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The Classical Limit of a Class of Quantum Dynamical Semigroups

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ABSTRACT. — A class of quantum dynamical semigroups is proved to possess a residual phase-space stochasticity, if a weak-coupling limit is performed, whereas the usual limit in which $\hbar \to 0$ is completely deterministic. As a consequence, solutions of the corresponding Chapman-Kolmogorov differential equation, with linear Liouville term, are constructed.

RÉSUMÉ. — Il est prouvé qu'une classe de semi-groupes d'évolution pour des systèmes quantiques possède des propriétés stochastiques résiduelles dans l'espace des phases, lorsque l'on effectue une limite de faible couplage, alors que la limite $\hbar \to 0$ est totalement déterministe. Comme conséquence, il est possible de construire explicitement des solutions à l'équation différentielle de Chapman-Kolmogorov correspondant à ces évolutions et contenant un terme de Liouville linéaire.

1. INTRODUCTION

We shall consider a non-relativistic one-dimensional one-particle quantum system described by a state operator, or density matrix, $\hat{\rho}$ and a

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Hamiltonian $\hat{H}_\hbar$ acting on a separale Hilberrt space $\mathscr{H}$; $\hat{q}_\hbar$, $\hat{p}_\hbar$ will denote position and momentum operators fulfilling $[\hat{q}_\hbar, \hat{p}_\hbar]=i\hbar$. Beside evolving unitarily, the system is assumed to be randomly spatially localized with accuracy $\sqrt{\alpha}$, according to:

$$(1.1)\quad \hat{\rho}\to T[\hat{\rho}]=\sqrt{\frac{\alpha}{\pi}}\int_{\mathbf{R}} \exp\left(-\frac{\alpha}{2}(\hat{q}_\hbar-x^2)^2\right)\hat{\rho}\exp\left(-\frac{\alpha}{2}(\hat{q}_\hbar-x)^2\right)dx,$$

the process $\hat{\rho}\to T[\hat{\rho}]$ occuring with mean frequency λ. Conservation of probability in the course of evolution is accounted for by:

$$(1.2)\qquad \partial_t\hat{\rho}_t=-\frac{i}{\hbar}[\hat{H}_\hbar, \hat{\rho}_t]-\lambda\hat{\rho}_t+\lambda T[\hat{\rho}_t],$$

this is the ensemble evolution equation of the so called Ghirardi-Rimini-Weber (G.R.W.-model) ([1], [2]). The r.h.s. of (1.2) is the Lindblad type generator [3] of a quantum dynamical semigroup $\{\gamma_t\}_{t\geq 0}$, whose properties are quite well-known ([3], [4]) and suffices here to mention that it consists of completely positive, trace preserving, contractions on the trace-class operators. The term

$$(1.3)\qquad \langle q|T[\hat{\rho}]|\bar{q}\rangle=\exp\left(-\frac{\alpha}{4}(q-\bar{q})^2\right)\langle q|\hat{\rho}|\bar{q}\rangle,$$

appeared already in the literature [5] as a model of position measuring gaussian device. From (1.3) we deduce that states which are largely delocalized with respect to $\dfrac{1}{\sqrt{\alpha}}$ have off-diagonal elements which are damped by the presence of $-\lambda\hat{\rho}_t$ in (1.2). On the other hand those for which $|q-\bar{q}|\simeq\dfrac{1}{\sqrt{\alpha}}$ are nearly unaffected. Far-away localized states become thus disentangled as time passes and this mechanism is the starting point of a sensible theory that tries to overcome, on entirely physical grounds, the puzzling situations connected with the broad concept of reduction of the wave packet ([1], [2]). At most quadratic Hamiltonians have been used in [6], [7] to give explicit solutions of the G.R.W-model and provide quite a similarity with a phase space markoffian stochastic process. It is preferable to introduce here some notations and results that serve to make the point clear. Let

$$(1.4)\qquad \hat{W}_\hbar(q, p)=e^{(i/\hbar)(q\hat{p}_\hbar+p\hat{q}_\hbar)}=e^{(i/\hbar)q\hat{p}_\hbar}e^{(i/\hbar)p\hat{q}_\hbar}e^{(i/2\hbar)qp}$$

be Weyl operators satisfying the algebraic rules:

$$(1.5)\quad \hat{W}_\hbar(q_1, p_1)\hat{W}_\hbar(q_2, p_2)=\hat{W}_\hbar(q_1+q_2, p_1+p_2)e^{(i/\hbar)(q_1p_2-p_1q_2)}$$

and giving:

$$(1.6) \qquad (\hat{W}_\hbar(q,p)\psi)(x) = e^{(i/\hbar)((qp/2)+px)}\psi(q+x)$$

when represented a la' Schrödinger on $L^2(R, dx)$.

