ANNALES DE L'I. H. P., SECTION A

P. STEFANOV
G. VODEV

Distribution of resonances for the Neumann problem in linear elasticity outside a ball

Annales de l'I. H. P., section A, tome 60, nº 3 (1994), p. 303-321
<http://www.numdam.org/item?id=AIHPA_1994__60_3_303_0>

Ann. Inst. Henri Poincaré,

Vol. 60, n° 3, 1994, p. 303-321. *Physique théorique*

Distribution of resonances for the Neumann problem in linear elasticity outside a ball

by

P. STEFANOV and G. VODEV

Institute of Mathematics, Bulgarian Academy of Sciences,
1090 Sofia, Bulgaria

ABSTRACT. – For the Neumann problem in linear elasticity outside a ball in $\mathbf{R}^3$ existence of a sequence of resonances converging exponentially fast to the real axis is proven. It is also shown that there are no other resonances below some cubic parabola.

RÉSUMÉ. – Pour le problème de Neumann en élasticité linéaire dans l'extérieur d'une boule de $\mathbf{R}^3$ nous prouvons l'existence d'une suite de résonances qui convergent exponentiellement vite vers l'axe réel. Nous prouvons aussi qu'il n'y a pas d'autres résonances sous une certaine parabole cubique.

1. INTRODUCTION

The aim of this work is to study the location of the resonances for the elasticity operator $H\vec{u} = -\mu\Delta\vec{u} - (\lambda + \mu)\,\text{grad}\,\text{div}\,\vec{u}$ in the exterior in $\mathbf{R}^3$ of a ball with Neumann boundary conditions. It is well known that for the Neumann problem in linear elasticity there are Rayleigh surface waves moving along the boundary (*see* e.g. [R], [A], [CP], [G], [T]) and as a consequence, according to [IN], [K] the local energy for the corresponding elastic wave equation does not decay exponentially. This implies that we can expect resonances near the real axis. Let us recall that for the Laplace operator with Dirichlet or Neumann boundary condition in the exterior of a convex (or more generally a non-trapping) obstacle resonances are separated from the real axis by a strip (*see* [LP]). We show that there exists a sequence of resonances of H converging exponentially fast to the real axis. This sequence consists of zeros of a determinant found by Ikehata and Nakamura [IN] in their study of the local energy nondecay. We show that these zeros are in fact resonances and we prove the exponential decay of their imaginary parts. A similar sequence has been shown to exists in the case of $\mathbf{R}^2$ in [P]. Next, we show that there are no other resonances below some cubic parabola $\Im z \leqq C_1 |z|^{1/3} + C_2$, $C_1 > 0$. Recall that in the case of the Laplace operator in the exterior of a nontrapping obstacle with analytic boundary all the resonances are situated above such a parabola (*see* [BLR]).

2. NOTATIONS AND MAIN RESULT

Denote by $B = \{\vec{x} \in \mathbf{R}^3;\ |\vec{x}| \leqq r\}$ a ball in $\mathbf{R}^3$ and consider the elasticity operator $H\vec{u} = -\mu\Delta\vec{u} - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u}$, where $\vec{u} = {}^t(u_1, u_2, u_3)$ is a vector-valued function in $\Omega = \mathbf{R}^3 \setminus B$. Here λ and μ are the Lamé constants and we assume that

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \lambda \neq 0. \tag{1}$$

The Neumann boundary condition for the elasticity problem requires that the normal components of the stress tensor vanish

$$\sum_{i=1}^{3} n_i \sigma_{ij} = 0 \quad \text{on } \partial\Omega = S_r^2, \qquad j = 1, 2, 3, \tag{2}$$

where $\vec{n} = {}^t(n_1, n_2, n_3)$ is the outer normal to $\partial\Omega$ and

$$\sigma_{ij} = \lambda\,(\mathrm{div}\,\vec{u})\,\delta_{ij} + \mu\,(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$$

is the stress tensor. Since the boundary condition (2) is coercive (e.g. *see* [T], the proof there works under the conditions (1) as well), the operator H defined on the set $\{\vec{u} \in C_0^\infty(\bar{\Omega}; \mathbf{C}^3);\ \vec{u} \text{ satisfies (2)}\}$ has a self-adjoint non-negative extension in $L^2(\Omega; \mathbf{C}^3)$ which we will denote again by H. For any z with $\Im z < 0$ the resolvent $R(z) = (H - z^2)^{-1}$ is well defined as a holomorphic function of z. It is known (*see* e.g. [V], [IS]) that the cutoff resolvent $R_\chi(z) = \chi R(z) \chi$, χ being an arbitrary cutoff C_0^∞-function equal to 1 near the boundary, admits a meromorphic extension to the entire complex plane with possible poles in the upper half-plane $\{z \in \mathbf{C};\ \Im z > 0\}$.

DEFINITION 1. — The poles of $R_\chi(z)$ are called *resonances* of H.

We refer to [SZ] for a definition of the multiplicity of a resonance. Our main theorem studies the distribution of resonances near the real axis.

THEOREM 1. – *Assume* (1). *Then for the resonances of the operator* H *defined above we have*:

(*a*) *There exist two sequences of resonances* $\{z_n\}_{n=1}^{\infty}$, $\{-\bar{z}_n\}_{n=1}^{\infty}$, *of the form* $z_n = d_0 n + d_1 + O(n^{-1})$, *where* $d_0, d_1 \in \mathbf{R}$, *such that*

$$0 < \Im z_n \leqq C e^{-\gamma n} \text{ with some } C > 0, \qquad \gamma > 0.$$

Here $d_0 = C_R/r$, *where* C_R *is the Rayleigh speed given by* $C_R = \mu^{1/2} a_0$, a_0 *being the unique root of the Rayleigh function* [see (28)] *in* (0, 1). *Moreover, the multiplicity of* z_n *is* $2n+1$.

(*b*) *For any* $C_1 < \mu^{1/3} 3^{1/2} 2^{-4/3} \alpha_1 r^{-2/3}$ (*here* $-\alpha_1$ *is the first zero of the Airy function*) *there exists a constant* C_2, *such that there are no other resonances in the domain* $\Im z < C_1 |z|^{1/3} + C_2$.

Remark. – Theorem 1 indicates that the leading term of the resonances z_n is $C_R \omega_n$, where $\omega_n = (n(n+1))^{1/2}/r$ is the n-th eigenvalue of the square root of the Laplace-Beltrami operator on S_r^2 and moreover z_n and ω_n have the same multiplicities.

3. REDUCTION OF THE PROBLEM

First we are going to make some reduction of the problem which is more or less standard. Without loss of generality we may assume that $r=1$, *i.e.* that B is the unit ball. We will show that instead of studying the poles of $R_\chi(z)$ we can consider the poles of the following boundary value problem

$$\begin{cases} (-\Delta^* - z^2)\vec{w} = 0 & \text{in } \Omega, \\ N\vec{w} = \vec{g} & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Delta^* = \mu\Delta + (\lambda+\mu)\,\mathrm{grad\,div}$ and N is the operator given by the left-hand side of (2). Assume $\Im z < 0$ and $\vec{g} \in H^{1/2}(\partial\Omega)$. It is easy to see that (3) is uniquely solvable in $L^2(\Omega)$ (the solution in fact belongs to H^2). Indeed, it can be verified that (*see* e.g. [IN])

$$(N\vec{w})_i = \sum_{j,k,l=1}^{3} c_{ijkl} n_j (n_l \partial_{\vec{n}} w_k + \partial_{\vec{\tau}_l} w_k),$$

where $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the elastic tensor, $\vec{\tau}_l = \vec{e}_l - n_l \vec{n}$, $\vec{e}_l$, $l=1, 2, 3$ being the unit vectors $\vec{e}_1 = (1, 0, 0)$, etc. Hence the matrix $\sum_{j,l} c_{ijkl} n_j n_l$ is invertible. Let us look for a solution to the equation $N\vec{v} = \vec{g}$ (which is not unique). We can set $\vec{v} = 0$ on $\partial\Omega$ and then the equation

$N\vec{v}=\vec{g}$ determines uniquely the normal derivative of $\vec{v}$ as an element of $H^{1/2}(\partial\Omega)$. It is known [RS] that there exists a function $\vec{v}\in H^2(\Omega)$ with the so prescribed normal derivative and the support of $\vec{v}$ can be made arbitrary close to the boundary. Let us denote the operator $H^{1/2}(\partial\Omega)\ni\vec{g}\mapsto\vec{v}\in H^2(\Omega)$ by V (with some fixed choice of the extension map). So, $\vec{v}=V\vec{g}$. Now, let us look for a solution to (3) of the form $\vec{w}=\vec{u}+\vec{v}$. Since $N\vec{v}=\vec{g}$, $\vec{u}$ solves the problem

$$\begin{cases}(-\Delta^*-z^2)\vec{u}=(\Delta^*+z^2)\vec{v} & \text{in } \Omega,\\ N\vec{u}=0 & \text{on } \partial\Omega.\end{cases}$$

