ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 58, nº 2 (1993), p. 127-153
<http://www.numdam.org/item?id=AIHPA_1993__58_2_127_0>

Ann. Inst. Henri Poincaré,
Vol. 58, n° 2, 1993, p. 127-153. *Physique théorique*

The Fermion Number Processes as a Functional Central Limit of Quantum Hamiltonian Models

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ABSTRACT. – In the present paper, we investigate, in the Fermion case, how the number processes arise from a limit of a quantum Hamiltonian model. Our conclusion is that the time evolution of a certain quantum Hamiltonian model tends to the solution of a quantum stochastic differential equation driven by the Fermion number processes.

RÉSUMÉ. – Dans cet article nous étudions comment le processus de nombre de fermions apparaît comme une limite dans un modèle Hamiltonien quantique. Notre conclusion est que l'évolution temporelle d'un certain modèle Hamiltonien converge vers la solution d'une équation différentielle stochastique avec une source qui est le processus du nombre de fermions.

1. INTRODUCTION

In the series papers ([1], . . ., [6]), we have investigated in the Boson case the low density limit of a quantum Hamiltonian system and shown

(*) On leave of absence from Beijing Normal University.

Vol. 58/93/02/127/27/$4,70/© Gauthier-Villars

that the time evolution of the quantum Hamiltonian system tends to a quantum stochastic process which satisfies a quantum stochastic differential equation diven by quantum Poisson processes.

The present paper is devoted to the Fermion analogue of [1] and for sake of brevity we shall omit here the motivations of the problem and refer the reader to the "Introduction" of [1], [6].

Following the pattern of [3], we formulate the problem for a general quasi-free state and we prove the convergence of the kinematical process of the collective **number** vectors to Fermion Brownian motion in the general case. Starting from Section 3 we restrict our attention to the Fock case.

Let H_0 denote the system Hilbert space; H_1 the one particle reservoir Hilbert space and $W(H_1)$ the CAR-algebra on H_1, *i. e.* the algebra generated by the set

$$\{A(f) : f \in H_1\} \tag{1.1}$$

where, $A(f)$ is the Fermion annihilation operator. Let H be a self-adjoint bounded below operator on H_1 and z, β positive real numbers interpreted respectively as density of the reservoir particles and inverse temperature. Denote φ the Fock state characterized by the condition:

$$\varphi(A^+(f)A(g)) = \langle f, g \rangle, \qquad \forall f \in H_1 \tag{1.2}$$

and let $\{\mathscr{H}, \pi, \Phi\}$ be the GNS-triple of $\{W(H_1), \varphi\}$, so that

$$\langle \Phi, \pi(A^+(f))\pi(A(g))\Phi \rangle = \varphi(A^+(f)A(g)) \tag{1.3}$$

We whall write A (resp. A^+) for $\pi \circ A$ (resp. $\pi \circ A^+$). Let S_t be a unitary group on $B(H_1)$ (the one particle free evolution of the reservoir). The second quantization of S_t, denoted $\Gamma(S_t)$, leaves φ invariant hence it is implemented, in the GNS representation, by a 1-parameter unitary group V_t whose generator H_R is called the free Hamiltonian of the reservoir. As in [3] we assume that there exists a non zero subspace K of H_1 (in all the examples it is a dense subspace) such that

$$\int_{\mathbb{R}} |\langle f, S_t g \rangle| \, dt < \infty, \qquad \forall f, g \in K \tag{1.4}$$

Let be given a self-adjoint operator H_S on the system space H_0, called the system Hamiltonian. The total free Hamiltonian is defined to be

$$H^{(0)} := H_S \otimes 1 + 1 \otimes H_R \tag{1.5}$$

We define the interaction Hamiltonian V as in [1] i. e., we fix two functions $g_1, g_0 \in K$ and define

$$V := i(D \otimes A^+(g_0).A(g_1) - D^+ \otimes A^+(g_1).A(g_0)) = i \sum_{\varepsilon \in \{0, 1\}} D_\varepsilon \otimes A^+(g_\varepsilon).A(g_{1-\varepsilon}) \tag{1.6}$$

with the notations

$$D_0 = D, \qquad D_1 = -D^+ \tag{1.7}$$

and where D is a bounded operator on H_0 satisfying

$$\exp(-it H_S) . D . \exp(it H_S) = D \tag{1.8}$$

Moreover we assume that g_0 and g_1 have **disjoint energy spectra,** *i.e.*

$$\langle g_0, S_t g_1 \rangle = 0, \qquad \forall t \in \mathbf{R} \tag{1.9}$$

More general interactions will be discussed in subsequent papers.