Let $U_\hbar(t) = e^{(i/\hbar)\hat{H}_\hbar t}$ be the unitary evolutor corresponding to a Hamiltonian that sends $(\hat{q}_\hbar, \hat{p}_\hbar)$ into $(a(t)\hat{q}_\hbar + b(t)\hat{p}_\hbar,\ c(t)\hat{q}_\hbar + d(t)\hat{p}_\hbar)$, then [7] the solution of (1.3) with initial condition $\hat{\rho}$ is:

$$(1.7) \quad \hat{\rho}_t = \frac{1}{(2\pi\hbar)^2}\int_{R^2} dx\, dy\, d\xi\, d\pi \\ \times \{ e^{(i/\hbar)(\xi y - \pi x)} F(\alpha, \xi, \pi, t)\hat{W}_\hbar(x, -y)U_\hbar(t)\hat{\rho}U_\hbar(-t)\hat{W}_\hbar(-x, y)\}$$

where

$$(1.8) \quad F(\alpha, \xi, \pi, t) = \exp\left\{-\lambda t + \lambda \int_0^t ds \exp\left(-\frac{\alpha}{4}\xi^2_{-s}(\xi,\pi)\right)\right\}$$

and $\xi_{-s}(\xi, \pi) = a(-s)\xi + b(-s)\pi$.

By duality we easily derive from $\operatorname{Tr}\hat{\rho}_t\hat{W}_\hbar(q,p) = \operatorname{Tr}\hat{\rho}\hat{W}^t_\hbar(q,p)$ that:

$$(1.9) \qquad \hat{W}^t_\hbar(q, -p) = F(\alpha, q, p, t)\hat{W}_\hbar(q_{-t}(q,p), -p_{-t}(q,p)).$$

Because of linearity we can now try what kind of evolution equation is satisfied by mean values of $\hat{W}^t_\hbar(q, -p)$ in vector states, thus following the old Ehrenfest's argument. For future purposes we introduce the class of coherent states ([8], [9], [10]) based on the minimal indeterminacy state

$$\psi^\hbar_0(x) = \frac{1}{\sqrt[4]{\pi\hbar}}e^{-(x^2/2\hbar)}$$

given by:

$$(1.10) \qquad |Q, P\rangle = \hat{W}_\hbar(-Q, P)|\psi^\hbar_0\rangle$$

and satisfying:

$$(1.11) \quad \langle Q_1, P_1 | Q_2, P_2\rangle = e^{(i/2\hbar)(Q_1P_2 - P_1Q_2)}e^{-(1/4\hbar)[(Q_1-Q_2)^2+(P_1-P_2)^2]}.$$

Then we easily calculate

$$\langle Q, P|\hat{W}^t_\hbar(q, -p)|Q, P\rangle \equiv E^\hbar_{q,p}(Q, P, t) \\ = F(\alpha, q, p, t)e^{(i/\hbar)(qP_t(Q,P) - pQ_t(Q,P))}e^{-(1/4\hbar)[q^2_{-t}(q,p)+p^2_{-t}q,p)]}$$

and

$$(1.12) \quad \partial_t E^\hbar_{q,p}(Q, P, t) = -\lambda E^\hbar_{q,p}(Q, P, t) + \lambda e^{-(\alpha/4)q^2_{-t}(q,p)}E^\hbar_{q,p}(Q, P, t) \\ + \{E^\hbar_{q,p}(Q, P, t), H(Q, P)\} \\ + \{\partial_t e^{-(1/4\hbar)[q^2_{-t}(q,p)+p^2_{-t}(q,p)]}\}F(\alpha, q, p, t)E^\hbar_{q,p}(Q, P, t)$$

where the first term in the second line is he Poisson bracket with the classical Hamiltonian. We make now the following

Remarks 1.1. – 1. The term $e^{(-(\alpha/4)\, q^2_{-t}(q,\, p))}\, E^{\hbar}_{q,\, p}(Q,\, P,\, t)$, by Fourier transforming, reads as:

$$\sqrt{\frac{1}{\alpha\pi\hbar^2}}\int_{\mathbb{R}} dy\, e^{-(1/\alpha\hbar^2)\, y^2}\, E^{\hbar}_{q,\, p}(Q,\, P+y).$$

To the evolution of the $\hbar$-dependent phase space function $E^{\hbar}_{q,\, p}(Q,\, P,\, t)$ various are the contributions, among them a gaussianly distributed momentum kick that, together with the damping term, forms the momentum jump process

$$\sqrt{\frac{1}{\alpha\pi\hbar^2}}\int_{\mathbb{R}} dy\, e^{-(1/\alpha\hbar^2)\, y^2}\, [E^{\hbar}_{q,\, p}(Q,\, P+y,\, t)-E^{\hbar}_{q,\, p}(Q,\, P,\, t)].$$

2. Suppose $\lambda=0$, then the quantum evolution (1.2) becomes purely Hamiltonian and this is indeed reflected by the Poisson bracket in (1.12). The classical limit $\hbar\to 0$ should get rid of the unwanted third line and at the same time provide a meaningful $E^{\hbar\to 0}_{q,\, p}(Q,\, P,\, t)$. For instance we could consider rescaled translations, $\hat{W}_{\hbar}(\hbar q,\, \hbar p)$, instead of $\hat{W}_{\hbar}(q,\, p)$ so that:

$$\begin{aligned} E^{\hbar}_{\hbar q,\, \hbar p}(Q,\, P,\, t) &= F(\alpha\hbar^2,\, q,\, p,\, t)\, e^{i\,[q\, P_t(Q,\, P)-p\, Q_t(Q,\, P)]}\, e^{-(\hbar/4)\,[q^2_{-t}(q,\, p)+p^2_{-t}(q,\, p)]} \\ &\to e^{i\,[qP_t(Q,\, P)-pQ_t(Q,\, P)]}\equiv E^{t}_{q,\, p}(Q,\, P) \end{aligned}$$

Through the linearity of the classical (and quantum) evolution: $\Phi^{H}_t(Q,\, P)=(Q_t(Q,\, P),\, P_t(Q,\, P)$, and the above rescaling we are thus led to a pure Liouvillian equation of motion.

