JOÃO-PAULO DIAS
MARIO FIGUEIRA
JEFFREY RAUCH

**Scattering for a one-sided Klein-Gordon
equation in quantum gravity**

Scattering for a one-sided Klein-Gordon equation in quantum gravity

by

João-Paulo DIAS, Mario FIGUEIRA

CMAF, University of Lisdon, Av. Prof. Gama Pinto,
1699 Lisboa Codex, Portugal

and

Jeffrey RAUCH

Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109-1003, U.S.A.

ABSTRACT. – In this paper we consider a Klein-Gordon equation in one space dimension with a class of smooth positive increasing potentials V motivated by a model in quantum gravity. For large times the wave propagate freely to the left. We prove decay in time estimates and asymptotic completeness justifying this phrase.

RÉSUMÉ. – Dans cet article nous considérons une équation de Klein-Gordon dans une dimension d'espace avec une classe de potentiels réguliers positifs et croissants V motivée par un modèle en gravité quantique. Pour des grandes valeurs du temps l'onde se propage librement à gauche. Nous donnons des estimations de décroissance temporelle et prouvons la complétude asymptotique en justifiant cette phrase.

1. INTRODUCTION AND MAIN RESULTS

This paper is motivated by the results obtained in [2] on the decay of the solutions of the following equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{1}{4}(p-1)^2 u + e^{4z} u = 0,$$

where $p \in \mathbb{R}$ is a constant and $(z, y) \in \mathbb{R}^2$.

This equation is obtained, by the transformation $u(z, y) = x^{\frac{p-1}{2}} \psi(x, y)$, $z = \log x$, $x > 0$, from the simplified Wheeler-DeWitt equation with a massless single scalar field y (cf. [4])

$$\frac{\partial^2 \psi}{\partial y^2} - x^2 \frac{\partial^2 \psi}{\partial x^2} - px \frac{\partial \psi}{\partial x} + x^4 \psi = 0 \tag{1.1}$$

where $x \in \mathbb{R}_+$ is a scale factor (radius of the Universe), $p \in \mathbb{R}$ is a constant which reflects the factor-ordering ambiguity and $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ is the wave function of the Universe for the minisuperspace model.

As a consequence of the results in this paper we conclude that, for large values of y, the wave function ψ propagates freely near the origin, that is the $x^4 \psi$ term is negligable, and is very small for large values of x.

We study the asymptotic behaviour and scattering properties of the solutions of the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + V u = 0, \qquad (x, t) \in \mathbb{R}^2 \tag{1.2}$$

where the potential V satisfies

$$V \in C^3(\mathbb{R}), \qquad \dot{V} \geqq 0,$$

$$\left. \begin{array}{l} 0 < \dfrac{dV}{dx} \leqq c(1+V), \qquad \left| \dfrac{d^n V}{dx^n} \right| \leqq c_1 \dfrac{dV}{dx}, \\[2mm] n = 2, 3, \quad c, c_1 \text{ positive constants}, \\[1mm] V = 0\left(|x|^{-1-\varepsilon}\right), \qquad \varepsilon > 0, \quad \text{when } x \to -\infty. \end{array} \right\} \tag{1.3}$$

It is well known that the operator $A_0 : D(A_0) \to L^2(\mathbb{R})$ defined by

$$A_0 u = -\frac{d^2 u}{dx^2} + u + V u \tag{1.4}$$

with domain

$$D(A_0) = \left\{ u \in H^1(\mathbb{R}) | V^{\frac{1}{2}} u \in L^2(\mathbb{R}) \text{ and } -\frac{d^2 u}{dx^2} + V u \in L^2(\mathbb{R}) \right\},$$

is self-adjoint in $L^2(\mathbb{R})$.

Let $H_V^1 = \{ u \in H^1(\mathbb{R}) | V^{\frac{1}{2}} u \in L^2(\mathbb{R}) \}$ with its natural norm and put $\mathcal{H} = H_V^1 \times L^2$, $D(A) = D(A_0) \times H_V^1$,

$$A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.5}$$

It is easy to see that the operator A, with domain $D(A)$, is self-adjoint in $\mathcal{H}$.

Denote by e^{-itA} the unitary group in $\mathcal{H}$ generated by $-iA$.

