξ_1 in the complex region: $|\mathrm{Im}\,\xi_1| \le \frac{1}{2}\,|\mathrm{Re}\ \xi_1|$, one can make a changement of the integral countor of variable in ξ_1 to show $U(\theta)\,\sqrt{m^2-\Delta}\,U(\theta)^{-1}$

is a standard ps. d. o. and

$$\begin{aligned} &U(\theta)\sqrt{m^2-\Delta}\,U(\theta)^{-1} f(x) \\ &= \frac{1}{(2\pi)^n}\int\int e^{i(x-y)\cdot\xi}\sqrt{m^2+(\xi_1/\tilde{\psi}_\theta(x_1, y_1))^2+\xi'^2} \\ &\times \frac{|\phi'_\theta(x)|^{1/2}\,|\phi'_\theta(y)|^{1/2}}{\tilde{\psi}_\theta(x_1, y_1)} f(y)\,d\xi\,dy. \end{aligned} \tag{2.4}$$

The analyticity of $U(\theta)\sqrt{m^2-\Delta}\,U(\theta)^{-1}$ in $\theta\in\Omega$ follows from the analyticity of $\tilde{\psi}_\theta(x_1, y_1)$ and (2.4).

THEOREM 2.1. – *For $\theta\in\Omega$, $p_\theta(x, D)$ defined on $H^1\cap D(x_1)$ is closed in $L^2(R^n)$ and is a holomorphic family of type (A).*

Proof. – For $\operatorname{Im} z>0$ sufficiently large, one has

$$|p_\theta(x, \xi)-z|\geq C(\langle\xi\rangle+|\beta x_1|+\operatorname{Im} z).$$

Let $b_\theta(x, \xi)=(p_\theta(x, \xi)-z)^{-1}$. By the calculus of pseudodifferential operators (*cf.* [CM]), one has

$$(p_\theta(x, D)-z)\,b_\theta(x, D)=I+R_1(x, D),$$

and

$$b_\theta(x, D)\,(p_\theta(x, D)-z)=I+R_2(x, D),$$

where $R_i(x, D)$ satisfy the estimates

$$||R_i(x, D)||\leq\frac{C}{\operatorname{Im} z},\quad i=1,\ 2.$$

By these estimates, one can prove easily that $p_\theta(x, D)$ is a closed operator. The analyticity of $(p_\theta(x, D)\,u, u)$ in $\theta\in\Omega$ follows from the assumption (H1) on $v(x)$ and (2.4). ■

THEOREM 2.2. – *For $\theta\in\Omega$, $\sigma_{\text{ess}}(p_\theta(x, D))\subset E(\theta, \delta)$.*

Proof. – One can choose a cut off function χ_t, such that $\operatorname{supp}\chi_t\subset B_{2t}$ and $|v(\phi_\theta(x)|\leq\frac{\delta}{4}$, $x\notin B_t$. Let $\tilde{p}_\theta(x, D)=p_\theta(x, D)-\chi_t.v(\phi_\theta(x))$. Then $\sigma_{\text{ess}}(\tilde{p}_\theta(x, D))=\sigma_{\text{ess}}(p_\theta(x, D))$. In order to characterize the essential spectrum of $\tilde{p}_\theta(x, D)$, we first calculate the numerical range of $\tilde{p}_\theta(x, D)$ on $L^2(R^n_+)\oplus L^2(R^n_-)$, where $R^n_+=\{x=(x_1, x'),\ x_1>(\lambda+2\delta)/\beta\}$, $R^n_-=\{x=(x_1, x'),\ x_1<(\lambda+2\delta)/\beta\}$.

For u with $\operatorname{supp} u \subset \left\{x,\ x_1 < \frac{\lambda + 2\delta}{\beta}\right\}$, when ρ, β are sufficiently small, one has

$$(\tilde{p}_\theta(x, D)\, u,\ u) \in E(\theta, \delta).$$

For u with $\operatorname{supp} u \subset \left\{x,\ x_1 > \frac{\lambda + 2\delta}{\beta}\right\}$.

