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Semiclassical and weak-magnetic-field eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential

by

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ABSTRACT. – We consider the discrete spectrum of the Schrödinger operator $\mathfrak{H}_{h,\mu} := (ih\nabla + \mu A)^2 - V$ where A is the magnetic potential, $-V$ is the electric potential, h is the Planck constant, and μ is the magnetic-field coupling constant. We study the asymptotic behaviour of the number of the eigenvalues of $\mathfrak{H}_{h,\mu}$ smaller than $\lambda \leq 0$ as $h \downarrow 0$, $\mu > 0$ being fixed, or $\mu \downarrow 0$, $h > 0$ being fixed.

RÉSUMÉ. – On considère le spectre discret de l'opérateur de Schrödinger $\mathfrak{H}_{h,\mu} := (ih\nabla + \mu A)^2 - V$ où A est le potentiel magnétique, $-V$ est le potentiel électrique, h est la constante de Planck, et μ est la constante du couplage du champ magnétique. On étudie le comportement asymptotique du nombre des valeurs propres de $\mathfrak{H}_{h,\mu}$ plus petites que $\lambda \leq 0$ pour $h \downarrow 0$, $\mu > 0$ étant fixée, ou pour $\mu \downarrow 0$, $h > 0$ étant fixée.

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0. INTRODUCTION

For $u \in C_0^\infty(\mathbb{R}^m)$, $m \geq 2$, introduce the real-valued quadratic form

$$\mathfrak{h}_{h,\mu}[u] := \int_{\mathbb{R}^m} (|ih\nabla u + \mu A u|^2 - V|u|^2)\, dx$$

where $A : \mathbb{R}^m \to \mathbb{R}^m$ is the magnetic potential, $-V : \mathbb{R}^m \to \mathbb{R}$ is the electric potential, $h > 0$ is the Planck constant, and $\mu \geq 0$ is the magnetic-field coupling constant. We assume $A \in L^2_{\text{loc}}(\mathbb{R}^m)^m$. Moreover, we suppose that the multiplier by $V_+ := \max\{V, 0\}$ is -Δ-form-bounded with zero relative form bound, and $V_- := V_+ - V \in L^1_{\text{loc}}(\mathbb{R}^m)$. In the formulation of our main results we shall impose more restrictive assumptions on A and V which will guarantee, in particular, the validity of these general conditions.

It is well-known that under these hypotheses the quadratic form $\mathfrak{h}_{h,\mu}$ is lower-bounded and closable in $L^2(\mathbb{R}^m)$ (*see* e.g. [Av.Her.Sim 1], Theorem 2.5). Define the Schrödinger operator $\mathfrak{H}_{h,\mu}$ as the unique selfadjoint operator generated by the closed quadratic form $\mathfrak{h}_{h,\mu}$.

In the present paper we study the asymptotic behaviour of the discrete spectrum of $\mathfrak{H}_{h,\mu}$ as $h \downarrow 0$, μ being fixed or as $\mu \downarrow 0$, h being fixed.

The paper is organized as follows. In section 1 we introduce the basic notations used throughout the paper. Section 2 contains semiclassical eigenvalue asymptotics for the operator $\mathfrak{H}_{h,\mu}$, i.e. the asymptotics of the discrete spectrum of $\mathfrak{H}_{h,\mu}$ as $h \downarrow 0$, the number $\mu > 0$ being fixed. First, we consider the case of quite arbitrary magnetic potentials A and electric potentials V which decay rapidly at infinity in a certain sense. Next, we study the case of magnetic potentials A associated with magnetic fields

$$\left.\begin{array}{c} B \equiv \operatorname{curl} A := \{B_{jk}\}_{j,k=1}^m, \\ B_{jk} := \partial A_k/\partial x_j - \partial A_j/\partial x_k, \\ j, k = 1, \ldots, m, \end{array}\right\} \tag{0.1}$$

which are constant with respect to $x \in \mathbb{R}^m$, and electric potentials which decay slowly at infinity [i.e. $V(x)$ behaves like $|x|^{-\alpha}$, $\alpha \in (0, 2]$, as $|x| \to \infty$]. For approximately the same two classes of potentials (A, V), in section 3 we investigate the weak-magnetic-field eigenvalue asymptotics, i.e. the asymptotics of the eigenvalues of $\mathfrak{H}_{h,\mu}$ as $\mu \downarrow 0$, the number $h > 0$ being fixed.

Related problems (which however differ essentially from the ones considered here) have been treated in [Ale], [Av.Her.Sim. 1], Section 6, [Com.Sch.Sei], [Av.Her.Sim 2], Section 7, [Hel.Sjö 1, 2] and [Ivr 1-4].

The results of the paper are obtained by means of a variational technique of Weyl-Courant type (*see* [Bir.Sol 2] or [Ree.Sim], Ch. XIII). In particular, we use essentially some spectral estimates due to E. Lieb and Y. Colin de Verdière. In section 3 we also apply the approach of M. Kac, W. L. Murdock and G. Szegö to the study of the semiclassical eigenvalue asymptotics of compact pseudodifferential operators (*see* [Gre.Sze], Section 7.1). Here the Feynman-Kac-Itô formula for the resolvent of the magnetic Schrödinger operator also plays an important rôle.

A weaker version of the present results has been announced in the author's short communication [Rai 4]. Here the minor errors made there have been corrected, and the unnecessary assumptions have been cancelled.

1. NOTATIONS AND PRELIMINARIES

1.1. Let T be a selfadjoint operator in a Hilbert space. Then $\sigma(T)$ is the spectrum of T, and $\sigma_{\text{ess}}(T)$ is its essential spectrum. Moreover, if $(\lambda, \mu) \subseteq \mathbb{R}$, then $P_{(\lambda, \mu)}(T)$ denotes the spectral projection of T corresponding to the open interval (λ, μ). Put

$$\mathcal{N}(\lambda, \mu|T) = \operatorname{rank} \mathcal{P}_{(\lambda, \mu)}(T),$$
$$\mathcal{N}(\lambda; T) = \mathcal{N}(-\infty, \lambda|T), \qquad \lambda \in \mathbb{R},$$
$$n(\lambda; T) = \mathcal{N}(\lambda, \infty|T), \qquad \lambda > 0.$$

1.2. Let $\Omega \subseteq \mathbb{R}^m$, $m \geq 2$, be an open set. By $W_p^q(\Omega)$, $q \in [1, \infty]$, $p \in \mathbb{N}_+ := \{1, 2, \dots, \}$, we denote the standard Sobolev spaces, and by $\overset{\circ}{W}{}_p^q(\Omega)$–the closure of $C_0^\infty(\Omega)$ in the $W_p^q(\Omega)$-norm. Denote by $-\Delta_\Omega^D$ the operator generated in $L^2(\Omega)$ by the closure of the quadratic form $\int_\Omega |\nabla u|^2 \, dx$, $u \in C_0^\infty(\Omega)$. Suppose that the multiplier by the function $\mathcal{V}_+ := \Omega \to \mathbb{R}_+$ is $-\Delta_\Omega^D$-norm-bounded with zero relative form bound, and the function $\mathcal{V}_- := \Omega \to \mathbb{R}_+$ is in $L^1_{\text{loc}}(\Omega)$. Set $\mathcal{V} := \mathcal{V}_+ - \mathcal{V}_-$. Let $\mathcal{A} \in L^2_{\text{loc}}(\Omega)^m$. Introduce the quadratic form

$$\int_\Omega (|i \nabla u + \mathcal{A} u|^2 - \mathcal{V}|u|^2) \, dx, \qquad u \in C_0^\infty(\Omega). \tag{1.1}$$

Denote by $H_\Omega^D(\mathcal{A}, \mathcal{V})$ the operator generated in $L^2(\Omega)$ by its closure. If $\Omega = \mathbb{R}^m$, we write $H(\mathcal{A}, \mathcal{V})$ instead of $H_{\mathbb{R}^m}^D(\mathcal{A}, \mathcal{V})$. In particular, we have $\mathfrak{H}_{h, \mu} = h^2 H(h^{-1} \mu A, h^{-2} V)$.

Now, assume that $\Omega \subset \mathbb{R}^m$, $m \geq 2$, is a bounded domain with Lipschitz boundary. Let $\mathcal{A} \in L^p(\Omega; \mathbb{R}^m)^m$, where $p = m$ if $m \geq 3$, $p > 2$ if $m = 2$, and $\mathcal{V} \in L^q(\Omega; \mathbb{R})$, where $q = m/2$ if $m \geq 3$, $q > 1$ if $m = 2$. On $C^\infty(\Omega)$ introduce a quadratic form analogous to (1.1), and denote by $H_\Omega^N(\mathcal{A}, \mathcal{V})$ the operator generated in $L^2(\Omega)$ by its closure.

1.3. Let $\mathcal{L}$ be a finite or a countable set. We shall say that the family $\{\varphi_l\}_{l\in\mathcal{L}}$ is a partition of unity over $\mathbb{R}^m$ if and only if the following conditions are satisfied:

(i) $\varphi_l \in C^\infty(\mathbb{R}^m)$, $\forall l \in \mathcal{L}$;

(ii) $0 \leq \varphi_l(x) \leq 1$, $\forall l \in \mathcal{L}$, $\forall x \in \mathbb{R}^m$;

(iii) $\sum_{l\in\mathcal{L}} \varphi_l^2(x) = 1$, $\forall x \in \mathbb{R}^m$;

(iv) for any given compact subset $K \subseteq \mathbb{R}^m$ the intersection $K \cap \operatorname{supp} \varphi_l$ may be nonempty just for a finite set of indices $l \in \mathcal{L}$;

(v) we have $\sup_{x\in\mathbb{R}^m} \sum_{l\in\mathcal{L}} |\nabla \varphi_l(x)|^2 < \infty$.