For $\varphi = (u_0, u_1) \in D(A)$, denote by $u(t) = e^{-itA} \varphi$ the corresponding solution of the Cauchy problem $\left(u(0) = u_0, \dfrac{\partial u}{\partial t}(0) = u_1 \right)$ for the equation (1.2).

Let $\mathcal{H}_0 = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $B : D(B) \to \mathcal{H}_0$, $D(B) = H^2 \times H^1$, where B is the self-adjoint operator in $\mathcal{H}_0$ defined by

$$B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ \dfrac{d^2}{dx^2} - 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.6}$$

Denote by e^{-itB} the (free) unitary group in $\mathcal{H}_0$ generated by $-iB$. The operator B has constant coefficients, $D(A) \subset D(B)$ and, on $D(A)$,

$$A - B = i \begin{pmatrix} 0 & 0 \\ -V & 0 \end{pmatrix}.$$

The estimates of [2] show that the solutions of (1.2) are small in $x > 0$ for $t \to \infty$. Our main result asserts that the solutions are asymptotically equal to solutions of the free Klein-Gordon equation which travel to the left. For $t \to \infty$, it is free waves travelling to the right (for time increasing) with intervenue. In a sense this is a three space scattering theory.

Write the finite energy free solutions as

$$\sum_{+,-} \int a_\pm(\xi) e^{i(x\xi \pm t\langle\xi\rangle \operatorname{sgn}(\xi))} d\xi, \qquad \text{where} \quad \langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

The $\pm$ term has group velocity equal to $\mp|\xi|/\langle\xi\rangle$. The waves with plus sign travel to the left and those with a minus to the right. For example, if $a_+ \in C_0^\infty(\mathbb{R} \setminus 0)$ then the plus term has $L^2(]0, +\infty[)$ norm which tends

to zero faster than any power of $1/t$ as t tends to $+\infty$. The same is true for all derivatives (apply the results in app. 1 to XI.3 in [6]).

The energy in $\mathcal{H}_0$ is equal to

$$\sum_{\pm} \int |\langle \xi \rangle \, a_{\pm}(\xi)|^2 \, d\xi.$$

Thus $\mathcal{H}_0 = \mathcal{H}_+ \oplus_{\perp} \mathcal{H}_-$ where $\mathcal{H}_{\pm}$ denotes the space of solutions with $a_{\mp} \equiv 0$. One has

$$\hat{u}(0, \xi) = (2\pi)^{-\frac{1}{2}} \, (a_+(\xi) + a_-(\xi)),$$

$$\hat{u}_t(0, \xi) = (2\pi)^{-\frac{1}{2}} \langle \xi \rangle \, (\text{sgn}(\xi) \, a_+(\xi) - \text{sgn}(\xi) \, a_-(\xi))/i.$$

Thus $(u_0, u_1) \in \mathcal{H}_+$ iff $u_1 = -i \langle D \rangle \, \text{sgn}(D) \, u_0$. Note the familiar Hilbert transform $\text{sgn}(D)$.

The following result has a proof similar to the proof of theorem 2 in [2].

THEOREM 1. – *Assume* $\varphi = (u_0, u_1) \in D(A^2)$. *Then*

$$\lim_{t \to \infty} \left\| \left(\frac{dV}{dx} \right)^{\frac{1}{2}} \left(|u(\cdot, t)| + \left| \frac{\partial u}{\partial t}(\cdot, t) \right| \right) \right\|_{L^2(\mathbb{R})} = 0.$$

By an adaptation of the proof of the theorem 3 in [2] we will prove the following:

THEOREM 2. – *Assume* $\varphi = (u_0, u_1) \in (\mathcal{D}(\mathbb{R}))^2$. *Then* $u \in C^4(\mathbb{R}^2)$, $u(\cdot, t) \in C_0^4(\mathbb{R})$, $\dfrac{\partial u}{\partial t}(\cdot, t) \in C_0^3(\mathbb{R})$ *and*

$$\lim_{t \to \infty} \left\| \left(\frac{dV}{dx} \right)^{\frac{1}{2}} \frac{\partial u}{\partial x}(\cdot, t) \right\|_{L^2(\mathbb{R})} = 0.$$

In addition,

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}} \frac{dV}{dx} \left(|u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right) dx \, dt < +\infty \qquad (1.7)$$

COROLLARY. – (i) *If* $\varphi \in (\mathcal{D}(\mathbb{R}))^2$ *and, for* $a \in \mathbb{R}$, $H_a = H^1(]a, +\infty[) \times L^2(]a, +\infty[)$, *then* $\lim\limits_{t \to \infty} \|e^{-itA} \varphi\|_{H_a} = 0$. (ii) *If* $P_{ac} = P_{ac}(A)$ *denote the orthogonal projection of* $\mathcal{H}$ *onto the subspace of absolute continuity for* A, *then* $P_{ac} = \text{Id}$.