$$\begin{aligned}(\tilde{p}_\theta(x, D)\, u,\ u) &= \int\!\!\int \tilde{p}_\theta(x, \xi)\, W(u)(x, \xi)\, dx d\xi \\ &= \int\!\!\int (p_0(x, \xi) - \chi_t\, v(\phi_\theta(x)) + R_\theta(x, \xi))\, W(u)(x, \xi)\, dx d\xi \\ &= \int\!\!\int (\sqrt{m^2 + \xi^2}\,(1 + C_\theta(x, \xi)) + R_\theta(x, \xi) \\ &\quad + (1 - \chi_t(x))\, v(\phi_\theta(x)) + e^{\theta_\chi(x)}\, \beta\, x_1)\, W(u)(x, \xi)\, dx d\xi,\end{aligned}$$

where $W(u)(x, \xi)$ is the Wigner distribution of u (*cf.* [Fo]), $C_\theta(x, \xi) \in S^0$, and $|C_\theta(x, \xi)| \leq C|\theta|$. Therefore, when ρ, β are sufficiently small, we have $(\tilde{p}_\theta(x, D)u, u) \in E(\theta, \delta)$. We proved that $\sigma_{\text{ess}}(\tilde{p}_\theta(x, D)) \subset E(\theta, \delta)$ on $L^2(R^n_+) \oplus L^2(R^n_-)$. Since the $\sigma_{\text{ess}}(\tilde{p}_\theta(x, D))$ on $L^2(R^n)$ is the same as on $L^2(R^n_+) \oplus L^2(R^n_-)$ (*cf.* [K]). This completes the proof. ■

The spectra of $p_\theta(x, D)$ in $C \backslash E(\theta, \delta)$ are discrete eigenvalues. We call these eigenvalues the resonances of $p(x, D)$.

THEOREM 2.3. – *The eigenvalues of $p_\theta(x, D)$ are essentially independent of $\theta \in \Omega$. More precisely, for every compact $K \subset\subset \Omega$, there exists $\beta_0 > 0$, such that for $0 < \beta < \beta_0$, $z \in C \backslash E(\theta, \delta)$, $z \in \sigma(p_\theta(x, D))$, for some $\theta \in K$, then $z \in \sigma(p_\theta(x, D))$ for any $\theta \in K$.*

The proof of the theorem is standard, we omit it here.

Remark. – The resonances of $p(x, D)$ are independent of the choice of χ used in the definition of $U(\theta)$ (*cf.* [Wa1]).

3. DECAY PROPERTIES OF EIGENFUNCTIONS

In this section, we will estimate the decay properties of eigenfunctions of various operators. By the estimates, we can prove the stability theorem of resonances. Some of the results in this section were obtained by Carmona-Master-Simon in [CMS], by probability method. In the following, we will use the commutator method, as in [Wa2], to study the decay properties of eigenfunctions.

Let $H = \sqrt{m^2 - \Delta} - m + v(x)$, and $H(x, \xi) = \sqrt{m^2 + \xi^2} - m + v(x)$.

PROPOSITION 3.1. – *Assume that* $d(x) \in C^\infty(R^n)$, $0 \leq d(x) \leq C|x|$, $|d'(x)| < m$. $|\partial^\alpha d(x)| \leq C|x|^{1-|\alpha|}$, *for* $|x|$ *sufficiently large. Then the symbol,* $H_d(x, \xi)$, *of operator* $e^{d(x)} H(x, D) e^{-d(x)}$, *is given by*

$$H_d(x, \xi) = H(x, \xi + id'(x)) + H_0(x, \xi), \tag{3.1}$$

with $H_0(x, \xi) \in S^0$, *and*

$$|\partial_x^\alpha \partial_\xi^\beta H_0(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1} \langle \xi \rangle^{-|\beta|}. \tag{3.2}$$

Proof.

$$\begin{aligned} & e^{d(x)} H(x, D) e^{-d(x)} u(x) \\ &= (2\pi)^{-n} \int\int e^{i(x-y)\eta + (d(x) - d(y))} H\left(\frac{x+y}{2}, \eta\right) u(y)\, dy d\eta \\ &= (2\pi)^{-n} \int\int e^{i(x-y)(\eta - id'(\tilde{x}))} H\left(\frac{x+y}{2}, \eta\right) u(y)\, dy d\eta, \end{aligned}$$

where $d(x) - d(y) = (x-y) d'(\tilde{x})$. Since $H(x, \xi)$ is an analytic function in variable η in the complex region $\{\eta \in C^n, |\mathrm{Im}\,\eta| < m\}$, one can change the integral countor in η to show

$$\begin{aligned} & e^{d(x)} H(x, D) e^{-d(x)} u(x) \\ &= (2\pi)^{-n} \int\int e^{i(x-y)\xi} H\left(\frac{x+y}{2}, \xi + id'(\tilde{x})\right) u(y)\, dy d\xi. \end{aligned}$$