LEMMA 1.1. – *Let the family* $\{\varphi_l\}_{l\in\mathcal{L}}$ *be a partition of unity over* $\mathbb{R}^m$ *such that* $\operatorname{supp}\varphi_l$ *is contained in the open set* Ω_l. *Suppose that* $\mathcal{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^m)^m$, $\mathcal{V}_- \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ *and* $\mathcal{V}_+$ *is -Δ-form-bounded with zero relative form bound. Then we have*

$$\mathcal{N}(0; H(\mathcal{A}, \mathcal{V})) \leq \sum_{l\in\mathcal{L}} \mathcal{N}(0; H_{\Omega_l}^D(\mathcal{A}, \mathcal{V} + \sum_{s\in\mathcal{L}} |\nabla \varphi_s|^2)). \qquad (1.2)$$

Proof. – Write the "magnetic" version of the so-called IMS localization formula

$$\sum_{l\in\mathcal{L}} \{ (H(\mathcal{A}, \mathcal{V})\varphi_l u, \varphi_l u) - \sum_{s\in\mathcal{L}} (|\nabla \varphi_s|^2 \varphi_l u, \varphi_l u) \}$$
$$= (H(\mathcal{A}, \mathcal{V})u, u), \qquad \forall u \in C_0^\infty(\mathbb{R}^m),$$

(*see* [Cy.Fr.Ki.Sim], Section 3.1), which combined with the minimax principle entails (1.2). □

2. SEMICLASSICAL EIGENVALUE ASYMPTOTICS

In this section we discuss the behaviour of the quantity $\mathcal{N}(\lambda; \mathfrak{H}_{h,1})$ as $h \downarrow 0$, the number $\lambda \leq 0$ being fixed.

2.1. In the present subsection we deal with quite arbitrary magnetic potentials A and electric potentials V which decay rapidly at infinity.

THEOREM 2.1. – *Let $m \geq 3$. Suppose that $A \in L^m_{\rm loc}(\mathbb{R}^m)^m$, $V_- \in L^1_{\rm loc}(\mathbb{R}^m)$. Fix $\lambda \leq 0$ and assume that $(V+\lambda)_+ \in L^{m/2}(\mathbb{R}^m)$. Moreover, suppose that there exists an open set $\Omega_\lambda \subseteq \mathbb{R}^m$ such that $V(X)+\lambda > 0$ if $x \in \Omega_\lambda$, and $V(x)+\lambda \leq 0$ if $x \notin \Omega_\lambda$. Then we have*

$$\lim_{h \downarrow 0} h^m \mathcal{N}(\lambda;\, \mathfrak{H}_{h,1}) = \int_{\mathbb{R}^m} (V+\lambda)_+^{m/2}\, dx/(4\pi)^{m/2}\, \Gamma(1+m/2). \quad (2.1)$$

The hypotheses of Theorem 2.1 imply, in particular, that the multiplier by V_+ is -Δ-form-bounded with zero relative form bound. As a matter of fact, we have

$$V(x)_+ = (V(x)+\lambda)_+ - V(x)\,\chi_1(x;\,\lambda) - \lambda\chi_2(x;\,\lambda)$$

where $\chi_1(x;\,\lambda)$ is the characteristic function of the set $\{x \in \mathbb{R}^m : -\lambda \leq V(x) < 0\}$, and $\chi_2(x;\,\lambda)$ is the characteristic function of the set $\{\chi \in \mathbb{R}^m : V(x) > -\lambda\}$. The functions $V(x)\,\chi_1(x;\,\lambda)$ and $\chi_2(x;\,\lambda)$ are bounded, and the multiplier by $(V+\lambda)_+ \in L^{m/2}(\mathbb{R}^m)$ is -Δ-form-compact, so all the three terms in the representation of $V(x)_+$ are -Δ-form-bounded with zero relative form bound.

If $m = 2$, Theorem 2.1 is valid again but under more complicated assumptions. For example, (2.1) holds if $A \in L^p_{\rm loc}(\mathbb{R}^2)^2$, $p > 2$, $V_- \in L^1_{\rm loc}(\mathbb{R}^2)$, $(V+\lambda)_+ \in L^q(\mathbb{R}^2)$, $q > 1$, and there exists a *bounded* open set $\Omega_\lambda \subset \mathbb{R}^m$ such that $V(x)+\lambda > 0$ if $x \in \Omega_\lambda$ and $V(x)+\lambda \leq 0$ if $x \notin \Omega_\lambda$.

We should mention the formal similarity of Theorem 2.1 with the results of [Ale], Theorem 1.1, [Com.Sch.Seil], Corollary 3.2, [Ivr 1], Theorem 3, [Ivr 2], Theorem 6 (i), and some of the results in [Ivr 4], Chapters 6, 10 and 11. However, in [Ale] only potentials $A \in L^m(\mathbb{R}^m)^m$ are considered, while we assume just the validity of the *local* condition $A \in L^m_{\rm loc}(\mathbb{R}^m)^m$, and do not impose *any* restrictions on the behaviour of A at infinity. Further, the authors of [Com.Sch.Seil] investigate the semiclassical eigenvalue asymptotics for magnetic Schrödinger operators with compact resolvent, while the assumptions of Theorem 2.1 entail the discreteness of the spectrum of the operator $\mathfrak{H}_{h,1}$ only below the point $-\lambda \leq 0$. Finally, more precise versions of the asymptotic formula (2.1) can be found in [Ivr 1, 2, 4]; namely, these works contain a sharp estimate of the remainder, and, in some cases, even the second asymptotic term of $\mathcal{N}(\lambda;\mathfrak{H}_{h,1})$. However, the potentials (A, V) in [Ivr 1, 2, 4] are supposed to satisfy quite numerous conditions, by far more restrictive than our assumptions which are close to the minimal ones guaranteeing the finiteness of the right-hand-side of (2.1) and the self-adjointness of $\mathfrak{H}_{h,1}$ for all $h > 0$.

The proof of Theorem 2.1 essentially depends on the following auxiliary result.

LEMMA 2.2. – *Let* $m \geq 3$, $\mathcal{A} \in L^2_{\text{loc}}(\mathbb{R}^m)^m$, $\mathcal{V}_- \in L^1_{\text{loc}}(\mathbb{R}^m)$ *and* $\mathcal{V}_+ \in L^{m/2}(\mathbb{R}^m)$. *Then we have*

$$\mathcal{N}(0;\, H(\mathcal{A},\, \mathcal{V})) \leq c \int_{\mathbb{R}^m} \mathcal{V}(x)_+^{m/2}\, dx \tag{2.2}$$

where the constant c *depends only on the dimension* m.

The proof of the relation (2.2) which extends the famous Cwikel-Lieb-Rozenbljum estimate to the case $\mathcal{A} \neq 0$, can be found in [Av.Her.Sim 1], Theorem 2.15, and [Sim], Chapter V.

Proof of Theorem 2.1. – Our argument is similar to the one utilized in the proof of Theorem 1.1 in [Rai 3]. The asymptotics (2.1) will follow from the estimates

$$\limsup_{h \downarrow 0} \pm h^m \mathcal{N}(\lambda;\, \mathfrak{H}_{h,1}) \leq \pm \int_{\mathbb{R}^m} (V+\lambda)_+^{m/2}\, dx/(4\pi)^{m/2}\,\Gamma(1+m/2). \qquad (2.3)_\pm$$

First, we verify $(2.3)_+$ Obviously, we have

$$\mathcal{N}(\lambda;\, \mathfrak{H}_{h,1}) \leq \mathcal{N}(0;\, H(h^{-1}A,\, h^{-2}(V+\lambda)_+)). \tag{2.4}$$

Fix an arbitrary $\varepsilon > 0$ and write $(V+\lambda)_+ = V_1 + V_2$ where $V_1 \in C_0^\infty(\mathbb{R}^m)$ and V_2 satisfies the estimate

$$\int_{\mathbb{R}^m} |V_2|^{m/2}\, dx < \varepsilon. \tag{2.5}$$

The minimax principle yields

$$\begin{aligned} &\mathcal{N}(0;\, H(h^{-1}A,\, h^{-2}(V+\lambda)_+)) \\ &\quad \leq \mathcal{N}(0;\, H(h^{-1}A,\, (1-\tau)^{-1}h^{-2}V_1)) \\ &\qquad + \mathcal{N}(0;\, H(h^{-1}A,\, \tau^{-1}h^{-2}V_2)), \qquad \forall \tau \in (0,\, 1). \end{aligned} \tag{2.6}$$

The estimate (2.2) combined with (2.5) implies

$$\mathcal{N}(0;\, H(h^{-1}A,\, \tau^{-1}h^{-2}V_2)) \leq c\,\tau^{-m/2}h^{-m}\,\varepsilon. \tag{2.7}$$

Let $\mathfrak{B}$ be an open ball in $\mathbb{R}^m$ such that supp $V_1 \subset \mathfrak{B}$. By the minimax principle we have

$$\begin{aligned} &\mathcal{N}(0;\, H(h^{-1} A,\, (1-\tau)^{-1} h^{-2} V_1)) \\ &\quad\leq \mathcal{N}(0;\, H_{\mathfrak{B}}^{N}(h^{-1} A,\, (1-\tau)^{-1} h^{-2} V_1)), \qquad \forall \tau \in (0,\, 1). \end{aligned} \tag{2.8}$$

Employing the general variational methods developed in [Bir.Sol 2] and [Ale], we get the Weyl-type asymptotics

$$\left.\begin{aligned} &\mathcal{N}(0;\, H_{\mathfrak{B}}^{N}(h^{-1} A,\, (1-\tau)^{-1} h^{-2} V_1)) \\ &\quad = (2\pi)^{-m} \operatorname{vol}\{(x,\, \xi) \in T^* \mathfrak{B} :\, |h\xi - A(x)|^2 \\ &\quad - (1-\tau)^{-1} V_1(x) < 0\}(1 + o(1)), \\ &\quad h \downarrow 0, \qquad \forall \tau \in (0,\, 1). \end{aligned}\right\} \tag{2.9}$$

Obviously, we have

$$\begin{aligned} &(2\pi)^{-m} \operatorname{vol}\{(x,\, \xi) \in T^* \mathfrak{B} : |h\xi - A(x)|^2 - (1-\tau)^{-1} V_1(x) < 0\} \\ &\quad = h^{-m}(1-\tau)^{-m/2} \int_{\mathbb{R}^m} (V_1)_+^{m/2}\, dx/(4\pi)^{m/2}\,\Gamma(1+m/2). \end{aligned} \tag{2.10}$$

Combining (2.4) with (2.6)-(2.10), we obtain the estimate

$$\begin{aligned} &\limsup_{h \downarrow 0} h^m \mathcal{N}(\lambda;\, \mathfrak{H}_{h,1}) \leq (1-\tau)^{-m/2} \\ &\quad \times \int_{\mathbb{R}^m} (V_1)_+^{m/2}\, dx/(4\pi)^{m/2}\,\Gamma(1+m/2) + c\tau^{-m/2}\varepsilon, \\ &\qquad\qquad \forall \tau \in (0,\, 1), \quad \forall \varepsilon > 0. \end{aligned} \tag{2.11}$$

Letting consecutively $\varepsilon \downarrow 0$ and $\tau \downarrow 0$, we come to $(2.3)_+$.

