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A covariant and extended model for relativistic magnetofluiddynamics

by

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ABSTRACT. — A set S of nine partial differential equations (PDE) is here obtained for the determination of nine unknown independent variables. One of these variables is an auxiliary quantity x. The solutions of S with $x=0$ are those of relativistic magnetofluiddynamics. Moreover S is expressed in covariant form and is equivalent to a symmetric hyperbolic system.

The wave speeds for S are ± 1 and the well-known material waves, Alfvén waves and magnetoacoustic waves.

RÉSUMÉ. — On obtient un ensemble S de 9 équations aux dérivées partielles (PDE) pour la détermination de 9 variables inconnues. Une de ces variables est en quantité auxiliaire x. Les solutions de S avec $x=0$ sont celles de la magnétofluidodynamique relativiste. On peut exprimer S sous la forme covariante et il est équivalent à un système symétrique hyperbolique.

Les ondes de S sont donc les ondes à vitesse ± 1 et les ondes bien connues matérielles de Alfvén et magnétoacoustiques.

1. INTRODUCTION

Relativistic magnetofluiddynamics uses the nine field equations

$$\partial_\alpha (nu^\alpha)=0 \qquad \text{(conservation of mass)}$$

$$\partial_\alpha [(e+p+b^2) u^\alpha u^\beta + (p+b^2/2) g^{\alpha\beta} - b^\alpha b^\beta]$$
$$= 0 \qquad \text{(conservation of energy-momentum)} \quad (1)$$
$$\partial_\alpha (u^\alpha b^\beta - b^\alpha u^\beta) = 0 \qquad \text{(Maxwell's equations)}$$

to determine the rest-mass density n, the total energy-density e, the four-velocity u^α and the four-vector b^α related to the proper magnetic field H^α by $b^\alpha = \mu^{1/2} H^\alpha$ with $\mu > 0$ the (constant) magnetic permeability; these variables are constrained by

$$u^\alpha u_\alpha = -1; \qquad u^\alpha b_\alpha = 0 \qquad (2)$$

so that one has only eight independent variables.

Attempts have been made by Ruggeri and Strumia [1] and later by Anile and Pennisi [2] to prove that the system (1) is equivalent to a symmetric hyperbolic one. They have proved that one of the equations (1) is consequence of the others and of the initial conditions; this equation has been eliminated and the remaining system has been proved to be equivalent to a symmetric hyperbolic one. But unfortunately, this result has been achieved at the cost of losing manifest covariance (Even if an ortonormalized tetrad $\{\xi^\alpha, \zeta^\alpha_I\}$ of constant congruences may be employed, this is equivalent to choosing a particular reference frame). To eliminate this problem Strumia ([3], [4]) has elaborated a very elegant theory on conservation laws with constrained field variables, where he has used a constraint manifold C defined by $(n, e, u^\alpha, b^\alpha) \in \mathbb{R}^{10}$ satisfying eqs. (2). However, this method has the drawback that if one wants to apply the equations to some pratical case, he must choose eight independent vectors locally tangent to C; as consequence the covariance is lost again.

Here I propose an alternative method based on these ideas and on those of extended thermodynamics ([5], [6]), *i.e.* to introduce another independent variable x and to find a new system of equations that for $x=0$ reduces to (1). The effective procedure is exposed in Sec. 2 and the resulting system is

$$\left.\begin{aligned} &\partial_\alpha (n u^\alpha) = 0 \\ &\partial_\alpha [(e+p+b^2) u^\alpha u^\beta + (p+b^2/2) g^{\alpha\beta} - b^\alpha b^\beta \\ &\quad + x (u^\alpha b^\beta - b^\alpha u^\beta) - g^{\alpha\beta} x^2/2] = 0 \\ &\partial_\alpha (u^\alpha b^\beta - b^\alpha u^\beta - x g^{\alpha\beta}) = 0 \end{aligned}\right\} \qquad (3)$$

in the nine unknowns n, e, u^α, b^α, x.

It is evident that for $x=0$ the system (3) reduces itself to (1). To this end it is not necessary to impose $x=0$ only after that the system (3) has been solved; it is instead sufficient to impose $x=0$ and the Maxwell's equations on a given initial hypersurface ortogonal to a time-like 4-vector ξ_α with $\xi_\alpha \xi^\alpha = -1$. In fact eq. $(3)_3$ contracted by ξ_β gives

$$\xi^\alpha \partial_\alpha x = \xi_\beta \partial_\gamma (u^\alpha b^\beta - b^\alpha u^\beta)(g^\gamma_\alpha + \xi_\alpha \xi^\gamma)$$

because

$$\xi_\alpha \xi_\beta \xi^\gamma \partial_\gamma (u^\alpha b^\beta - b^\alpha u^\beta) = 0.$$

Moreover system (3) is expressed in the covariant form and is equivalent to a symmetric hyperbolic system at least in a neighbourhood of $x=0$, as it will be proved in Sec. 2 and 3. Its hyperbolicity is proved also directly, in Sec. 4, obtaining explicit and covariant expressions for the eigenvectors and the corresponding wave speeds. These last ones are ± 1 and those already found in ref. [2], *i.e.* the material, Alfvén and magneto-acoustic waves.

I conclude this section noticing that a similar covariant and extended approach was tried also in the first part of paper [2], where 10 field equations were used and the constraints (2) were not taken into account, in order to have 10 independent variables. The results were only partially succesful because the conservative form was lost and the hyperbolicity did not hold in some special cases. Therefore I consider the results of the present paper a great improvement over those already known.

2. EXTENDED RELATIVISTIC MAGNETOFLUIDDYNAMICS

Let Q^α, $T^{\alpha\beta}$, $\psi^{\alpha\beta}$ be particular functions of n, e, u^α, b^α and of an auxiliary variable x, such that the system

$$\partial_\alpha Q^\alpha = 0; \qquad \partial_\alpha T^{\alpha\beta} = 0; \qquad \partial_\alpha \psi^{\alpha\beta} = 0 \tag{4}$$

reduces for $x=0$ to the system (1).

Moreover I want that an entropy principle holds for the system (4); in other words I suppose that a four-vectorial function h_α exists such that the inequality $\partial_\alpha h^\alpha \geqq 0$ holds for all solutions of the system (4). The mathematical exploitation of this principle is easier if we use Liu's paper [7]; here he proved that this statement is equivalent to assuming the existence of Lagrange multipliers λ, λ_α, ψ_α such that the inequality

$$\partial_\alpha h^\alpha + \lambda \partial_\alpha Q^\alpha + \lambda_\beta \partial_\alpha T^{\alpha\beta} + \psi_\beta \partial_\alpha \psi^{\alpha\beta} \geqq 0 \tag{5}$$

holds for all values of the independent variables. This property has been proved also by Friedrichs ([8], [9]).

Now I suppose that the functions λ, λ_β, ψ_β are invertible, I take them as independent variables and define

$$h'^\alpha = h^\alpha + \lambda Q^\alpha + \lambda_\beta T^{\alpha\beta} + \psi_\beta \psi^{\alpha\beta}. \tag{6}$$

In this way I am following an idea developed by Boillat, Ruggeri and Strumia which they applied in different physical contexts concerning both the classical [10] and the relativistic case ([11], [12]). As consequence the inequality (5) assumes the form

$$\partial_\alpha h'^\alpha - Q^\alpha \partial_\alpha \lambda - T^{\alpha\beta} \partial_\alpha \lambda_\beta - \psi^{\alpha\beta} \partial_\alpha \psi_\beta \geqq 0$$

for all values of λ, λ_β, ψ_β. Clearly this condition is equivalent to

$$Q^\alpha = \frac{\partial h'^\alpha}{\partial \lambda}; \qquad T^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \lambda_\beta}; \qquad \psi^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \psi_\beta} \tag{7}$$

and therefore the system (4) assumes the symmetric form.