The first part of the corollary is an immediate consequence of theorems 1 and 2, since $\dfrac{dV}{dx} \geq a_1 > 0$ in $[a, +\infty[$ [by (1.3)]. The second part will be provided as a consequence of (1.7).

Similar results can be proved for the corresponding Schrödinger equation $i\dfrac{\partial v}{\partial t} + \dfrac{\partial^2 v}{\partial x^2} - V v = 0$ (*cf.* [3] for the special case $V(x) = e^{4x}$).
Choose

$$\left. \begin{array}{l} \chi \in C^\infty(\mathbb{R}), \qquad 0 \leqq \chi \leqq 1, \\[2mm] \begin{cases} \chi = 1 & \text{in }]-\infty, b], \\ \chi = 0 & \text{in } [b_1, +\infty[, \qquad b_1 > b \end{cases} \end{array} \right\} \tag{1.8}$$

It is easy to see that

$$\mathcal{H}_\mp = \left\{ \varphi \in \mathcal{H}_0 \, | \, \|(1-\chi)\, e^{-itB}\, \varphi\|_{\mathcal{H}_0} \xrightarrow[t \to \mp\infty]{} 0 \right\}$$

$$= \left\{ \varphi \in \mathcal{H}_0 \, | \, \varphi = \lim_{t \to \mp\infty} e^{itB}\, \chi\, e^{-itB}\, \varphi \text{ in } \mathcal{H}_0 \right\}.$$

Our main results are

THEOREM 3 (Existence of the wave operators). – *For each $\varphi \in \mathcal{H}_\mp$ there exists a unique φ_- (resp. φ_+) in $\mathcal{H}$ such that*

$$\lim_{t \to \mp\infty} \|\chi\, e^{-iBt}\, \varphi - e^{-iAt}\, \varphi_\mp\|_{\mathcal{H}} = 0 \tag{1.9}$$

We have $\varphi_\mp = M^\mp \varphi = \lim\limits_{t \to \mp\infty} e^{iAt} \chi\, e^{-iBt}\, \varphi$ in $\mathcal{H}$ and the maps $M^\mp : \mathcal{H}_\mp \to \mathcal{H}$ are bounded.

THEOREM 4 (Completeness of the wave operators). – *For each $\psi \in \mathcal{H}$ there exists a unique ψ_- (resp. ψ_+) in $\mathcal{H}_\mp$ such that*

$$\lim_{t \to \mp\infty} \|e^{-iAt}\, \psi - \chi\, e^{-iBt}\, \psi_\mp\|_{\mathcal{H}_0} = 0 \tag{1.10}$$

We have $\psi_\mp = \lim\limits_{t \to \mp\infty} e^{iBt}\, e^{-iAt}\, \psi$ in $\mathcal{H}_0$, and the maps $\psi \to \psi_\mp$ are bounded inverses of the maps $M^\mp$ defined in theorem 3. In addition, if $V(x) = e^{\lambda x}$, $\lambda > 0$, then the map $S = (M^+)^{-1} M^- : \mathcal{H}_- \to \mathcal{H}_+$ is an isometry.

For the proof of theorems 3, 4, we use the asymetric Birman-Kato theory, more precisely the theorem in [5], and (for theorem 4) the corollary of theorems 1 and 2. Note that the asymmetry of the problem forces us to use the Birman-Kato theory two times, once for existence and, in a different way, for completeness. Analogous results can be proved for the Schrödinger equation

$$i\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} - V v = 0$$

using decay estimates extending those of [3] proved for the special case $V(x) = e^{4x}$.