Finally, we just outline the demonstration of $(2.3)_-$. Fix $\varepsilon > 0$ and write again $(V+\lambda)_+ = V_1 + V_2$, where V_1 and V_2 have the same meaning as above. In this case, however, we assume without any loss of generality supp $V_1 \subset \Omega_\lambda$, where Ω_λ is the set described in the hypotheses of Theorem 2.1. The minimax principle entails the inequalities

$$\begin{aligned} \mathcal{N}(\lambda;\, \mathfrak{H}_{h,1}) &\geq \mathcal{N}(0;\, H_{\Omega_\lambda}^{D}(h^{-1} A,\, h^{-2}(V+\lambda)_+)) \\ &\geq \mathcal{N}(0;\, H_{\Omega_\lambda}^{D}(h^{-1} A,\, (1+\tau)^{-1} h^{-2} V_1)) \\ &\quad - \mathcal{N}(0;\, H_{\Omega_\lambda}^{D}(h^{-1} A,\, -\tau^{-1} h^{-2} V_2)), \qquad \forall \tau > 0. \end{aligned} \tag{2.12}$$

Further the derivation of $(2.3)_-$ from (2.12) is quite similar to the derivation of $(2.3)_+$ from (2.4) and (2.6). $\square$

2.2. In the subsection we deal with constant magnetic fields B and electric potentials V which decay slowly at infinity.

Suppose that we have

$$B_{jk} = \text{const.}, \qquad \forall j,\, k = 1, \ldots, m, \quad B \not\equiv 0, \tag{2.13}$$

where the magnetic-field B is defined in (0.1). Whenever (2.13) holds, we assume without any loss of generality that the potential A has components $A_j = \frac{1}{2} \sum_{l=1}^{m} B_{lj}\, x_l$, $j = 1, \ldots, m$; in particular, div $A = 0$. Moreover, the spectrum of the skew-symmetric matrix B is a subset of the imaginary axis which is symmetric with respect to the origin. Let $b_1 \geq \ldots \geq b_d > 0$ be such numbers that the nonzero eigenvalues of B coincide together with the multiplicities with the imaginary numbers $-ib_j$ and ib_j, $j = 1, \ldots, d$. Thus we have $2\,d = \operatorname{rank} B$ and $0 < 2\,d \leq m$. Set $k := m - 2\,d \equiv \dim \operatorname{Ker} B$.

Further, we shall say that V satisfies the condition $\mathcal{D}_\alpha$, $\alpha > 0$, if and only if $V \in C^1(\mathbb{R}^m)$ and the estimates

$$C^{-1} \langle x \rangle^{-\alpha} \leq V(x) \leq C \langle x \rangle^{-\alpha},$$
$$|\nabla V(x)| \leq C \langle x \rangle^{-\alpha-1}, \qquad \langle x \rangle := (1 + |x|^2)^{1/2},$$

hold for each $x \in \mathbb{R}^m$ and some constant $C \geq 1$.

Assume that (2.13) is valid and V satisfies $\mathcal{D}_\alpha$ with any $\alpha > 0$. Then the lower bound of $\sigma_{\text{ess}}(\mathfrak{H}_{h,\mu})$ coincides with $h\,\mu\Lambda$ where

$$\Lambda := \sum_{j=1}^{d} b_j$$

(*see* [Rai 2]). For $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, $k > 0$, set $\theta_k(t) = t_+^{k/2}$; respectively, $\theta_0(t) = 1$, if $t > 1$, and $\theta_0(t) = 0$, if $t \leq 0$. Further, for $t \in \mathbb{R}$ introduce the quantity

$$\Theta(t) \equiv \Theta(t;\, B) := \mathcal{C}_m(\mathbf{b}) \sum_{\mathbf{n} \in \mathbb{N}^d} \theta_k(t - 2\,\mathbf{n}.\mathbf{b} - \Lambda)$$

where

$$\mathbb{N} := \{0, 1, 2, \ldots\}, \qquad \mathbf{b} := (b_1, \ldots, b_d)$$

and

$$\mathcal{C}_m(\mathbf{b}) = b_1 \ldots b_d / 2^{d+k}\, \pi^{m/2}\, \Gamma(1 + k/2).$$

Assume that V satisfies the condition $\mathcal{D}_\alpha$, $\alpha > 0$. For $s > 0$ set

$$\psi(s) := \operatorname{vol}\{x \in \mathbb{R}^m : V(x) > s\}.$$

We shall say that the potential V satisfies condition $\mathcal{T}$ if and only if we have

$$\lim_{\delta\downarrow 0} \limsup_{s\downarrow 0} \psi((1-\delta)s)/\psi(s) = 1. \tag{2.14}$$

The condition $\mathcal{T}$ is valid if the estimate

$$-(x.\nabla V(x)) \geq c\langle x\rangle^{-\alpha}, \qquad c > 0,$$

holds for sufficiently large $|x|$. As a matter in this case the function $\psi(s)$ is differentiable for $s \in (0, s_0]$ and $s_0 > 0$ small enough, and we have

$$-s\,\psi'(s) \leq c\,\psi(s)$$

which immediately entails (2.14) (*see* [Dau.Rob]). Another sufficient condition which guarantees the validity of (2.14) is the asymptotic relation

$$V(x) = v(\hat{x})\,|x|^{-\alpha}(1 + o(1)), \qquad \hat{x} := x/|x|, \quad |x| \to \infty, \tag{2.15}$$

where $v \in C(\mathbb{S}^{m-1})$ is a strictly positive function. In this case we have

$$\psi(s) = \frac{s^{-m/\alpha}}{m}\int_{\mathbb{S}^{m-1}} v(\omega)^{m/\alpha}\, dS(\omega)\,(1 + o(1)), \qquad s \downarrow 0.$$

Let V satisfy $\mathcal{D}_\alpha$, $\alpha \in (0, 2]$. Assume that (2.13) holds, and for $g > 0$ put

$$\nu_0(g) := \int_{\mathbb{R}^m} \Theta(g\,V(x);\, B)\, dx.$$

Obviously, the estimates

$$\nu_0(g) \asymp g^{m/\alpha}, \qquad \alpha \in (0, 2), \tag{2.16}$$

$$\nu_0(g) \asymp g^{m/2}\log g, \qquad \alpha = 2, \tag{2.17}$$

hold as $g \to \infty$. Moreover, if V obeys the asymptotics (2.15) we have

$$\lim_{g\to\infty} g^{-m/\alpha}\,\nu_0(g) = \mathcal{C}_m(\mathbf{b})\,\frac{\Gamma(m/\alpha - k/2)}{\alpha\Gamma(1 + m/\alpha)}\,\Gamma(1 + k/2)$$

$$\sum_{\mathbf{n}\in\mathbb{N}^d} (\Lambda + 2\,\mathbf{n}.\mathbf{b})^{k/2 - m/\alpha}\int_{\mathbb{S}^{m-1}} v(\omega)^{m/\alpha}\, dS(\omega), \qquad \alpha \in (0, 2),$$

$$\lim_{g\to\infty} g^{-m/2}(\log g)^{-1}\,\nu_0(g)$$

$$= \int_{\mathbb{S}^{m-1}} v(\omega)^{m/2}\, dS(\omega)/2\,(4\,\pi)^{m/2}\,\Gamma(1 + m/2), \qquad \alpha = 2.$$

LEMMA 2.3. – *Assume that (2.13) holds and V satisfies the condition $\mathcal{D}_\alpha$ with $\alpha \in (0, 2]$. If $k = 0$ and $\alpha \in (0, 2)$, assume in addition that V satisfies the condition $\mathcal{T}$. Then we have*

$$\lim_{\delta \downarrow 0} \limsup_{g \to \infty} \nu_0 ((1 + \delta) g) / \nu_0 (g) = 1. \tag{2.18}$$

The proof of the lemma can be found in the Appendix.

THEOREM 2.4. – *Assume that the hypotheses of Lemma 2.3 hold. Set $\nu_1 (h) := h^{-m/2} \nu_0 (h^{-1})$. Then we have*

$$\mathcal{N} (0; \mathfrak{H}_{h,1}) = \nu_1 (h) (1 + o(1)), \qquad h \downarrow 0. \tag{2.19}$$

Note that if V satisfies $\mathcal{D}_\alpha$ with $\alpha > 2$, then $V \in L^{m/2} (\mathbb{R}^m)$ so that in this case Theorem 2.1 is valid (provided $m \geq 3$).

Remark 2.5. – Assume that the potential U satisfies the condition $\mathcal{D}_\alpha$ with $\alpha \in (0, 2]$; if $k = 0$ and $\alpha \neq 2$, assume in addition that U satisfies $\mathcal{T}$. If $m \geq 3$, suppose $W \in L^{m/2} (\mathbb{R}^m; \mathbb{R})$; if $m = 2$ suppose that the support of W is compact and, moreover, $W \in L^q (\mathbb{R}^2; \mathbb{R})$ for some $q > 1$. Then the asymptotics (2.19) is valid for $V = U + W$. Note that in this case the main asymptotic term of $\nu_1 (h)$ as $h \downarrow 0$ depends only on U but not on W.

Similarly to the case of Theorem 2.1, the results of [Ivr 2], Theorem 6 (ii)-(iii), and some of the results of [Ivr 4], Chapters 10 and 11, contain more precise versions of (2.19) but the assumptions about V are more restrictive than ours.

In the demonstration of the asymptotics (2.19) we shall use systematically the following important technical result due to Y. Colin de Verdière (*see* [CdV], Theorem 3.1).