By Taylor formula, one has

$$\begin{aligned} H\left(\frac{x+y}{2}, \xi + id'(\tilde{x})\right) &= H\left(\frac{x+y}{2}, \xi + id'\left(\frac{x+y}{2}\right)\right) \\ &+ H'_\xi\left(\frac{x+y}{2}, \xi + i\tilde{d}'(x, y)\right)\left(d'(\tilde{x}) - d'\left(\frac{x+y}{2}\right)\right), \end{aligned}$$

where $\tilde{d}'(x, y) = \tau d'\left(\frac{x+y}{2}\right) + (1-\tau) d'(\tilde{x})$, $0 < \tau < 1$. Therefore, we have

$$H_d(x, \xi) = H(x, \xi + id'(x)) + H_0(x, \xi),$$

and

$$|\partial_x^\alpha \partial_\xi^\beta H_0(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1} \langle \xi \rangle^{-|\beta|}.$$

This completes the proof. ■

PROPOSITION 3.2. – *Assume that* $d(x) \in C^\infty(R^n)$, $0 \le d(x) \le C|x|$, $|d'(x)| < C$, $C < m$ *and* $|\partial^\alpha d(x)| \le C_\alpha |x|^{1-|\alpha|}$, *as* $|x|$ *sufficiently large. For* $E \in R$, *we assume that*

$$|H(x, \xi + id'(x)) - E| \ge C\langle \xi \rangle, \qquad \textit{for} \quad \langle x \rangle + \langle \xi \rangle \ge C', \tag{3.3}$$

where C, C' *are constants. Then there is a* $Q(x, \xi) \in S^{-1}$, *such that*

$$Q(x, D)(H_d(x, D) - E) = I + R_d(x, D), \tag{3.4}$$

with $R_d(x, \xi) \in S^{-1}$, *and satisfy*

$$|\partial_\xi^\beta \partial_x^\alpha R_d(x, \xi)| \le C_{\alpha\beta}\langle x \rangle^{-1}\langle \xi \rangle^{-1-\beta}, \tag{3.5}$$

where the constants $C_{\alpha\beta}$ *are uniformly bounded with respect to the* C^k *semi-norms of* $d(x)$.

Proof. – By the assumption (3.3), one can choose $Q(x, \xi) \in S^{-1}$, such that $Q(x, \xi) = (H(x, \xi + id'(x)) - E)^{-1}$, for $\langle x \rangle + \langle \xi \rangle \ge C'$. By the calculus of pseudodifferential operators, one can get easily

$$Q(x, D)(H_d(x, D) - E) = I + R_d(x, D),$$

with $R_d(x, \xi)$ satisfying the estimate (3.5). ■

For $E \in R$, $E < 0$, let $m(E) = \sqrt{2m|E| - E^2}$, for $|E| \le m$, $m(E) = m$, for $|E| > m$. In the following, we will choose $d^\varepsilon(x) \in C^\infty$, $d^\varepsilon(x) = (1-\varepsilon)\,m(E)|x|$, for $\varepsilon > 0$, $|x|$ sufficiently large. One can get the following estimate

$$|\sqrt{m^2 + (\xi + id^{\varepsilon'}(x))^2} - m - E| \ge C\langle \xi \rangle, \tag{3.6}$$

as $|x|$ sufficiently large.

THEOREM 3.3. – *Assume* $v(x)$ *satisfy the conditions* (H1) *and* (H2). *Let* f *be an eigenfunction of operator* $\sqrt{m^2 - \Delta} - m + v(x)$ *with eigenvalue* E, $E \in R$, $E < 0$, *and* $d^\varepsilon(x)$ *chosen as above. Then for any* $\varepsilon > 0$, $e^{d^\varepsilon(x)} f(x) \in L^2(R^n)$.

Proof. – If $e^{d^\varepsilon(x)} f(x) \notin L^2(R^n)$, we can choose $\chi_\lambda(t) = t$, for $t \le \lambda$, and $\chi_\lambda(t) = \lambda + 1$, for $t \ge \lambda + 2$, with $\chi_\lambda \ge 0$, $0 < \chi'_\lambda(t) \le 1$ and $|\chi_\lambda^{(k)}(t)| \le C$, for any λ, t. Let $d_\lambda(x) = \chi_\lambda(d^\varepsilon(x))$, and $f_\lambda(x) = e^{d_\lambda(x)} f(x)/\|e^{d_\lambda} f(x)\|$. Then