The condition (1.9) is natural and has already been used in the literature on the weak coupling limit (*cf.* [8], [8 *a*], [8 *b*]). With the condition (1.9), the condition (1.8) is also natural since a typical example for D in quantum optics is $D = |0\rangle\langle 1|$, where $|1\rangle$, $|0\rangle$ are eigenvectors of the system Hamiltonian H_0 (rotating wave approximation). This corresponds to $[H_0, D] = (\omega_1 - \omega_0) D$ (ω_1, ω_0 are the eigenvalues). The condition (1.8) corresponds to taking $\omega_1 = \omega_0$, but the choice $\omega_1 \neq \omega_0$ results only in a trivial shift in the one particle reservoir Hamiltonian (*cf.* Section 5 in [6] for the detail). Also from the point of view of mathematics, the difference between the condition (1.8) and the general N-level case is as we have shown in [3], only to applying (many times) Reimann-Lebesgue Lemma – of course a different quantum process is obtained in the N-levels case but the difference is not fundamental (*cf.* [3], for the weak coupling case [9]).

With these notations, the total Hamiltonian is

$$H_{\text{total}} := H_S \otimes 1 + 1 \otimes H_R + V \tag{1.10}$$

and the wave operator at time t is defined by

$$U_t := \exp(-it H^{(0)}) . \exp(it H_{\text{total}}) \tag{1.11}$$

Therefore we have the equation

$$\frac{d}{dt} U_t = \frac{1}{i} V(t) U_t; \qquad U(0) = 1 \tag{1.12}$$

where,

$$V(t) := \exp(-it H_S \otimes 1) V \exp(it H_S \otimes 1)$$
$$= i \sum_{\varepsilon \in \{0, 1\}} D_\varepsilon \otimes A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}) \tag{1.13}$$

Moreover the solution of (1.12) is given by the iterated series

$$U_t = \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n (-i)^n V(t_1) \dots V(t_n) \tag{1.14}$$

which is norm convergent since the field operators are bounded.

An important role in the present paper will be played by the **collective number** vectors defined by

$$u \otimes \Phi_N\left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \ k=1, \ldots, N\right) \tag{1.15}$$

where $u \in H_0$, and for each $n \in N, f_1, \ldots, f_n \in H_1$

$$\Phi_n(f_k, k \in \{1, \ldots, n\}) := A^+(f_1) \ldots A^+(f_n) \Phi \tag{1.16}$$

From Lemma (3.2) of [10], we know that the assumption (1.4) implies that the sesquilinear form $(.|.) : K \times K \to \mathscr{C}$ defined by

$$(f|g) := \int_{\mathbf{R}} \langle f, S_t g \rangle \, dt, \qquad f, g \in K \tag{1.17}$$

defines a pre-scalar product on K. We denote $\{K, (.|.)\}$, or simply K, the completion of the quotient of K by the zero $(.|.)$-norm elements.

The analogy with the new techniques, developed in [13], for the weak coupling limit, suggests to consider the limit, as $z \to 0$ of expressions of the form

$$\begin{aligned} &\langle u, (.)v \rangle \otimes \varphi_{Q_z}\left(\mathbf{1} \otimes \Phi_N\left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \ k=1, \ldots, N\right),\right. \\ &\quad \left. v \otimes U_{t/z^2}(X \otimes 1) U_{t/z^2}^+ \mathbf{1} \otimes \Phi_{N'}\left(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \ k=1, \ldots, N'\right)\right) \end{aligned} \tag{1.18}$$

In analogy with the strategy of [1], the first step in our investigation will be to control the following limit:

$$\begin{aligned} \lim_{z \to 0} &\left\langle u \otimes \Phi_N\left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \ k=1, \ldots, N\right),\right. \\ &\qquad \left. U_{t/z^2} v \otimes \Phi_{N'}\left(z \int_{R'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \ k=1, \ldots, N'\right)\right\rangle \\ &= \lim_{z \to 0} \left\langle u \otimes \Phi_N\left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \ k=1, \ldots, N\right),\right. \\ &\qquad \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n (-i)^n V(t_1) \ldots V(t_n) \\ &\qquad \left. \Phi_{N'}\left(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \ k=1, \ldots, N'\right)\right\rangle \end{aligned} \tag{1.19}$$

We first outline the common and different points between the present work and [1]: Due to the form (1.6) of the interaction, the Wick ordered from of the products

$$A^+(f_1) A(g_1) \ldots A^+(f_n) A(g_n) \tag{1.20}$$

is the main subject to the considered in the low density limit, both in the Boson and the Fermion cases. The only difference between the two cases is some power of (-1) which is due to the different commutation relations. Therefore one can hope that

(1) The negligible terms will be similar to the Boson case.

(2) The uniform estimate theorem in [1] can be used directly to the present situation.