Note that the right-hand-side of the equation above is in $L^2(\Omega)$. The solution $\vec{u}$ is given by $\vec{u}=R(z)(\Delta^*+z^2)\vec{v}$ (recall that $\Im z<0$). Therefore, if we denote by $S(z)$ the operator solving the problem (3) for $\Im z<0$ (*i.e.* $\vec{w}=S(z)\vec{g}$), we get

$$S(z)=V+R(z)(\Delta^*+z^2)V.$$

Since the range of V above is in a set of functions with uniformly bounded supports close to the boundary, we can replace $R(z)$ above by $R(z)\chi$. Therefore we see that

$$\chi S(z)=V+R_\chi(z)(\Delta^*+z^2)V \tag{4}$$

with suitable χ. Hence $\chi S(z)$ admits a meromorphic extension in **C** and the poles of $\chi S(z)$ are among those of $R_\chi(z)$. In order to prove the inverse, let us consider the problem

$$\begin{cases}(-\Delta^*-z^2)\vec{u}=\vec{f} & \text{in } \Omega,\\ N\vec{u}=0 & \text{on } \partial\Omega,\end{cases} \tag{5}$$

where $\vec{f}\in L^2_{comp}(\Omega)$. Given a function $\vec{f}\in L^2(\Omega)$ denote by $T\vec{f}\in L^2(\mathbf{R}^3)$ the function $\vec{f}$ continued as zero in $\mathbf{R}^3\setminus\Omega=B$. Let us look for a solution of (5) of the form $\vec{u}=\vec{v}+\vec{w}$, where $\vec{v}=R_0(z)T\vec{f}$, $R_0(z)=(-\Delta^*-z^2)^{-1}$ being the free resolvent in the entire space $\mathbf{R}^3$. For $\vec{w}$ we get

$$\begin{cases}(-\Delta^*-z^2)\vec{w}=0 & \text{in } \Omega,\\ N\vec{w}=-NR_0(z)T\vec{f} & \text{on } \partial\Omega.\end{cases}$$

Therefore, $\vec{w}=-S(z)NR_0(z)T\vec{f}$, thus

$$R(z)=R_0(z)T-S(z)NR_0(z)T. \tag{6}$$

Here $N:H^2(\mathbf{R}^3)\to H^{1/2}(\partial\Omega)$. Since $\chi R_0(z)\chi$ is an entire function of z, we see from above that any resonance is a pole of $\chi S(z)$, too. Moreover, it can be seen that the multiplicities of the poles of $\chi S(z)$ and $R_\chi(z)$ coincide. Therefore, combining (4) and (6) we have proven the following.

PROPOSITION 1. — *The resonances of* H *coincide* (*with multiplicities*) *with the poles of the operator* $\chi S(z): H^{1/2}(\partial\Omega)\to L^2(\Omega)$.

In order to find the poles of $\chi S(z)$, *i.e.* of the problem (3), we will solve (3) explicitly in terms of special functions. Our analysis here is close to that of [IN] and we will keep some of the notations from there (*see* also [MF]).

Let (r, θ, φ) be the polar coordinates in $\mathbf{R}^3$. Denote by $P_n^m(z)$, $n, m \in \mathbf{Z}_+$, $m \leq n$ Ferrer's function defined by

$$P_n^m(z) = (2^n n!)^{-1} (1-z^2)^{m/2} \left(\frac{\partial}{\partial z}\right)^{m+n} (z^2-1)^n.$$

Set

$$Y_{emn}(\vec{\omega}) = \Re\, e^{im\varphi} P_n^m(\cos\theta), \qquad Y_{omn}(\vec{\omega}) = \Im\, e^{im\varphi} P_n^m(\cos\theta),$$

where $\vec{\omega} = \vec{x}/r = (\sin\theta\, \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$. Define the vector spherical harmonics $\vec{P}_{nm}^\sigma$, $\vec{B}_{nm}^\sigma$, $\vec{C}_{nm}^\sigma$ $(m, n \in \mathbf{Z}_+, \sigma = e, o)$ by

$$\vec{P}_{nm}^\sigma(\vec{\omega}) = \vec{\omega}\, Y_{\sigma mn}(\vec{\omega}), \qquad \vec{B}_{nm}^\sigma(\vec{\omega}) = \vec{\omega} \times \vec{C}_{nm}^\sigma(\vec{\omega}),$$
$$\vec{C}_{nm}^\sigma(\vec{\omega}) = (n(n+1))^{-1/2} \operatorname{rot}[\vec{x}\, Y_{\sigma mn}(\vec{\omega})]\,|_{r=1}.$$

It is known [MF] that $\vec{P}_{nm}^\sigma$, $\vec{B}_{nm}^\sigma$, $\vec{C}_{nm}^\sigma$ form an orthogonal base in $L^2(S^2)$. Denote by $h_n^{(1)}(z) = (\pi/2z)^{1/2} H_{n+1/2}^{(1)}(z)$ the spherical Hankel function of first order and denote by $\vec{L}_{\sigma mn}$, $\vec{M}_{\sigma mn}$, $\vec{N}_{\sigma mn}$ the functions

$$\left.\begin{aligned} \vec{L}_{\sigma mn}(\vec{x}, k) &= k^{-1} \operatorname{grad}(Y_{\sigma mn}(\vec{\omega})\, h_n^{(1)}(kr)), \\ \vec{M}_{\sigma mn}(\vec{x}, k) &= \operatorname{rot}(\vec{x}\, Y_{\sigma mn}(\vec{\omega})\, h_n^{(1)}(kr)), \\ \vec{N}_{\sigma mn}(\vec{x}, k) &= k^{-1} \operatorname{rot} \vec{M}_{\sigma mn}(\vec{x}, k). \end{aligned}\right\} \tag{7}$$

Then it can be easily seen that the following lemma holds (*see* e.g. [MF]).

LEMMA 1. – (i) $\vec{L}_{\sigma mn}(\vec{x}, k)$, $\vec{M}_{\sigma mn}(\vec{x}, k)$, $\vec{N}_{\sigma mn}(\vec{x}, k)$ *solve the Helmholtz equation*

$$(\Delta + k^2)\vec{U} = 0 \quad \text{in } \mathbf{R}^3.$$

(ii) *Let* $k = -(\lambda + 2\mu)^{-1/2} z$, $K = -\mu^{-1/2} z$. *Then*

$$\vec{L}_{\sigma mn}(\vec{x}, k), \qquad \vec{M}_{\sigma mn}(\vec{x}, K), \qquad \vec{N}_{\sigma mn}(\vec{x}, K)$$

solve the equation

$$(\Delta^* + z^2)\vec{U} = 0 \quad \text{in } \mathbf{R}^3.$$

In order to solve (3) let us note that any $\vec{g} \in H^{1/2}(S^2)$ can be written in the form

$$\vec{g} = \Sigma\, \vec{g}_{\sigma mn}, \qquad \vec{g}_{\sigma mn} = \alpha_{\sigma mn} \vec{P}_{nm}^\sigma + \beta_{\sigma mn} \vec{B}_{nm}^\sigma + \gamma_{\sigma mn} \vec{C}_{nm}^\sigma. \tag{8}$$

We are looking for a solution to the problem $(\Delta^* + z^2)\vec{v} = 0$ $(\Im z < 0)$, such that $N\vec{v} = \vec{g}$ on S^2. Let us search that solution in the form

$$\vec{v} = \Sigma\, \vec{v}_{\sigma mn}, \qquad \vec{v}_{\sigma mn} = a_{\sigma mn} \vec{L}_{\sigma mn}(\vec{x}, k) + b_{\sigma mn} \vec{M}_{\sigma mn}(\vec{x}, K) + c_{\sigma mn} \vec{N}_{\sigma mn}(\vec{x}, K) \tag{9}$$

where

$$k = -(\lambda + 2\mu)^{-1/2} z, \qquad K = -\mu^{-1/2} z$$

(*see* Lemma 1) and $a_{\sigma mn}$, $b_{\sigma mn}$, $c_{\sigma mn}$ will depend on z. Going back to the definition (7) of $\vec{L}_{\sigma mn}(\vec{x}, k)$, $\vec{M}_{\sigma mn}(\vec{x}, K)$, $\vec{N}_{\sigma mn}(\vec{x}, K)$ we see that

$$\vec{v}_{\sigma mn} = \operatorname{grad} \phi_{\sigma mn} + \operatorname{rot} \vec{A}_{\sigma mn}, \tag{10}$$

where

$$\phi_{\sigma mn} = k^{-1} a_{\sigma mn} Y_{\sigma mn}(\vec{\omega}) h_n^{(1)}(kr), \tag{11}$$

$$\vec{A}_{\sigma mn} = K^{-2} b_{\sigma mn} \operatorname{rot} \vec{M}_{\sigma mn}(\vec{x}, K) + K^{-1} c_{\sigma mn} \vec{M}_{\sigma mn}(\vec{x}, K) \tag{12}$$

with $\operatorname{div} \vec{A}_{\sigma mn} = 0$. Here we have made use of the fact that

$$\operatorname{rot}\operatorname{rot} \vec{M}_{\sigma mn} = \nabla(\nabla . \vec{M}_{\sigma mn}) - \Delta \vec{M}_{\sigma mn} = -\Delta \vec{M}_{\sigma mn} = K^2 \vec{M}_{\sigma mn}.$$

The following lemma has been proven in [Gr] (*see* also [IN]).