LEMMA 2.6. – *Let $Q_R \subset \mathbb{R}^m$, $m \geq 2$, be any cube whose side length equals R. Assume that $B = \operatorname{curl} A$ satisfies (2.13). Then for each $\mu \in \mathbb{R}$, $R > 0$ and any $R_0 \in (0, R/2)$ we have*

$$\mathcal{N} (\mu; H^D_{Q_R} (A, 0)) \leq R^m \, \Theta (\mu; B),$$
$$\mathcal{N} (\mu; H^D_{Q_R} (A, 0)) \geq (R - R_0)^m \, \Theta (\mu - C_0 R_0^{-2}; B),$$

where the constant C_0 depends only on the dimension m.

Proof of Theorem 2.4. – Set $V_h (x) := h^{-1} V (h^{1/2} x)$ and change the variables $x \to h^{1/2} x$ in order to verify the identity

$$\mathcal{N} (0; \mathfrak{H}_{h,1}) = \mathcal{N} (0; H (A, V_h)). \tag{2.20}$$

Further, for a fixed sufficiently small $\delta > 0$ introduce a disjoint covering of $\mathbb{R}^m$ by open cubes $Q_l \equiv Q_l(r_l; x_l)$, $l \geq 1$, with centres at the points x_l and side lengths r_l satisfying

$$C^{-1}\,\delta\,(1+|x_l|) \leq r_l \leq C\,\delta\,(1+|x_l|)$$

where the constant $C > 1$ is independent of l and δ. The existence of such a covering can be verified if we modify in a straightforward manner the argument in the proof of Lemma 4 in [Roz]. Introduce a partition of unity $\{\chi_l\}_{l=1}^{\infty}$ such that the function χ_l is supported on $\tilde{Q}_{l,\delta} := Q_l\,((1+\delta)\,r_l;\, x_l)$ and the estimates

$$|D^{\gamma}\,\chi_l| \leq c_{\gamma}\,(\delta r_l)^{-|\gamma|} \tag{2.21}$$

hold for each multiindex γ and some constants c_{γ} which are independent of r_l and δ. The quantity $\#\{j\,:\,\operatorname{supp}\chi_j \cap \operatorname{supp}\chi_l \neq \emptyset\}$ is uniformly bounded with respect to l and δ. Moreover, the ratios $(1+|x_l|)/(1+|x_j|)$ are uniformly positive and bounded with respect to the pairs $(l,\, j)$ for which supp $\chi_l \cap \operatorname{supp}\chi_j \neq \emptyset$. Applying Lemma 1.1 and the estimates (2.21) with $|\gamma| = 1$, we get

$$\mathcal{N}(0;\, H(A,\, V_h)) \leq \sum_{l=1}^{\infty} \mathcal{N}(C_1\,\delta^{-2}\,r_l^{-2};\, H^{D}_{\tilde{Q}_{l,\delta}}(A,\, V_h)) \tag{2.22}$$

where the constant C_1 is independent of l and δ. Put

$$V^{+}_{h,\,l} = \sup_{x \in \tilde{Q}_{l,\,\delta}} V_h(x).$$

Using Lemma 2.6, we obtain the estimate

$$\sum_{l=1}^{\infty} \mathcal{N}(C_1\,\delta^{-2}\,r_l^{-2};\, H^{D}_{\tilde{Q}_{l,\delta}}(A,\, V_h)) \leq (1+\delta)^m \operatorname{vol} Q_l\,\Theta\,(V^{+}_{h,\,l} + C_1\,\delta^{-2}\,r_l^{-2};\, B). \tag{2.23}$$

The condition $\mathcal{D}_{\alpha}$ with $\alpha \in (0,\, 2]$ implies that for a given $\delta > 0$ and sufficiently small $h > 0$, we have $V^{+}_{h,\,l} + C_1\,\delta^{-2}\,r_l^{-2} \leq (1+\delta)\,V_h(x)$ for each $x \in Q_l$ and every $l \geq 1$. Therefore, combining (2.22) and (2.23), we get

$$\mathcal{N}(0;\, H(A,\, V_h)) \leq (1+\delta)^m \int_{\mathbb{R}^m} \Theta\,((1+\delta)\,V_h(x);\, B)\,dx. \tag{2.24}$$

In view of Lemma 2.3, we have

$$\lim_{\delta\downarrow 0}\limsup_{h\downarrow 0}(1+\delta)^m \int_{\mathbb{R}^m} \Theta\left((1+\delta)V_h(x);\, B\right)\mathbf{d}x/\nu_1(h) \leq 1. \quad (2.25)$$

Hence, the estimates (2.20), (2.24) and (2.25) imply

$$\limsup_{h\downarrow 0} \mathcal{N}(0;\, \mathfrak{H}_{h,1})/\nu_1(h) \leq 1. \quad (2.26)$$

Further, by the minimax principle, we have

$$\mathcal{N}(0;\, H(A, V_h)) \geq \sum_{l=1}^{\infty} \mathcal{N}(0;\, H^D_{Q_l}(A, V_h)).$$

Applying Lemma 2.6, and mimicking the derivation of (2.24), we get

$$\mathcal{N}(0;\, H(A, V_h)) \geq (1-\delta)^m \int_{\mathbb{R}^m} \Theta\left((1-\delta)V_h(x);\, B\right) dx$$

which entails

$$\liminf_{h\downarrow 0} \mathcal{N}(0;\, \mathfrak{H}_{h,1})/\nu_1(h) \geq 1. \quad (2.27)$$

Putting together (2.26) and (2.27), we come to (2.19). □

The proof of Theorem 2.4 is inspired by the proof of Theorem 1 (i) in [Tam] and is quite similar to the proof of Theorem 2.1 in [Rai 3]. Note that the explicit assumption that V satisfies the condition $\mathcal{T}$, if $k = 0$ and $\alpha \in (0, 2)$, has been omitted in the hypotheses of Theorem 2.1 in [Rai 3] although this assumption is necessary (*see* the Appendix).

3. WEAK-MAGNETIC-FIELD EIGENVALUE ASYMPTOTICS

The results of this section concern the behaviour of the quantity $\mathcal{N}(-\lambda;\, \mathfrak{H}_{1,\mu})$ as $\mu \downarrow 0$, the number $-\lambda \leq 0$ being fixed.

3.1. In this subsection we deal with electric potentials V which decay rapidly at infinity in a certain sense.

We shall write that $V \in \mathcal{K}_j$, $j = 0, 1$, if and only if for each $\varepsilon > 0$ we can represent V in the form

$$V = V_1 + V_2 \quad (3.1)$$

where $V_1 \in C_0^\infty(\mathbb{R}^m)$, and V_2 satisfies the inequality

$$\int_{\mathbb{R}^m} |V_2|\,|u|^2\,dx \le \varepsilon \int_{\mathbb{R}^m} (|\nabla u|^2 + j\,|u|^2)\,dx, \qquad \forall\, u \in C_0^\infty(\mathbb{R}^m). \tag{3.2}$$

The class $\mathcal{K}_0$ will be considered only in the case $m \ge 3$.

If $V \in \mathcal{K}_1$, then the negative spectrum of the operator $\mathfrak{H}_{1,0} \equiv -\Delta + V$ is purely discrete and, hence, the quantity $\mathcal{N}(-\lambda;\, \mathfrak{H}_{1,0})$ is finite for each $-\lambda < 0$. Moreover, if V satisfies $\mathcal{K}_0$, then the negative eigenvalues of $\mathfrak{H}_{1,0}$ do not accumulate to the origin, i. e. we have $\mathcal{N}(0;\, \mathfrak{H}_{1,0}) < \infty$ (*see* [Bir]).

The following proposition which can be proved using the methods of [Bir] and [Bir.Sol 2] contains some *sufficient* conditions which guarantee $V \in \mathcal{K}_j$, $j = 0, 1$.

PROPOSITION 3.1. – (i) *Let* $q = m/2$ *if* $m \ge 3$, *and* $q > 1$ *if* $m = 2$. *Assume* $\mathcal{V} \in L^q_{\text{loc}}(\mathbb{R}^m)$ *and* $\int_{|y-x|<1} |\mathcal{V}(x)|^q\,dx \to 0$ *as* $|y| \to \infty$. *Then we have* $\mathcal{V} \in \mathcal{K}_1$.

(ii) *Let* $m \ge 3$ *and* $\mathcal{V} \in L^{m/2}(\mathbb{R}^m)$. *Then we have* $\mathcal{V} \in \mathcal{K}_0$.

THEOREM 3.2. – *Let* $A \in L^2_{\text{loc}}(\mathbb{R}^m)^m$, $m \ge 2$.

a) Assume $V \in \mathcal{K}_1$. *Suppose that the number* $-\lambda < 0$ *is not an eigenvalue of the operator* $\mathfrak{H}_{1,0}$. *Then we have*

$$\mathcal{N}(-\lambda;\, \mathfrak{H}_{1,\mu}) \underset{\mu \downarrow 0}{\to} \mathcal{N}(-\lambda;\, \mathfrak{H}_{1,0}). \tag{3.3}$$

b) Let $m \ge 3$. *Assume* $V \in \mathcal{K}_0$. *Suppose that the zero is not an eigenvalue of the operator* $\mathfrak{H}_{1,0}$. *Then we have*

$$\mathcal{N}(0;\, \mathfrak{H}_{1,\mu}) \underset{\mu \downarrow 0}{\to} \mathcal{N}(0;\, \mathfrak{H}_{1,0}).$$

COROLLARY 3.3. – Assume $V \in \mathcal{K}_1$. Suppose that the negative number $-\lambda$ is an eigenvalue of the operator $\mathfrak{H}_{1,0}$ of multiplicity κ. Then under the hypotheses of Theorem 3.2 *a)* we have

$$\mathcal{N}(-\lambda - \varepsilon,\, -\lambda + \varepsilon | \mathfrak{H}_{1,\mu}) = \kappa$$

for both $\varepsilon > 0$ and $\mu > 0$ small enough.