Moreover, from the relativity principle it follows that the function h'^α assumes the form

$$h'^\alpha = H \lambda^\alpha + K \psi^\alpha \tag{8}$$

with H and K functions of λ, $G_1 = \lambda^\alpha \lambda_\alpha$, $G_2 = \psi_\alpha \psi^\alpha$, $G = \lambda_\alpha \psi^\alpha$.

This property may be proved directly or by applying the most general results found by Pennisi and Trovato [13]. As a consequence the relations (7) become

$$Q^\alpha = \frac{\partial H}{\partial \lambda} \lambda^\alpha + \frac{\partial K}{\partial \lambda} \psi^\alpha \tag{9}$$

$$T^{\alpha\beta} = 2 \frac{\partial H}{\partial G_1} \lambda^\alpha \lambda^\beta + \frac{\partial H}{\partial G} \psi^\beta \lambda^\alpha + 2 \frac{\partial K}{\partial G_1} \lambda^\beta \psi^\alpha + \frac{\partial K}{\partial G} \psi^\alpha \psi^\beta + H g^{\alpha\beta} \tag{10}$$

$$\psi^{\alpha\beta} = \frac{\partial H}{\partial G} \lambda^\alpha \lambda^\beta + 2 \frac{\partial H}{\partial G_2} \psi^\beta \lambda^\alpha + \frac{\partial K}{\partial G} \lambda^\beta \psi^\alpha + 2 \frac{\partial K}{\partial G_2} \psi^\alpha \psi^\beta + k g^{\alpha\beta} \tag{11}$$

I want that for $x=0$ the system (4) reduces to (1); for the system (1) Strumia [4] has proved that $\lambda_\beta = L u_\beta$, $\psi_\beta = P b_\beta$ with L and P scalar functions. As a consequence we have that $G=0$ corresponds to $x=0$ and moreover

$$\left\{ \begin{array}{l} \left(\dfrac{\partial H}{\partial G} \right)_{G=0} = 0; \qquad (K)_{G=0} = 0; \\ \left(2 \dfrac{\partial H}{\partial G_2} + \dfrac{\partial K}{\partial G} \right)_{G=0} = 0 \end{array} \right. \tag{12}$$

in order that for $x=0$ the expressions (9)-(11) reduce to

$$(Q^\alpha)_{x=0} = n u^\alpha \tag{13}$$

$$(T^{\alpha\beta})_{x=0} = (e + p + b^2) u^\alpha u^\beta + (p + b^2/2) g^{\alpha\beta} - b^\alpha b^\beta; \tag{14}$$

$$(\psi^{\alpha\beta})_{x=0} = u^\alpha b^\beta - b^\alpha u^\beta. \tag{15}$$

The eqs. (12) have solution

$$H = H_0(\lambda, G_1, G_2) + G^2 H_1(\lambda, G_1, G_2, G) \tag{16}$$

$$K = -2G \frac{\partial H_0(\lambda, G_1, G_2)}{\partial G_2} + G^2 K_1(\lambda, G_1, G_2, G) \tag{17}$$

with $H_0(\lambda, G_1, G_2)$, $H_1(\lambda, G_1, G_2, G)$, $K_1(\lambda, G_1, G_2, G)$ arbitrary functions. After that the eqs. (9)-(11) calculated for $G=0$ become

$$(Q^\alpha)_{G=0}=\frac{\partial H_0}{\partial \lambda}\lambda^\alpha$$

$$(T^{\alpha\beta})_{G=0}=2\frac{\partial H_0}{\partial G_1}\lambda^\alpha\lambda^\beta-2\frac{\partial H_0}{\partial G_2}\psi^\alpha\psi^\beta+H_0 g^{\alpha\beta}$$

$$(\psi^{\alpha\beta})_{G=0}=2\frac{\partial H_0}{\partial G_2}(\lambda^\alpha\psi^\beta-\psi^\alpha\lambda^\beta)$$

that, compared with (13)-(15) give

$$n=\left|\frac{\partial H_0}{\partial \lambda}\right|\sqrt{-G_1};\qquad u^\alpha=\frac{\partial H_0}{\partial \lambda}\left|\frac{\partial H_0}{\partial \lambda}\right|^{-1}(-G_1)^{-1/2}\lambda^\alpha;$$

$$b^\alpha=2\frac{\partial H_0}{\partial G_2}\frac{\partial H_0}{\partial \lambda}\left|\frac{\partial H_0}{\partial \lambda}\right|^{-1}(-G_1)^{1/2}\psi^\alpha;$$

$$p=H_0+2G_1\left(\frac{\partial H_0}{\partial G_2}\right)^2 G_2;$$

$$e=-2\frac{\partial H_0}{\partial G_1}G_1-H_0+2G_1\left(\frac{\partial H_0}{\partial G_2}\right)^2 G_2-4G_1\left(\frac{\partial H_0}{\partial G_2}\right)-2.$$

From these eqs. one obtains

$$H_0=-\frac{1}{2}G_2 G_1^{-1}+H_2(G_1, \lambda) \tag{18}$$

with $H_2(G_1, \lambda)$ a function such that $\frac{\partial H_2}{\partial \lambda}$ has an assigned sign; I suppose that $\frac{\partial H_2}{\partial \lambda}>0$ because the other case can be investigated with the same procedure with $-H_2$ instead of H_2. After that the above eqs. give

$$\left.\begin{aligned} &n=\frac{\partial H_2}{\partial \lambda}\sqrt{-G_1};\qquad u^\alpha=(-G_1)^{-1/2}\lambda^\alpha;\qquad b^\alpha=(-G_1)^{-1/2}\psi^\alpha;\\ &p=H_2;\qquad e=-H_2-2\frac{\partial H_2}{\partial G_1}G_1.\end{aligned}\right\} \tag{19}$$

By using these expressions we obtain

$$\frac{1}{n}de+(e+p)\,d\left(\frac{1}{n}\right)=(-G_1)^{-1/2}\,d\left[\frac{e+p}{n}(-G_1)^{1/2}-\lambda\right]$$

for all values of G_1, λ; this is the Gibbs relation with

$$\left.\begin{aligned} &T=(-G_1)^{-1/2}\,(\text{temperature}) \\ &\text{and} \\ &S=\frac{e+p}{nT}-\lambda\,(\text{specific entropy}). \end{aligned}\right\} \tag{20}$$

The expression (18) for H_0 has been obtained for $G=0$. But $H_0(\lambda, G_1, G_2)$ does not depend on G; therefore eq. (18) is valid also in the general case so that eqs. (16), (17) give

$$H=-\frac{1}{2}G_2 G_1^{-1}+H_2(G_1, \lambda)+G^2 H_1(\lambda, G_1, G_2, G) \tag{21}$$

$$K=GG_1^{-1}+G^2 K_1(\lambda, G_1, G_2, G). \tag{22}$$

In the next section it will be proved that eqs. (21), (22), (8), (7), (4) give a symmetric hyperbolic system in a neighbourhood of $x=0$, provided that

$$K_1(\lambda, G_1, G_2, 0)=0; \qquad T^4+2H_1(\lambda, G_1, G_2, 0)>0 \text{ hold}. \tag{23}$$

Then a particular case is $K_1=H_1=0$, λ, λ^α, ψ^α related to T, n, x, u^α, b^α by

$$\lambda=-S+\frac{e+p}{nT}; \qquad \lambda^\alpha=\frac{u^\alpha}{T}; \qquad \psi^\alpha=\frac{1}{T}(b^\alpha-xu^\alpha)$$

and

$$H_2(G_1, \lambda)=p\,[n(G_1, \lambda), e(G_1, \lambda)].$$

These relations for $x=0$ give eqs. (20), $(19)_{2,3}$ and $G=0$; eqs. $(19)_{1,4,5}$ hold also for $x\neq 0$ because λ and λ^α does not depend on x. By using all these expressions, we can find H, K from (21)-(22), h'^α from eq. (8) and finally Q^α, $T^{\alpha\beta}$, $\psi^{\alpha\beta}$ from (7); then the resulting system (4) assumes the preannounced expression (3). The function h^α can be obtained from eq. (6) and reads $h^\alpha=nSu^\alpha$.