$$f_\lambda \to 0, \quad \text{weakly, as } \lambda \to \infty.$$

For $|x|$ sufficiently large, one has

$$|\sqrt{m^2 + \xi^2} - m - E + v(x)| \ge C\langle \xi \rangle.$$

By the estimate (3.6), one can verify easily

$$| H(x, \xi + i d'_\lambda(x)) - E | \geq C \langle \xi \rangle.$$

The constant C is independent of λ. By Proposition (3.2), there is a $Q_\lambda(x, \xi) \in S^{-1}$, such that

$$Q_\lambda(x, D)(H_{d_\lambda}(x, D) - E) = I + R_{d_\lambda}(x, D)$$

and $R_{d_\lambda}(x, \xi)$ satisfies the estimate (3.5) uniformly with respect to λ. Therefore $R_{d_\lambda}(x, D)$ are compact operators and

$$| \partial_\xi^\beta \partial_x^\alpha R_{d_\lambda}(x, \xi) | \leq C_{\alpha\beta} \langle x \rangle^{-1} \langle \xi \rangle^{-\beta}, \tag{3.7}$$

Since $(H_{d_\lambda}(x, D) - E) f_\lambda = 0$, one has

$$(I + R_{d_\lambda}(x, D)) f_\lambda = 0 \tag{3.8}$$

From (3.7), one has

$$\lim_{\lambda \to \infty} R_{d_\lambda}(x, D) f_\lambda = 0.$$

By (3.8), one has $f_\lambda \to 0$, strongly. This is a contradiction. ■

Remark. – The analyticity assumption on $v(x)$ is not necessary in theorem 3.3. In fact, if $v(x) \to 0$, as $|x| \to \infty$, and $| \partial^\alpha v(x) | \leq C_\alpha \langle x \rangle^{-|\alpha|}$, then all results in this section hold. This decay results has been obtained by probability method in [CMS], and in [HP], Helffer and Parrisse studied the decay properties of eigenfunctions of operator $\sqrt{1 - h^2 \Delta} + v(x)$, in the semiclassical limit $h \to 0$.

For $\eta > 0$, $\eta < -E - \delta$, one can choose $\chi_\eta \in C^\infty(R)$, such that $\chi_\eta(t) = 1$, for $t > -\eta$; $\chi_\eta(t) = 0$, for $t < -2\eta$. Let

$$\tilde{H}(\beta) = \sqrt{m^2 - \Delta} - m + v(x) + \beta x_1 \chi_\eta(\beta x_1).$$

THEOREM 3.4. – *Let $f_\beta(x)$ be a normalized eigenfunction of $\tilde{H}(\beta)$ with eigenvalue in $I(\beta) = (E - C\beta^{1/2}, E + C\beta^{1/2})$, $C > 0$ is a constant. Then, for any $\varepsilon > 0$, there exists a $\beta_\varepsilon > 0$, such that*

$$\sup_{0 < \beta < \beta_\varepsilon} \| e^{d^\varepsilon(x)} f_\beta(x) \| < \infty \tag{3.9}$$

Proof. – For any $\varepsilon > 0$, one can choose $\eta > 0$, $\beta_\varepsilon > 0$ sufficiently small such that $(1 - \varepsilon) m(E) \leq (1 - \varepsilon') m(E + \eta + \beta_\varepsilon)$, for some $\varepsilon' > 0$. For $|x|$ sufficiently large, one can get easily the following estimate

$$| \sqrt{m^2 + (\xi + i d^{\varepsilon'}(x))^2} - m - E(\beta) + v(x) + \beta x_1 \chi_\eta(\beta x_1) | \geq C \langle \xi \rangle.$$

By this estimate and the proof processes in Theorem 3.1 and Theorem 3.3, one can prove (3.9). ■

THEOREM 3.5. – *Let E_0 be a discrete eigenvalue of H with multiplicity N, and $I = (E_0 - C(\beta), E_0 + C(\beta))$, $C(\beta) = C\beta^{1/2}$, $C > 0$. There exists a $\beta_0 > 0$, such that for $0 < \beta < \beta_0$, there are exactly N eigenvalues of $\tilde{H}(\beta)$, repeated according to their multiplicity in I. Let $\mu_1, \ldots, \mu_N$, denote these eigenvalues. Then we have $|\mu_j - E_0| < C\beta$, $0 < \beta \leq \beta_0$, $j = 1, \ldots, N$, and moreover*

$$\mu_l(\beta) \sim E_0 + \sum_{j=1}^{\infty} \lambda_j^l \beta^j, \qquad l = 1, \ldots, N. \tag{3.10}$$