Theorem 3.2 and Corollary 3.3 treat the stability of the isolated eigenvalues of the operator $\mathfrak{H}_{1,0}$ with respect to a perturbation by a weak magnetic field. Related results can be found in [Av.Her.Sim 1], Section 6, and [Av.Her.Sim 2], Section 7. The authors of [Av.Her.Sim 1-2], however, consider just the case of constant magnetic fields B, and a class of electric potentials V which is narrower than the one we study in Theorem 3.2.

On the other hand, in [Av.Her.Sim 1-2] the analyticity with respect to small μ is proved, while we just obtain limiting relations of the type of (3.3). Moreover, in [Av.Her.Sim 2], Sect. 7, the many-particle Schrödinger operator is considered.

In the sequel we denote by $\mathfrak{S}_q$, $q \in [1, \infty)$, the spaces of linear compact operators with norm $\|T\|_q := (\operatorname{Tr}|T|^q)^{1/q}$ (*see* e.g. [Bir.Sol 3], Ch. 11). The proof of Theorem 3.2 relies substantially on the following lemma due to Kac-Murdock-Szegö.

LEMMA 3.4. – *Let T_μ, $\mu \geq 0$, be a family of linear compact operators such that $\|T_\mu\| \leq t_0$, $\forall \mu \geq 0$, and $T_\mu \in \mathfrak{S}_q$, $q \geq 1$. Let the positive numbers t_j, $j = 1, 2$, be not eigenvalues of the operator T_0. Then the limiting relations*

$$\operatorname{Tr} T_\mu^n \underset{\mu \downarrow 0}{\to} \operatorname{Tr} T_0^n, \qquad \forall n \in \mathbb{N}, \quad n \geq q,$$

imply

$$\mathcal{N}(t_1, t_2 | T_\mu) \underset{\mu \downarrow 0}{\to} \mathcal{N}(t_1, t_2 | T_0).$$

The simple proof of the lemma employs the ideas used in [Gre.Sze], Section 7.1.

Proof of Theorem 3.2. – For definiteness we prove the first assertion of the theorem. We assume $\operatorname{div} A = 0$ in the distribution sense since we can always achieve this property by means of a gauge transform (*see* [Lei], Lemma 1.1 and Theorem 1.2).

Let the multiplier by the real function $W : \mathbb{R}^m \to \mathbb{R}$ be $-\Delta$-form-compact. Define the "magnetic" Birman-Schwinger operator

$$T_{\mu, \lambda}(W) := (H(\mu A, 0) + \lambda)^{-1/2} W (H(\mu A, 0) + \lambda)^{-1/2},$$
$$\mu \geq 0, \quad \lambda > 0.$$

Note that $-\lambda \notin \sigma(\mathfrak{H}_{1,0})$ implies $1 \notin \sigma(T_{0,\lambda}(V))$. Fix $\delta \in (0, 1/2)$ in such a way that the inequality

$$2\delta < \operatorname{dist}\{1, \sigma(T_{0,\lambda}(V))\} \tag{3.4}$$

holds, set $\varepsilon = \delta \min\{1, \lambda\}$, and write V in the form (3.1). Then, in view of the diamagnetic inequality (*see* [Av.Her.Sim 1], Theorem 2.3), we get

$$\|T_{\mu,\lambda}(V_1)\| \leq t_0 := \| |V_1|^{1/2} (-\Delta + \lambda)^{-1/2} \|^2, \qquad \forall \mu \geq 0. \tag{3.5}$$

Further, the estimate (3.2), the diamagnetic inequality, and the relation between ε and δ entail

$$\|T_{\mu,\lambda}(V_2)\| \leq \delta, \qquad \forall \mu \geq 0. \tag{3.6}$$

Hence, we have

$$\begin{aligned} \pm \mathcal{N}(-\lambda; \mathfrak{H}_{1,\mu}) &= \pm n(1; T_{\mu,\lambda}(V)) \\ &\leq \pm n(1 \mp \delta; T_{\mu,\lambda}(V_1)) = \pm \mathcal{N}(1 \mp \delta, \tau | T_{\mu,\lambda}(V_1)) \end{aligned} \qquad (3.7)_\pm$$

where the number τ is strictly greather than t_0 (and $1+\delta$).

Since the support of V_1 is compact, we have $T_{0,\lambda}(|V_1|) \in \mathfrak{S}_{2p}$ provided that $p \in \mathbb{N}, p > m/4$. Hence, we have $T_{\mu,\lambda}(|V_1|) \in \mathfrak{S}_{2p}$ for each $\mu \geq 0$ and each $p \in \mathbb{N}$ such that $p > m/4$ (*see* [Av.Her.Sim 1], p. 850). By virtue of the minimax principle, the same is valid for the operator $T_{\mu,\lambda}(V_1)$, $\mu \geq 0$. The inequality (3.4) [resp. (3.5)] entails $1 \mp \delta \notin \sigma(T_{0,\lambda}(V_1))$ (resp. $\tau \notin \sigma(T_{0,\lambda}(V_1))$). Therefore, Lemma 3.4 implies that it suffices to verify the limiting relations

$$\operatorname{Tr} T_{\mu,\lambda}(V_1)^n \underset{\mu \downarrow 0}{\to} \operatorname{Tr} T_{0,\lambda}(V_1)^n, \quad \forall n \in \mathbb{N}, \quad n \geq 2p, \tag{3.8}$$

in order to conclude that

$$\mathcal{N}(1 \mp \delta, \tau | T_{\mu,\lambda}(V_1)) \underset{\mu \downarrow 0}{\to} \mathcal{N}(1 \mp \delta, \tau | T_{0,\lambda}(V_1)). \qquad (3.9)_\pm$$

If $S, T \in \mathfrak{S}_2$ are integral operators acting in $L^2(\mathbb{R}^m)$ with kernels $s(x, y)$ and $t(x, y)$, then $ST \in \mathfrak{S}_1$ and we have

$$\operatorname{Tr} ST = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} s(x, y)\, t(y, x)\, dx\, dy,$$

the integral at the right-hand side being absolutely convergent (*see* e.g. [Bir.Sol 1], § 8). Since we have $T_{\mu,\lambda}(V_1)^p \in \mathfrak{S}_2, p > m/4$, $T_{\mu,\lambda}(V_1)^{n-p} \in \mathfrak{S}_2$, $n \geq 2p$, it is not difficult to verify the validity of the formula

$$\begin{aligned} &\operatorname{Tr} T_{\mu,\lambda}(V_1)^n \\ &\quad = \int_{\mathbb{R}^{nm}} V_1(x_1)\, \mathcal{R}(x_1, x_2; \lambda, \mu A)\, V_1(x_2) \\ &\qquad \ldots V_1(x_n)\, \mathcal{R}(x_n, x_1; \lambda, \mu A)\, dx_1\, dx_2 \ldots dx_n, \qquad n \geq 2p, \end{aligned} \tag{3.10}$$

where $\mathcal{R}(x, y; \lambda, \mu A)$ is the distribution kernel of the operator $(H(\mu A, 0) + \lambda)^{-1}$, $\lambda > 0$. Since $\operatorname{div} A = 0$, we can write the Feynman-Kac-Itô formula in the form

$$\mathcal{R}(x, y; \lambda, \mu A) = \int_0^\infty dt\, e^{-\lambda t} \int dE_{0, x; t, y}(\omega(s)) \times \exp\left\{ i\mu \int_0^t A(\omega(s)).d\omega \right\}, \qquad \lambda > 0,$$

where $\omega(s)$ are the Wiener paths, and $dE_{0, x; t, y}(\omega(s))$ is the conditional Wiener measure (*see* [Sim], Section 15). Hence, in particular we have

$$|\mathcal{R}(x, y; \lambda, \mu A)| \leq \mathcal{R}(x, y;; \lambda, 0), \qquad \forall \mu \geq 0,$$

for almost every $(x, y) \in \mathbb{R}^{2m}$. Thus we obtain

$$|\operatorname{Tr} T_{\mu, \lambda}(V_1)^n| \leq \operatorname{Tr} T_{0, \lambda}(|V_1|)^n \equiv \|T_{0, \lambda}(|V_1|)\|_n^n < \infty,$$
$$n \in \mathbb{N}, \quad n \geq 2p.$$

Moreover we have

$$\mathcal{R}_1(x, y; \lambda, \mu A) \underset{\mu \downarrow 0}{\rightarrow} \mathcal{R}_1(x, y; \lambda, 0)$$

for almost every $(x, y) \in \mathbb{R}^{2m}$. Consequently, we find that the integrand in (3.10) tends as $\mu \downarrow 0$ to its value at $\mu = 0$ for almost every $(x_1, ..., x_n) \in \mathbb{R}^{nm}$. Bearing in mind the formula (3.10), and applying the dominated convergence theorem we come to (3.8), and whence to $(3.9)_\pm$. The estimates (3.4)-(3.6) and the Birman-Schwinger principle entail

$$\begin{aligned} \pm\mathcal{N}(1 \mp \delta, \tau \,|\, T_{0, \lambda}(V_1)) &= \pm n(1 \mp \delta; T_{0, \lambda}(V_1)) \\ &\leq \pm n(1 \mp 2\delta; T_{0, \lambda}(V)) \\ &= \pm n(1; T_{0, \lambda}(V)) \\ &= \pm\mathcal{N}(-\lambda; \mathfrak{H}_{1, 0}). \end{aligned} \qquad (3.11)_\pm$$

Putting together $(3.7)_\pm$, (3.8) and $(3.11)_\pm$, we come to (3.3). □

3.2. In this subsection we consider constant magnetic fields and electric potentials which decay slowly at infinity.

THEOREM 3.5. – *Suppose that (2.13) holds and V satisfies the condition* $\mathcal{D}_\alpha$ *with* $\alpha \in (0, 2)$. *If* $k = 0$, *assume in addition that V satisfies the condition* $\mathcal{T}$. *For* $\mu > 0$ *put* $\nu_2(\mu) = \mu^{m/2}\nu_0(\mu^{-1})$. *Then we have*

$$\mathcal{N}(0; \mathfrak{H}_{1, \mu}) = \nu_2(\mu)(1 + o(1)), \qquad \mu \downarrow 0.$$

Suppose that the assumptions of Remark 2.5 are fulfilled for $\alpha \in (0, 2)$. Then Theorem 3.5 remains valid for $V = U + W$, and the main asymptotic term of $\nu_2(\mu)$ as $\mu \downarrow 0$ again depends only on U but not on W.