3. HYPERBOLICITY OF THE SYSTEM (4) WITH Q^α, $T^{\alpha\beta}$, $\psi^{\alpha\beta}$ GIVEN BY (7), (8), (21), (22)

By using eqs. (7), the system (4) assumes the symmetric form. Therefore it is hyperbolic in the time-like direction ξ_α iff the convexity of entropy holds, *i.e.*

$$\xi_\alpha \frac{\partial^2 h'^\alpha}{\partial F_A \partial F_B}\partial F_A \delta F_B \geqq 0, \quad \forall \delta F_A, \qquad \text{where} \quad F_A \equiv (\lambda, \lambda_\beta, \psi_\beta)$$

and $\xi_\alpha \xi^\alpha=-1$. By using eqs. (7) this inequality becomes

$$\xi_\alpha[\delta Q^\alpha \delta\lambda+\delta T^{\alpha\beta}\delta\lambda_\beta+\delta\psi^{\alpha\beta}\delta\psi_\beta]\geqq 0, \qquad \forall\,\delta\lambda, \delta\lambda_\beta, \delta\psi_\beta. \tag{24}$$

If we introduce the transformation of variables from λ, λ_α, ψ_α to n, T, G, u_α, b_α defined by

$$\lambda^\alpha = \mathrm{T}^{-1} u^\alpha; \qquad \lambda = \frac{e+p}{n\mathrm{T}} - \mathrm{S}; \qquad \psi^\alpha = \mathrm{T}^{-1} b^\alpha - \mathrm{GT}\, u^\alpha,$$

the property (24) becomes

$$\xi_\alpha \Big\{ (2\,\bar{\mathrm{K}}_1\, \mathrm{T}^{-1} b^\alpha + (2\,\bar{\mathrm{H}}_1\, \mathrm{T}^{-1} + \mathrm{T}^3)\, u^\alpha]\, (\delta \mathrm{G})^2 - 2\,\mathrm{T}\, h^{\alpha\beta}\, \delta b_\beta\, \delta \mathrm{G} + \delta (n u^\alpha)\, \delta\left(\frac{e+p}{n\mathrm{T}} - \mathrm{S} \right) + \delta [(e+p+b^2)\, u^\alpha u^\beta + (p + b^2/2)\, g^{\alpha\beta} - b^\alpha b^\beta]\, \delta\left(\frac{u_\beta}{\mathrm{T}} \right) + \delta (u^\alpha h^\beta - b^\alpha u^\beta)\, \delta\left(\frac{b_\beta}{\mathrm{T}} \right) \Big\} \geqq 0$$

for all values of δG, δn, δT, δu^α, δb^α restricted only by $u^\alpha\, \delta u_\alpha = 0$, $b^\alpha\, \delta u_\alpha + u^\alpha\, \delta b_\alpha = 0$. In the above inequality the coefficients have been calculated in $\mathrm{G}=0$ because I am looking for hyperbolicity in a neighbourhood of $\mathrm{G}=0$; moreover I have posed

$$\bar{\mathrm{K}}_1 = K_1 (\lambda, \mathrm{G}_1, \mathrm{G}_2, 0); \qquad \bar{\mathrm{H}}_1 = \mathrm{H}_1 (\lambda, \mathrm{G}_1, \mathrm{G}_2, 0); \qquad h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta.$$

We can evaluate this inequality in the reference frame Σ characterized by $u^\alpha \equiv (1, 0, 0, 0)$; $b^\alpha \equiv (0, b, 0, 0)$; $\xi^\alpha \equiv (\xi^0, \xi^1, \xi^2, 0)$; it becomes

$$\xi_0 \left\{ \mathrm{A}_{11} (\delta \mathrm{Y}_1)^2 + 2\, \mathrm{A}_{12}\, \delta \mathrm{Y}_1 \delta \mathrm{Y}_2 + \mathrm{A}_{22} (\delta \mathrm{Y}_2)^2 + \sum_{i=1}^{7} \mathrm{A}_i (\delta \mathrm{X}_i)^2 \right\} \geqq 0 \quad (26)$$

$\forall\, \delta \mathrm{X}_i$, $\delta \mathrm{Y}_j$ where $\delta \mathrm{Y}_1 = \delta u_1$; $\delta \mathrm{Y}_2 = \delta u_2$; $\delta \mathrm{X}_1 = \delta u_3$;

$$\delta \mathrm{X}_2 = \delta \mathrm{T} + \frac{\mathrm{TP}_T}{e_T} \xi_0^{-1} (\xi_1\, \delta u^1 + \xi_2\, \delta u^2);$$

$$\delta \mathrm{X}_3 = \delta n + n\, \xi_0^{-1} (\xi_1\, \delta u^1 + \xi_2\, \delta u^2);$$

$$\delta \mathrm{X}_4 = \delta b_1 - (b\, \xi_2\, \delta u_2 + \mathrm{T}^2\, \xi_1\, \delta \mathrm{G})\, \xi_0^{-1};$$

$$\delta \mathrm{X}_5 = \delta b_2 - (b\, \xi_1\, \delta u_2 + \mathrm{T}^2\, \xi_2\, \delta \mathrm{G})\, \xi_0^{-1};$$

$$\delta \mathrm{X}_6 = \delta b_3 - b\, \xi_1\, \delta u_3\, \xi_0^{-1}; \qquad \delta \mathrm{X}_7 = \delta \mathrm{G}$$

$$\mathrm{A}_{11} = \frac{1}{\mathrm{T}} \left[e + p - \left(n p_n + \frac{\mathrm{T} (\mathrm{P}_T)^2}{e_T} \right) \xi_1^2 (1 + \xi_1^2 + \xi_2^2)^{-1} \right];$$

$$\mathrm{A}_{12} = -\frac{1}{\mathrm{T}} \left(n p_n + \frac{\mathrm{T} (\mathrm{P}_T)^2}{e_T} \right) \xi_0^{-2}\, \xi_1\, \xi_2;$$

$$\mathrm{A}_{22} = \frac{1}{\mathrm{T}} \left[e + p + b^2\, \xi_0^{-2} - \left(n p_n + \frac{\mathrm{T} (\mathrm{P}_T)^2}{e_T} \right) \xi_0^{-2}\, \xi_2^2 \right];$$

$$\mathrm{A}_1 = \frac{1}{\mathrm{T}} \left[e + p + b^2\, \xi_0^{-2} (1 + \xi_2^2) \right];$$

$$A_2=\frac{e_T}{T^2};\qquad A_3=\frac{P_n}{nT};\qquad A_4=A_5=A_6=\frac{1}{T};$$
$$A_7=T^{-1}\xi_0^{-2}[T^4+2\bar{H}_1\xi_0^2+2\bar{K}_3 b\xi_0\xi_1].$$

In evaluating eq. (26) I have used

$$\frac{\partial S}{\partial n}=\frac{e_n}{nT}-\frac{e+p}{n^2T};\qquad \frac{\partial S}{\partial T}=\frac{e_T}{nT} \tag{27}$$

which are consequence of the Gibbs relation and whose symmetry condition $\frac{\partial^2 S}{\partial T\,\partial n}=\frac{\partial^2 S}{\partial n\,\partial T}$ is

$$e_n=\frac{e+p}{n}-\frac{Tp_T}{n} \tag{28}$$

from which $(27)_1$ becomes

$$\frac{\partial S}{\partial n}=-\frac{p_T}{n^2}. \tag{29}$$

Moreover one has $\xi_0^2=1+\xi_1^2+\xi_2^2$; $\delta b_0=-b\,\delta u_1$; $\delta u_0=0$.