LEMMA 2 ([Gr]). – *Let*

$$\vec{v} = \operatorname{grad} \phi + \operatorname{rot} \vec{A}$$

with ϕ, $\vec{A}$ *satisfying the equations*

$$(\Delta + k^2)\phi = 0, \qquad (\Delta + K^2)\vec{A} = 0, \qquad \operatorname{div} \vec{A} = 0 \quad \text{in } \Omega.$$

Then the boundary condition $N\vec{v} = \vec{g}$ *on the boundary can be rewritten as follows* (*here* k, K *are the same as in Lemma* 1):

$$(K^2 - 2k^2) z^2 \phi \vec{\omega} - 2 z^2 \partial_r (\operatorname{grad} \phi) + K^2 z^2 \vec{A} \times \vec{\omega} - 2 z^2 \partial_r (\operatorname{rot} \vec{A}) = -K^2 \vec{g} \quad \text{on } S^2. \tag{13}$$

Since in (10) we have represented $\vec{v}_{\sigma mn}$ in the form required by Lemma 2, we can use (13) to compute the boundary conditions for $\vec{v}_{\sigma mn}$. Let us compute the left-hand-side of (13) with $\phi = \phi_{\sigma mn}$, $\vec{A} = \vec{A}_{\sigma mn}$. We will omit the calculations which are tedious but elementary and will present only the final results. For the first and the second term in (13) we get

$$(K^2 - 2k^2) z^2 \phi_{\sigma mn} \vec{\omega} = a_{\sigma mn} z^2 k^{-1} (K^2 - 2k^2) h_n^{(1)}(k) \vec{P}^{\sigma}_{nm}(\vec{\omega}),$$

$$-2 z^2 \partial_r (\operatorname{grad} \phi_{\sigma mn}) = 2 (n(n+1))^{1/2} a_{\sigma mn} z^2 k^{-1} \times (h_n^{(1)}(k) - k (h_n^{(1)})'(k)) \vec{B}^{\sigma}_{nm}(\vec{\omega}) - 2 a_{\sigma mn} z^2 k (h_n^{(1)})''(k) \vec{P}^{\sigma}_{nm}(\vec{\omega}).$$

The third and the fourth term in (13) for

$$\vec{A} = \vec{A}^1_{\sigma mn} := K^{-2} b_{\sigma mn} \operatorname{rot} \vec{M}_{\sigma mn}(\vec{x}, K)$$

[*see* (12)] become

$$K^2 z^2 \vec{A}^1_{\sigma mn} \times \vec{\omega} - 2 z^2 \partial_r (\operatorname{rot} \vec{A}^1_{\sigma mn}) = -b_{\sigma mn} z^2 (n(n+1))^{1/2} (K (h_n^{(1)})'(K) - h_n^{(1)}(K)) \vec{C}^{\sigma}_{nm}(\vec{\omega}).$$

Finally, for the third and the fourth term in (13) for

$$\vec{A} = \vec{A}^2_{\sigma mn} := K^{-1} c_{\sigma mn} \vec{M}_{\sigma mn}(\vec{x}, K),$$

we get

$$K^2 z^2 \vec{A}^2_{\sigma mn} \times \vec{\omega} - 2 z^2 \partial_r (\operatorname{rot} \vec{A}^2_{\sigma mn})$$
$$= c_{\sigma mn} z^2 K^{-1} [2 (n(n+1)) (h_n^{(1)}(K) - K (h_n^{(1)})'(K)) \vec{P}^\sigma_{nm}(\vec{\omega})$$
$$- (n(n+1))^{1/2} (K^2 h_n^{(1)}(K) + 2(-h_n^{(1)}(K)$$
$$+ K (h_n^{(1)})'(K) + K^2 (h_n^{(1)})''(K))) \vec{B}^\sigma_{nm}(\vec{\omega})].$$

Combining the equalities above we see that the equality $N \vec{v}_{\sigma mn} = \vec{g}_{\sigma mn}$ on S^2 [*see* (8), (9)] is equivalent to the linear system

$$T(n, z)^t (a_{\sigma mn}, c_{\sigma mn}, b_{\sigma mn}) = -K^{2\, t} (\alpha_{\sigma mn}, \beta_{\sigma mn}, \gamma_{\sigma mn}), \tag{14}$$

where $T(n, z)$ is a (3×3) matrix with elements

$$T_{11} = z^2 [k^{-1} (K^2 - 2k^2) h_n^{(1)}(k) - 2k (h_n^{(1)})''(k)],$$
$$T_{12} = 2 z^2 n(n+1) K^{-1} (h_n^{(1)}(K) - K (h_n^{(1)})'(K)),$$
$$T_{13} = T_{23} = T_{31} = T_{32} = 0,$$
$$T_{21} = 2 z^2 (n(n+1))^{1/2} k^{-1} (h_n^{(1)}(k) - k (h_n^{(1)})'(k)),$$
$$T_{22} = -z^2 (n(n+1))^{1/2} K^{-1} [K^2 h_n^{(1)}(K) + 2(-h_n^{(1)}(K)$$
$$+ K (h_n^{(1)})'(K) + K^2 (h_n^{(1)})''(K))],$$
$$T_{33} = z^2 (n(n+1))^{1/2} (h_n^{(1)}(K) - K (h_n^{(1)})'(K)).$$

Note that the submatrix $\{T_{ij}\}_{i,j=1}^2$ has been found by Ikehata and Nakamura [IN]. We will show that the resonances of H coincide with the union U of the zeros of $\det T(n, z)$, $n = 1, 2, \ldots$ (except $z = 0$). First, note that

$$\det T(n, z) = c_n z^2 (h_n^{(1)}(K) - K (h_n^{(1)})'(K)) \Delta_n(z),$$

where $\Delta_n(z) = \det \{T_{ij}\}_{i,j=1}^2 = T_{11} T_{22} - T_{12} T_{21}$. The coefficients $a_{\sigma mn}$, $b_{\sigma mn}$, $c_{\sigma mn}$ obtained by solving (14) are meromorphic functions of z with possible poles in U. The corresponding solution $\vec{v}_{\sigma mn}$ given by (9) with so computed $a_{\sigma mn}$, $b_{\sigma mn}$, $c_{\sigma mn}$ has the same property. Moreover, for $\Im z < 0$, $\vec{v}_{\sigma mn} \in L^2(\Omega)$ because of the exponential decay of $h_n^{(1)}(kr)$, $h_n^{(1)}(Kr)$. Therefore, $\vec{v}_{\sigma mn} = S(z) \vec{g}_{\sigma mn}$ Therefore, $S(z)\vec{g}$ is given by (9) with $a_{\sigma mn}$, $b_{\sigma mn}$, $c_{\sigma mn}$ obtained by solving (14) at least for any $\vec{g}$ with finite Fourier expansion (8). In the sequel we will overcome the question whether the series (9) is convergent with $a_{\sigma mn}$, $b_{\sigma mn}$, $c_{\sigma mn}$ computed as above. We will show that in fact it suffices to work with finite sums in (8),(9).

Assume that z_0 is a resonance. Then, according to Proposition 1 for some $d > 0$ and $\vec{g}_0 \in H^{1/2}(\partial\Omega)$ we have for $|z - z_0| \ll 1$

$$0 < c_0 \leqq \| \, |z - z_0|^d \chi S(z) \vec{g}_0 \|, \ \| \, |z - z_0|^d \chi S(z) \|_{\mathscr{L}(H^{1/2}(\partial\Omega), L^2(\Omega))} < c_1 < \infty.$$

It follows immediately that if $\vec{g} = \vec{g}_0 + \vec{g}_\varepsilon$, $\|\vec{g}_\varepsilon\|_{H^{1/2}} < \varepsilon$, then

$$\| \, |z - z_0|^d \chi S(z) \vec{g} \| \geqq c_0 - \varepsilon c_1 > 0$$

if $\varepsilon > 0$ is small enough. Therefore, $\chi S(z) \vec{g}$ has a pole at $z = z_0$ for all $\vec{g}$ sufficiently close to $\vec{g}_0$. Since the finite linear combinations of

$\vec{P}^{\sigma}_{nm}$, $\vec{B}^{\sigma}_{nm}$, $\vec{C}^{\sigma}_{nm}$ form a dense set in $H^{1/2}(\partial\Omega)$, we see that z_0 is a pole of any solution $\vec{v}=S(z)\vec{g}$ corresponding to some $\vec{g}$ with finite Fourier expansion. Since for such $\vec{v}$ we have already seen that they are holomorphic outside U, we conclude that $z_0 \in U$.