(3) The limit of the non-negligible terms is similar to the Boson case.

Exactly as in the Boson case, the estimates needed to solve this problem will allow, with minor modifications, to control more general situations (*cf.* [1]). In order to formulate our result, let us recall from [13] the definition of the Fermion Brownian Motion:

DEFINITION (1.1). – Let $\mathscr{K}$ be a Hilbert space, T an interval in **R**. Let $0 \leqq Q \leqq 1$ be a self-adjoint operator on $\mathscr{K}$ and let

$$\{\mathscr{H}_Q, \pi_Q, \Phi_Q\} \tag{1.21}$$

denote the GNS representation of the CAR algebra over $L^2(T, dt; \mathscr{K})$ with respect to the quasi-free state φ_Q on $W(L^2(T, dt; \mathscr{K}))$ characterized by

$$\varphi_Q(A^+(\xi)A(\xi')) = \left\langle \xi, \frac{1\otimes(1-Q)}{2}\xi' \right\rangle; \quad \xi, \xi' \in L^2(T, dt; \mathscr{K}) \tag{1.22}$$

The quantum stochastic process

$$\{\Gamma(L^2(T, dt; \mathscr{K})), A(\chi_{(s,t]}\otimes f), A^+(\chi_{(s,t]}\otimes f); (s, t] \subseteqq T, f \in \mathscr{K}\} \tag{1.23}$$

where $A(.)$, $A^+(.)$ denote respectively the annihilation and creation fields in the representation (1.23), is called the **Q-Fermion Brownian Motion** on $L^2(T, dt; \mathscr{K})$. The Fock Fermion Brownian Motion corresponds to the choice of $Q=1$.

Our main result in this paper is to prove that, the limit (1.19) exists and is equal to

$$\langle u \otimes \Psi_N(\chi_{[S_k, T_k]} \otimes f_k, k=1, \ldots, N), U(t) \times v \otimes \Psi_{N'}(\chi_{[S'_k, T'_k]} \otimes f'_k, k=1, \ldots, N') \tag{1.24}$$

where $\{\mathscr{H}, A, A^+, \Psi\}$ is the Fock Brownian motion on

$$L^2(\mathbf{R}, dt; K) \cong L^2(R) \otimes K$$

and $U(t)$ satisfies a quantum stochastic differential equation driven by purely discontinuous noises in the sense of [14] and [15], whose form is given by (5.28).

2. THE NOISE SPACE

We know from [1] that for each S, T, S′, T′ ∈ **R**, and $f, f' \in K$ satisfying (1.6), one has

$$\lim_{z \to 0} \left\langle z \int_{S/z^2}^{T/z^2} S_u f\, du,\ z \int_{S'/z^2}^{T'/z^2} S_u f'\, du \right\rangle = \langle \chi_{[S,T]}, \chi_{[S',T']} \rangle_{L^2(\mathbf{R})} . (f \mid f') \quad (2.1)$$

Moreover, the limit is uniform for S, T, S′, T′ in a bounded set in **R**.

THEOREM (2.1). – *For each* N, N′ ∈ **N**, $f_1, \ldots, f_N, f'_1, \ldots, f'_{N'} \subset K$, $\{S_h, T_h\}_{h=1}^{N}$, $\{S'_h, T'_h\}_{h=1}^{N'} \subset \mathbf{R}$

$$\lim_{z \to 0} \left\langle \Phi_N\left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k\, du,\ k=1, \ldots, N \right), \Phi_{N'}\left(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\, du,\ k=1, \ldots, N' \right) \right\rangle$$
$$= \langle \Psi_N(\chi_{[S_k, T_k]} \otimes f_k,\ k=1, \ldots, N), \Psi_{N'}(\chi_{[S'_k, T'_k]} \otimes f'_k,\ k=1, \ldots, N') \rangle \quad (2.2)$$

Proof. – The proof is similar to that of Lemma (2.1) of [1]. The only difference being that now we have number rather then coherent collective vectors.

By expanding the scalar product in the left hand side of (2.2) and using the CAR, one finds

$$\delta_{N,N'} \sum_{\sigma, \varepsilon \in \mathscr{S}_N} (-1)^{\|\varepsilon\| + \|\sigma\|} \prod_{k=1}^{N} z^2 \int_{S_{\sigma(k)}/z^2}^{T_{\sigma(k)}/z^2} du \int_{S'_{\varepsilon(k)}/z^2}^{T'_{\varepsilon(k)}/z^2} \times dv \langle S_u Q_+ f_{\sigma(k)}, S_v Q_+ f'_{\varepsilon(k)} \rangle \quad (2.3)$$