We omit the proof of Theorem 3.5 since it is quite the same as the proof of Theorem 2.4.

3.3. In this subsection we consider the case where $V(x)$ behaves like $|x|^{-2}$ as $|x| \to \infty$, i. e. the border-line case between Theorem 3.5 and Theorem 3.2 *b*). More precisely, we assume the relation (2.15) holds with $\alpha = 2$. Denote by $\{-\lambda_l(v)\}_{l \geq 1}$ the nondecreasing sequence of the negative eigenvalues of the operator

$$\mathcal{S}(v) = -\Delta_s - v$$

where Δ_s is the Laplace-Beltrami operator defined in $L^2(\mathbb{S}^{m-1})$. Evidently, the set $\{\lambda_l(v)\}_{l \geq 1}$ is finite and not empty.

THEOREM 3.6. – *Assume that (2.13) holds, and $V \in L^\infty(\mathbb{R}^m)$ satisfies (2.15) with $\alpha = 2$. Then we have*

$$\lim_{\mu \downarrow 0} |\log \mu|^{-1} \mathcal{N}(0; \mathfrak{H}_{1,\mu}) = \frac{1}{2\pi} \sum_{l \geq 1} \left(\lambda_l(v) - \frac{(m-2)^2}{4} \right)_+^{1/2}. \quad (3.12)$$

Moreover, if $\lambda_1(v) < (m-2)^2/4$, we have

$$\mathcal{N}(0; \mathfrak{H}_{1,\mu}) = 0(1), \qquad \mu \downarrow 0. \quad (3.13)$$

Under the hypotheses of Theorem 3.6 the negative spectrum of the operator $\mathfrak{H}_{1,0}$ is discrete. Moreover, the quantity $\mathcal{N}(0, H(0, V))$ is finite if $\lambda_1(v) < (m-2)^2/4$, and infinite if $\lambda_1(v) > (m-2)^2/4$.

Proof of Theorem 3.6. – For $\varepsilon \in (-1, 1)$ and $\mu \geq 0$ set

$$V_0(x; \varepsilon, \mu) = (1+\varepsilon) v(\hat{x}) (\mu + |x|^2)^{-1}.$$

Applying a standard variational technique (*cf.* [Rai 1], Lemma 4.1), we obtain the estimates

$$\pm \mathcal{N}(0; \mathfrak{H}_{1,\mu}) \leq \pm \mathcal{N}(0; H(\mu A, V_0(\pm\varepsilon, 1))) + 0(1),$$
$$\mu \downarrow 0, \quad \forall \varepsilon \in (0, 1) \quad (3.14)_\pm$$

Changing the variables $x \to \mu^{1/2} x$, we get

$$\mathcal{N}(0; H(\mu A, V_0(\varepsilon, 1)))$$
$$= \mathcal{N}(0; H(A, V_0(\varepsilon, \mu))), \qquad \varepsilon \in (-1, 1), \quad \mu > 0. \quad (3.15)$$

Let $\Omega := \{x \in \mathbb{R}^m : |x| < 1\}$. Then for each $\varepsilon \in (0, 1)$ and $\varepsilon' \in (\varepsilon, 1)$ we have

$$\pm \mathcal{N}(0; H(A, V_0(\pm\varepsilon, \mu))) \leq \pm\mathcal{N}(0; H_\Omega^D(0, V_0(\pm\varepsilon', \mu))) + 0(1), \qquad \mu \downarrow 0. \tag{3.16$_\pm$}$$

In order to verify $(3.16)_+$, put $\mathcal{O} = \{x \in \mathbb{R}^m : |x| > 1/2\}$ and introduce a partition of unity $\{\varphi_l\}_{l=1}^2$ over $\mathbb{R}^m$ such that $\operatorname{supp} \varphi_1 \subset \Omega$, $\operatorname{supp} \varphi_2 \subset \mathcal{O}$. By Lemma 1.1 we get

$$\left.\begin{aligned} &\mathcal{N}(0; H(A, V_0(\varepsilon, \mu))) \\ &\quad\leq \mathcal{N}\left(0; H_\Omega^{\mathcal{D}}\left(A, V_0(\varepsilon, \mu) + \sum_{l=1}^{2} |\nabla\varphi_l|^2\right)\right) \\ &\quad + \mathcal{N}\left(0; H_{\mathcal{O}}^D\left(A, V_0(\varepsilon, \mu) + \sum_{l=1}^{2} |\nabla\varphi_l|^2\right)\right), \\ &\qquad\qquad \varepsilon \in (0, 1), \quad \mu > 0. \end{aligned}\right\} \tag{3.17}$$

Obviously we have

$$\mathcal{N}\left(0; H_{\mathcal{O}}^D\left(A, V_0(\varepsilon, \mu) + \sum_{l=1}^{2} |\nabla\varphi_l|^2\right)\right) \leq \mathcal{N}\left(0; H_{\mathcal{O}}^D\left(A, V_0(\varepsilon, 0) + \sum_{l=1}^{2} |\nabla\varphi_l|^2\right)\right), \qquad \forall \mu \geq 0.$$

Since $\inf\, \sigma_{\text{ess}}(H_{\mathcal{O}}^D(A, 0))$ is strictly positive and the multiplier by $V_0(\varepsilon, 0) + \sum_{l=1}^{2} |\nabla\varphi_l|^2$ is a relatively compact perturbation of the operator $H_{\mathcal{O}}^D(A, 0)$, the second term at the right-hand side of (3.17) remains uniformly bounded as $\mu \downarrow 0$. Further, the minimax principle entails

$$\begin{aligned} &\mathcal{N}\left(0; H_\Omega^D\left(A, V_0(\varepsilon, \mu) + \sum_{l=1}^{2} |\nabla\varphi_l|^2\right)\right) \\ &\quad\leq \mathcal{N}(0; H_\Omega^D(0, (1-\tau)^{-1} V_0(\varepsilon, \mu))) \\ &\qquad + \mathcal{N}\left(0; -\tau\Delta_\Omega^D + 2i\,A.\nabla + |A|^2 - \sum_{l=1}^{2} |\nabla\varphi_l|^2\right), \\ &\qquad\qquad \forall \tau \in (0, 1). \end{aligned} \tag{3.18}$$

Note that the second term at the right-hand side of (3.18) is independent of μ and finite for each $\tau > 0$. Choosing τ so that $1 + \varepsilon = (1 - \tau)(1 + \varepsilon')$ and combining (3.17) with (3.18), we come to $(3.16)_+$. The estimate $(3.16)_-$ can be verified in a similar (and simpler) manner.

Now, assume $\mu < 1$ and put $\Omega_1 \equiv \Omega_1(\mu) := \{x \in \mathbb{R}^m : |x| < \sqrt{\mu}\}$, $\Omega_2 \equiv \Omega_2(\mu) := \Omega \backslash \overline{\Omega_1(\mu)} \equiv \{x \in \mathbb{R}^m : \sqrt{\mu} < |x| < 1\}$.The minimax principle entails the inequality

$$\mathcal{N}(0;\, H_\Omega^D(0,\, V_0(\varepsilon,\, \mu))) \leq \mathcal{N}(0;\, H_{\Omega_1(\mu)}^N(0,\, V_0(\varepsilon,\, \mu))) + \mathcal{N}\Big(0;\, H_{\Omega_2(\mu)}^N(0,\, V_0(\varepsilon,\, 0))\Big), \qquad \forall \varepsilon > 0, \quad \forall \mu > 0. \tag{3.19}$$

Changing the variables $x \to \sqrt{\mu}\, x$, we establish the estimate

$$\mathcal{N}(0;\, H_{\Omega_1(\mu)}^N(0,\, V_0(\varepsilon,\, \mu))) = \mathcal{N}(0;\, H_{\Omega_1(1)}^N(0,\, V_0(\varepsilon,\, 1))) < \infty. \tag{3.20}$$

Further, set $\mathfrak{N} \equiv \mathfrak{N}(\varepsilon) := \mathcal{N}(0; \mathcal{S}((1 + \varepsilon)v))$ and denote by $\mathfrak{X}_l^N(\varepsilon,\, \mu)$ [resp. by $\mathfrak{X}_l^D(\varepsilon,\, \mu)$], $l = 1, \ldots, \mathfrak{N}$, the operator generated in $L^2[(\sqrt{\mu},\, 1);\, r^{m-1}\, dr]$ by the closed quadratic form

$$\int_{\sqrt{\mu}}^1 \{|du/dr|^2 - \lambda_l((1 + \varepsilon)v)\, r^{-2}\, |u|^2\}\, r^{m-1}\, dr, \quad \varepsilon \in (-1,\, 1), \tag{3.21}$$

with domain $W_1^2(\sqrt{\mu},\, 1)$ [or, respectively, $\overset{\circ}{W}{}_1^2(\sqrt{\mu},\, 1)$].