Proof. – Let $f_1, \ldots, f_N$ be the orthonormalized eigenfunctions of H, $H f_j = E_0 f_j$, $j = 1, \ldots, N$. Put

$$g_j(\beta) = \chi_\eta\left(\frac{x_1}{\beta}\right) f_j,$$

where $\chi_\eta \in C^\infty(R)$, such that $\chi_\eta(t) = 1$, for $t > -\eta$; $\chi_\eta(t) = 0$, for $t < -2\eta$ then, from Theorem 3.3, we have $g_j \in D(\tilde{H}(\beta))$ and

$$\tilde{H}(\beta) g_j = E_0 g_j + O(\beta), \quad \text{in } L^2(R^n). \tag{3.11}$$

Now let u_j be the normalized eigenfunctions of $\tilde{H}(\beta)$ associated with the eigenvalue $\mu_j(\beta)$, $j = 1, \ldots, \tilde{N}$. Put $\nu_j = \chi_\eta\left(\frac{x_1}{\beta}\right) u_j$. From Theorem 3.4, we obtain

$$H\nu_j = \mu_j \nu_j + O(\beta), \quad \text{in } L^2(R^n). \tag{3.12}$$

By the estimates in Theorem 3.3 and Theorem 3.4, one knows that the matrices $((g_i, g_j))$ and $((\nu_i, \nu_j))$ are close to the identity matrices in $R^{N\times N}$, $R^{\tilde{N}\times\tilde{N}}$ respectively, as $\beta \to 0$. So $\{g_1, \ldots, g_N\}$ and $\{\nu_1, \ldots, \nu_{\tilde{N}}\}$ are two linearly independent sets of $L^2(R^n)$, for $0 < \beta < \beta_0$. From this and estimates (3.11), (3.12), one can prove easily that $N = \tilde{N}$, $|\mu_j - E_0| \leq C\beta$. The detailed process can be found in [Wa1]. The proof of the asymptotic formula (3.10) is similar to that in [S]. ■

4. LOCATIONS OF RESONANCES

In this section, we will study the locations of resonances of the operator $H(\beta)$. That is to study the positions of eigenvalues of operator $p_\theta(x, D)$. We also assume $v(x)$ satisfies the condition (H1), (H2). Let E_0 be a discrete eigenvalue of H, with multiplicity N. In order to estimate the width of resonances, in the definition of $U(\theta)$ in section 2, we will choose $\lambda = E_0$, $\delta > 0$ sufficiently small, and in the definition of $\tilde{H}(\beta)$ in section 3, we will choose $\eta < -E_0 - \delta$. By Theorem 3.5, there exist exactly N eigenvalues of $\tilde{H}(\beta)$ near E_0. In order to study the eigenfunctions of $p_\theta(x, D)$, we will construct an approximate inverse of $(p_\theta(x, D) - z)^{-1}$ out side $\{x, x_1 < (E_0 + \delta)/\beta\}$.

Let $\tilde{p}_\theta(x, \xi) = p_\theta(x, \xi) - \chi_\eta(\beta |x|) v(\phi_\theta(x))$, and $S(E_0) = \{z \in C, |z - E_0| < C\beta^{1/2}\}$. Then for $z \in S(E_0)$, one has the following estimates,

$$|\tilde{p}_\theta(x, \xi) - E| \geq C\delta |\theta| (|\xi| + \beta |x_1| + 1) \tag{4.1}$$

$$|\partial_x^\alpha \tilde{p}_\theta(x, \xi)| \leq C\beta, \qquad |\alpha| \geq 1 \tag{4.2}$$

$$|\partial_\xi^\alpha \tilde{p}_\theta(x, \xi)| \leq C\langle\xi\rangle^{1-|\alpha|}, \qquad |\alpha| \geq 1 \tag{4.3}$$

The estimate (4.2) comes from $|\chi'(x)| \leq c\frac{\beta}{\delta}$, (2.3) and the property of $v(x)$. By these estimates and the calculus of pseudodifferential operators (*cf.* [CM]), we have the following proposition.

PROPOSITION 4.1. – *There exists a $\beta_0 > 0$, such that for $\theta \in \Omega^+$, $E \in S(E_0)$, $\tilde{p}_\theta(x, D) - E$ is invertible and moreover we have*

$$\|(\tilde{p}_\theta(x, D) - E)^{-1}\| \leq C\delta^{-k} |\theta|^{-k},$$

for any $E \in S(E_0)$, where k is a constant.