Remembering that $p_n>0$, $e_T>0$ are the classical stability conditions on compressibility and specific heat, it is clear that (26) is satisfied iff

$$A_7>0;\qquad A_{11}>0;\qquad \begin{vmatrix} A_{11} & A_{12}\\ A_{12} & A_{22}\end{vmatrix}>0. \tag{30}$$

When $\xi_\alpha=u_\alpha$ these conditions are satisfied iff $(23)_2$ holds. But it is important to have hyperbolicity for *all* time-like ξ_α because this property assures that the speeds of the shocks can not exceed the speed of light, as shown by Strumia [4]. We have that the condition $(30)_1$ can hold for all ξ_α only if $\bar{K}_1=0$. In fact, if $\bar{K}_1\xi_0>0$, then $(30)_1$ is violated for $\xi_1\to-\infty$; similarly if $\bar{K}_1\xi_0<0$, then $(30)_1$ does not hold for $\xi_1\to+\infty$. In this way I have proved eqs. $(23)_{1,2}$.

Now we have that $\sup\frac{\xi_1^2}{1+\xi_1^2+\xi_2^2}=1$; as a consequence eq. $(30)_2$ holds for all ξ_α iff

$$e+p\geqq np_n+\frac{T(T_T)^2}{e_T}. \tag{31}$$

After that also $(30)_3$ is satisfied because it reads explicitly

$$(np_n e_T+T(p_T)^2)\xi_0^{-2}[e+p+b^2\xi_0^{-2}(1+\xi_2^2)]$$
$$+(e+p+b^2\xi_0^{-2})e_T\left(e+p-np_n-\frac{T(P_T)^2}{e_T}\right)>0.$$

The condition (31) is not a further restriction due to this extended model; it already appears in ref. [2] where it assumes the form $e_p\geqq 1$, because p

and S are taken as independent variables. In fact the derivatives of $p=p[n(p, \mathrm{S})], \mathrm{T}(p, \mathrm{S})]$ and $\mathrm{S}=\mathrm{S}[n(p, \mathrm{S})], \mathrm{T}(p, \mathrm{S})]$ with respect to p and S give $p_n n_p + p_T \mathrm{T}_p = 1$; $\mathrm{S}_n n_p + \mathrm{S}_T \mathrm{T}_p = 0$ from which

$$n_p = \frac{\mathrm{S}_T}{\begin{vmatrix} p_n & p_T \\ \mathrm{S}_n & \mathrm{S}_T \end{vmatrix}}; \qquad \mathrm{T}_p = \frac{-\mathrm{S}_n}{\begin{vmatrix} p_n & p_T \\ \mathrm{S}_n & \mathrm{S}_T \end{vmatrix}}.$$

By using $(27)_2$ and (29) we obtain

$$n_p = (np_n e_T + \mathrm{T}(p_T)^2)^{-1} ne_T; \qquad \mathrm{T}_p = (np_n e_T + \mathrm{T}(p_T)^2)^{-1} \mathrm{T}\mathrm{P}_T;$$

therefore, by using also (28), we have

$$e_p = e_n n_p + e_T \mathrm{T}_p = (n p_n e_T + \mathrm{T}(p_T)^2)^{-1} e_T (\mathrm{e}+p)$$

from which it is clear that $e_p - 1 \geqq 0$ is equivalent to eq. (31). This condition (31) is present also in refs. [1], [14] where it is expressed as (32) $0 < p_e \leqq 1$ in the independent variables e, S; now the derivative with respect to p of $p=p[e(p, \mathrm{S}), \mathrm{S}]$ gives $1 = p_e e_p$ from which $e_p = 1/p_e$ and consequently eq. (32) is equivalent to $e_p \geqq 1$ which I have already proved is equivalent to eq. (31). From refs. [1], [14] we read also the physical significance of eq. (31): it expresses the requirement that the sound velocity must be smaller than that of light in vacuo.

At last, we can see how this condition appears also in a different physical context [15].

In the next section, I come back to the system (3) giving explicit expressions for the eigenvecors of the characteristic matrix and obtaining also the wave speeds.

4. DETERMINATION OF WAVE SPEEDS AND EIGENVECTORS OF THE CHARACTERISTIC MATRIX

The system (3) can be written in matrix formulation

$$\mathrm{A}_{\mathrm{B}}^{\alpha \mathrm{A}} \partial_\alpha \mathrm{U}^{\mathrm{B}} = 0; \qquad \mathrm{A}, \mathrm{B} = 1, \ldots, 9.$$

This system is hyperbolic, in the sense of Friedrichs [16], in the time-direction ξ_α such that $\xi_\alpha \xi^\alpha = -1$, if the following two conditions hold

1) $\det(\mathrm{A}^\alpha \xi_\alpha) \neq 0$;

2) for any ζ_α such that $\zeta_\alpha \xi^\alpha = 0$, $\zeta_\alpha \zeta^\alpha = 1$, the eigenvalue problem

$$\mathrm{A}^\alpha (\zeta_\alpha - \lambda \xi_\alpha) \underline{d} = 0 \tag{33}$$

has only real eigenvalues λ and nine linear independent eigenvectors $\underline{d}$.

If we use the notation $\underline{d}=(\delta x, \delta T, \delta n, \delta u^\alpha, \delta b^\alpha)$, $\varphi_\alpha=\zeta_\alpha-\lambda\xi_\alpha$, the system (33) becomes $\varphi_\alpha \delta(nu^\alpha)=0$, $\varphi_\alpha \delta T^{\alpha\beta}=0$, $\varphi_\alpha \delta\psi^{\alpha\beta}=0$, i. e.

$$\left.\begin{aligned}
&\varphi_\alpha u^\alpha \delta n + n\varphi_\alpha \delta u^\alpha = 0\\
&\varphi_\alpha \{ [(e_T+p_T) u^\alpha u^\beta + p_T g^{\alpha\beta}] \delta T + [(e_n+p_n) u^\alpha u^\beta + p_n g^{\alpha\beta}] \delta n\\
&\quad + (u^\alpha b^\beta - u^\beta b^\alpha) \delta x + (e+p+b^2)(u^\alpha \delta u^\beta + u^\beta \delta u^\alpha)\\
&\quad + (2u^\alpha u^\beta + g^{\alpha\beta}) b^\gamma \delta b_\gamma - b^\alpha \delta b^\beta - b^\beta \delta b^\alpha \} = 0\\
&\varphi_\alpha (u^\alpha \delta b^\beta + b^\beta \delta u^\alpha - b^\alpha \delta u^\beta - u^\beta \delta b^\alpha - g^{\alpha\beta} \delta x) = 0
\end{aligned}\right\} \quad (34)$$

where to the second equation I have added te third one multiplied by $-x$.

It will be evident from the considerations exposed below that the eigenvalues of system (34) are the roots of

$$\left.\begin{aligned}
&a(\lambda)=\varphi_\alpha u^\alpha = 0, \qquad G(\lambda)=\varphi_\alpha \varphi^\alpha = 0 \qquad (i.e.\, \lambda=\pm 1),\\
&A(\lambda)=(e+p+b^2)(\varphi_\alpha u^\alpha)^2-(\varphi_\alpha b^\alpha)^2=0,\\
&N_4(\lambda)=(e+p)\left(e+p-np_n-\frac{T(P_T)^2}{e_T}\right)(\varphi_\alpha u^\alpha)^4\\
&\quad -(e+p)\left(np_n+\frac{T(P_T)^2}{e_T}+b^2\right)(\varphi_\alpha u^\alpha)^2 \varphi_\beta \varphi^\beta\\
&\quad +\left(np_n+\frac{T(P_T)^2}{e_T}\right)(\varphi_\alpha b^\alpha)^2 \varphi_\beta \varphi^\beta = 0.
\end{aligned}\right\} \quad (35)$$

The roots of $(35)_{1,3,4}$ correspond to material waves, Alfvén waves and magnetoacoustic waves respectively; they have been found also in Ref. [2] as it can be seen by using also $e_p=(e+p)\left(np_n+\dfrac{T(P_T)^2}{e_T}\right)^{-1}$ that was found in sect. 3.