Conversely, let $0 \neq z_0 \in U$. This means that $z=z_0$ is a zero of $\det T(n, z)=c_n z^2 (h_n^{(1)}(K)-K(h_n^{(1)})'(K))\Delta_n(z)$ for some n. Let us first assume that z_0 is a zero of $h_n^{(1)}(K)-K(h_n^{(1)})'(K)$ (recall that $K=-\mu^{-1/2}z$). Then from (14) we see that

$$b_{\sigma mn}=-\frac{\gamma_{\sigma mn}}{\mu(n(n+1))^{1/2}(h_n^{(1)}(K)-K(h_n^{(1)})'(K))},$$

therefore $b_{\sigma mn}(z)$ has a pole at $z=z_0$. It is easy to show that this implies the existence of a pole of $\chi S(z)\vec{C}^{\sigma}_{nm}=\chi(\vec{x})\, b_{\sigma mn}(z)\vec{M}_{\sigma mn}(\vec{x}, K)$ at $z=z_0$ as well. Therefore, z_0 is a resonance. Secondly, let z_0 be not a zero of $h_n^{(1)}(K)-K(h_n^{(1)})'(K)$, but $z_0 \in U$. Then $\Delta_n(z_0)=0$. Then by solving (14) with $\alpha_{\sigma mn}=0$, $\beta_{\sigma mn}=1$, $\gamma_{\sigma mn}=0$, for $c_{\sigma mn}$ we get

$$c_{\sigma mn}(z)=C_n z^2 K(h_n^{(1)}(K)-K(h_n^{(1)})'(K))/\Delta_n(z).$$

Since z_0 is not a zero of the denominator, we see that $c_{\sigma mn}(z)$ has a pole at $z=z_0$. This leads easily to the conclusion that z_0 is a pole of $\chi S(z)\vec{B}^{\sigma}_{nm}$, too, *i.e.* z_0 is a resonance. As for $z=0$ we note that in [IS] it is proven that $z=0$ is not a resonance. Therefore, we have proven the following.

PROPOSITION 2. – *Resonances of* H *coincide with the union of the zeros of the functions* $\Delta_n(z)=T_{11}T_{22}-T_{12}T_{21}$ *and* $h_n^{(1)}(K)-K(h_n^{(1)})'(K)$, $n=1, 2, \ldots$ *Moreover, if* $\Delta_{n_0}(z_0)=0$, *then the resonance* z_0 *has multiplicity* $2n_0+1$.

4. THE ZEROS OF Δ_n AND $h_n^{(1)}-K(h_n^{(1)})'$

Let us proceed with the study of the zeros of the functions given in Proposition 2. We are interested only in the zeros lying in a region below some cubic parabola of the type $\Im z < C_1|z|^{1/3}+C_2$, $C_1>0$, It is known that $h_n^{(1)}(-z)$, $n=1, 2, \ldots$ do not vanish in such a region provided that C_1, C_2 are suitable chosen, so $h_n^{(1)}(k)$, $h_n^{(1)}(K)$ do not have zeros z there. We will use this observation latter. The same is true for the function $h_n^{(1)}(K)-K(h_n^{(1)})'(K)$ as proven by Tokita [To]. In fact, the zeros of these functions are connected with the resonances of the exterior problem for the Helmholtz equation in Ω with Dirichlet and Robin boundary conditions, respectively. This leads to another proof of the absence of zeros of these functions below some cubic parabola. Therefore, what we have to study is the distribution of the zeros of $\Delta_n(z)$.

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of distinct zeros of the family $\Delta_n(z)$, $n=1, 2, \ldots$, lying below some parabola $\Im z < C_1 |z|^{1/3} + C_2$, $C_1 > 0$. Since the domain $\Im z \leqq 0$ is free of poles, for some sequence n_j we have

$$\Delta_{n_j}(z_j)=0, \qquad j=1, 2, \ldots, \tag{15}$$

$$0 < \Im z_j < C_1 |z_j|^{1/3} + C_2. \tag{16}$$

Since any Δ_n has a finite number of zeros, we have $n_j \to \infty$. Our strategy below is to use the results of Olver [O1], [O2], [O3] on the asymptotics of $H_\nu^{(1)}(\nu a)$ and its derivatives as $\nu \to \infty$, which lead to similar asymptotics for $h_n^{(1)}((n+1/2)a)$. For a similar approach, see also [To], [IN]. These asymptotics are different when a belongs to different sets in $\mathbf{C}$ and that is why we will consider several cases below. So, we are interested in the quantity $z_j/(n_j+1/2)$.

Case A. – Let z_j/n_j be unbounded. By choosing a subsequence we may assume that $|z_j/n_j| \to \infty$. The same holds for the corresponding sequences k_j, K_j. Let us set for convenience

$$\nu = n + \frac{1}{2}. \tag{17}$$

As noted above, $h_n^{(1)}(K)$ and $h_n^{(1)}(k)$ do not vanish in the region (16) if C_1, C_2 are properly chosen, so we may divide by $h_n^{(1)}(K)$, $h_n^{(1)}(k)$ freely. Note that $\Delta_n(z)$ can be rewritten in the form:

$$\Delta_n(z) = -z^4\left(\nu^2 - \frac{1}{4}\right)^{1/2} k\,K\,h_n^{(1)}(k)\,h_n^{(1)}(K)\left\{\left[\left(\frac{K}{k}\right)^2 - 2 - 2\frac{(h_n^{(1)})''(k)}{h_n^{(1)}(k)}\right]\right.$$
$$\times\left[1 + 2\left(-\frac{1}{K^2} + \frac{(h_n^{(1)})'(K)}{K\,h_n^{(1)}(K)} + \frac{(h_n^{(1)})''(K)}{h_n^{(1)}(K)}\right)\right]$$
$$\left. + 4\frac{\nu^2 - 1/4}{K k}\left[\frac{(h_n^{(1)})'(K)}{h_n^{(1)}(K)} - \frac{1}{K}\right]\left[\frac{(h_n^{(1)})'(k)}{h_n^{(1)}(k)} - \frac{1}{k}\right]\right\}.$$

We are going to study the limit $\Delta_n(\nu z)$ as $\nu \to \infty$ (recall (17)) by using the asymptotics of $H_\nu^{(1)}(\nu z)$ found by Olver [O1], [O2], [O3]. Let us set

$$a = \frac{K}{\nu}, \qquad b = \frac{k}{\nu}. \tag{18}$$

Then we see that the equation $\Delta_n(z)=0$ in the region we are interested in is equivalent to

$$0 = \tilde{\Delta}_n(z) := \left[\frac{a^2}{b^2} - 2 - 2\frac{(h_n^{(1)})''(k)}{h_n^{(1)}(k)}\right]$$
$$\times\left[1 + \frac{2}{K}\left(-\frac{1}{K} + \frac{(h_n^{(1)})'(K)}{h_n^{(1)}(K)}\right) + 2\frac{(h_n^{(1)})''(K)}{h_n^{(1)}(K)}\right]$$
$$+ 4\frac{\nu^2 - 1/4}{a b \nu^2}\left[\frac{(h_n^{(1)})'(K)}{h_n^{(1)}(K)} - \frac{1}{K}\right]\left[\frac{(h_n^{(1)})'(k)}{h_n^{(1)}(k)} - \frac{1}{k}\right]. \tag{19}$$

We get from [O3, pp. 374-380] that for $|a|>1$, $|\arg a|<\pi$ we have

$$\frac{(h_n^{(1)})'(\nu a)}{h_n^{(1)}(\nu a)}=\frac{i\sqrt{a^2-1}}{a}+O(\nu^{-1}), \tag{20}$$

$$\frac{(h_n^{(1)})''(\nu a)}{h_n^{(1)}(\nu a)}=-\frac{a^2-1}{a^2}+O(\nu^{-1}), \tag{21}$$

and the remainder is uniform with respect to a in any region $|a|\geqq 1+\varepsilon$, $|\arg a|<\pi-\varepsilon$. In fact, (20) is fulfilled in a larger region (a complement of an eye-shaped domain K, see [O3, p. 380]) and (21) follows from (20) and from the differential equation satisfied by $h_n^{(1)}(\nu a)$, namely

$$\frac{(h_n^{(1)})''}{h_n^{(1)}}(\nu a)=\frac{1-a^2}{a^2}-\frac{2}{\nu a}\frac{(h_n^{(1)})'}{h_n^{(1)}}(\nu a)-\frac{1}{4\nu^2 a^2}. \tag{22}$$