Pass to spherical coordinates in $\Omega_2(\mu)$, and decompose the trial function u in the domain of the quadratic form of the operator $H_{\Omega_2(\mu)}^N(0,\, V_0(\varepsilon,\, 0))$ in a series with respect to the eigenfunctions of the operator $\mathcal{S}((1 + \varepsilon)v)$. Thus we obtain

$$\mathcal{N}(0;\, H_{\Omega_2(\mu)}^N(0,\, V_0(\varepsilon,\, 0))) = \sum_{l=1}^{\mathfrak{N}} \mathcal{N}(0;\, \mathfrak{X}_l^N(\varepsilon,\, \mu)). \tag{3.22}$$

Recalling that $\dim W_1^2(\sqrt{\mu},\, 1) \ominus \overset{\circ}{W}{}_1^2(\sqrt{\mu},\, 1) = 2$, we come to the estimate

$$\sum_{l=1}^{\mathfrak{N}} \mathcal{N}(0;\, \mathfrak{X}_l^N(\varepsilon,\, \mu)) \leq \sum_{l=1}^{\mathfrak{N}} \mathcal{N}(0;\, \mathfrak{X}_l^D(\varepsilon,\, \mu)) + 2\,\mathfrak{N}(\varepsilon). \tag{3.23}$$

Fix $\delta > 0$ and assume $\mu < \delta$. Then the minimax principle implies

$$\left.\begin{aligned} &\mathcal{N}(0;\, H_\Omega^D(0,\, V_0(-\varepsilon,\, \mu)))\\ &\quad\geq \mathcal{N}(0;\, H_{\Omega_2(\mu/\delta)}^D(0,\, V_0(-\varepsilon,\, \mu)))\\ &\quad\geq \mathcal{N}(0;\, H_{\Omega_2(\mu/\delta)}^D(A,\, V_0(-\varepsilon',\, 0))), \end{aligned}\right\} \tag{3.24}$$

with $\varepsilon \in (0,\, 1)$, $\varepsilon' \in (\varepsilon,\, 1)$ and $\delta > 0$ connected by $1-\varepsilon = (1-\varepsilon')(1+\delta)$. By analogy with (3.22) we get

$$\begin{aligned} &\mathcal{N}(0;\, H_{\Omega_2(\mu/\delta)}^D(A,\, V_0(-\varepsilon,\, 0)))\\ &\quad= \sum_{l=1}^{\mathfrak{N}} \mathcal{N}(0;\, \mathfrak{X}_l^D(-\varepsilon,\, \mu/\delta)), \qquad \forall \varepsilon \in (0,\, 1). \end{aligned} \tag{3.25}$$

Now, substitute the trial function $u \in \overset{\circ}{W}{}_1^2(\sqrt{\mu},\, 1)$ according to the formula $u \longrightarrow r^{(2-m)/2}u$, and then change the variable $r \longrightarrow t = -\log r/\log\sqrt{\mu}$. Bearing in mind (3.21), we find that the operator $\mathfrak{X}_l^D(\varepsilon,\, \mu)$, $\varepsilon \in (-1,\, 1)$ is unitarily equivalent to the operator generated by the quadratic form

$$\int_0^1 \{4|\log\mu|^{-2}\, |du/dt|^2 - (\lambda_l((1+\varepsilon)\, v) - (m-2)^2/4)|u|^2\}\, dt,$$

with domain $\overset{\circ}{W}{}_1^2\,(0,\, 1)$. Applying an elementary semiclassical asymptotic formula for the eigenvalues of this operator, we get

$$\left.\begin{aligned} &\lim_{\mu\downarrow 0}\, |\log\mu|^{-1}\, \mathcal{N}(0;\, \mathfrak{X}_l^D(\varepsilon,\, \mu))\\ &\quad= \frac{1}{2\pi}\left(\lambda_l((1+\varepsilon)\, v) - \frac{(m-2)^2}{4}\right)_+^{1/2},\\ &\forall \varepsilon \in (-1,\, 1), \qquad l = 1, \ldots, \mathfrak{N}. \end{aligned}\right\} \tag{3.26}$$

Putting together $(3.14)_\pm$, (3.15), $(3.16)_\pm$, (3.19), (3.20), (3.22)-(3.26), taking into account the continuity for small $|\varepsilon|$ of the quantities $\lambda_l((1+\varepsilon)\, v)$, $l = 1, \ldots, \mathfrak{N}$, and utilizing the relation $\lim_{\mu\downarrow 0} |\log\mu|^{-1}\, |\log(\mu/\delta)| = 1$, we come to (3.12).

Finally assume that $\lambda_1(v) < (m-2)^2/4$. Then for $\varepsilon > 0$ small enough the quantity $\lambda_1((1+\varepsilon)v)$ does not exceed $(m-2)^2/4$ as well. Hence, we get

$$\mathcal{N}(0;\, \mathfrak{X}_l^D(\varepsilon,\, \mu)) = 0, \qquad l = 1, \ldots, \mathfrak{N}, \qquad \forall \mu > 0. \tag{3.27}$$

The combination of $(3.14)_+$ (3.15), $(3.16)_+$, (3.19), (3.20), (3.22), (3.23) and (3.27) yields (3.13). □

APPENDIX: PROOF OF LEMMA 2.3

In view of (2.16)-(2.17), it suffices to verify the relations

$$\lim_{\delta \downarrow 0} \limsup_{g \to \infty} g^{-m/\alpha} \{\nu_0((1+\delta)g) - \nu_0(g)\} = 0, \quad \alpha \in (0,\, 2), \tag{A.1}$$

or

$$\lim_{\delta \downarrow 0} \limsup_{g \to \infty} g^{-m/2} (\log g)^{-1} \{\nu_0((1+\delta)g) - \nu_0(g)\} = 0, \quad \alpha = 2, \tag{A.2}$$

in order to prove (2.18).

First, we assume that $\alpha \in (0,\, 2)$ and verify (A.1). For $k \geq 0$, $\lambda > 0$, $g > 0$, put

$$\Psi_k(\lambda;\, g) := \int_{\mathbb{R}^m} \theta_k(g\, V(x) - \lambda)\, dx \equiv -g^{k/2} \int_{\lambda/g}^{\infty} (s - \lambda/g)^{k/2}\, d\psi(s).$$

Then we have

$$\begin{aligned} & g^{-m/\alpha} \{\nu_0((1+\delta)g) - \nu_0(g)\} \\ & \quad = g^{-m/\alpha}\, \mathcal{C}_m(\mathbf{b}) \sum_{\mathbf{n} \in \mathbb{N}^d} \Psi_k(2\,\mathbf{n}.\mathbf{b} + \Lambda;\, g) \\ & \quad \times \left\{ \frac{\Psi_k(2\,\mathbf{n}.\mathbf{b} + \Lambda;\, (1+\delta)g)}{\Psi_k(2\,\mathbf{n}.\mathbf{b} + \Lambda;\, g)} - 1 \right\} \end{aligned} \tag{A.3}$$

It is easy to check the estimate

$$g^{-m/\alpha}\, \Psi_k(2\,\mathbf{n}.\mathbf{b} + \Lambda;\, g) \leq c\,(1 + |\mathbf{n}|)^{k/2 - m/\alpha}, \qquad \mathbf{n} \in \mathbb{N}^d,$$

where the constant c is independent of $\mathbf{n}$ and g. Note that the series $\sum_{\mathbf{n}\in\mathbb{N}^d} (1+|\mathbf{n}|)^{k/2-m/\alpha}$ is convergent if $\alpha \in (0, 2)$. Hence, applying the identity (A.3), we find that the relation (A.1) would follow from the estimate

$$\left.\begin{aligned}\lim_{\delta\downarrow 0}\limsup_{g\to\infty}[\Psi_k(\lambda;\,(1+\delta)\,g) - \Psi_k(\lambda;\,g)]/\Psi_k(\lambda;\,g) = 0,\\ \forall\lambda > 0,\quad \forall k \geq 0.\end{aligned}\right\} \tag{A.4}$$

Since we have $\Psi_k(\lambda;\,g) \underset{\cap}{\cup} g^{m/\alpha-k/2}$, $g \to \infty$, the estimate (A.4) would follow from the estimate

$$\lim_{\tau\downarrow 0}\limsup_{\varepsilon\downarrow 0}\varepsilon^{m/\alpha-k/2} \times \left\{\int_{\mathbb{R}^m}\{\theta_k(V(x)-(1-\tau)\,\varepsilon) - \theta_k(V(x)-\varepsilon)\}\,dx\right\} = 0. \tag{A.5}$$

Let $k \geq 2$. Then we have

$$\int_{\mathbb{R}^m}\{\theta_k(V(x)-(1-\tau)\,\varepsilon) - \theta_k(V(x)-\varepsilon)\}\,dx = \frac{k}{2}\int_{(1-\tau)\,\varepsilon}^{\varepsilon} dt \int_{\mathbb{R}^m}\theta_{k-2}(V(x)-t)\,dx. \tag{A.6}$$

Since the estimate $V(x) \leq c'\,|x|^{-\alpha}$ holds, the right-hand-side of (A.6) is upper bounded by

$$\frac{k}{2}\int_{(1-\tau)\,\varepsilon}^{\varepsilon} dt \int_0^{(c'/t)^{-1/\alpha}} (c'\,r^{-\alpha}-t)^{k/2-1}\,r^{m-1}\,dr \leq c''\,\varepsilon^{k/2-m/\alpha}\,[1-(1-\tau)^{k/2-m/\alpha}].$$

Thus (A.6) entails (A.5) if $k \geq 2$.

Let $k = 1$. It is easy to check that we have

$$\begin{aligned}\int_{\mathbb{R}^m}\{\theta_1(V(x)-(1-\tau)\,\varepsilon) - \theta_1(V(x)-\varepsilon)\}\,dx &= \frac{1}{2}\int_{(1-\tau)\,\varepsilon}^{\infty}(s-(1-\tau)\,\varepsilon)^{-1/2}\,\psi(s)\,ds\\ &\quad - \frac{1}{2}\int_{\varepsilon}^{\infty}(s-\varepsilon)^{-1/2}\,\psi(s)\,ds\\ &\leq \frac{1}{2}\int_{(1-\tau)\,\varepsilon}^{\varepsilon}(s-\varepsilon)^{-1/2}\,\psi(s)\,ds.\end{aligned} \tag{A.7}$$

Since the estimate $\psi(s) \leq c' s^{-m/\alpha}$, $s > 0$, holds, the rightmost quantity in (A.7) is upper bounded by $c'' \varepsilon^{1/2-m/\alpha} [1 - (1-\tau)^{1/2-m/\alpha}]$. Hence in the case $k = 1$, the relation (A.1) holds again.

Assume $k = 0$. Then the quantity $\Psi_0(\lambda; g)$ coincides with $\psi(\lambda/g)$, $\lambda > 0$, $g > 0$. Hence, in the case $k = 0$ the relation (A.1) is implied directly by the condition $\mathcal{T}$ satisfied by V according to the hypotheses of Lemma 2.3. Thus, we have completed the proof of (A.1) for all values of $k \geq 0$.