2. PROOF OF THEOREM 2

Assume $\varphi = (u_0,\, u_1) = \left(u(0),\, \dfrac{\partial u}{\partial t}(0) \right) \in (\mathcal{D}(\mathbb{R}))^2 \subset D(A^4)$. Since $V \in C^3$ it is easy to see that the corresponding solution of the Cauchy problem u belongs to $C^4(\mathbb{R}^2)$ and that for each t, $u(\cdot,\, t) \in C_0^4(\mathbb{R})$, $\dfrac{\partial u}{\partial t}(\cdot,\, t) \in C_0^3(\mathbb{R})$. We have, for $v = \dfrac{\partial u}{\partial x}$,

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v + Vv + \frac{dV}{dx}u = 0 \tag{2.1}$$

Multiply the equation (2.1) by $\dfrac{\partial \bar{v}}{\partial t}$, then integrate in $x \in \mathbb{R}$ and take the real part. This yields, with

$$E(v(t)) = \frac{1}{2} \int_{\mathbb{R}} \left[\left| \frac{\partial v}{\partial t} \right|^2 + \left| \frac{\partial v}{\partial x} \right|^2 + |v|^2 + V|v|^2 \right](x,\, t)\, dx,$$

$$\frac{\partial}{\partial t} E(v(t)) + \mathrm{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial \bar{v}}{\partial t}\, dx = 0,$$

$$\frac{\partial}{\partial t} \left[E(v(t)) - \frac{1}{2} \int_{\mathbb{R}} \frac{d^2 V}{dx^2} |u|^2\, dx \right] = \mathrm{Re} \int_{\mathbb{R}} \frac{dV}{dx} \frac{\partial u}{\partial t} \frac{\partial \bar{u}}{\partial x}\, dx.$$

Thus, for $t \geqq 0$, $\delta > 0$,

$$E(v(t)) \leqq E(v(0)) + c_2 E(u_0) + \delta \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |v|^2\, dx\, ds$$

$$+ c(\delta) \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} \left| \frac{\partial u}{\partial t} \right|^2\, dx\, ds \tag{2.2}$$

Now, multiply the equation (2.1) by $\dfrac{\partial \bar{v}}{\partial x}$, integrate in $x \in \mathbb{R}$ and take the real part to find

$$\operatorname{Re} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial x} \, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{dV}{dx} \left| \frac{\partial u}{\partial x} \right|^2 + \operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial^2 \bar{u}}{\partial x^2} \, dx = 0$$

and

$$\operatorname{Re} \int_{\mathbb{R}} \frac{dV}{dx} u \frac{\partial^2 \bar{u}}{\partial x^2} \, dx = - \int_{\mathbb{R}} \frac{dV}{dx} \left| \frac{\partial u}{\partial x} \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}} \frac{d^3 V}{dx^3} |u|^2 \, dx.$$

Hence,

$$\operatorname{Re} \frac{\partial}{\partial t} \int_{\mathbb{R}} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial x} \, dx = \frac{3}{2} \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{d^3 V}{dx^3} |u|^2 \, dx$$

and so, by (1.3),

$$\int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx \, ds \leqq c_3 \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} |u|^2 \, dx$$
$$+ c_4 \, E \left(v \left(0 \right) \right) + c_4 \, E \left(v \left(t \right) \right) \qquad (2.3)$$

Now, (2.2) and (2.3) yield, with a suitable $\delta > 0$,

$$E \left(v \left(t \right) \right) \leqq c_5 + c_6 \int_0^t \int_{\mathbb{R}} \frac{dV}{dx} \left[|u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] dx \, ds \qquad (2.4)$$

On the other hand, it is easy to see that the proof of proposition 3.1 in [2] yields

$$\int_0^{+\infty} \int_{\mathbb{R}} \frac{dV}{dx} \left(|u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \right) dx \, ds$$
$$\leqq 4 \left(E \left(u_0 \right) + E \left(u_1 \right) + E \left(\frac{\partial^2 u}{\partial t^2} \left(0 \right) \right) \right) \qquad (2.5)$$

Hence, using (2.3), (2.4) and (2.5) we obtain

$$\int_0^{\infty} \int_{\mathbb{R}} \frac{dV}{dx} |v|^2 \, dx \, ds < +\infty \qquad (2.6)$$

Taking the t derivative in equation (2.1), replacing v by $\dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial t}\right)$, and imitating the proof using (2.5), yields

$$\int_0^\infty \int_{\mathbb{R}} \frac{dV}{dx} \left|\frac{\partial v}{\partial t}\right|^2 dx\, ds < +\infty \qquad (2.7)$$