In the following considerations I shall use the functions defined by

$$f(\xi_\alpha, V_\alpha)=\begin{vmatrix} V^\alpha V_\alpha & V^\alpha u_\alpha & V^\alpha b_\alpha & V^\alpha \xi_\alpha\\ u^\alpha V_\alpha & -1 & 0 & u^\alpha \xi_\alpha\\ b^\alpha V_\alpha & 0 & b^2 & b^\alpha \xi_\alpha\\ \xi^\alpha V_\alpha & u^\alpha \xi_\alpha & \xi^\alpha b_\alpha & -1 \end{vmatrix}$$

$$f(V_\alpha)=\begin{vmatrix} V^\alpha V_\alpha & V^\alpha u_\alpha & V^\alpha b_\alpha\\ u^\alpha V_\alpha & -1 & 0\\ b^\alpha V_\alpha & 0 & b^2 \end{vmatrix}; \qquad g(\xi_\alpha, V^\alpha)=\begin{vmatrix} \xi_\alpha u^\alpha & \xi_\alpha b^\alpha\\ \zeta^\alpha u_\alpha & \zeta^\alpha b^\alpha \end{vmatrix};$$

$$t_1(\lambda)=-\frac{TP_T}{e_T}\varphi^\mu u_\mu (e+p)[(e+p+b^2)(\varphi_\alpha u^\alpha)^2-(\varphi_\alpha b^\alpha)^2];$$

$$N_1(\lambda)=-n\varphi^\mu u_\mu (e+p)[(e+p+b^2)(\varphi_\alpha u^\alpha)^2-(\varphi_\alpha b^\alpha)^2];$$

$$U_1^\gamma(\lambda)=b^\gamma\left(-e-p+np_n+\frac{T(P_T)^2}{e_T}\right)(\varphi_\mu u^\mu)^2 \varphi_\beta b^\beta$$

$$+\varphi_\mu h^{\mu\gamma}\left\{\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)[(e+p+b^2)(\varphi_\alpha u^\alpha)^2-(\varphi_\alpha b^\alpha)^2] + (e+p)\,b^2(\varphi_\alpha u^\alpha)^2\right\};$$

$$\mathrm{B}_1^\gamma(\lambda)=\varphi_\mu b^\mu \mathrm{U}_1^\gamma(\lambda)(\varphi_\nu \upsilon^\nu)^{-1}+\left[\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\varphi_\beta b^\beta u^\gamma - \varphi_\beta u^\beta b^\gamma(e+p)\right][(e+p+b^2)(\varphi_\alpha u^\alpha)^2-(\varphi_\alpha b^\alpha)^2];$$

$$t_2(\lambda)=\frac{\mathrm{TP}_T}{e_T}\varphi^\mu u_\mu[b^2\varphi^\gamma\varphi_\gamma+b^2(\varphi^\gamma u_\gamma)^2-(\varphi^\gamma b_\gamma)^2];$$

$$\mathrm{N}_2(\lambda)=n\,\varphi^\mu u_\mu[b^2\varphi^\gamma\varphi_\gamma+b^2(\varphi^\gamma u_\gamma)^2-(\varphi^\gamma b_\gamma)^2];$$

$$\mathrm{U}_2^\alpha(\lambda)=-\varphi^\mu\varphi_\mu\varphi^\nu b_\nu b^\alpha+[-b^2(\varphi^\gamma u_\gamma)^2+(\varphi^\gamma b_\gamma)^2]\varphi_\beta h^{\beta\alpha};$$

$$\mathrm{B}_2^\alpha(\lambda)=\varphi^\mu b_\mu(\varphi^\nu u_\nu)^{-1}\mathrm{U}_2^\alpha(\lambda) + (\varphi^\mu u_\mu b^\alpha-\varphi^\mu b_\mu u^\alpha)[b^2\varphi^\gamma\varphi_\gamma+b^2(\varphi^\gamma u_\gamma)^2-(\varphi^\gamma b_\gamma)^2].$$

It is clear that, in the reference frame Σ defined in the previous section, we have

$$f(\mathrm{V}^\alpha)=-b^2[(\mathrm{V}_2)^2+(\mathrm{V}_3)^2],$$

$$g(\xi_\alpha,\zeta_\alpha)=b(\xi_0\zeta_1-\zeta_0\xi_1);\qquad f(\xi_\alpha,\mathrm{V}_\alpha)=-b^2(\xi_2)^2(\mathrm{V}_3)^2.$$

I start investigating condition 2) of hyperbolicity in the cases in which a root of eq. $(35)_i$ coincides with one of $(35)_j$ for $i, j=1, \ldots, 4$; $i\neq j$. It is clear that $(35)_1$ and $(35)_2$ have no common root otherwise $\varphi_\alpha=0$ from which the absurd $1=\zeta_\alpha\zeta^\alpha=\lambda^2\xi_\alpha\xi^\alpha=-\lambda^2$; $(35)_1$ and $(35)_{3,4}$ have a common root if and only if

Case 1: $g(\xi_\alpha,\zeta_\alpha)=0$.

The roots of eq. $(35)_4$ are those of $(35)_1$ and those of

$$f(\lambda)=(\zeta_0-\lambda\xi_0)^2\left[(e+p)(1+\xi_2^2)+\xi_1^2\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\right][(e+p)\xi_0^2 + b^2(1+\xi_2^2)]-\xi_0^2[(\zeta_2-\lambda\xi_2)^2+\zeta_3^2]\left[\xi_0^2(e+p)\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right) + b^2(e+p)(1+\xi_2^2)+b^2\xi_1^2\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\right]=0$$

which is expressed in the frame Σ.

Now we can see that $f(\zeta_0/\xi_0)<0$ and the coefficient of λ^2 in $f(\lambda)$ is positive; in fact it is

$$\xi_0^2\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)[(e+p)\xi_0^2+b^2(1+\xi_2^2)] \\ +\xi_0^4\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)[(e+p)\xi_0^2+b^2]>0.$$

Therefore $(35)_4$ has two other real roots μ_1, μ_2 distinct from $\lambda=\zeta_\alpha u^\alpha/\xi_\beta u^\beta$. It can be seen by direct calculation that

$$(0,\, p_n,\, -p_T,\, 0^\alpha,\, 0^\alpha);\qquad (0,\, 0,\, -b^2,\, 0^\alpha,\, p_n b^\alpha);$$
$$(0,\, 0,\, 0,\, b^\alpha,\, b^2 u^\alpha);\qquad (0,\, 0,\, 0,\, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \varphi^\delta,\, 0^\alpha);$$
$$(0,\, 0,\, 0,\, 0^\alpha,\, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \varphi^\delta)$$

are eigenvectors for the system (34) corresponding to the eigenvalue $\lambda=\zeta_\alpha u^\alpha/\xi_\beta u^\beta$;

$$(u_\mu \varphi^\mu(\pm 1),\, 0,\, 0,\, 0^\alpha,\, h^\alpha_\mu \varphi^\mu(\pm 1))$$

are eigenvectors corresponding to $\lambda=\pm 1$;

$$(0,\, t_1(\mu_i),\, \mathrm{N}_1(\mu_i),\, \mathrm{U}^\alpha_1(\mu_i),\, \mathrm{B}^\alpha_1(\mu_i))$$

for $i=1, 2$ are eigenvectors corresponding to the other two roots of $(35)_4$.

Obviously all these eigenvectors are linearly independent, $\varepsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol and $h^\alpha_\mu=g^\alpha_\mu+u^\alpha u_\mu$. Consequently we have that condition 2) of hyperbolicity is satisfied in this case. In the following cases we have $g(\xi_\alpha, \zeta_\alpha)\neq 0$ and therefore the root of $(35)_1$ is distinct from those of $(35)_{2,3,4}$. In these cases eq. $(35)_3$ has two real and distinct roots. In fact in Σ we have $\mathrm{A}(\zeta_0/\xi_0)<0$ and the coefficient of λ^2 is $(e+p)\xi_0^2+b^2(1+\xi_2^2)>0$; henceforth I shall refer to them as λ_1 and λ_2 and they are such that $\lambda_1<\zeta_0/\xi_0<\lambda_2$.