which, as $z \to 0$, (2.7), by formula (2.1), tends to

$$\delta_{N,N'} \sum_{\sigma, \varepsilon \in \mathscr{S}_N} (-1)^{\|\varepsilon\| + \|\sigma\|} \times \prod_{k=1}^{N} \langle \chi_{[S_{\sigma(k)}, T_{\sigma(k)}]}, \chi_{[S'_{\varepsilon(k)}, T'_{\varepsilon(k)}]} \rangle_{L^2(\mathbf{R})} \cdot (f_{\sigma(k)} \mid f'_{\varepsilon(k)})$$
$$= \langle \Psi_N(\chi_{[S_k, T_k]} \otimes f_k,\ k=1, \ldots, N), \Psi_{N'}(\chi_{[S'_k, T'_k]} \otimes f'_k,\ k=1, \ldots, N') \rangle \quad (2.4)$$

Since the limit (2.1) corresponds to the 0-th term in the expansion (1.14), Theorem (2.1) shows that our limit processes, if it exists, lives on the Hilbert space $\Gamma(L^2(\mathbf{R}, dt) \otimes K)$)-the Fermi Fock space over $L^2(\mathbf{R}) \otimes K$, *i.e.* the space of the Fermi Fock Brownian Motion.

3. THE COLLECTIVE TERMS AND THE NEGLIGIBLE TERMS

Starting from the iterated series (1.14) and using (1.13), one has

$$(-i)^n V(t_1) \dots V(t_n) = \sum_{\varepsilon \in \{0,1\}^n} D_{\varepsilon(1)} \dots D_{\varepsilon(n)} \otimes A^+(S_{t_1} g_{\varepsilon(1)}) A(S_{t_1} g_{1-\varepsilon(1)}) \dots A^+(S_{t_n} g_{\varepsilon(n)}) A(S_{t_n} g_{1-\varepsilon(n)}). \quad (3.1)$$

In the right hand side of (3.1) the operator on the system space is rather simple and the most important thing is to know what is the contribution of the product of creation and annihilation operators. In order to do this, as usual, we shall bring that product to the normal ordered form. This is done in the following:

THEOREM (3.1). — *For each* $n \in \mathbf{N}$, *the normal ordered from of the product*

$$A^+(S_{t_1} g_{\varepsilon(1)}) A(S_{t_1} g_{1-\varepsilon(1)}) \dots A^+(S_{t_n} g_{\varepsilon(n)}) A(S_{t_n} g_{1-\varepsilon(n)}) \quad (3.2)$$

is equal to

$$\sum_{m=0}^{n-1} \sum_{2 \leqq q_1 < \dots < q_m \leqq n} (-1)^{(n, \{q_h\}_{h=1}^m)} \prod_{h=1}^m \langle S_{t_{q_h - 1}} g_{1-\varepsilon(q_h - 1)}, S_{t_{q_h}} g_{\varepsilon(q_h)} \rangle \prod_{\alpha \in \{1, \dots, n\} \setminus \{q_h\}_{h=1}^m} A^+(S_{t_\alpha} g_{\varepsilon(\alpha)}) \cdot \prod_{\alpha \in \{1, \dots, n\} \setminus \{q_h - 1\}_{h=1}^m} A(S_{t_\alpha} g_{\varepsilon(\alpha)})$$
$$+ \sum_{m=0}^{n-1} \sum_{2 \leqq q_1 < \dots < q_m \leqq n} \sum_{(p_1, q_1, \dots, p_m, q_m)}' \vartheta \prod_{h=1}^m \langle S_{t_{p_h}} g_{1-\varepsilon(p_h)}, S_{t_{q_h}} g_{\varepsilon(q_h)} \rangle \prod_{\alpha \in \{1, \dots, n\} \setminus \{q_h\}_{h=1}^m} A^+(S_{t_\alpha} g_{\varepsilon(\alpha)}) \cdot \prod_{\alpha \in \{1, \dots, n\} \setminus \{p_h\}_{h=1}^m} A^+(S_{t_\alpha} g_{\varepsilon(\alpha)})$$
$$=: I_n(\varepsilon) + II_n(\varepsilon) \quad (3.3)$$

where, $\vartheta \mp 1$ and $(n, \{q_h\}_{h=1}^m)$ is defined as

$$\sum_{\alpha \in \{1, \dots, n\} \setminus \{q_h - 1\}_{h=1}^m} |\{j, j > \alpha, j \in \{1, \dots, n\} \setminus \{q_h\}_{h=1}^m| \quad (3.4)$$

The sum $\sum'_{(p_1, q_1, \dots, p_m, q_m)}$ means the sum for all $1 \leqq p_1, \dots, p_m \leqq n$ satysfying $|\{p_h\}_{h=1}^m| = m$ (the cardinality of the set $\{p_h\}_{h=1}^m$ is equal to m), $p_h < q_h$ for any $h = 1, \dots, m$ and $p_h < q_h - 1$ for some $h = 1, \dots, m$.