The two inequalities (2.6), (2.7) with a classical argument employed for example in the proof of the theorem 2 in [2], imply

$$\lim_{t\to+\infty} \left\|\left(\frac{dV}{dx}\right)^{\frac{1}{2}} \frac{\partial u}{\partial x}(\cdot,\, t)\right\|_{L^2(\mathbb{R})} = 0.$$

Time reversibility completes the proof of theorem 2. ■

PROOF OF PART (ii) OF THE COROLLARY. – For $\varphi \in (\mathcal{D}(\mathbb{R}))^2$ a dense subset of $\mathcal{H}$, we have

$$(\varphi,\, e^{-itA}\,\varphi)_{\mathcal{H}} = \int_{\mathbb{R}} e^{-it\lambda}\, d\,(E_\lambda\,\varphi,\, \varphi).$$

The second part of theorem 2 implies that the l.h.s. belongs to $L^2(\mathbb{R})$. Therefore, the Fourier transform of the bounded measure $d\,(E_\lambda\,\varphi,\,\varphi)$ belongs to $L^2(\mathbb{R})$. Thus the measure is absolutely continuous so $P_{\text{ac}}(\varphi) = \varphi$. ■

3. PROOF OF THEOREMS 3 AND 4

PROOF OF THEOREM 3. – Let $\mathcal{H}_1 = H^2(\mathbb{R}) \times H^1(\mathbb{R})$, $\mathcal{H}_2 = \mathcal{H} = H^1_V \times L^2$, $H_1 = B_{|H^3 \times H^2}$, $H_2 = A$ and $J : \mathcal{H}_1 \to \mathcal{H}_2$ the linear continuous operator defined by $J = \chi\, H_1^{-1}$, where χ is the function defined in (1.8). We have, for $\varphi \in H^3 \times H^2$,

$$e^{iH_2 t}\,\chi\, e^{-iH_1 t}\,\varphi = e^{iH_2 t}\,\chi\, H_1^{-1}\, e^{-iH_1 t}\, H_1\,\varphi$$
$$= e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi,$$

with $\psi = H_1\,\varphi \in \mathcal{H}_1$.

Furthermore, $J\,D\,(H_1) \subset D\,(H_2)$ and

$$H_2\, J - J H_1 = i \begin{pmatrix} 0 & 0 \\ \dfrac{d^2\chi}{dx^2} + 2\,\dfrac{d\chi}{dx}\,\dfrac{d}{dx} - V\,\chi & 0 \end{pmatrix} H_1^{-1}$$

[extended to $\mathcal{H}_1$, as an element of $\mathcal{B}\,(\mathcal{H}_1,\, \mathcal{H}_2)$].

On the other hand, we have

$$
H_1^{-1} = \begin{pmatrix} 0 & i\left(-\dfrac{d^2}{dx^2}+1\right)^{-1} \\ -i & 0 \end{pmatrix}.
$$

Thus, if $\psi = (u,\, v) \in \mathcal{H}_1$, we get

$$
(H_2\, J - JH_1)\, \psi = i\left[\begin{pmatrix} 0 \\ \left(\dfrac{d^2\,\chi}{dx^2} - V\,\chi\right)\left(\dfrac{d^2}{dx^2}+1\right)^{-1}(iv) \end{pmatrix} \right.
$$

$$
\left. + \begin{pmatrix} 0 \\ 2\dfrac{d\chi}{dx}\left(-\dfrac{d^2}{dx^2}+1\right)^{-1}\left(i\dfrac{dv}{dx}\right) \end{pmatrix} \right].
$$

Theorem 3.2 in [1] implies that the operator $H_2\, J - JH_1$ is a trace class operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. This and the theorem in [5] imply that the limits

$$
\lim_{t\to\pm\infty} e^{iAt}\, \chi\, e^{-iBt}\varphi = \lim_{t\to\pm\infty} e^{iH_2 t}\, J\, e^{-iH_1 t}\, \psi
$$

exist in $\mathcal{H}_2 = \mathcal{H}$. By density we can take $\varphi \in \mathcal{H}_0 = H^1 \times L^2$, and it follows immediatelly that, if $\varphi_\pm = \lim\limits_{t\to\pm\infty} e^{iAt}\, \chi\, e^{-iBt}\, \varphi$, then $\|\varphi_\pm\|_{\mathcal{H}} \leqq c\|\varphi\|_{\mathcal{H}_0}$.