Eqs. $(35)_2$ and $(35)_3$ have no common root, otherwise in Σ we can obtain $(\varphi_0)^2$ from $(35)_2$ and substitute it in $(35)_3$ that becomes $(e+p+b^2)(\varphi_2^2+\varphi_3^2)+(e+p)\varphi_1^2\neq 0$. Eqs. $(35)_2$ and $(35)_4$ have a common root only if $e+p=np_n+\dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}$; in this case the roots of $(35)_4$ are ± 1 and λ_1, λ_2. I divide this case in the following subcases $2a$, $2b$, $2c$.

Case $2a$: $g(\xi_\alpha, \zeta_\alpha)\neq 0$; $e+p=np_n+\dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}$;

$$f[\varphi^\alpha(\lambda_1)]\, f[\varphi^\alpha(\lambda_2)]\neq 0.$$

In this case we have the following eigenvectors

$$(0,\, p_n,\, -p_T,\, 0^\alpha,\, 0^\alpha)\qquad \text{corresponding to } \zeta_\alpha u^\alpha/\xi_\beta u^\beta;$$
$$(u^\mu \varphi_\mu(\pm 1),\, 0,\, 0,\, 0^\alpha,\, h^\alpha_\mu \varphi^\mu(\pm 1)),$$
$$(0,\, t_1(\pm 1),\, \mathrm{N}_1(\pm 1),\, \mathrm{U}^\alpha_1(\pm 1),\, \mathrm{B}^\alpha_1(\pm 1))\qquad \text{corresponding to } \lambda=\pm 1;$$

$$(0,\ t_2(\lambda_i),\ N_2(\lambda_i),\ U_2^\alpha(\lambda_i),\ B_2^\alpha(\lambda_i)),$$
$$(0,\ 0,\ 0,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \varphi^\delta(\lambda_i)\, \varphi^\mu(\lambda_i)\, u_\mu,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \varphi^\delta(\lambda_i)\, \varphi^\mu(\lambda_i)\, b_\mu),$$

for $i=1, 2$.

They are *l.i.* because u^β, b^β, $\varphi^\beta(\lambda_i)$ are *l.i.* as consequence of the condition $f[\varphi^\alpha(\lambda_i)]\neq 0$.

Case 2 *b* : $g(\xi_\alpha, \zeta_\alpha)\neq 0;\ e+p=np_n+\dfrac{T(P_T)^2}{e_T};\ f(\xi_\alpha)\neq 0;$

$$f[\varphi^\alpha(\lambda_1)]\, f[\varphi^\alpha(\lambda_2)]=0.$$

In the reference Σ these hypotesis give that $\zeta_3=0$, $\xi_2\neq 0$ and moreover ζ_2/ξ_2 coincides with λ_1 or λ_2. Let us indicate λ_1, λ_2 with ζ_2/ξ_2, λ^*; therefore we obtain $f[\varphi^\alpha(\lambda^*)]\neq 0$.

The eigenvectors are

$$(0,\ p_n,\ -p_T,\ 0^\alpha,\ 0^\alpha) \qquad \text{corresponding to } \zeta_\alpha u^\alpha/\xi_\beta u^\beta;$$
$$(u^\mu \varphi_\mu(\pm 1),\ 0,\ 0,\ 0^\alpha,\ h^\alpha_\mu \varphi^\mu(\pm 1)),$$
$$(0,\ t_1(\pm 1),\ N_1(\pm 1),\ U_1^\alpha(\pm 1),\ B_1^\alpha(\pm 1)) \qquad \text{corresponding to } \lambda=\pm 1;$$
$$(0,\ t_2(\lambda^*),\ N_2(\lambda^*),\ U_2^\alpha(\lambda^*),\ B_2^\alpha(\lambda^*)),$$
$$(0,\ 0,\ 0,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda^*)\, u_\mu,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda^*)\, b_\mu),$$

corresponding to $\lambda=\lambda^*$;

$$(0,\ 0,\ 0,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \xi^\delta \varphi^\mu(\zeta_2/\xi_2)\, u_\mu,\ \varepsilon^\alpha_{\beta\gamma\delta}\, u^\beta b^\gamma \xi^\delta \varphi^\mu(\zeta_2/\xi_2)\, b_\mu),$$
$$(0,\ 0,\ 0,\ (\xi^\mu h^\alpha_\mu - b^{-2}\xi^\mu b_\mu b^\alpha)\, u_\nu \varphi^\nu(\zeta_2/\xi_2), (\xi^\mu h^\alpha_\mu - b^{-2}\xi^\mu b_\mu b^\alpha)\, b_\nu \varphi^\nu(\zeta_2/\xi_2))$$

corresponding to $\lambda=\zeta_2/\xi_2$.

Case 2 *c*: $g(\xi_\alpha, \zeta_\alpha)\neq 0;\ e+p=np_n+\dfrac{T(P_T)^2}{e_T};\ f(\xi_\alpha)=0;$

$$f[\varphi^\alpha(\lambda_1)]\, f[\varphi^\alpha(\lambda_2)]=0.$$

In the reference Σ these hypotesis give $\xi_2=0$, $\zeta_2=\zeta_3=0$; therefore this is a particular case of case (3 *a*) to which I defer.

Let us now see when eqs. $(35)_3$ and $(35)_4$ have a common root. To this end we can consider the identity that in the reference Σ is expressed by

$$\begin{aligned} b^2 N_4(\lambda) = {} & [A(\lambda)]^2\left(np_n+\frac{T(P_T)^2}{e_T}\right) \\ & + A(\lambda)\left\{\varphi_0^2\left[b^2(e+p)-\left(np_n+\frac{T(P_T)^2}{e_T}\right)(e+p+b^2)\right]\right. \\ & \left. - b^2\left(np_n+\frac{T(P_T)^2}{e_T}\right)(\varphi_2^2+\zeta_3^2)\right\} \\ & - b^4\left(e+p-np_n-\frac{T(P_T)^2}{e_T}\right)\varphi_0^2(\varphi_2^2+\zeta_3^2); \end{aligned} \tag{36}$$

from this expression it can be seen that another case in which $(35)_3$ and $(35)_4$ have a common root is

Case 3 a: $g(\xi_\alpha, \zeta_\alpha) \neq 0; f(\xi_\alpha) = 0; f(\zeta_\alpha) = 0$.

In the reference Σ we have $\xi_2 = \zeta_2 = \zeta_3 = 0$; $\zeta_0 \xi_1 - \xi_0 \zeta_1 \neq 0$; the roots of $(35)_4$ are those of $(35)_3$ and those of

$$(e+p) b^2 (\varphi^\alpha u_\alpha)^2 - \left(np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T} \right) (\varphi^\alpha b_\alpha)^2 = 0.$$

This is a second degree equation in the unknown λ with

$$\Delta/4 = (e+p) b^4 \left(np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T} \right) (\zeta_0 \xi_1 - \xi_0 \zeta_1)^2 > 0$$

and therefore it has two real and distinct roots λ_3, λ_4. The eigenvectors are

$$(0, p_n, -p_T, 0^\alpha, 0^\alpha) \qquad \text{corresponding to } \zeta_\alpha u^\alpha / \xi_\beta u^\beta;$$

$$(u^\mu \varphi_\mu (\pm 1), 0, 0, 0^\alpha, h^\alpha_\mu \varphi^\mu (\pm 1)), \qquad \text{corresponding to } \lambda = \pm 1;$$

$$(0, -\mathrm{T} p_T \varphi^\mu (\lambda_i) b_\mu, -ne_\mathrm{T} \varphi^\mu (\lambda_i) b_\mu, e_\mathrm{T} \varphi^\mu (\lambda_i) u_\mu b^\alpha, b^2 e_\mathrm{T} \varphi^\mu (\lambda_1) u_\mu \mathrm{u}^\alpha)$$

for $i = 3, 4$.

$$(0, 0, 0, \varphi^\mu (\lambda_i) u_\mu d^\alpha_1, \varphi^\mu (\lambda_i) b_\mu d^\alpha_1),$$

$$(0, 0, 0, \varphi^\mu (\lambda_i) u_\mu d^\alpha_2, \varphi^\mu (\lambda_i) b_\mu d^\alpha_2), \qquad \text{for } i = 1, 2$$

where d^α_1, d^α_2 are two linearly independent solutions of $d^\nu b_\nu = d^\nu u_\nu = 0$.