Remark. — In the second term of (3.3) (type II), the value of ϑ is not relevant because we shall majorize the modulus of the sum with the sum of the moduli (for which the value of ϑ is irrelevant) and then we prove that the latter tends to zero.

Proof. – The only difference between the proof of this Lemma and that of Lemma (3.1) in [6] is the precise computation of the exponent of (-1) in the type I term, *i.e.* of the quantity (3.4). This is achieved as follows: by bringing to normal form the products of the creation and annihilation operators in (3.1) and arguing as in Lemma (3.1) of [6], one arrives to an expression which differs from (3.2) only by the replacement of the power of (-1) and by an unknown factor ϑ.

In order to compute this factor, denote

$$\{\beta_h\}_{h=1}^{n-m}=\{1, \ldots, n\}\setminus\{q_h-1\}_{h=1}^{m}$$

with

$$\beta_1<\ldots<\beta_{n-m} \tag{3.5}$$

the indices which label the annihilators which have not been used to produce scalar products. Then notice that to move $A(S_{t_{\beta_{n-m}}}g_{1-\varepsilon(\beta_{n-m})})$ to the right hand side of $A^+(S_{t_n}g_{\varepsilon(n)})$ one needs to exchange $A(S_{t_{\beta_{n-m}}}g_{1-\varepsilon(\beta_{n-m})})$ with the creators which are the right hand side of it and have not not used to produce scalar products, *i.e.* $A^+(S_{t_j}g_{\varepsilon(j)})$ for $j>\beta_{n-m}$, and $j\in\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^{m}$, so one gets a factor

$$(-1)^{|\{j,\, j>\beta_{n-m},\, j\in\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^{m}\}|} \tag{3.6}$$

The same argument shows that to move $A(S_{t_{\beta_{n-m-1}}}g_{1-\varepsilon(\beta_{n-m-1})})$ to the left hand side of $A(S_{t_{\beta_{n-m}}}g_{1-\varepsilon(\beta_{n-m})})$ one needs to exchange

$$A(S_{t_{\beta_{n-m-1}}}g_{1-\varepsilon(\beta_{n-m-1})})$$

with $A^+(S_{t_j}g_{\varepsilon(j)})$ for $j>\beta_{n-m-1}$, and $j\in\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^{m}$. So, one gets a factor

$$(-1)^{|\{j,\, j>\beta_{n-m-1},\, j\in\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^{m}\}|} \tag{3.7}$$

Repeating the argument $n-m$ times [*i.e.* once for each of the β_j in (3.5)], the factor (-1) to the power (3.4) arises.

Now let investigate the contributions of terms $I_n(\varepsilon)$ and $II_n(\varepsilon)$. First of all we have:

THEOREM (3.2). – *For each* N, $N'\in\mathbf{N}$, $n\in\mathbf{N}$, $f_1, \ldots, f_N$, $f'_1, \ldots, f'_{N'}\in K$, $\{S_h, T_h\}_{h=1}^{N}$, $\{S'_h, T'_h\}_{h=1}^{N'}\subset\mathbf{R}$, $\varepsilon\in\{0, 1\}$, *let the term* $II_n(\varepsilon)$ *be defined by* (3.3). *Then*

$$\lim_{z\to 0}\left\langle \Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1, \ldots, N\right), \int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\right.$$
$$\left. II_n(\varepsilon)\,\Phi_{N'}\left(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1, \ldots, N'\right)\right\rangle=0 \tag{3.8}$$

Proof. – By the definition of $II_n(\varepsilon)$ [*see* (3.3)] and letting the creation, annihilation operators in $II_n(\varepsilon)$ act on the number vectors in the left hand side of (3.8), one shows that the module of the scalar product in the left hand side of (3.8) is dominated by

$$\sum_{m=0}^{n-1} \sum_{2 \leq q_1 < \ldots < q_m \leq n} \sum'_{(p_1, q_1, \ldots, p_m, q_m)} z^{2(n-m)} \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$
$$\prod_{h=1}^{m} |\langle S_{t_{p_h}} g_{1-\varepsilon(p_h)}, S_{t_{q_h}} g_{\varepsilon(q_h)} \rangle|$$
$$\times \sum_{1 \leq \alpha_1, \ldots, \alpha_m \leq N,\, 1 \leq \beta_1, \ldots, \beta_m \leq N'} \sigma(\{\alpha_j, \beta_j\}, N, N', m)$$
$$\prod_{h=1}^{n-m} \int_{S_{\alpha_h}/z^2}^{T_{\alpha_h}/z^2} |\langle S_u f_{\alpha_h}, S_{t_{\alpha_h}} g_{\varepsilon(\alpha_h)} \rangle|$$
$$\times \prod_{h=1}^{n-m} \left| \left\langle S_{t_{\beta_h}} g_{1-\varepsilon(\alpha_h)}, \int_{S'_{\beta_k}/z^2}^{T'_{\beta_h}/z^2} S_u f'_{\beta_h} \right\rangle \right| \tag{3.9}$$