LEMMA 4.2. – *The integral kernel $K_\theta(x, y)$ of the operator $(\tilde{p}_\theta(x, D) - E)^{-1}$ satisfies the estimate*

$$K_\theta(x, y) = O(e^{-\gamma d(x-y)}), \tag{4.4}$$

for some $\gamma > 0$, and uniformly for $E \in S(E_0)$, i.e. *for $r > 0$, $y_0 \in R^n$, let $U(y_0) = \{y \in R^n, |y - y_0| < r\}$, one has*

$$\|e^{\gamma d(x-y_0)}(\tilde{p}_\theta(x, D) - E)^{-1} g\| \leq C(r) \|g\|_{L^2(U(y_0))}. \; E \in S(E_0).$$

where $d(x) \in C^\infty(R^n)$, $0 \leq d(x) \leq C\langle x\rangle$, $|\partial^\alpha d(x)| \leq C_\alpha \langle x\rangle^{1-|\alpha|}$.

Proof. – Let $\tilde{p}_{\theta,\gamma}(x, D) = e^{\gamma d(x-y_0)} \tilde{p}_\theta(x, D) e^{-\gamma d(x-y_0)}$, and

$$\tilde{R}_{\theta,\gamma}(x, D) = \tilde{p}_{\theta,\gamma}(x, D) - \tilde{p}_\theta(x, D).$$

By (2.4), one knows that $\tilde{p}_\theta(x, \xi)$ is an analytic function in variable ξ in the complex region: $|\mathrm{Im}\,\xi| \leq C|\mathrm{Re}\,\xi|$, where C is a small constant. As in the proof of proposition 3.1, for γ sufficiently small, one can make a changement of the integral countor of variable in ξ to show that $\tilde{p}_{\theta,\gamma}(x, D)$ is a standard ps. d. o. and

$$\tilde{p}_{\theta,\gamma}(x, D) f(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi} \tilde{p}_\theta \left(\frac{x+y}{2}, \xi + i\gamma d'_{y_0}(\tilde{x})\right) f(y)\, dy d\xi,$$

where $d_{y_0}(x) = d(x - y_0)$. and $d_{y_0}(x) - d_{y_0}(y) = (x-y)\, d'_{y_0}(\tilde{x})$. By Taylor expansion, one can get the following estimate easily.

$$\|\tilde{R}_{\theta,\gamma}(x, D)\| \leq C\gamma.$$

When γ sufficiently small one has

$$(\tilde{p}_{\theta,\gamma}(x, D) - E)^{-1} = (1 + (\tilde{p}_\theta(x, D) - E)^{-1} \times \tilde{R}_{\theta,\gamma}(x, D))^{-1} (\tilde{p}_\theta(x, D) - E)^{-1}$$

Let $g \in L^2(R^n)$, and $\operatorname{supp} g \subset U(y_0)$. Put

$$f(x) = \int_{R^n} K_\theta(x, y)\, g(y)\, dy.$$

Then $(\tilde{p}_\theta(x, D) - E) f = g$, and $(\tilde{p}_{\theta,\gamma}(x, D) - E)\, e^{\gamma d_{y_0}} f = e^{\gamma d_{y_0}} g$. One has

$$e^{\gamma d_{y_0}} f = (\tilde{p}_{\theta,\gamma} - E)^{-1} (e^{\gamma d_{y_0}} g)$$

Therefore

$$\| e^{\gamma d_{y_0}} f \|_{L^2} \leq C(r) \| g \|.$$

This completes the proof. ■

In the following, we will outline the construction of Grushin problem for $p_\theta(x, D) - z$, $z \in S(E_0)$, the detailed process, *see* [HS] and [Wa1]. Let $\mu_1(\beta), \ldots, \mu_N(\beta)$ be the eigenvalues of $\tilde{H}(\beta)$ in the interval $I(\beta)$ and $u_1(\beta), \ldots, u_N(\beta)$ be the associated normalized eigenfunctions. Define the maps.

$$R_0^+ : L^2(R^n) \to C^N, \quad \text{by } (R_0^+ u)_j = (u, u_j)$$

$$R_0^- : C^N \to L^2(R^n), \quad \text{by } (R_0^- \nu) = \sum_{i=1}^N \nu_j u_j$$

For $z \in S(E_0)$, consider

$$\begin{pmatrix} \tilde{H}(\beta) - z & R_0^- \\ R_0^+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u^- \end{pmatrix} = \begin{pmatrix} v \\ v^+ \end{pmatrix} \tag{4.5}$$

with $(v, v^+) \in L^2(R^n) \times C^N$. Let $E' = \operatorname{Span}\{u_1, \ldots, u_N\}$, $E'' = E'^{\perp}$, $\tilde{H}'(\beta) = \tilde{H}(\beta)|_{E''}$. By theorem 3.5, $\tilde{H}'(\beta)$ has no spectrum in $S(E_0)$, one has the following estimate.