It can be easily seen that these 9 eigenvectors are *l. i.* even if λ_3, λ_4 may coincide with λ_1, λ_2. This eventuality occurs when

$$b^2 \left(e + p - np_n - \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T} \right) = (e+p) \left(np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T} \right).$$

In the present case $3a$ we have never used the hypothesis $e + p > np_n + \dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}$; consequently case $2c$ has been also proved as a particular case of this one.

The other cases remain with $e + p > np_n + \dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}$.

From eq. (36) we have also that eqs. $(35)_3$ and $(35)_4$ have a common root in the above considered cases $2a$, $2b$, $2c$, $3a$ and in the forthcoming cases $3b$, $3c$.

Case 3 b:

$$g(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad (e+p) b^2 = \left(np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T} \right) (e + p + b^2);$$

$$f(\xi_\alpha) \neq 0; \qquad f(\xi_\alpha, \zeta_\alpha) = 0; \qquad f[\varphi^\alpha (\lambda_1)] f[\varphi^\alpha (\lambda_2)] = 0.$$

In the reference Σ we have $\xi_2 \neq 0$; $\zeta_3 = 0$; $\zeta_0 \xi_1 - \xi_0 \zeta_1 \neq 0$; one of the roots λ_1, λ_2 is equal to ζ_2/ξ_2 and let $\lambda_2 = \zeta_2/\xi_2$. Eq. $(35)_4$ has the root λ_2 with multiplicity 2 and the solutions of

$$\mathrm{P}(\lambda) = [(e+p)\xi_0^2 + b^2(1+\xi_2^2)]^2(\lambda_1 - \lambda)^2 - b^2\xi_2^2(e+p)(\zeta_0 - \lambda\xi_0)^2 \\ - \xi_2 b^2 [(e+p)\xi_0^2 + b^2(1+\xi_2^2)](\zeta_2 - \lambda\xi_2)(\lambda_1 - \lambda) = 0.$$

By using $\lambda_2 = \zeta_2/\xi_2$, we can obtain λ_1 from $(35)_3$ and after that

$$\lambda_1 - \lambda_2 = 2\xi_2^{-1}[(e+p)\xi_0^2 + b^2(1+\xi_2^2)]^{-1}[\xi_0(\zeta_0\xi_2 - \zeta_2\xi_0)(e+p+b^2) \\ - \xi_1 b^2(\zeta_1\xi_2 - \zeta_2\xi_1)];$$

this relation can be used to find $\mathrm{P}(\lambda_2)$ where we can substitute

$$b(\zeta_1\xi_2 - \zeta_2\xi_1) = \pm(\zeta_0\xi_2 - \zeta_2\xi_0)(e+p+b^2)^{1/2} \text{ [see } (35)_3],$$

obtaining

$$\xi_2^2 \mathrm{P}(\lambda_2) = (\zeta_0\xi_2 - \zeta_2\xi_0)^2(\mathrm{M}+\mathrm{N})$$

with

$$\mathrm{M} = 4\xi_0^2(e+p)^2 + 4\xi_0^2 b^4 + 7\xi_0^2(e+p)b^2 + 4\xi_1^2 b^2(e+p+b^2) \\ + (1+\xi_1^2)(e+p)b^2; \quad (37)$$

$$\mathrm{N} = \mp 8(e+p+b^2)^{3/2}\xi_0\xi_1 b. \quad (38)$$

By using (37), (38) and $\xi_0^2 = 1 + \xi_1^2 + \xi_2^2$ we find

$$\mathrm{M}^2 - \mathrm{N}^2 = 16(e+p+b^2)^4 + 8\xi_2^2(e+p+b^2)[4(e+p)^2 + 4b^4 + 7(e+p)b^2] \\ + 32\xi_1^2(e+p)(e+p+b^2)^3 + \xi_2^4\{b^4(e+p)^2 + 8(e+p+b^2)^2[2(e+p)^2 \\ + 2b^4 + 3(e+p)b^2]\} + 8\xi_1^2\xi_2^2(e+p)(e+p+b^2)[2b^4 + 4(e+p)^2 \\ + 7b^2(e+p)] + 16\xi_1^4(e+p)^2(e+p+b^2)^2 > 0$$

from which $\mathrm{M} > |\mathrm{N}|$, $\mathrm{P}(\lambda_2) > 0$ and therefore λ_2 is not a root of $\mathrm{P}(\lambda)$; it has only multiplicity 2 for the equation $(35)_4$. Moreover we have $\mathrm{P}(\lambda_1) = -b^2(e+p)(\zeta_0\xi_2 - \zeta_2\xi_0)^2 < 0$ and that the coefficient of λ^2 in $\mathrm{P}(\lambda)$ is $[(e+p)\xi_0^2 + b^2]^2 + b^4\xi_2^2 > 0$.

Therefore $\mathrm{P}(\lambda)$ has two real and distinct roots λ_3, λ_4 which are distinct also from λ_1, λ_2. The eigenvectors are

$$(0, p_n, -p_T, 0^\alpha, 0^\alpha) \qquad \text{corresponding to } \zeta_\alpha u^\alpha/\xi_\beta u^\beta;$$

$$(u^\mu \varphi_\mu(\pm 1), 0, 0, 0^\alpha, h_\mu^\alpha \varphi^\mu(\pm 1)), \qquad \text{corresponding to } \lambda = \pm 1;$$

$$(0, 0, 0, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda_i) u_\mu, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda_i) b_\mu),$$

corresponding to $\lambda = \lambda_1, \lambda_2$;

$$(0, 0, 0, (\xi^\mu h_\mu^\alpha - b^{-2}\xi^\mu b_\mu b^\alpha) u_\nu \varphi^\nu(\lambda_2), (\xi^\mu h_\mu^\alpha - b^{-2}\xi^\mu b_\mu b^\alpha) b_\nu \varphi^\nu(\lambda_2))$$

$$(0, -\mathrm{T} p_T \varphi_\mu(\lambda_2) b^\mu, -n e_T \varphi_\mu(\lambda_2) b^\mu, e_T \varphi_\mu(\lambda_2) u^\mu b^\alpha, b^2 e_T \varphi_\mu(\lambda_2) u^\mu u^\alpha)$$

corresponding to $\lambda = \lambda_2$;

$$(0, t_1(\lambda_i), \mathrm{N}_1(\lambda_i), \mathrm{U}_1^\alpha(\lambda_i), \mathrm{B}_1^\alpha(\lambda_i)), \qquad \text{for} \quad i = 3, 4.$$

Case 3 *c*:

$$\left.\begin{array}{c} g(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad f(\xi_\alpha) \neq 0; \\ f(\xi_\alpha, \zeta_\alpha) = 0; \qquad e+p > np_n + \dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}; \\ (e+p+b^2)\left(np_n + \dfrac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right) \neq b^2(e+p); \\ f[\varphi^\alpha(\lambda_1)]\, f[\varphi^\alpha(\lambda_2)] = 0. \end{array}\right\} \tag{39}$$