3. In the above procedure the damping function $F(\alpha,\, q,\, p,\, t)$ as well as the jump process disappear: we can attempt the following interpretation.

The localization process of which $\hat{W}^{t}_{\hbar}(q,\, -p)$ is a result, the greater is the accuracy $\sqrt{\alpha}$, the more feels the distance between the mean positions around which the gaussian states $|Q,\, P\rangle$ and $\hat{W}_{\hbar}(q,\, -p)\,|Q,\, P\rangle$ are localized. To remain relevant along with $\hat{W}_{\hbar}(\hbar q,\, \hbar p)\,|Q,\, P\rangle\to|Q,\, P\rangle$ when $\hbar\to 0$, the mechanism should accordingly increase its accuracy as $\hbar^{-1}\sqrt{\alpha}$ as can be grasped by looking either at the damping function or at the jump process in the first of these remarks.

The rest of this note aims at showing that the stochastic behaviour inherent in the Lindblad generator of (1.2) can be kept indeed at the classical level through a weak-coupling limit, a fact that meets a general result obtained by Davies [11]. The Chapman-Kolmogorow equation for the phase space distribution that results will contain a jump process giving rise to a gaussianly randomized momentum, in natural agreement with the localizing properties of the quantum counterpart.

2. CLASSICAL LIMIT: GENERAL CONSIDERATIONS

Having noticed that the limit $\hbar \to 0$ alone does eliminate the stochastic behaviour of (1.2), which we want to keep instead, we introduce the symmetrically rescaled representation

$$\left\{\begin{array}{l}\hat{q}_g = g\sqrt{\hbar}\,\hat{q}\\ \hat{p}_g = g\sqrt{\hbar}\,\hat{p}\end{array}\right. \tag{2.1}$$

in which

$$\left\{\begin{array}{l}(\hat{q}\,\psi)(x) = x\,\psi(x)\\ (\hat{p}\,\psi)(x) = -i\,\psi'(x).\end{array}\right. \tag{2.2}$$

on $L^2(\mathrm{R}, dx)$. Lengths and momenta become then g times greater than what they used to be in the physical units, $\hat{\mathrm{H}}_\hbar = \mathrm{H}(\hat{q}_\hbar, \hat{p}_\hbar)$ is to be replaced by $\hat{\mathrm{H}}_g = \mathrm{H}(\hat{q}_g, \hat{p}_g)$, $\hbar$ by $\hbar g^2$ wherever it appears and the accuracy $\sqrt{\alpha}$ by $g^{-1}\sqrt{\alpha}$, thus diverging in the classical limit which now corresponds to $g \to 0$ (*see* [9] p. 410). As the evolutor $e^{i/\hbar\,\hat{\mathrm{H}}_\hbar t}$ goes over into $e^{i/\hbar g^2 t\,\hat{\mathrm{H}}_g}$, the above can be seen as a weak-coupling limit in which the time has been rescaled as tg^{-2} so that to smaller g correspond larger physical times. In the chosen representation the Weyl operators $\hat{\mathrm{W}}(q, p)$ and the minimal indeterminacy state $|\psi_0\rangle$ do not depend on $\hbar$ anymore and to the properties and results in the introduction we only add the overcompleteness of the coherent states $|q, p\rangle = \hat{\mathrm{W}}(q, -p)|\psi_0\rangle$:

$$\int_{\mathrm{R}^2} |q, p\rangle\langle q, p\ \frac{dq\,dp}{2\pi} = 1 \tag{2.3}$$

and the following pretty obvious consequence:

$$\text{(i)}\quad \lim_{g\to 0}\left\langle \frac{\mathrm{Q}}{g\sqrt{\hbar}}, \frac{\mathrm{P}}{g\sqrt{\hbar}}\right|\hat{\mathrm{W}}(g\sqrt{\hbar}\,q, -g\sqrt{\hbar}\,p)\left|\frac{\mathrm{Q}}{g\sqrt{\hbar}}, \frac{\mathrm{P}}{g\sqrt{\hbar}}\right\rangle = e^{i(\mathrm{P}q - \mathrm{Q}p)} \equiv \mathrm{E}_{q,p}(\mathrm{Q}, \mathrm{P})$$

from which we deduce

$$\text{(ii)}\quad \lim_{g\to 0} \mathrm{E}^{\hbar g^2}_{q,p}(\mathrm{Q}, \mathrm{P}, t) = \mathrm{E}^t_{q,p}(\mathrm{Q}, \mathrm{P}) \lim_{g\to 0} \mathrm{F}(\alpha\hbar^2 g^2, q, p, t)$$