Let us apply these results to our case. Without loss of generality we may assume that $\Re z_j\leqq 0$. Set $a_j=\mathrm{K}_j/\nu_j(=-\mu^{-1/2}z_j/(n_j+1/2))$, $b_j=k_j/\nu_j(=-(\lambda+2\mu)^{-1/2}z_j/(n_j+1/2))$ [*see* (18)]. Since $|a_j|\to\infty$, $|b_j|\to\infty$, $|\arg a_j|\leqq\pi/2$, $|\arg b_j|\leqq\pi/2$, for any $\varepsilon>0$ we get

$$\frac{(h_{n_j}^{(1)})'}{h_{n_j}^{(1)}}(\nu_j a_j)\to i,\qquad \frac{(h_{n_j}^{(1)})''}{h_{n_j}^{(1)}}(\nu_j a_j)\to -1,\qquad \text{as}\quad j\to\infty$$

and the same remains true if a_j is replaced by b_j. Passing to the limit $z=z_j$, $j\to\infty$ in (19), we get $-(\lambda+2\mu)/\mu=0$, which is a contradiction to (1).

Case B. – Let z_j/n_j be bounded. By choosing a subsequence we can assume that

$$a_j=\frac{\mathrm{K}_j}{\nu_j}=-\frac{z_j}{\mu^{1/2}\nu_j}\to\tilde a,\qquad b_j=\frac{k_j}{\nu_j}=-\frac{z_j}{(\lambda+2\mu)^{1/2}\nu_j}\to\tilde b, \tag{23}$$

with some $\tilde a\in\mathbf{C}$, $\tilde b\in\mathbf{C}$. Obviously, $b_j/a_j=\tilde b/\tilde a=\sqrt{\mu/(\lambda+2\mu)}$. By dividing (16) by ν_j and using (23) we see that

$$0\leqq-\Im a_j\leqq\frac{\mathrm{C}_3}{\nu_j^{2/3}}, \tag{24}$$

where $\mathrm{C}_3>\mathrm{C}_1/\mu^{1/3}$. Therefore $\Im\tilde a=0$, *i.e.* $\tilde a\in\mathbf{R}$, $\tilde b\in\mathbf{R}$. Without loss of generality we may assume that $\tilde a\geqq 0$. We will consider several possibilities for $\tilde a$ (and $\tilde b$) because of the fact that asymptotics of the type (20), (21) look different when a is close to $a=1$ and when a belongs to a neighborhood in $\mathbf{C}$ of the intervals $(0,1-\varepsilon)$ and $(1+\varepsilon,\infty)$, respectively.

Case B1. – Let $0<\tilde a<1$, so $0<\tilde b<1$ as well. In this case our analysis is close to that of Ikehata and Nakamura [IN]. Consider (19). According to [O1], [O2], [O3] we have

$$\frac{(h_n^{(1)})'(\nu a)}{h_n^{(1)}(\nu a)} = -\frac{\sqrt{1-a^2}}{a} + O(\nu^{-1}), \tag{25}$$

$$\frac{(h_n^{(1)})''(\nu a)}{h_n^{(1)}(\nu a)} = \frac{1-a^2}{a^2} + O(\nu^{-1}) \tag{26}$$

and the remainder is uniform in a provided that a belongs to some complex neighborhood of $\tilde{a}$. In fact, the asymptotics (25), (26) hold for a belonging to an eye-shaped domain K containing the open interval $(-1, 1)$ (*see* [O3, pp. 374-380]). Since $0<\tilde{a}<1$, for j sufficiently large a_j belong to the domain K and so do $b_j = k_j/\nu_j$ because $0 < b_j/a_j = \sqrt{\mu/(\lambda+2\mu)} < 1$. Therefore, we can apply (25), (26) to (19) to get

$$0 = -\frac{\mu a_j^4}{\lambda+2\mu} \tilde{\Delta}_{n_j}(z_j) = \mathrm{R}(a_j) + O(j^{-1}), \tag{27}$$

where $\mathrm{R}(a)$ is the Rayleigh function

$$\mathrm{R}(a) = (2-a^2)^2 - 4(1-a^2)^{1/2}\left(1 - \frac{\mu}{\lambda+2\mu} a^2\right)^{1/2}. \tag{28}$$

It is known and can be easily seen that in the interval $0<a<1$ there is exactly one simple zero $a=a_0$ of $\mathrm{R}(a)$. Therefore, if such a sequence of poles exists then we must have $\tilde{a}=a_0$. It is not hard to prove that such a sequence does exist (*see* also [IN], where this sequence appears in the proof of a non-decaying property of the local energy for the elastic wave equation). Indeed, let us look for zeros of $\tilde{\Delta}_n$ of the type $z_n = \nu\mu^{1/2} a_n$. Set $s=\nu^{-1}$. Then we can write the equation $\tilde{\Delta}_n(\nu\mu^{1/2} a) = 0$ in the form $f(a, s) = 0$, where $f(a, s) = \mathrm{R}(a) + O(s)$ uniformly in a in a neighborhood of a_0 because of the limit

$$-\frac{\mu}{\lambda+2\mu} a^4 \tilde{\Delta}_n(\nu\mu^{1/2} a) = \mathrm{R}(a) + O(\nu^{-1}) \tag{29}$$

(compare with (27)). Applying the implicit function theorem to $f(a, s)$ in a neighborhood of $a=a_0$, we see that $f(a, s)=0$ has a root of the type $a(s) = a_0 + O(s)$ which is unique because (29) can be differentiated termwise. We can use the asymptotics in [O3] in order to calculate the next terms in (25), (26). The calculations show that they are real. Thus it can be proven that the zero $a(s)$ admits a full expansion $a(s) = a_0 + s a^{(1)} + s^2 a^{(2)} + \ldots$ modulo $O(s^\infty)$ with real $a^{(k)}$. This shows that there exists a sequence of resonances with asymptotics $z_n = n d_0 + d_1 + n^{-1} d_2 + \ldots$, $n \geqq n_0$ with $d_0 = \mu^{1/2} a_0$, d_k real. Without loss of generality we can assume $n = 1, 2, \ldots$ In the next section we will prove that $\Im z_n$ decays exponentially in n.

Case B2. – Let $\tilde{a}=0$. In this case we need the behaviour of $\tilde{\Delta}_n(\nu\mu^{1/2}a)$ near $a=0$. Let us note that the asymptotics (25), (26) remain true near $a=0$ with little modifications, namely

$$\frac{(h_n^{(1)})'(\nu a)}{h_n^{(1)}(\nu a)}=-\frac{\sqrt{1-a^2}}{a}+\frac{1}{a}O(\nu^{-1}), \tag{30}$$

$$\frac{(h_n^{(1)})''(\nu a)}{h_n^{(1)}(\nu a)}=\frac{1-a^2}{a^2}+\frac{1}{a^2}O(\nu^{-1}), \tag{31}$$

where the remainders are uniform in a in a complex neighborhood of $a=0$. Therefore, we get as above that (29) holds uniformly if a is as above. Since $R(0)=0$, the expansion (29) is not sufficient to get a contradiction. Let us compute the second term in (29). Using Olver's expansions [O1], we see that

$$\frac{(h_n^{(1)})'(\nu a)}{h_n^{(1)}(\nu a)}-\frac{1}{\nu a}=-\frac{\sqrt{1-a^2}}{a}+\frac{4a^2-3}{2a(1-a^2)}\nu^{-1}+\frac{1}{a}O(\nu^{-2}),$$

$$\frac{(h_n^{(1)})''(\nu a)}{h_n^{(1)}(\nu a)}=\frac{1-a^2}{a^2}+\frac{2\sqrt{1-a^2}}{a^2}\nu^{-1}+\frac{1}{a^2}O(\nu^{-2}),$$

uniformly near $a=0$, which is a refinement of (30), (31). Using these formulae one can see that in (29) we actually have

$$-\frac{\mu}{\lambda+2\mu}a^4\tilde{\Delta}_n(\nu\mu^{1/2}a)=R(a)+Q(a)\nu^{-1}+O(\nu^{-2}),$$

where $R(a)=a^2R_1(a)$, $Q(a)=a^2Q_1(a)$, $R_1(0)=-2+2\mu/(\lambda+2\mu)<0$ and $R_1(a)$, $Q_1(a)$ are analytic near $a=0$. Therefore, it follows from the assumption $\tilde{\Delta}_{n_j}(\nu_j\mu^{1/2}a_j)=0$ that

$$R_1(a_j)+\nu_j^{-1}Q_1(a_j)=a_j^{-2}O(\nu_j^{-2}).$$

Since $\nu_j a_j=K_j$ and $|K_j|\to\infty$, we see that the right hand side above vanishes as $j\to\infty$ and since $a_j\to 0$, we get $R_1(0)=0$, which is a contradiction. Therefore, $\tilde{a}$ cannot be zero.