To prove unicity, suppose that $\varphi \in \mathcal{H}_+$ and $0 = \varphi_+ = \lim\limits_{t\to+\infty} e^{iAt}\, \chi\, e^{-iBt}\, \varphi$ in $\mathcal{H}$. Then

$$
\|\chi\, e^{-itB}\, \varphi\|_{\mathcal{H}_0} \leqq \|\chi\, e^{-itB}\, \varphi\|_{\mathcal{H}} \xrightarrow[t\to+\infty]{} 0.
$$

Moreover,

$$
\|(1-\chi)\, e^{-itB}\, \varphi\|_{\mathcal{H}_0} \xrightarrow[t\to+\infty]{} 0.
$$

Hence,

$$
\|\varphi\|_{\mathcal{H}_0} = \|\, e^{-itB}\, \varphi\|_{\mathcal{H}_0} \xrightarrow[t\to+\infty]{} 0. \quad \blacksquare
$$

PROOF OF THEOREM 4. – Let $\mathcal{H}_1 = D(A^2)$, $\mathcal{H}_2 = \mathcal{H}_0 = H^1 \times L^2$, $H_1 = A_{|D(A^3)}$, $H_2 = B$ and $J : \mathcal{H}_1 \to \mathcal{H}_2$ defined by $J = \chi\, H_1^{-1}$, where

$$
H_1^{-1} = \begin{pmatrix} 0 & i\left(-\dfrac{d^2}{dx^2}+1+V\right)^{-1} \\ -i & 0 \end{pmatrix}.
$$

Since the projection (in $\mathcal{H}_1$) onto the subspace of absolute continuity for H_1 is $P_{\text{ac}|\mathcal{H}_1} = \text{Id}$, we have, for $\varphi \in D\left(A^3\right)$,

$$e^{iH_2 t} \chi\, e^{-iH_1 t}\varphi = e^{iH_2 t}\chi\, H_1^{-1}\, e^{-iH_1 t} H_1\,\varphi = e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi,$$

with $\psi = H_1\,\varphi \in \mathcal{H}_1$.

Furthermore, for $\psi_1 = (u,\,v) \in \mathcal{H}_1$ we obtain (by extending $H_2\,J - JH_1$ to $\mathcal{H}_1$):

$$(H_2\,J - JH_1)\,\psi_1 = i\left[\left(\begin{array}{c} 0 \\ \left(\dfrac{d^2\chi}{dx^2} - V\chi\right)\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1}(iv) \end{array}\right)\right.$$
$$\left. + \left(\begin{array}{c} 0 \\ 2\,\dfrac{d\chi}{dx}\,\dfrac{d}{dx}\left(\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1}(iv)\right) \end{array}\right)\right].$$

To estimate the last term write

$$\frac{d\chi}{dx}\,\frac{d}{dx}\left(\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}(v)\right)$$

$$= \frac{d\chi}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}\frac{dv}{dx}$$

$$- \frac{d\chi}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}\left(\frac{dV}{dx}\left(-\frac{d^2}{dx^2} + 1 + V\right)^{-1}v\right),$$

where we have used the fact that $v \in D\left(-\dfrac{d^2}{dx^2} + 1 + V\right)$ implies that $\dfrac{dV}{dx}\left(-\dfrac{d^2}{dx^2} + 1 + V\right)^{-1}v \in L^2$ and the map from $D\left(-\dfrac{d^2}{dx^2} + 1 + V\right)$ to L^2 so defined is continuous (the proof is exactly as for the case $V(x) = e^{4x}$ which is presented in section 3 of [3]). We conclude, by the theorem 3.2 in [1] that the operator $H_2\,J - JH_1$ is a trace class operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. Hence, by the theorem in [5] we conclude that the limits

$$\lim_{t \to \pm\infty} e^{iBt}\chi\, e^{-iAt}\varphi = \lim_{t \to \pm\infty} e^{iH_2 t}\, J\, e^{-iH_1 t}\,\psi$$

exist in $\mathcal{H}_2 = \mathcal{H}_0$. By density, the same limits exist for

$$\varphi \in \mathcal{H} = H_V^1 \times L^2.$$

For such φ, let

$$\varphi_+ = \lim_{t \to +\infty} e^{iBt} \chi e^{-iAt} \varphi \quad \text{in} \quad \mathcal{H}_0.$$

The next step is to prove that

$$\varphi_+ = \lim_{t \to +\infty} e^{iBt} e^{-iAt} \varphi \quad \text{in} \quad \mathcal{H}_0.$$