see Remark 1.1.2. Indeed, $\hat{\mathrm{W}}_\hbar(q, -p)$ has to be replaced by

$$\hat{\mathrm{W}}_{g^2\hbar}(q, -p) = \hat{\mathrm{W}}\left(\frac{q}{g\sqrt{\hbar}}, -\frac{p}{g\sqrt{\hbar}}\right),$$

so that $\hat{\mathrm{W}}(qg\sqrt{\hbar}, -pg\sqrt{\hbar})$ actually corresponds to $\hat{\mathrm{W}}_{g^2\hbar}(qg^2\hbar, -pg^2\hbar)$ and this must be taken into account in the computation together with the fact that in the new units the accuracy is $\sqrt{\alpha_g} = g^{-1}\sqrt{\alpha}$.

$\alpha \hbar^2 g^2 = \beta$ would then make the stochastic features of the quantum evolution survive the classical limit and we are thus prompted to the choice $g^2 = \frac{\beta}{\alpha \hbar^2}$ and, according to the interpretation given in Remark 1.1.1, $\sqrt{\beta}$ is chosen to have the dimension of a momentum.

Remarks 2.1. — 1. It is to be noticed that the g-rescaled representation allowed us to replace the joint limit $\hbar \to 0$, $\alpha \to +\infty$, $\hbar^2 \alpha = \beta$ with a weak-coupling limit in which is the accuracy $\sqrt{\alpha}$ of the quantum localizing mechanism that diverges together with vanishing distances. Our choice $\sqrt{\beta}$ is, on the other hand, the natural momentum associated with the intrinsic length scale $\frac{1}{\sqrt{\alpha}}$ via the indeterminacy relations, we thus see that the classical stochasticity is a memory of quantal effects.

2. With $g = \frac{1}{\hbar}\sqrt{\frac{\beta}{\alpha}}$ we attack the introductory question and get:

$$E^{\beta}_{q,p}(Q, P, t) \equiv \lim_{\alpha \to +\infty} E^{\beta/\alpha\hbar}_{q,p}(Q, P, t) = F(\beta, q, p, t)\, E^{t}_{q,p}(Q, P)$$

which in turn solves

$$\partial_t E^{\beta}_{q,p}(Q, P, t) = \{ E^{\beta}_{q,p}(Q, P, t), H(Q, p) \} \\ - \lambda E^{\beta}_{q,p}(Q, P, t) + \lambda \frac{1}{\sqrt{\pi\beta}} \int_{-\infty}^{+\infty} dy\, e^{-y^2/\beta} E^{\beta}_{q,p}(Q, P+y, t),$$

with initial condition $E^{\beta}_{q,p}(Q, P, t=0) = E_{q,p}(Q, P)$, *see* (ii) above.

3. By the duality

$$\int_{R^2} dQ\, dP\, \rho_t(Q, P)\, E^{\beta}_{q,p}(Q, P) = \int_{R^2} dQ\, dP\, \rho(Q, P)\, E^{\beta}_{q,p}(Q, P, t)$$

and the density of the exponential fuctions among the continuous functions on R^2 vanishing at infinity ($C_0(R^2)$), we infer the Chapman-Kolmogorov differential equation

$$(24) \quad \partial_t \rho_t(Q, P) = \{ H(Q, P), \rho_t(Q, P) \} \\ - \lambda \rho_t(Q, P) + \lambda \frac{1}{\sqrt{\pi\beta}} \int_{-\infty}^{+\infty} dy\, e^{-y^2/\beta} \rho_t(Q, P+y)$$

on the state space of $L^1(R^2)$-probability distributions [w^*-dense in the dual of $C_0(R^2)$]. (2.4) generates a homogeneous Markov process with transition probability $P(Q, P, t\,|\,Q_0, P_0, t_0)$ satysfying the Markov relation:

$$P(Q, P, t\,|\,Q_0, P_0, t_0) = \int_{R^2} dQ_1\, dP_1\, P(Q, P, t\,|\,Q_1, P_1, t_1)\, P(Q_1, P_1, t_1\,|\,Q_0, P_0, t_0).$$

Note that for small β (2.4) is approximated by the Fokker-Planck equation:

$$\partial_t \rho_t(Q, P) = \{H(Q, P), \rho_t(Q, P)\} + \frac{\lambda\beta}{4}\,\partial_p^2 \rho_t(Q, P)$$

which possesses no stationary solution.

4. From (1.7), after going to the g-rescaled representation and some harmless manipulations, we derive:

$$\left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right| \hat{\rho}_t \left| \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right\rangle = \frac{1}{(2\pi)^2} \int_{R^4} dx\, dy\, d\xi\, d\pi \left\{ e^{i(\xi(y-P)-\pi(x-Q))} F(\beta, \xi, \pi, t) \left\langle \frac{x}{g\sqrt{\hbar}}, \frac{y}{g\sqrt{\hbar}} \right| e^{-(i/\hbar g^2)t\hat{H}_g}\, \hat{\rho}\, e^{-(i/\hbar g^2)t\hat{H}_g} \left| \frac{x}{g\sqrt{\hbar}}, \frac{y}{g\sqrt{\hbar}} \right\rangle \right\}.$$

Because of overcompleteness of coherent states and the positivity and trace preserving properties of quantum dynamical semigroups

$$\rho^g \equiv \left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right| \frac{1}{2\pi\hbar g^2}\, \hat{\rho} \left| \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right\rangle$$

is a g-dependent $\left(g = \frac{1}{\hbar}\sqrt{\frac{\beta}{\alpha}}\right)$ $L^1(R^2, dx\, dy)$-probability distribution.