Now we assume $\alpha = 2$, and prove (A.2). First of all note that the set $\{2\,\mathbf{n}.\mathbf{b}\}_{\mathbf{n}\in\mathbb{N}^d}$ coincides with the nondecreasing sequence $\{\Lambda_j\}_{j=1}^{\infty}$ of the eigenvalues of the selfadjoint operator

$$\sum_{j=1}^{d} b_j (-\partial^2/\partial x_j^2 + x_j^2) - \Lambda$$

which is essentially selfadjoint on $C_0^\infty(\mathbb{R}^m)$. Then the function $\nu_0(g)$ can be written in the form

$$\nu_0(g) = \mathcal{C}_m(\mathbf{b}) \sum_{j=1}^{\infty} \int_{\mathbb{R}^m} \theta_k (g\,V(x) - \Lambda_j - \Lambda)\,dx, \qquad g > 0. \tag{A.8}$$

It is well-known that the eigenvalues Λ_j obey the asymptotics

$$\Lambda_j = C_d\, j^{1/d} (1 + o(1)), \qquad j \to \infty,$$

with $C_d := \left(2^d\, d!\, b_1 ... b_d\right)^{1/d}$. On the other hand, it is easy to check that each individual term in (A.8) has order $O(g^{m/2})$ as $g \to \infty$. Thus we obtain the asymptotic estimates

$$\begin{aligned} &\nu_0((1+\delta)\,g) \\ &\quad\leq \mathcal{C}_m(\mathbf{b}) \sum_{j=1}^{\infty} \int_{\mathbb{R}^m} \theta_k((1+\delta)\,g\,V(x) - (1-\eta)\,C_d\,j^{1/d} - \Lambda)\,dx \\ &\qquad + O(g^{m/2}), \end{aligned} \tag{A.9}$$

$$\begin{aligned} \nu_0(g) \geq \mathcal{C}_m(\mathbf{b}) &\sum_{j=1}^{\infty} \int_{\mathbb{R}^m} \theta_k (g\,V(x) - (1+\eta)\,C_d\,j^{1/d} - \Lambda)\,dx \\ &+ O(g^{m/2}), \end{aligned} \tag{A.10}$$

which hold for $g \to \infty$ and each $\eta \in (0, 1)$.

Note the elementary inequalities

$$\sum_{j=1}^{\infty} f(j) \leq \int_0^{\infty} f(t)\, dt \leq \sum_{j=0}^{\infty} f(j) \tag{A.11}$$

where $f(t) := \theta_k(\gamma_1 - \gamma_2 t)$, and γ_j, $j = 1, 2$, are positive parameters. Hence, (A.9)-(A.10) entail

$$\begin{aligned} &\nu_0((1+\delta)g) \\ &\quad \leq (1-\eta)^{-d} \int_{\mathbb{R}^m} \theta_m((1+\delta)g V(x) - \Lambda)\, dx/(4\pi)^{m/2}\,\Gamma(1+m/2) \\ &\quad + O(g^{m/2}), \\ &\nu_0(g) \\ &\quad \geq (1-\eta)^{-d} \int_{\mathbb{R}^m} \theta_m(g V(x) - \Lambda)\, dx/(4\pi)^{m/2}\,\Gamma(1+m/2) \\ &\quad + O(g^{m/2}), \end{aligned}$$

which hold for $g \to \infty$ and each $\eta \in (0,1)$. Thus we get

$$\begin{aligned} &g^{-m/2}(4\pi)^{m/2}\,\Gamma(m/2)\,\{\nu_0((1+\delta)g) - \nu_0(g)\} \\ &\quad \leq (1-\eta)^{-d}\,[(1+\delta)^{m/2} - 1] \int_0^{\infty} \theta_{m-2}(s - \Lambda/(1+\delta)g)\,\psi(s)\, ds \\ &\quad + (1-\eta)^{-d} \int_{\Lambda/(1+\delta)g}^{\Lambda/g} dt \int_{\mathbb{R}^m} \theta_{m-2}(V(x) - t)\, dx \\ &\quad + [(1-\eta)^{-d} - (1+\eta)^{-d}] \\ &\quad \times \int_0^{\infty} \theta_{m-2}(s - \Lambda/g)\,\psi(s)\, ds + O(1), \quad g \to \infty, \end{aligned} \tag{A.12}$$

Since the function $\psi(s)$ vanishes identically for s large enough, and admits the estimate $\psi(s) \leq cs^{-m/2}$ for sufficiently small $s > 0$, the integrals of the type $\int_0^{\infty} \theta_{m-2}(s - \varepsilon)\,\psi(s)\, ds$, $\varepsilon > 0$, occurring in the first and the third term at the right-hand side of (A.12) has order $O(|\log \varepsilon|)$ as $\varepsilon \downarrow 0$. Further, since we have $V(x) \leq c\,|x|^{-2}$, $x \in \mathbb{R}^m$, we easily find that the second term of the right-hand side of (A.12) has order $O(1)$ as $g \to \infty$. Finally, since $\eta > 0$ (and, hence, $(1-\eta)^{-d} - (1+\eta)^{-d}$) can chosen as small as needed, we can conclude that (A.2) is valid.

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REFERENCES

[Ale] A. B. ALEKSEEV, On the spectral asymptotics of differential operators with polynomial occurrence of a small parameter, *Probl. Mat. Fiz.*, Vol. **8**, 1976, pp.3-15 (in Russian).

[Av.Her.Sim 1] J. AVRON, I. HERBST and B. SIMON, Schrödinger operators with magnetic fields. I. General interactions, *Duke Math. J.*, Vol. **45**, 1978, pp. 847-883.

[Av.Her.Sim 2] J. AVRON, I. HERBST and B. SIMON, Schrödinger operators with magnetic fields. III. Atoms in homogeneous magnetic field, *Commun. Math. Phys.*, Vol. **79**, 1981, pp. 529-572.

[Bir] M. S. BIRMAN, On the spectrum of singular boundary value problems, *AMS Translations*, (2), Vol. **53**, 1966, pp. 23-80.

[Bir.Sol 1] M. S. BIRMAN and M. Z. SOLOMJAK, Estimates of the singular numbers of integral operators, *Russian Math. Surveys*, Vol. **32**, 1977, pp. 15-89.

[Bir.Sol 2] M. S. BIRMAN and M. Z. SOLOMJAK, Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory, *AMS Translations* (2), Vol. **114**, 1980.

[Bir.Sol 3] M. S. BIRMAN and M. Z. SOLOMJAK, Spectral Theory of Selfadjoint Operators in Hilbert Space, D. REIDEL, Dordrecht, 1987.

[CdV] Y. COLIN DE VERDIÈRE, L'asymptotique de Weyl pour les bouteilles magnétiques, *Comm. Math. Phys.*, Vol. **105**, 1986, pp. 327-335.

[Com.Sch.Sei] J.M. COMBES, R. SCHRADER and R. SEILER, Classical bounds and limits for energy distributions of Hamiltonian operators in electromagnetic fields, *Ann. Physics*, Vol. **111**, 1978, pp. 1-18.

[Cy.Fr.Ki.Sim] H. L. CYCON, R.G. FROESE, W. KIRSCH and B. SIMON, Schrödinger operators with Application to Quantum Mechanics and Global Geometry. Springer, Berlin, 1987.

[Dau.Rob] M. DAUGE and D. ROBERT, Weyl's formula for a class of pseudodifferential operators of negative order on $L^2(\mathbb{R}^n)$, *Lect. Notes Mah.*, Vol. **1256**, 1987, pp. 91-122.

[Gre.Sze] U. GRENANDER and G. SZEGÖ, Toeplitz Forms and Their Applications, University of California Press, Berkeley-Los Angeles, 1958.

[Hel.Sjö 1] B. HELFFER and J. SJÖSTRAND, Équation de Schrödinger avec champ magnétique et équation de Harper, II. Approximation champ magnétique faible: substitution de Peierls. In: Schrödinger operators, *Lect. Notes Phys.*, Vol. **345**, 1988, pp. 138-191.

[Hel.Sjö 2] B. HELFFER and J. SJÖSTRAND, On the link between the Schrödinger operator with magnetic field and Harper's equation, In: *Proceedings of Symp. Part. Diff. Equ.*, HOLZHAU, 1988. *Teubner-Texte zur Mathematik*, Vol. **112**, 1989, pp. 139-156.

[Ivr 1] V. Ya. IVRII, Estimations pour le nombre de valeurs propres négatives de l'opérateur de Schrödinger avec potentiels singuliers, *C. R. Acad. Sci. Paris, Sér. I*, Vol. **302**, 1986, pp. 467-470.

[Ivr 2] V. Ya. Ivrii, Estimates of the number of negative eigenvalues of the Schrödinger operator with a strong magnetic field, *Soviet Math. Dokl.*, Vol. **36**, 1988, pp. 561-564.

[Ivr 3] V. Ya. Ivrii, Sharp spectral asymptotics for the two-dimensional Schrödinger operator with a strong magnetic field, *Soviet Math. Dokl.*, Vol. **39**, 1989, pp. 437-441.

[Ivr 4] V. Ya. Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics, Preprints 1-9, École Polytechnique, 1990-1992.

[Lei] H. Leinfelder, Gauge invariance of Schrödinger operators and related spectral properties, *J. Opt. Theory*, Vol. **9**, 1983, pp. 163-179.

[Rai 1] G. D. Raikov, Spectral asymptotics for the Schrödinger operator with potential which steadies at infinity, *Comm. Math. Phys.*, Vol. **124**, 1989, pp. 665-685.

[Rai 2] G. D. Raikov, Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips, *Comm. P.D.E.*, Vol. **15**, 1990, pp. 407-434.

[Rai 3] G. D. Raikov, Strong electric field eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, *Letters Math. Phys.*, Vol. **21**, 1991, pp. 41-49.

[Rai 4] G. D. Raikov, Semiclassical and weak-magnetic-field eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, *C. R. Acad. Sci. Bulg.*, Vol. **44**, 1991, No. 1, pp.15-18.

[Ree.Sim] M. Reed and B. Simon, Methods of Modern Mathematical Physics. IV. Analysis of Operators, Academic Press, London, 1978.

[Roz] G. V. Rozenbljum, An asymptotic of the negative discrete spectrum of the Schrödinger operator, *Math. Notes U.S.S.R.*, Vol. **21**, 1977, pp. 222-227.

[Sim] B. Simon, Functional Integration and Quantum Physics, Academic Press, London, 1979.

[Tam] H. Tamura, Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields, *Osaka J. Math.*, Vol.**25**, 1988, pp. 633-647.

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