This implies that

$$\| e^{-iAt} \varphi - \chi e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \xrightarrow[t \to +\infty]{} 0$$

and so $\varphi_+ \in \mathcal{H}_+$ and $\|\varphi_+\|_{\mathcal{H}_0} = \lim_{t \to +\infty} \|e^{iBt} e^{-iAt} \varphi\|_{\mathcal{H}_0} \leqq \|\varphi\|_{\mathcal{H}}.$

Choose $\varphi_n \in (\mathcal{D}(\mathbb{R}))^2$, $n = 1, 2, \cdots$, such that $\varphi_n \xrightarrow[n \to \infty]{} \varphi$ in $\mathcal{H}$. We have

$$\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \leqq \|\chi e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0}$$
$$+ \|(1 - \chi) e^{-iAt} \varphi \|_{\mathcal{H}_0}$$
$$\leqq \|\chi e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0}$$
$$+ \|(1 - \chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} + c_2 \|\varphi - \varphi_n\|_{\mathcal{H}}.$$

Since, for each n, and by the corollary of theorems 1 and 2, part (i), $\|(1 - \chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} \xrightarrow[t \to +\infty]{} 0$, we conclude, by standard arguments, that $\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \xrightarrow[t \to +\infty]{} 0.$

Now, we derive

$$\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \leqq \|\chi (e^{-iAt} \varphi - e^{-iBt} \varphi_+) \|_{\mathcal{H}_0}$$
$$+ \|(1 - \chi) e^{-iAt} \varphi \|_{\mathcal{H}_0}$$
$$\leqq c_3 \|e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0}$$
$$+ \|(1 - \chi) e^{-iAt} \varphi_n \|_{\mathcal{H}_0} + c_2 \|\varphi - \varphi_n\|_{\mathcal{H}},$$

are therefore, by similar arguments,

$$\| e^{-iAt} \varphi - e^{-iBt} \varphi_+ \|_{\mathcal{H}_0} \xrightarrow[t \to +\infty]{} 0.$$

Finally, let $S = (M^+)^{-1} M^- : \mathcal{H}_- \to \mathcal{H}_+$ which is bijective and bicontinuous. Assume $V(x) = e^{\lambda x}$, $\lambda > 0$. Let $\psi = M^- \varphi_-$, $\varphi_+ = (M^+)^{-1} \psi$. Since $D(A^2)$ is dense in $\mathcal{H}$ we can assume $\psi \in D(A^2)$. Hence, by theorem 1 and since $\dfrac{dV}{dx} = \lambda V$, we get

$$\|\psi\|_{\mathcal{H}} = \lim_{t \to \infty} e^{-itA} \psi\|_{\mathcal{H}} = \lim_{t \to \infty} \|e^{iBt} \psi\|_{\mathcal{H}_0}.$$

Moreover, we have $\varphi_\pm = \lim\limits_{t\to+\infty} e^{itB} e^{-itA} \psi$ in $\mathcal{H}_0$ and so

$$\|\varphi_\pm\|_{\mathcal{H}_0} = \lim\limits_{t\to\pm\infty} \| e^{-itA} \psi \|_{\mathcal{H}_0} = \|\psi\|_{\mathcal{H}}. \quad \blacksquare$$

REFERENCES

[1] E. B. DAVIES and B. SIMON, Scattering Theory for Systems with Different Spatial Asymptotics on the Left and Right, *Commun. Math. Phys.,* Vol. **63**, 1978, pp. 277-301.
[2] J. P. DIAS and M. FIGUEIRA, On a class of solutions for the simplified Wheeler-DeWitt equation with a massless single scalar field, to appear in *Ricerche di Matematica.*
[3] J. P. DIAS and M. FIGUEIRA, On a singular limit for a class of solutions for the simplified Wheeler-DeWitt equation with a massless single scalar field, *Rendic. di Mat.,* Vol. **13**, 1993, pp. 529-542.
[4] G. W. GIBBONS and L. P. GRISHCHUK, What is a typical wave function for the Universe?, *Nucl. Phys. B,* Vol. **313**, 1989, pp. 736-748.
[5] D. B. PEARSON, A Generalization of the Birman Trace Theorem, *J. Funct. Anal.,* Vol. **28**, 1978, pp. 182-186.
[6] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, III: Scattering Theory,* Academic Press, New York, 1979.