5. In the following we shall justify the statement that, when

$$\rho(Q, P) = \lim_{\alpha \to +\infty} \rho^g(Q, P)$$

is well defined, then:

$$(2.5)\quad \rho_t(Q, P) \equiv \lim_{\alpha \to +\infty} \rho_t^g(Q, P) = \frac{1}{(2\pi)^2} \int_{R^4} dx\, dy\, d\xi\, d\pi \times \left\{ e^{i[\xi(y-P_{-t}(Q, P))-\pi(x-Q_{-t}(Q, P))]} F(\beta, \xi_t(\xi, \pi), \pi_t(\xi, \pi), t)\, \rho(x, y) \right\}.$$

It is indeed easy to verify that (2.5) solves

$$(2.6)\quad \partial_t \rho_t(Q, P) = \{H(Q, P), \rho_t(Q, P)\} - \lambda\rho_t(Q, P) + \frac{\lambda}{\sqrt{\pi\beta}} \int_{-\infty}^{+\infty} dy\, e^{-(y^2/\beta)}\, \rho_t(Q, P+y)$$

with $\rho_{t=0}(Q, P) = \rho(Q, P)$ as initial condition, and that the transition probability

$$P(Q, P, t \mid Q_0, P_0, t_0) = \frac{1}{(2\pi)^2} \int_{R^2} dQ\, dP$$
$$\times \left\{ e^{i[\xi(P_0 - P_{t_0 - t}(Q, P)) - \pi(Q_0 - Q_{t_0 - t}(Q, P))]} F(\beta, \xi_{t-t_0}(\xi, \pi), \pi_{t-t_0}(\xi, \pi), t - t_0) \right\}$$

satisfies the Markov relation: it is sufficient to point out that

$$F(\beta, \xi_{t-t_0}(\xi, \pi), \pi_{t-t_0}(\xi, \pi), t - t_0)$$
$$= \exp\left\{ -\lambda(t - t_0) + \lambda \int_{t_0}^{t} ds\, e^{-\beta/4\, \xi^2_{s-s_0}(\xi, \pi)} \right\}$$

and that, by assumption, the diffeomorphism Φ_t^H is linear.

6. As to (2.6), we expect it to hold for a large class of Hamiltonians and density matrices. In so far the classical limit is concerned the following result is the best available. Let Φ_t^H be the flow of diffeomorphisms on R^2 generated by the Hamiltonian vector field $X_H = \left(\frac{p}{m}, -V'(q)\right)$, then a more general restatement of the fact that the mean values $\left\langle \frac{Q}{\hbar}, \frac{P}{\hbar} \middle| (\hat{q}_\hbar(t), \hat{p}_\hbar(t)) \middle| \frac{Q}{\hbar}, \frac{P}{\hbar} \right\rangle$ go into the classical solutions $\Phi_t^H(q, p) = (q(t), p(t))$ when $\hbar \to 0$ is the following

THEOREM 2.2 [8]. – *As long as the local flow of diffeomorphisms* $\{\Phi_t^H\}$ *exists and the potential* $V(q)$ *is* δ*-Hölder continuous about the classical trajectory and*

$$\int_R |V(q)| e^{\rho q^2} dq < +\infty \quad \textit{for some } 0 < \rho < \infty,$$

then the following strong operator limit exists:

$$(2.21)\quad s - \lim_{\hbar \to 0} \hat{W}\left(\frac{Q}{\sqrt{\hbar}}, -\frac{P}{\sqrt{\hbar}}\right) e^{(i/\hbar)\hat{H}_\hbar t}$$
$$\times \hat{W}(\sqrt{\hbar}\, q, -\sqrt{\hbar}\, p) e^{-(i/\hbar)\hat{H}_\hbar t} \hat{W}\left(-\frac{Q}{\sqrt{\hbar}}, \frac{P}{\sqrt{\hbar}}\right) = \exp(i[q(t)\pi - p(t)\xi]).$$