where, $\sigma(\{\alpha_j, \beta_j\}, N, N', m)$ is the modulus of the scalar product of a pair of collective number vectors, *i.e.*

$$\left| \left\langle \Phi_{N-m}\left(z \int_{S_\alpha/z^2}^{T_\alpha/z^2} S_u f_\alpha \, du,\ \alpha \in \{1, \ldots, N\} \setminus \{\alpha_h\}_{h=1}^m \right), \right.\right.$$
$$\left.\left. \Phi_{N'-m}\left(z \int_{R'_\beta/z^2}^{T'_\beta/z^2} S_u \, du \, f'_\beta,\ \beta \in \{1, \ldots, N'\} \setminus \{\beta_h\}_{h=1}^m \right) \right\rangle \right|$$

hence, by Theorem (2.1) a convergent, and therefore bounded quantity, as $z \to 0$.

The factor in the last line of (3.9) is majorized by

$$\left(\max_{F \in \{f_h\}_{h=1}^N \cup \{f'_h\}_{h=1}^{N'},\, G \in \{g_0, g_1\}} \int_{-\infty}^{\infty} |\langle F, S_t G \rangle| \, dt \right)^{2(n-m)} \tag{3.10}$$

The factor given by the first two lines in (3.9), up to a constant, is the same as the right hand side of (3.16) of [1] and there we have proved that it tends, as $z \to 0$, to zero. Thus the thesis follows.

In order to compute the limit of the type I terms we rewrite the term $I_n(\varepsilon)$ in another form in which the exponent of the factor -1 has an expression much clearer than formula (3.3).

LEMMA (3.3). – *For each* $n\in\mathbf{N}$, $\varepsilon\in\{0,1\}^n$

$$\mathrm{I}_n(\varepsilon)=\sum_{m=1}^{n}\sum_{1=q_1<q_2<\ldots<q_m\leq n}(-1)^{(1/2)m(m-1)}$$
$$\times\prod_{\alpha\in\{1,\ldots,n\}\setminus\{q_h\}_{h=1}^{m}}\langle \mathrm{S}_{t_{\alpha-1}}g_{1-\varepsilon(\alpha-1)},\mathrm{S}_{t_\alpha}g_{\varepsilon(\alpha)}\rangle$$
$$\prod_{h=1}^{m}\mathrm{A}^+(S_{t_{q_h}}g_{\varepsilon(q_h)})\cdot\prod_{h=1}^{m}\mathrm{A}(\mathrm{S}_{t_{q_{h+1}-1}}g_{1-\varepsilon(q_{h+1}-1)})\quad(3.11)$$

where, $q_{m+1}:=n+1$.

Proof. – It is clear that when we bring the product (3.2) to the normal ordered form, there will exist $m(\leq n)$ creators not used to produce the scalar products with annihilators. Moreover in the product (3.2), $\mathrm{A}^+(\mathrm{S}_{t_1}g_{\varepsilon(1)})$ is in ordered position, so $m\geq 1$. Label the *remaining* creators with $\{q_h\}_{h=1}^{m}$. This means that the creators

$$\{\mathrm{A}^+(\mathrm{S}_{t_\alpha}g_{\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus\{q_h\}_{h=1}^{m}\}\quad(3.12)$$

have been used to produce scalar products with the annihilators

$$\{\mathrm{A}(\mathrm{S}_{t_{\alpha-1}}g_{1-\varepsilon(\alpha-1)});\ \alpha\in\{1,\ldots,n\}\setminus\{q_h\}_{h=1}^{m}\}$$
$$=\{\mathrm{A}(\mathrm{S}_{t_\alpha}g_{1-\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus\{q_h\}_{h=1}^{m}\}\quad(3.13)$$

i.e. the **remaining** annihilators are

$$\{\mathrm{A}(\mathrm{S}_{t_\alpha}g_{1-\varepsilon(\alpha)});\ \alpha\in\{1,\ldots,n\}\setminus\{\{1,\ldots,n\}\setminus\{q_h\}_{h=1}^{m}-1\}\}$$
$$=\{\mathrm{A}(\mathrm{S}_{t_{q_{h+1}-1}}g_{1-\varepsilon(q_{h+1}-1)})\}_{h=1}^{m}\quad(3.14)$$