Case B3. – Let $\tilde{a}=1$. According to [O1], [O2], [O3] the asymptotics (20), (25) are not valid in a neighborhood of $a=1$. In this case they have a more complicated form including the Airy function. Let $\mathrm{Ai}(z)$ be the Airy function and set $\mathrm{Ai}_-(z)=\mathrm{Ai}(ze^{2\pi i/3})$. Recall that $\mathrm{Ai}_-(z)$ has zeros of the type $z_j=\alpha_j e^{\pi i/3}$, where $0<\alpha_1<\alpha_2<\dots$ Following [O1], [O2], [O3], introduce the function $\zeta=\zeta(a)$ by the equality

$$\frac{2}{3}\zeta^{3/2}=\ln\frac{1+(1-a^2)^{1/2}}{a}-(1-a^2)^{1/2},$$

where the branches take their principal values when $z\in(0,1)$, $\zeta\in(0,\infty)$ and are continuous elsewhere. We refer to [O3, pp. 420-421] for more details. It appears that ζ is a holomorphic function of a within $|\arg a| < \pi$ so in particular this is true near $a=1$. According to [O2, p. 338, p. 342] (*see* also [O3]), the following asymptotic formulae hold for $H_\nu^{(1)}(\nu a)$, $(H_\nu^{(1)})'(\nu a)$ as $\nu \to \infty$:

$$H_\nu^{(1)}(\nu a) \sim \frac{2e^{-\pi i/3}}{\nu^{1/3}} \phi(\zeta) \left\{ \frac{\mathrm{Ai}_-(\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{Ai}'_-(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^{2s}} \right\}, \quad (32)$$

$$(H_\nu^{(1)})'(\nu a) \sim -\frac{2e^{-\pi i/3}}{\nu^{1/3}} \psi(\zeta) \left\{ \frac{\mathrm{Ai}_-(\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{s=0}^{\infty} \frac{C_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{Ai}'_-(\nu^{2/3}\zeta)}{\nu^{2/3}} \sum_{s=0}^{\infty} \frac{D_s(\zeta)}{\nu^{2s}} \right\}, \quad (33)$$

where

$$\phi(\zeta) = \left(\frac{4\zeta}{1-a^2} \right)^{1/4}, \qquad \psi(\zeta) = \frac{2}{a\phi(\zeta)},$$

and $A_s(\zeta)$, $B_s(\zeta)$, $C_s(\zeta)$, $D_s(\zeta)$ are recursively defined functions, $A_0 = D_0 = 1$. In order to get a limit similar to (20), (25) in a neighborhood of $a=1$, let us note that from (32), (33) it follows that

$$\frac{(H_\nu^{(1)})'(\nu a)}{H_\nu^{(1)}(\nu a)} = -\frac{2}{a} \left(\frac{1-a^2}{4\zeta} \right)^{1/2} \frac{\mathrm{Ai}'_-(\nu^{2/3}\zeta)}{\nu^{1/3} \mathrm{Ai}_-(\nu^{2/3}\zeta)} + \text{lower order terms.} \quad (34)$$

In the case we study we have $a_j \to 1$ and $-\Im a_j \leqq C_3/\nu_j^{2/3}$ by (24). We will show that under these conditions

$$\frac{\mathrm{Ai}'_-(\nu_j^{2/3}\zeta_j)}{\nu_j^{1/3} \mathrm{Ai}_-(\nu_j^{2/3}\zeta_j)} \to 0. \quad (35)$$

Let us note that because of the analyticity of ζ near $a=1$ we have for large j

$$0 < C' \leqq \left| \frac{1-a_j}{\zeta_j} \right| \leqq C''. \quad (36)$$

Moreover, we have $\zeta = 2^{1/3}(1-a) + O(|1-a|^2)$, therefore by (24)

$$\Im \nu_j^{2/3} \zeta_j \leqq 2^{1/3} C_3 + C''' \nu_j^{2/3} |1-a_j|^2. \quad (37)$$

Assume first that $\nu_j^{2/3}|1-a_j| < C < \infty$. Then

$$\Im \nu_j^{2/3} \zeta_j \leqq 2^{1/3} C_3 + CC''' |1-a_j|,$$

therefore we can arrange the inequality

$$\Im \nu_j^{2/3} \zeta_j \leqq \Im \alpha_1 e^{\pi i/3} - \varepsilon = \alpha_1 \frac{\sqrt{3}}{2} - \varepsilon, \quad (38)$$

(recall that $\alpha_1 e^{\pi i/3}$ is the first zero of $\mathrm{Ai}_-(z)$) provided that C_3 and therefore C_1 and C_2 [*see* (16)] are suitably chosen. To this end it suffices to pick $C_3 < 2^{-4/3} 3^{1/2} \alpha_1$, *i.e.* [*see* (24)]

$$C_1 < \mu^{1/3} 2^{-4/3} 3^{1/2} \alpha_1.$$

By (36) inequality (38) means that the sequence $\nu_j^{2/3} \zeta_j$ forms a bounded set away from the zeros of $\mathrm{Ai}_-(z)$. Therefore $\mathrm{Ai}'_-(\nu_j^{2/3}\zeta_j)/\mathrm{Ai}_-(\nu_j^{2/3}\zeta_j)$ is bounded and (35) trivially holds in this case.

Assume next that $\nu_j^{2/3} |1-a_j|$ is unbounded. Without loss of generality we may assume that $\nu_j^{2/3} |1-a_j| \to \infty$. Let $\theta_j = \arg \zeta_j$. Then by (36) and (37)

$$\sin\theta_j = \frac{\Im \zeta_j}{|\zeta_j|} \leqq \frac{2^{1/3} C_3}{|\zeta_j| \nu_j^{2/3}} + C''' \frac{|1-a_j|^2}{|\zeta_j|} \leqq \frac{2^{1/3} C_3}{|\zeta_j| \nu_j^{2/3}} + C'' C''' |1-a_j| \to 0$$

because of the assumptions $\nu_j^{2/3} |1-a_j| \to \infty$, $a_j \to 1$. Therefore $|\arg \nu_j^{2/3} \zeta_j - \pi/3| > \pi/6$ for large j. Thus we can apply the well-known asymptotic formula (*see* e.g. [O3])

$$\frac{\mathrm{Ai}'(\eta)}{\mathrm{Ai}(\eta)} \sim -\eta^{1/2} \quad \text{as } |\eta| \to \infty,\ |\arg \eta| < \pi - \varepsilon$$

that in our case gives

$$\frac{\mathrm{Ai}'_-(\nu_j^{2/3}\zeta_j)}{\nu_j^{1/3} \mathrm{Ai}_-(\nu_j^{2/3}\zeta_j)} \sim \zeta_j^{1/2}$$

with $\zeta^{1/2}$ taking its principal branch as $\zeta > 0$. Since $\zeta_j \to 0$, we get (35) in this case as well.

Let us observe that the term $((1-a^2)/(4\zeta))^{1/2}$ in (34) remains bounded as $a = a_j \to 1$. Therefore by (35) we get that the first term in the expansion of $(H_{\nu_j}^{(1)})'(\nu_j a_j)/H_{\nu_j}^{(1)}(\nu_j a_j)$ vanishes. Moreover, if we go back to (32), (33), we see that (35) justifies the choice of the first term in (34). Thus

$$\frac{(H_{\nu_j}^{(1)})'(\nu_j a_j)}{H_{\nu_j}^{(1)}(\nu_j a_j)} \to 0 \quad \text{as } j \to \infty.$$

Since

$$\frac{(h_n^{(1)})'}{h_n^{(1)}}(\nu a) = \frac{(H_\nu^{(1)})'}{H_\nu^{(1)}}(\nu a) - \frac{1}{2\nu a},$$

we get

$$\frac{(h_{n_j}^{(1)})'(\nu_j a_j)}{h_{n_j}^{(1)}(\nu_j a_j)} \to 0 \quad \text{as } j \to \infty \tag{39}$$

under the assumptions $a_j \to 1$ and (16) with suitably chosen C_1, C_2. We note that the result is the same as if we set formally $a = 1$ in (20) or (25).