3. CLASSICAL LIMIT: DENSITY MATRICES

In order to face the problem of connecting the quantum evolution of states as given in (1.2) to the Chapman-Kolmogorov equation (2.6) for probabilities distributions on R^2, we solve formally (1.2) in the g-rescaled units where g is understood to be $\frac{1}{\hbar}\sqrt{\frac{\beta}{\alpha}}$. According to [7] we have:

$$(3.1)\quad \hat{\rho}_t = e^{-\lambda t}\,\hat{U}_g(-t)\left\{\hat{\rho} + \lambda\int_0^t ds\,\hat{U}_g(s)\,T[\hat{\rho}_s]\,\hat{U}_g(-s)\right\}\hat{U}_g(t)$$

$$= e^{-\lambda t}\,\hat{U}_g(-t)\left\{\sum_{k=0}^{+\infty}\lambda^k\int_0^t ds_{\varkappa}\,\hat{\tau}^k_{s_k}[\hat{\rho}]\right\}\hat{U}_g(t),$$

where:

$$(3.2)\quad \begin{cases} \text{(i)}\ \hat{U}_g(t) = e^{(i/\hbar g^2)\,t\hat{H}_g} \\ \text{(ii)}\ T[\hat{\rho}_s] = \dfrac{1}{\sqrt{\beta\pi}}\displaystyle\int_{\mathbb{R}} dy\, e^{-(y^2/\beta)}\,\hat{W}\left(0, -\frac{y}{g\sqrt{\hbar}}\right)\hat{\rho}_s\,\hat{W}\left(0, \frac{y}{g\sqrt{\hbar}}\right) \\ \text{(iii)}\ \displaystyle\int_0^t ds_0\,\hat{\tau}^0_{s_0}[\hat{\rho}] = \hat{\rho} \\ \text{(iv)}\ \hat{\tau}^k_{s_k}[\hat{\rho}] = \displaystyle\int_0^{s_k} ds_{k-1}\,\hat{U}_g(s_k)\,T[\hat{U}_g(s_k)\,\hat{\tau}^{k-1}_{s_{k-1}}[\hat{\rho}]\,\hat{U}_g(s_k)]\,\hat{U}_g(s_k) \\ \text{(v)}\ \left\|\displaystyle\int_0^t ds_k\,\hat{U}_g(t)\,\hat{\tau}^k_{s_k}[\hat{\rho}]\,\hat{U}_g(t)\right\|_1 \leq \dfrac{t^k}{k!}. \end{cases}$$

Remarks 3.1. – 1. Written in the form 3.1 (ii), which can be checked by Fourier transform, the localization mechanism $\hat{\rho} \to T[\hat{\rho}]$, directly involves a gaussian randomization of the momentum variable and gets to the classical jump whenever the limit of $\left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\middle|\hat{\rho}\middle|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle$ has some meaning for $g \to 0$.

2. 3.1 (iv), (v) tell that the formal series is convergent to a density matrix in the trace-norm topology, thence the following identification makes sense:

$$(3.3)\quad \rho_g(Q, P, t) \equiv \left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\middle|\hat{\rho}_t\middle|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle$$

$$= e^{-\lambda t}\sum_{k=0}^{+\infty}\lambda^k\int_0^t ds_k\left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\middle|\tau^k_{s_k}[\hat{\rho}]\middle|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle.$$

3. $\rho_g(Q, P, t)$ is a g-dependent probability distribution and we look at $g \to 0$ not uniformly in Q, P, but with respect to functions in $C_0(\mathbb{R}^2)$, namely we consider w^*-limits. The very fact that makes $\rho_g(Q, P, t)$ a probability distribution (over-completeness of coherent states), guarantees that the w^*-convergence of the partial sums to it is uniform in g and that

we can, in turn, interchange the limit and the sum in (3.3):

$$(3.4)\quad w^*-\lim_{g\to 0}\rho_g(Q, P, t)=e^{-\lambda t}\sum_{k=0}^{+\infty}\lambda^k\int_0^t ds_k$$
$$\times\left\{w^*-\lim_{g\to 0}\left\langle\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right|\frac{1}{2\pi\hbar g^2}\tau_{s_k}^k[\hat\rho]\left|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle\right\}.$$

To deal with the w^*-limit of the summands in the above series we resort to the following class of density matrices:

$$(3.5)\qquad \hat\rho_\mu=\int_{\mathbb{R}^2}dQ\,dP\,\mu(Q, P)\left|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle\left\langle\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right|$$

with any probability distribution $\mu(Q, P)$ in $C_0(\mathbb{R}^2)$. We shall use their Weyl-transforms [10]:

$$(3.6)\quad \hat\rho_\mu=\int_{\mathbb{R}^2}dQ\,dP\,e^{-(\hbar g^2/4)(Q^2+P^2)}\hat\mu(Q, P)\hat W(g\sqrt{\hbar}Q, -g\sqrt{\hbar}P)$$

where

$$\hat\mu(Q, P)=\frac{1}{2\pi}\int_{\mathbb{R}^2}dx\,dy\,e^{i(xP-yQ)}\mu(x, y).$$

Remarks 3.2. – 1. From the properties of coherent states we get that

$$\rho_\mu^g(Q, P)=\left\langle\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right|\frac{1}{2\pi\hbar g^2}\hat\rho_\mu\left|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle$$

w^*-converges to $\mu(Q, P)$.