$$||(\tilde{H}'(\beta) - z)^{-1}|| \leq C, \qquad z \in S(E_0), \quad 0 < \beta < \beta_0.$$

For $z \in S(E_0)$, then $H(\beta) - z$ is invertible, one has

$$\begin{pmatrix} u \\ u^- \end{pmatrix} = P_0(z) \begin{pmatrix} v \\ v^+ \end{pmatrix},$$

where

$$P_0(z) = \begin{pmatrix} R_0(z) & R_0^+(z) \\ R_0^-(z) & R_0^{-+}(z) \end{pmatrix}$$

$$R_0(z) = (\tilde{H}'(\beta) - z)^{-1}, \qquad R_0^{-+}(z) = \operatorname{diag}(z - \mu_j).$$

Let $\rho \in C^\infty(R)$, $\rho(t) = 0$, $t < (E_0 + \delta/2)/\beta$; $\rho(t) = 1$, $t > (E_0 + \delta)/\beta$, and $\psi \in C^\infty(R)$, $\psi(t) = 0$, $t < t_0$, $\psi(t) = 1$, $t > \tilde{t}_0$. Put

$$R^- = \rho R_0^-, \qquad R^+ = R_0^+ \rho.$$

Now we consider

$$P(\beta, z) = \begin{pmatrix} p_\theta(x, D) - z & R^- \\ R^+ & 0 \end{pmatrix}$$

Define

$$F(z) = \begin{pmatrix} \rho R_0(z) \psi + (\tilde{p}_\theta(x, D) - z)^{-1}(1 - \psi) & \rho R_0^+(z) \\ R_0^-(z) \psi & R_0^{-+}(z) \end{pmatrix}$$

Then $P(\beta, z) F(z) = I + K(z)$, where $K(z) = (K_{ij}(z))_{2\times 2}$

$$\begin{aligned} K_{11} &= (p_\theta(x, D) - z) \rho R_0(z) \psi - \psi \\ &\quad + \chi_\eta v(x) (\tilde{p}_\theta(x, D) - z)^{-1}(1 - \psi) + \sum (., \psi u_j) \rho u_j \\ K_{12} &= (p_\theta(x, D) - z) \rho R_0^+(z) + \rho R_0^+(z) R_0^{-+}(z) \\ K_{21} &= R_0^+(\rho^2 - 1) R_0(z) \psi + R_0^+(\tilde{p}_\theta(x, D) - z)^{-1}(1 - \psi) \\ K_{22} &= R_0^+(\rho^2 - 1) R_0^-. \end{aligned}$$

In order to estimate the operator $K(z)$, we will need the following lemma.

LEMMA 4.2. – *With the notations above, for any* $\varepsilon > 0$, *there are* $\gamma_0, \gamma_+, \gamma_- > 0$ *such that*

$$K_{R_0(z)}(x, y) = O(e^{-\gamma_0 d^\varepsilon (x-y)}), \tag{4.6}$$

$$K_{R_0^+(z)}(x, j) = O(e^{-\gamma_+ d^\varepsilon (x)}), \tag{4.7}$$

$$K_{R_0^-(z)}(j, y) = O(e^{-\gamma_- d^\varepsilon (y)}), \tag{4.8}$$

where the meaning of (4.6) *is the same as in the Lemma 4.2.*

Proof. – (4.7) and (4.8) follow from (3.8). The proof of (4.6) is similar to the proof of (4.4). ■

LEMMA 4.3. – *One can choose* ρ, χ_η, ψ *such that the following estimates hold*

$$(p_\theta(x, D) - \tilde{H}(\beta)) R_0(z) \psi = O(\beta^\infty) \tag{4.9}$$

$$(p_\theta(x, D) - z)(\rho - 1) R_0(z) \psi = O(\beta^\infty) \tag{4.10}$$

$$(p_\theta(x, D) - \tilde{H}(\beta)) R_0^+(z) = O(\beta^\infty) \tag{4.11}$$

$$(p_\theta(x, D) - z)(\rho - 1) R_0^+(z) \psi = O(\beta^\infty) \tag{4.12}$$

Proof.