The factor $(-1)^{(1/2)m(m-1)}$ comes from the following exchanges:

$\mathrm{A}(\mathrm{S}_{t_{q_m-1}}g_{1-\varepsilon(q_m-1)})$ with $\mathrm{A}^+(\mathrm{S}_{t_{q_m}}g_{\varepsilon(q_m)})$ (this gives the factor -1);

$\mathrm{A}(\mathrm{S}_{t_{q_{m-1}-1}}g_{1-\varepsilon(q_{m-1}-1)})$ with $\mathrm{A}^+(\mathrm{S}_{t_{q_{m-1}}}g_{\varepsilon(q_{m-1})})\mathrm{A}^+(\mathrm{S}_{t_{q_m}}g_{\varepsilon(q_m)})$ (this gives the factor $(-1)^2$); ...;

$\mathrm{A}(\mathrm{S}_{t_{q_2-1}}g_{1-\varepsilon(q_2-1)})$ with $\mathrm{A}^+(\mathrm{S}_{t_{q_2}}g_{\varepsilon(q_2)})\ldots\mathrm{A}^+(S_{t_{q_m}}g_{\varepsilon(q_m)})$ (this gives the factor $(-1)^{m-1}$). ■

THEOREM (3.4). – *For each* N, $\mathrm{N}'\in\mathbf{N}$, $n\in\mathbf{N}$, $f_1,\ldots,f_\mathrm{N}$, $f'_1,\ldots,f'_{\mathrm{N}'}\in\mathrm{K}$, $\{\mathrm{S}_h,\mathrm{T}_h\}_{h=1}^{\mathrm{N}}$, $\{\mathrm{S}'_h,\mathrm{T}'_h\}_{h=1}^{\mathrm{N}'}\subset\mathbf{R}$, $\varepsilon\in\{0,1\}$, *the limit*

$$\lim_{z\to0}\left\langle\Phi_\mathrm{N}\left(z\int_{\mathrm{S}_k/z^2}^{\mathrm{T}_k/z^2}\mathrm{S}_u f_k\,du,\ k=1,\ldots,\mathrm{N}\right),\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\right.$$
$$\left.\mathrm{I}_n(\varepsilon)\,\Phi_{\mathrm{N}'}\left(z\int_{\mathrm{S}'_k/z^2}^{\mathrm{T}'_k/z^2}\mathrm{S}_u f'_k\,du,\ k=1,\ldots,\mathrm{N}'\right)\right\rangle\quad(3.15)$$

exists and is equal to

$$\sum_{m=1}^{n} \sum_{1=q_1<q_2<\ldots<q_m\leqq n} \prod_{r=1}^{m} \prod_{h=q_r+1}^{q_r+1-1} (g_{1-\varepsilon(h-1)} | g_{\varepsilon(h)})_-$$

$$\int_{0\leqq t_{q_m}\leqq t_{q_{m-1}}\leqq \ldots \leqq t_{q_1}\leqq t} dt_{q_m} \ldots dt_{q_1}$$

$$\sum_{\substack{1\leqq x_1, \ldots, x_m\leqq N \\ |\{x_h\}_{h=1}^m|=m}} \prod_{h=1}^{m} (-1)^{(x_h, N)} \chi_{[S_{x_h}, T_{x_h}]}(t_{q_h}) (f_{x_h} | g_{\varepsilon(q_h)})$$

$$\sum_{\substack{1\leqq y_1, \ldots, y_m\leqq N \\ |\{y_h\}_{h=1}^m|=m}} \prod_{h=1}^{m} (-1)^{(y_h, N')} \chi_{[S'_{y_h}, T'_{y_h}]}(t_{q_h}) (g_{1-\varepsilon(q_{h+1}-1)} | f'_{y_h})$$

$$\langle \Psi_{N-m}(\chi_{[S_\alpha, T_\alpha]} \otimes f_\alpha, \alpha\in\{1, \ldots, N\}\setminus\{x_h\}_{h=1}^m), \\ \Psi_{N'-m}(\chi_{[S'_\alpha, T'_\alpha]} \otimes f'_\alpha, \alpha\in\{1, \ldots, N'\}\setminus\{y_h\}_{h=1}^m) \rangle \tag{3.16}$$

where Ψ is the vacuum vector of $\Gamma(L^2(\mathbf{R}; dt, K))$,

$$(x_h, N) : = |\{1, \ldots, x_h\}\setminus\{x_\alpha\}_{\alpha=1}^{h-1}|$$

and for $f, g\in K$ the half-scalar product $(f|g)_-$ is defined by

$$(f|g)_- : = \int_{-\infty}^{0} dt \langle f, S_t g \rangle \tag{3.16 a}$$

Proof. – Clearly, for each $n\in\mathbf{N}$, $m\leqq n$, $1=q_1<q_2<\ldots<q_m\leqq n$, $q_{m+1} : =n+1$

$$\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^m = \bigcup_{r=1}^{m} \{q_r+1, \ldots, q_{r+1}-1\} \tag{3.17}$$