2. If $H(Q, P)$ is a Hamiltonian function on $\mathbb{R}^2$ which satisfies the conditions of Theorem 2.2 and has $\mu(Q, P)$ in its domain as a Liouville operator, then

$$\rho_\mu^g(Q, P, t)\equiv\left\langle\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right|e^{-(i/\hbar g^2)t\hat H_g}\frac{\hat\rho_\mu}{2\pi\hbar g^2}e^{(i/\hbar g^2)t\hat H_g}\left|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle$$

w^*-converges to $(\mu\cdot\Phi_{-t}^H)(Q, P)$, as can be seen by using the Weyl representation (3.6). This in turn means that, if we consider the Weyl transform $f_\mu^g(q, p, t)$ of

$$e^{-(i/\hbar g^2)t\hat H_g}\hat\rho_\mu e^{(i/\hbar g^2)t\hat H_g}=\int_{\mathbb{R}^2}dp\,dp\,f_\mu^g(q, p, t)\hat W(g\sqrt{\hbar}q, -g\sqrt{\hbar}p)$$

this w^*-converges to the Fourier transform of $(\mu\cdot\Phi_{-t}^H)(Q, P)$ which will be denoted by $\hat\mu_t(q, p)$.

The last remark is the key to the following:

PROPOSITION 3.3. – Let $\Phi_y^p(q, p)=(q, p+y)$ be the phase-space flow of momentum translations generated by the Lie-derivative

$$L_q[.]=\{.,q\}$$
$$(\mu\cdot\Phi_y^p)(Q, P)=(e^{-L_q[\cdot]}\mu)(Q, P)=\mu(Q, P+y),$$

then:

$$(3.3.1)\quad \lim_{\alpha\to+\infty}\int_0^t ds_k\left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right| \times e^{-(i/\hbar g^2)t\hat{H}_g}\hat{\tau}_{s_k}^k\left[\frac{1}{2\pi\hbar g^2}\hat{\rho}_\mu\right]e^{(i/\hbar g^2)t\hat{H}_g}\left|\frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}}\right\rangle$$
$$=\int_0^t ds_k\int_0^{s_k}ds_{k-1}\dots\int_0^{s_2}ds_1\int_R dy_k\dots\int_R dy_1$$
$$\times\left\{\frac{e^{-(1/\beta)\sum_{l=1}^k y_l^2}}{(\sqrt{\beta\pi})^k}\left[\mu.\left(\prod_{l=1}^k \Phi_{-s_l}^H.\Phi_{y_l}^p.\Phi_{s_l}^H\right).\Phi_{-t}^H\right](Q, P)\right\}\equiv\tau_{s_k}^k[\mu](Q, P)$$

with

$$\prod_{l=1}^k \Phi_{-s_l}^H.\Phi_{y_l}^p.\Phi_{s_l}^H=\left[\prod_{l=1}^{k-1}\Phi_{-s_l}^H.\Phi_{y_l}^p.\Phi_{s_l}^H\right]\Phi_{-s_k}^H.\Phi_{y_k}^p.\Phi_{s_k}^H.$$

Moreover

$$\mu^\beta(Q, P, t)=e^{-\lambda t}\sum_{k=0}^{+\infty}\lambda^k\tau_{s_k}^k[\mu](Q, P)$$

is the solution of the differential Chapman-Kolmogorov equation

$$\partial_t\mu^\beta(Q, P, t)=\{H(Q, P), \mu^\beta(Q, P, t)\}$$
$$-\lambda\mu^\beta(Q, P, t)+\frac{\lambda}{\sqrt{\beta\pi}}\int_R dy\, e^{-y^2/\beta}\mu^\beta(Q, P+y, t)$$

with initial condition $\mu^\beta(Q, P, t=0)=\mu(Q, P)$.

Proof. – By inserting the Weyl representation of $\hat{\rho}_\mu$ in the summand with $k=0$ of (3.4) we get, according to Remark 3.2.2, $(\rho\cdot\Phi_{-t}^H)(Q, P)$ in the limit $\alpha\to+\infty$. The term with $k=1$ is more interesting, indeed it reads

$$\int_0^t ds_1\,\hat{U}_g(-t)\hat{\tau}_{s_1}^1\left[\frac{1}{2\pi\hbar g^2}\hat{\rho}_\mu\right]\hat{e}^{(i/\hbar\hat{H}_{\hbar}t}(t)=\int_0^t ds_1\int_R dy_1$$
$$\times\left\{\frac{1}{\sqrt{\beta}}e^{-y_1^2/\beta}\hat{U}_g(s_1-t)\hat{W}\left(0, -\frac{y_1}{g\sqrt{\hbar}}\right)\hat{U}_g(-s_1)\right.$$
$$\left.\times\frac{1}{2\pi\hbar g^2}\hat{\rho}_\mu\hat{U}_g(s_1)\hat{W}\left(0, \frac{y_1}{g\sqrt{\hbar}}\right)\hat{U}_g(t-s_1)\right\},$$

and, following again Remark 3.2.2, we rewrite the integrand in the first line as:

$$\left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right| \hat{U}_g(-t)\hat{\tau}^1_{s_1}\left[\frac{1}{2\pi\hbar g^2}\hat{\rho}_\mu\right]\hat{U}_g(t)\left| \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right\rangle$$

$$= \int_{R^3} dy_1\, d\xi_1\, d\pi_1 \left\{ \frac{1}{\sqrt{\beta}} e^{-(y_1^2/\beta)} e^{-iy_1\xi_1} f^g_\mu(\xi, \pi, s_1) \right.$$

$$\times \left\langle \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right| \hat{U}_g(s_1-t)\hat{W}(-g\sqrt{\hbar}\xi_1, g\sqrt{\hbar}\pi_1)$$

$$\left. \hat{U}_g(t-s_1)\left| \frac{Q}{g\sqrt{\hbar}}, \frac{P}{g\sqrt{\hbar}} \right\rangle \right\}.$$