In the reference Σ we have $\xi_2 \neq 0$, $\zeta_3 = 0$ and moreover $\mathrm{A}(\zeta_2/\xi_2) = 0$; $\mathrm{N}_4(\zeta_2/\xi_2) = 0$. By using eq. (36) and condition (39) we have that ζ_2/ξ_2 is not a double root of $\mathrm{N}_4(\lambda)$. If we call λ^* the root of $\mathrm{A}(\lambda)$ distinct from ζ_2/ξ_2, we have $\mathrm{N}_4(\zeta_0/\xi_0) > 0$; $\mathrm{N}_4(\lambda^*) < 0$ and $\lim_{\lambda \to \pm\infty} \mathrm{N}_4(\lambda) = +\infty$; as consequence there are two real and distinct roots of $\mathrm{N}_4(\lambda)$ that are lying in the half line delimited by ζ_0/ξ_0 and where λ^* lies. In the other half line there is the root ζ_2/ξ_2 with multiplicity 1 and therefore another simple root of $\mathrm{N}_4(\lambda)$ lies in this half line. Complessively, we have found that $\mathrm{N}_4(\lambda)$ has the real and distinct roots ζ_2/ξ_2, λ_3, λ_4, λ_5; $\mathrm{A}(\lambda)$ has the roots ζ_2/ξ_2, and λ^*. The eigenvectors are

$$(0, p_n, -p_T, 0^\alpha, 0^\alpha) \qquad \text{corresponding to } \zeta_\alpha u^\alpha/\xi_\beta u^\beta;$$

$$(u^\mu \varphi_\mu(\pm 1), 0, 0, 0^\alpha, h^\alpha_\mu \varphi^\mu(\pm 1)), \qquad \text{corresponding to } \lambda - \pm 1;$$

$$(0, t_1(\lambda_i), \mathrm{N}_1(\lambda_i), \mathrm{U}^\alpha_1(\lambda_i), \mathrm{B}^\alpha_1(\lambda_i)), \qquad \text{for } i = 3, 4, 5;$$

$$(0, 0, 0, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda_i) u_\mu, \varepsilon^\alpha_{\beta\gamma\delta} u^\beta b^\gamma \xi^\delta \varphi^\mu(\lambda_i) b_\mu),$$

corresponding to $\lambda = \lambda^*$, ζ_2/ξ_2;

$$(0, 0, 0, (\xi^\mu h^\alpha_\mu - b^{-2} \xi^\mu b_\mu b^\alpha) u_\nu \varphi^\nu(\zeta_2/\xi_2), (\xi^\mu h^\alpha_\mu - b^{-2} \xi^\mu b_\mu b^\alpha) b_\nu \varphi^\nu(\zeta_2/\xi_2))$$

corresponding to ζ_2/ξ_2.

There remain now the cases in which eqs. $(35)_i$ and $(35)_j$ have no common root for $i, j = 1, \ldots, 4$; $i \neq j$. They are

Case 4 *a:*

$$g(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad f(\xi_\alpha) = 0;$$
$$f(\zeta_\alpha) \neq 0; \qquad e+p > np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T};$$

Case 4 *b:*

$$g(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad f(\xi_\alpha) \neq 0;$$
$$f(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad e+p > np_n + \frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T};$$

Case 4 *c:*

$$g(\xi_\alpha, \zeta_\alpha) \neq 0; \qquad f(\xi_\alpha) \neq 0;$$
$$f(\xi_\alpha, \zeta_\alpha) = 0; \qquad e+p > np_n + \frac{\mathrm{T}(\mathrm{P}_\mathrm{T})^2}{e_\mathrm{T}};$$
$$f[\varphi^\alpha(\lambda_1)]\, f[\varphi^\alpha(\lambda_2)] \neq 0.$$

We have already proved that the roots of $(35)_{1,2,3}$ are all real and distinct; we have to examine now the roots of $(35)_4$. The coefficient of λ^4 in eq. $(35)_4$ is

$$\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)[(e+p)\xi_0^2+b^2\xi_1^2]+(e+p)b^2(1+\xi_2^2)+(e+p)\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\xi_0^2>0;$$

moreover we have that

$$\mathrm{N}_4(\zeta_0/\xi_0)=\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)b^2\xi_0^{-4}(\zeta_1\xi_0-\zeta_0\xi_1)^2[(\zeta_1\xi_0-\zeta_0\xi_1)^2+(\zeta_2\xi_0-\zeta_0\xi_2)^2+(\zeta_3\xi_0)^2]>0$$

and, from eq. (36) we have $\mathrm{N}_4(\lambda_{1,2})<0$. We have so obtained

$$\lim_{\lambda\to-\infty}\mathrm{N}_4(\lambda)=+\infty;\qquad \mathrm{N}_4(\lambda_1)<0;\qquad \mathrm{N}_4(\zeta_0/\xi_0)>0;$$
$$\mathrm{N}_4(\lambda_2)<0;\qquad \lim_{\lambda\to+\infty}\mathrm{N}_4(\lambda)=+\infty$$

and therefore $\mathrm{N}_4(\lambda)$ has four real and distinct roots λ_3, λ_4, λ_5, λ_6. The eigenvectors are

$$(0,\ p_n,\ -p_T,\ 0^\alpha,\ 0^\alpha)\qquad \text{corresponding to } \zeta_\alpha u^\alpha/\xi_\beta u^\beta;$$
$$(u^\mu\varphi_\mu(\pm 1),\ 0,\ 0,\ 0^\alpha,\ h^\alpha_\mu\varphi^\mu(\pm 1)),\qquad \text{corresponding to } \lambda=\pm 1;$$
$$(0,\ 0,\ 0,\ \varepsilon^\alpha_{\beta\gamma\delta}u^\beta b^\gamma\varphi^\delta(\lambda_i)\varphi^\mu(\lambda_i)u_\mu,\ \varepsilon^\alpha_{\beta\gamma\delta}u^\beta b^\gamma\varphi^\delta(\lambda_i)\varphi^\mu(\lambda_i)b_\mu),$$

for $i=1, 2$;

$$(0,\ t_1(\lambda_i),\ \mathrm{N}_1(\lambda_i),\ \mathrm{U}_1^\alpha(\lambda_i),\ \mathrm{B}_1^\alpha(\lambda_i)),\qquad \text{for}\quad i=3, 4, 5, 6.$$

Therefore the condition 2) of hyperbolicity is satisfied in every case.

From the above considerations it is also clear that $\det(\mathrm{A}^\alpha\varphi_\alpha)=0$ is equivalent to $a(\lambda)\,\mathrm{G}(\lambda)\,\mathrm{A}(\lambda)\,\mathrm{N}_4(\lambda)=0$; replacing φ_α with ξ_α we obtain that $\det(\mathrm{A}^\alpha\xi_\alpha)\neq 0$ is equivalent to

$$(\xi_\alpha u^\alpha)(-1)[(e+p+b^2)(\xi_\alpha u^\alpha)^2-(\xi_\alpha b^\alpha)^2]\times\left\{(e+p)\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)(\xi_\alpha u^\alpha)^4-(e+p)\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}+b^2\right)(\xi_\alpha u^\alpha)^2(-1)+\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)(\xi_\alpha b^\alpha)^2(-1)\right\}\neq 0.$$

and this is satisfied because the first member of this equation can be expressed as

$$-\xi_0\left[(e+p+b^2)(1+\xi_2^2)+(e+p)\xi_1^2\right] \times\left\{(e+p)\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\xi_0^4 + (e+p)\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\xi_0^2+b^2\left(e+p-np_n-\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)\xi_0^2 + b^2\left(np_n+\frac{\mathrm{T}(\mathrm{P}_T)^2}{e_T}\right)(1+\xi_2^2)\right\}\neq 0.$$

Therefore also the condition 1) of hyperbolicity is satisfied.

4. CONCLUSIONS

The results here obtained are very satisfactory; in fact I have found an hyperbolic system (3) which is also equivalent to a symmetric hyperbolic one. This property is very important because guarantees the well posedness of the Cauchy problem for smooth initial data, *i.e.* existence, uniqueness and continuous dependence in a neighbourhood of the initial manifold F [17]. Moreover if we impose on this manifold $x=0$ and the Maxwell's equations $\partial_i(u^i b^0 - b^i u^0)=\partial_\alpha(u^\alpha b^0 - b^\alpha u^0)=0$, we obtain that $x=0$ will propagate nicely off F and therefore the system 3) will have the same solutions of the ordinary system of equations (1) of relativistic magnetofluiddynamics.

I think also that a similar treatment can be used for many other systems with constrained fields ([3], [4]), by introducing an auxiliary independent variable for every constraint of the system.

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