$$(p_\theta(x, D) - \tilde{H}(\beta)) f(x)$$

$$= (p_\theta(x, D) - \tilde{H}(\beta))(\chi_\eta f)(x) + (p_\theta(x, D) - \tilde{H}(\beta))(1 - \chi_\eta) f(x)$$

$$(p_\theta(x, D) - \tilde{H}(\beta))\chi_\eta f(x) = U(\theta)^{-1} H(\beta) U(\theta) \chi_\eta f(x) - \tilde{H}(\beta) \chi_\eta f(x)$$
$$= U(\theta)^{-1} H(\beta) \chi_\eta f(x) - \tilde{H}(\beta) \chi_\eta f(x)$$

Therefore $(p_\theta(x, D) - \tilde{H}(\beta)) \chi_\eta f(x) = 0$, for $x_1 > (E_0 + 2\delta)/\beta$. For $x_1 < (E_0 + 2\delta)/\beta$.

$$\begin{aligned}(p_\theta(x, D) &- \tilde{H}(\beta)) \chi_\eta f(x) \\ &= (2\pi)^{-n} \int_{R^{2n}} e^{i(\phi_\theta(x) - y)\cdot\xi} \sqrt{m^2 + \xi^2}\, \chi_\eta(y) f(y)\, dy d\xi \\ &\quad - (2\pi)^{-n} \int_{R^{2n}} e^{i(x-y)\xi} \sqrt{m^2 + \xi^2}\, \chi_\eta(y) f(y)\, dy d\xi \\ &= O(\beta^\infty)\end{aligned} \tag{4.13}$$

By (4.6), one can prove that

$$(p_\theta(x, D) - \tilde{H}(\beta))(1 - \chi_\eta) R_0(z)\psi = O(\beta^\infty).$$

From the estimate (4.13) and the estimate above, we get the estimate (4.9). The proofs of the estimates (4.10), (4.11) and (4.12) are similar to this one. ■

Making use of Lemma 4.2 and Lemma 4.3, one can get the following estimate

$$K(z) = O(\beta^\infty), \tag{4.14}$$

in the sense of operator norm on $L^2(R^n) \times C^N$, uniformly for $z \in S(E_0)$.

As a consequence of (4.14), $P(\beta, z)$ is invertible for $\beta > 0$ sufficiently small. We write this inverse, $G(z)$, as

$$G(z) = \begin{pmatrix} R(z) & R^+(z) \\ R^-(z) & R^{-+}(z) \end{pmatrix}.$$

From (4.14) and the formula

$$G(z) = F(z)(I - K(z) + K(z)^2 + \cdots)$$

we can prove that

$$R^{-+}(z) = R_0^{-+}(z) + O(\beta^\infty) \tag{4.15}$$

THEOREM 4.4. – *Under the assumption above, $E_0 < 0$ is an eigenvalue of H with multiplicity N. Let $\mu_1(\beta), \ldots, \mu_N(\beta)$ be the eigenvalues of $\tilde{H}(\beta)$ in the interval $I(\beta)$. Let $\Gamma(\beta)$ denote the resonances of $H(\beta)$ in $S(E_0)$. Then there exists a bijection $b : \{\mu_1(\beta), \ldots, \mu_N(\beta)\} \to \Gamma(\beta)$ such that:*

$$|b(\mu) - \mu| = O(\beta^\infty), \qquad \beta \to 0_+.$$

Proof. – The proof of this theorem is standard now. The basic point is that $p_\theta(x, D) - z$ is invertible if and only if $R^{-+}(z)$ is bijective on C^N. Then we have the formula:

$$(p_\theta(x, D) - z)^{-1} = R(z) - R^+(z) R^{-+}(z)^{-1} R^-(z) \tag{4.16}$$

(*cf.* [HS]). By (4.16), we can show that the spectrum of $p_\theta(x, D)$ in $S(E_0)$ is in one-one correspondence with the zero of det $R^{-+}(z)$, even if we count the multiplicity of these elements. The desired result follows from (4.15). ■

Since the eigenvalues of $\tilde{H}(\beta)$ are real, from Theorem 4.4, one can get easily the following consequence.

COROLLARY 4.5. – *With the notations of Theorem* 4.4, *for* $\beta > 0$ *sufficiently small there exist exactly N resonances of* $H(\beta)$ *in* $S(E_0)$. *Let* $z_1(\beta), \ldots, z_N(\beta)$ *denote these resonances, repeated according to their multiplicity. Then we have*

$$\begin{aligned} z_j(\beta) &= E_0 + O(\beta), \qquad j = 1, \cdots, N \\ \operatorname{Im} z_j(\beta) &= O(\beta^\infty), \qquad \beta \to 0_+ \end{aligned}$$

Note that in the case of non-relativistic Schrödinger operator, the width of resonances in Stark effect is exponential small (*cf.* [Sig], [Wa1] and [Wa3]).

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