The limit $\alpha \to +\infty$ then gets:

$$\int_{R^3} dy_1\, d\xi_1\, d\pi_1$$

$$\times \left\{ \frac{1}{\sqrt{\beta}} e^{-y_1^2/\beta} e^{-iy_1\xi_1} e^{i[\pi_1 Q_{s_1-t}(Q,P)-\xi_1(y_1+P_{s_1-t}(Q,P))]} (\mu\cdot\hat{\Phi}^H_{-s_1})(\xi_1, \pi_1) \right\}$$

$$= \int_R dy_1 \frac{1}{\sqrt{\beta}} e^{-(y_1^2/\beta)} (\mu\cdot\Phi^H_{-s_1}\cdot\Phi^p_{y_1}\cdot\Phi^H_{s_1-t})(Q, P).$$

The proof of the first part of the theorem is then achieved by induction and that of the second half is completed by observing that the series solves

$$\mu^\beta(Q, P, t) = e^{-\lambda t}\left\{(\mu\cdot\Phi^H_{-t})(Q, P) \right.$$

$$\left. +\lambda\int_0^t ds\, e^{\lambda s}\int_R \frac{e^{-y^2/\beta}}{\sqrt{\beta\pi}} (\mu^\beta\cdot\Phi^p_y\cdot\Phi^H_{-t+s})(Q, P, s)\right\}.$$

the integral equation equivalent to the Chapman-Kolmogorov differential equation.

5. CONCLUSIONS

In [11] the general problem of studying the classical limit of quantum dynamical semigroups has been addressed by considering a generator

$$L[.] = \frac{1}{\lambda^2} L_0[.] + L_d[.]. \tag{5.1}$$

and the corresponding λ-dependent, one-parameter semigroup

$$\gamma_t^\lambda = \exp\left(t\left(\frac{1}{\lambda^2} L_0[.] + L_d[.]\right)\right) \tag{5.2}$$

on the state-space $B(H)_1^{s.a.}$ and by taking the limit $\lambda \to 0$ in, accordingly rescaled, vector states. The classical limit does then amount to a weak-coupling limit in which the generator $L_0[.]$ of the group of isometries, the Hamiltonian evolution in our case, is rescaled and long time behaviour is sought after. The G.R.W-model we have investigated, belongs to a particular class of quantum dynamical semigroups in which the scaling parameter appears in the dissipative term as well as in the Hamiltonian one. We have then showed that keeping the stochastic properties throughout the classical limit requires a joint limit worked out via a weak-coupling limit. We have been forced to do so in order that the localizing properties of the evolution equation (1.3) be felt on the background of $\hbar \to 0$. We observe that in [1] an attempt has been devoted to get a classical evolution not by letting $\hbar$ going to 0, but looking at a kind of long time regime where quantum fluctuations can be neglected, the result being a Fokker-Planck equation with diffusions both in position and momentum. With respect to [12], where a simpler quantum dynamical semigroup is studied, we observe that in our case the entire hierarchy of terms in the Kramers-Moyal expansion of the jump term in (2.6) can be made, heuristically, correspond to multiple commutators by the following rewriting of the localization mechanism (1.1), *see* also 3.2 (ii):

$$\begin{aligned} T[\hat{\rho}] &= \frac{1}{\pi \hbar^2 \alpha} \int_{\mathbb{R}} dy\, e^{-(y^2/\hbar^2\alpha)}\, e^{-(i/\hbar)\, y\hat{q}_\hbar}\, \hat{\rho}\, e^{(i/\hbar)\, y\hat{q}_\hbar} \\ &= \sum_{k=0}^{+\infty} \left(-\frac{i}{\hbar}\right)^k \frac{1}{\pi \hbar^2 \alpha} \int_{\mathbb{R}} dy\, e^{-(y^2/\hbar^2\alpha)}\, \underbrace{[\hat{q}_\hbar, [\hat{q}_\hbar, \ldots, [\hat{q}_\hbar,}_{k \text{ times}} \hat{\rho}\underbrace{]] \ldots]}_{k \text{ times}} \\ &= \exp\left(-\frac{\alpha}{4} [\hat{q}_\hbar, [\hat{q}_\hbar, .]]\right)[\hat{\rho}] = \exp\left(-\frac{\hbar^2\alpha}{4} \partial^2_{\hat{p}_\hbar}\right)[\hat{\rho}], \end{aligned}$$

from where we again see what the right rescaling of $\sqrt{\alpha}$ should be. Then the Chapman-Kolmogorov equation (2.6) fully corresponds to the modified quantum evolution (1.2) and (2.5) solves it when the Hamiltonian function has at most quadratic potential, whereas 3.3.2 is a formal solution in the more general case.

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