The reason for this is the condition (16). By (22) we also get

$$\frac{(h_{n_j}^{(1)})''(\nu_j a_j)}{h_{n_j}^{(1)}(\nu_j a_j)} \to 0 \quad \text{as } j \to \infty. \tag{40}$$

Since $\tilde{b} < \tilde{a} = 1$, for $(h_n^{(1)})'(k)/h_n^{(1)}(k)$, $(h_n^{(1)})''(k)/h_n^{(1)}(k)$ we have (25), (26) with a replaced by b. Taking the limit $j \to \infty$ in $\tilde{\Delta}_{n_j}(z_j) = 0$ [*see* (19)], we get similarly to (27)

$$0 = -\frac{\mu}{\lambda + 2\mu} a_j^4 \tilde{\Delta}_{n_j}(z_j) \to 1 \quad \text{as } j \to \infty,$$

which leads to contradiction.

Case B4. – Let $\tilde{a} > 1$. Then we can apply (20), (21) to handle the terms $(h_n^{(1)})'(\mathrm{K})/h_n^{(1)}(\mathrm{K})$, $(h_n^{(1)})''(\mathrm{K})/h_n^{(1)}(\mathrm{K})$ in (19). The asymptotics of $(h_n^{(1)})'(k)/h_n^{(1)}(k)$, $(h_n^{(1)})''(k)/h_n^{(1)}(k)$ depends on $\tilde{b} = \sqrt{\mu/(\lambda + 2\mu)}\,\tilde{a}$. If $\tilde{b} > 1$, then we can use (20), (21) with b instead of a and from (19) we get

$$(2 - \tilde{a}^2)^2 + 4(\tilde{a}^2 - 1)^{1/2}(\tilde{b}^2 - 1)^{1/2} = 0,$$

which is impossible when $\tilde{b} > 1$. For $\tilde{b} < 1$ we can use (25), (26) to estimate $(h_n^{(1)})'(k)/h_n^{(1)}(k)$, $(h_n^{(1)})''(k)/h_n^{(1)}(k)$. We get

$$(2 - \tilde{a}^2)^2 - 4i(\tilde{a}^2 - 1)^{1/2}(1 - \tilde{b}^2)^{1/2} = 0,$$

which cannot hold because $\tilde{a} \neq 1$, $\tilde{b} \neq 1$. It remains to consider the case $\tilde{b} = 1$. Then, according to (39), (40) we have

$$\frac{(h_{n_j}^{(1)})'(\nu_j b_j)}{h_{n_j}^{(1)}(\nu_j b_j)} \to 0 \qquad \frac{(h_{n_j}^{(1)})''(\nu_j b_j)}{h_{n_j}^{(1)}(\nu_j b_j)} \to 0 \quad \text{as } j \to \infty.$$

Taking the limit in (19) we get $(2 - \tilde{a}^2)^2 = 0$, therefore $\tilde{a}^2 = 2$. The equalities $\tilde{a}^2 = 2$, $\tilde{b}^2 = 1$ imply $\mu/(\lambda + 2\mu) = 1/2$, *i.e.* $\lambda = 0$. Since we have assumed $\lambda \neq 0$, this leads to a contradiction.

Therefore we proved all the assertions of Theorem 1 except that for the exponential decay which will be considered in the next section.

5. THE EXPONENTIAL DECAY

In this section we show that for the sequence of $z_n = d_0 n + d_1 + O(n^{-1})$ found to exists in Case B1, Section 4, we have

$$0 < \Im z_n \leqq \mathrm{C} e^{-\gamma n}, \qquad \mathrm{C} > 0, \qquad \gamma > 0. \tag{41}$$

Let us set as above $s = \nu^{-1}$ (recall (17)). Then z_n are solutions to the equation $\tilde{\Delta}_n(z) = 0$ which can be rewritten in the form $f(a, s) = 0$ after setting $a = \mathrm{K}/\nu = -z\mu^{-1/2}\nu^{-1}$, $f = -\mu/(\lambda + 2\mu) a^4 \tilde{\Delta}_n(\mu^{1/2}\nu a)$. By (29), $f(a, s) = \mathrm{R}(a) + O(s)$ which implies the existence of a root of $f(a, s)$ in

(0, 1). We wish to show that $\Im a(s) \leqq C e^{-\gamma/s}$ with some $C>0$, $\gamma>0$. To this end we will represent f in the form $f=f_1+if_2$ with f_1, f_2 analytic near $a=a_0$, $f_1=\Re f$, $f_2=\Im f$ for $a\in\mathbf{R}$, and we will show that f_2 admits an estimate of the type $|f_2|\leqq \tilde{C} e^{-\gamma/s}$.

Let us set $u_\nu(z)=zh_n^{(1)}(z)=zh_{\nu-1/2}^{(1)}(z)$ (recall (17)). Then u_ν satisfies the equation

$$u_\nu''=\left(\frac{\nu^2-1/4}{z^2}+1\right)u_\nu. \tag{42}$$

Set

$$w_\nu=\frac{u_\nu'}{u_\nu}, \qquad \eta_\nu=\frac{1}{2}(w_\nu(z)+\overline{w_\nu(\bar{z})}), \qquad \psi_\nu=\frac{1}{2i}(w_\nu(z)-\overline{w_\nu(\bar{z})}).$$

Note that η_ν, ψ_ν are analytic functions in $\mathbf{C}\setminus\{0\}$ and $\eta_\nu=\Re w_\nu$, $\psi_\nu=\Im w_\nu$ for z real.

LEMMA 3. – *For any* $\delta_1\in(0, 1/2)$ *there exists* $\delta_2>0$, *such that in the set* $A:=\{z\in\mathbf{C},\ \delta_1<\Re z<1-\delta_1,\ |\Im z|<\delta_2\}$ *we have*

$$|\psi_\nu(\nu z)|\leqq C e^{-\gamma\nu}, \qquad |\psi_\nu'(\nu z)|\leqq C e^{-\gamma\nu} \qquad \textit{for} \quad z\in A$$

with some $C>0$, $\gamma>0$.

Proof. – Suppose for a moment that $z\in\mathbf{R}$. Then by (42)

$$\psi_\nu'=\Im\left(\frac{u_\nu''}{u_\nu}-\left(\frac{u_\nu'}{u_\nu}\right)^2\right)=-2\eta_\nu\psi_\nu.$$

Therefore, for ψ_ν we get

$$\frac{d}{dz}\psi_\nu(\nu z)=-2\nu\eta_\nu(\nu z)\psi_\nu(\nu z), \tag{43}$$

hence

$$\frac{d}{dz}\left\{\psi_\nu(\nu z)\exp\left[2\nu\int_{z_0}^{z}\eta_\nu(\nu y)\,dy\right]\right\}=0.$$

This shows that

$$\psi_\nu(\nu z)=\psi_\nu(\nu z_0)\exp\left[-2\nu\int_{z_0}^{z}\eta_\nu(\nu y)\,dy\right]. \tag{44}$$

Here we can put $z_0=1-\delta_1/2$. Suppose now that $z\in A$, *i.e.* z is not necessarily real. Then (44) remains true and for any $z\in A$ we will regard $\int_{z_0}^{z}dy$ in (44) as an integral over the path $[1-\delta_1/2, \Re z]\cup[\Re z, z]$. Let us choose δ_2 small enough to ensure $-iA\subset K$, where K is the eye-shaped domain [O2], [O3] in which the asymptotics (25), (26) hold. By these asymptotics we have for w_ν

$$w_\nu(\nu z) = -\frac{\sqrt{1-z^2}}{z} + O(\nu^{-1})$$

uniformly in $z \in \mathrm{A}$. Therefore

$$\eta_\nu(\nu z) = -\frac{\sqrt{1-z^2}}{z} + O(\nu^{-1}), \qquad \psi_\nu(\nu z) = O(\nu^{-1}).$$

Thus in particular for $z \in \mathrm{A}$ and ν sufficiently large we have $|\eta_\nu(\nu z)| \leqq \mathrm{C}$. Moreover for real $z \in \mathrm{A}$ we have $\eta_\nu(\nu z) < -\delta_3 < 0$. Hence

$$\Re \int_{z_0}^{z} \eta_\nu(\nu y)\,dy \geqq \int_{z_0}^{\Re z} \eta_\nu(\nu y)\,dy - \left| \int_{\Re z}^{z} \eta_\nu(\nu y)\,dy \right|$$
$$\geqq -\int_{\Re z}^{1-\delta_1/2} \eta_\nu(\nu y)\,dy - \mathrm{C}\,\delta_2 > \delta_3\,\delta_1/2 - \mathrm{C}\,\delta_2 > \gamma/2 > 0,$$

where the last inequality holds with $\gamma = \delta_1\,\delta_3/2$ for $\delta_2 < \delta_1\,\delta_3/(4\,\mathrm{C})$. Comparing this result with (44), we see that

$$|\psi_\nu(\nu z)| \leqq \mathrm{C}\,e^{-\gamma\nu}.$$

A similar estimate holds for $\psi'_\nu(\nu z)$ because of the differential equation (43) satisfied by $\psi_\nu(\nu z)$. This completes the proof of the lemma.

As noted above, by setting $f(a, s) = -\frac{\mu}{\lambda + 2\mu}\, a^4\,\tilde{\Delta}_n(\mu^{1/2}\nu a)$, $s = 1/\nu$, we can write the equation $\tilde{\Delta}_n(\mu^{1/2}\nu a) = 0$ in the form $f(a, s) = 0$, where $f(a, s) = \mathrm{R}(a) + O(s)$. Let us represent f as $f = f_1 + i f_2$, where $f_1 = \frac{1}{2}(f(z) + \overline{f(\bar z)})$, $f_2 = \frac{1}{2i}(f(z) - \overline{f(\bar z)})$. In particular, $f_1 = \Re f$, $f_2 = \Im f$ as $a \in \mathbf{R}$. Let us note that Lemma 3 and (19) together imply that for s sufficiently small

$$|f_2(a, s)| \leqq \mathrm{C}\,e^{-\gamma/s}, \qquad a \in \mathrm{A}. \tag{45}$$

For $f_1(a, s)$ we have

$$f_1(a, s) = \mathrm{R}(a) + g(a, s), \qquad |g(a, s)| \leqq \mathrm{C}\,s, \qquad |g'_a(a, s)| \leqq \mathrm{C}\,s, \qquad a \in \mathrm{A} \tag{46}$$

for s in some neighborhood of the origin. The last inequality in (46) holds because the asymptotics of $h_n^{(1)}(\nu a)$, $(h_n^{(1)})'(\nu a)$, $(h_n^{(1)})''(\nu a)$ can be differentiated termwise, therefore $\partial_a(g + i f_2)$ can be estimated from above by $\mathrm{C}\,s$ and by the second inequality of Lemma 3 the same holds for $\partial_a f_2$. By the implicit function theorem there exist $a_1(s)$, $a(s)$, such that

$$\left.\begin{aligned} f_1(a_1(s), s) = 0, \qquad a_1(s) = a_0 + O(s), \qquad a_1(s) \in \mathbf{R}, \\ f(a(s), s) = 0, \qquad a(s) = a_0 + O(s). \end{aligned}\right\} \tag{47}$$

We wish to estimate $\Im a(s)$. To this end we are going to apply the Rouché theorem for the pair f_1, $f=f_1+if_2$. Set $U=\{a\in \mathbf{C};\ |a-a_1(s)|<M e^{-\gamma/s}\}$ with γ the same as in (45). Then in U we have

$$c_0\,|a-a_1(s)|\leqq |f_1(a, s)| \tag{48}$$

with some $c_0>0$. Indeed, since $f_1(a_1(s), s)=0$, we have

$$f_1(a, s)=(a-a_1(s))\int_0^1 (\partial_a f_1)(a_1(s)+t(a-a_1(s)), s)\,dt.$$

Here $\partial_a f_1(a, s)=R'(a)+g'_a(a, s)$. Taking advantage of the inequality $R'(a_0)>0$, (46), (47), we get

$$\int_0^1 (\partial_a f_1)(a_1(s)+t(a-a_1(s)), s)\,dt=R'(a_0)+O(s),$$

therefore (48) holds with $c_0=R'(a_0)/2$ for s sufficiently small. By (48) we see that on ∂U we have

$$|f_1(a, s)|\geqq c_0 M e^{-\gamma/s}, \qquad a\in\partial U. \tag{49}$$

Thus according to (45) we get

$$|f_2(a, s)|\leqq \frac{C}{c_0 M}|f_1(a, s)|, \qquad a\in\partial U.$$

Choosing M large enough we get

$$|f_2(a, s)|<|f_1(a, s)|, \qquad a\in\partial U.$$

Since f_1 has a root $a_1(s)$ in U, by the Rouché theorem $f=f_1+if_2$ must have a root in U as well. Therefore

$$|a(s)-a_1(s)|\leqq M e^{-\gamma/s} \tag{50}$$

and since $a_1(s)$ is real, we get

$$|\Im a(s)|\leqq M e^{-\gamma/s}.$$

This shows that we have a similar estimate for $\Im z_n$, because $z_n=\nu\mu^{1/2}a(\nu^{-1})$. The proof of (41) is complete. This completes the proof of Theorem 1.

ACKNOWLEDGEMENTS

This work has been written while the first author was a post-doc fellow at the University of Bordeaux and next a research fellow at the University of Helsinki and the second author was a post-doc fellow at the University of Pairs-Sud. The authors thank BSF (Grant MM8/91) for support.

REFERENCES

[A] J. D. ACHENBACH, *Wave Propagation in Elastic Solid*, North Holland, New York, 1973.

[BLR] C. BARDOS, G. LEBEAU and J. RAUCH, Scattering Frequencies and Gevrey 3 Singularities, *Invent. Math.*, Vol. **90**, 1987, pp. 77-114.

[CP] F. CARDOSO and G. POPOV, Rayleigh Quasimodes in Linear Elasticity, *Comm. P.D.E.*, Vol. **17**, 1992, pp. 1327-1367.

[Gr] R. GREGORY, The Propagation of Rayleigh Waves over Curved Surfaces at High Frequency, *Proc. Cambridge Philos. Soc.*, Vol. **70**, 1971, pp. 103-121.

[G] J. C. GUILLOT, Existence and Uniqueness of a Rayleigh Surface Wave Propagating along the Free Boundary of a Transversely Isotropic Elastic Half Space, *Math. Meth. Appl. Sci.*, Vol. **8**, 1986, pp. 289-310.

[IN] M. IKEHATA and G. NAKAMURA, Decaying and Nondecaying Properties of the Local Energy of an Elastic Wave outside an Obstacle, *Japan J. Appl. Math.*, Vol. **6**, 1989, pp. 83-95.

[IS] H. IWASHITA and Y. SHIBATA, On the Analyticity of Spectral Function for Exterior Boundary Value Problems, *Glas. Mat. Ser. III*, 23, Vol. **43**, 1988, pp. 291-313.

[K] M. KAWASHITA, On the Local-Energy Decay Property for the Elastic Wave Equation with the Neumann Boundary Conditions, *Duke Math. J.*, Vol. **67**, 1992, pp. 333-351.

[LP] P. D. LAX and R. S. PHILLIPS, *Scattering Theory*, New York, Academic Press, 1967.

[MF] P. MORSE and H. FESHBACH, Methods of Theoretical Physics, McGrow-Hill, New York, 1953.

[O1] F. OLVER, The Asymptotic Solution of Linear Differential Equations of Second Order for Large Values of a Parameter, *Phylos. Trans. Royal Soc. London Ser. A.*, Vol. **247**, pp. 307-327.

[O2] F. OLVER, The Asymptotic Expansion of Bessel Functions of Large Order, *Phylos. Trans. Royal Soc. London Ser. A.*, Vol. **247**, pp. 328-368.

[O3] F. OLVER, Asymptotics and Special Functions, Academic Press, New York, London, 1974.

[P] O. POISSON, Calculs des pôles de résonance associés à la diffraction d'ondes acoustiques et élastiques en dimension 2, thèse, Université Paris-IX.

[R] Lord RAYLEIGH, On Waves Propagated along Plane Surface of an Elastic Solid, *Proc. London Math. Soc.*, Vol. **17**, 1885, pp.4-11.

[RS] M. REED and B. SIMON, Methods of Modern Mathematical Physics II, Academic Press, New York, 1975.

[SZ] J. SJÖSTRAND and M. ZWORSKI, Complex scaling and the distribution of scattering poles, *J. Amer. Math. Soc.*, Vol. **4**, 4, 1991, pp. 729-769.

[SS] Y. SHIBATA and H. SOGA, Scattering Theory for the Elastic Wave Equation, *Publ. R.I.M.S.*, Vol. **25**, 1989, pp. 861-887.

[T] M. TAYLOR, Rayleigh Waves in Linear Elasticity as a Propagation of Singularities Phenomenon, in *Proc. Conf. on P.D.E. and Geometry*, Marcel Dekker, New York, 1979, pp. 273-291.

[To] T. TOKITA, Exponential Decay of Solutions for the Wave Equation in the Exterior Domain with Spherical Boundary, *J. Math. Kyoto Univ.*, Vol. **12**, 1972, pp. 413-430.

[V] B. R. VAINBERG, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach sci. publ., New York, 1988.

(*Manuscript received December* 11, 1992;
revised version received September 2, 1993.)