So, one can rewite the product of scalar products in the right hand side of (3.11) in the form:

$$\prod_{\alpha\in\{1, \ldots, n\}\setminus\{q_h\}_{h=1}^m} \langle S_{t_{\alpha-1}} g_{1-\varepsilon(\alpha-1)}, S_{t_\alpha} g_{\varepsilon(\alpha)} \rangle \\ = \prod_{r=1}^{m} \prod_{h=q_r+1}^{q_r+1-1} \langle S_{t_{h-1}} g_{1-\varepsilon(h-1)}, S_{t_h} g_{\varepsilon(h)} \rangle \tag{3.18}$$

Using (3.11) and (3.18) in (3.15) one finds that the limit (3.15) is equal to the limit of

$$\sum_{m=1}^{n} \sum_{1=q_1<q_2<\ldots<q_m\leqq n} (-1)^{(1/2)m(m-1)} z^{2m} \int_0^{t/z^2} dt_1 \int_0^{t_2} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\prod_{r=1}^{m} \prod_{h=q_r+1}^{q_r+1-1} \langle S_{t_{h-1}} g_{1-\varepsilon(h-1)}, S_{t_h} g_{\varepsilon(h)} \rangle$$

$$\left\langle \Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1,\ \dots,\ N\right),\ \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \prod_{h=1}^{m} A^+(S_{t_{q_h}} g_{\varepsilon(q_h)}) \cdot \prod_{h=1}^{m} A(S_{t_{q_{h+1}-1}} g_{1-\varepsilon(q_{h+1}-1)}) \Phi_{N'}\left(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1,\ \dots,\ N'\right)\right\rangle \tag{3.19}$$

Letting the creators in (3.19) act on the number vectors in the left hand side of the scalar product, one has

$$\prod_{h=m}^{1} A^+(S_{t_{q_h}} g_{\varepsilon(q_h)}) \Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1,\ \dots,\ N\right) = \sum_{\substack{1\leq x_1,\dots,x_m\leq N \\ |\{x_h\}_{h=1}^m|=m}} \prod_{h=1}^{m} (-1)^{(x_h,N)} \int_{S_k/z^2}^{T_k/z^2} \langle S_u f_{x_h},\ S_{t_{q_h}} g_{\varepsilon(q_h)}\rangle\,du \; \Phi_{N-m}\left(z\int_{S_\alpha/z^2}^{T_\alpha/z^2} S_u f_\alpha\,du,\ \alpha\in\{1,\ \dots,\ N\}\setminus\{x_h\}_{h=1}^m\right) \tag{3.20}$$

where we have used the symbol $\prod_{h=m}^{1}$ to denote the product of operators with decreasing time-indices. Similarly, letting the annihilators in (3.19) act on the $\Phi_{N'}$-number vectors and changing their order, using the CAR, so to obtain a sequence of decreasing time-indices, we obtain the expression

$$(-1)^{m(m-1)/2} \prod_{h=m}^{1} A(S_{t_{q_{q+1}-1}} g_{1-\varepsilon(q_{h+1}-1)}) \tag{3.21}$$

acting on the $\Phi_{N'}$ – number vectors in (3.19). Now we can apply (3.20) and this leads to the result:

$$(-1)^{(1/2)m(m-1)} \sum_{\substack{1\leq y_1,\dots,y_m\leq N \\ |\{y_h\}_{h=1}^m|=m}} \prod_{h=1}^{m} (-1)^{(y_h,N')} \int_{S'_{y_h}/z^2}^{T'_{y_h}/z^2} \times \langle S_{t_{q_{h+1}-1}} g_{\varepsilon(q_{h+1}-1)},\ S_v f'_{x_h}\rangle\,dv \left\langle \Phi_{N'-m}\left(z\int_{S'_\beta/z^2}^{T'_\beta/z^2} S_u f'_\beta\,du,\ \beta\in\{1,\ \dots,\ N'\}\setminus\{y_h\}_{h=1}^m\right)\right\rangle \tag{3.22}$$

Summing up, (3.19) is equal to

$$\sum_{m=1}^{n} \sum_{1=q_1<q_2<\ldots<q_m\le n} (-1)^{1/2m(m-1)} z^{2m} \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\prod_{r=1}^{m} \prod_{h=q_r+1}^{q_{r+1}-1} \langle S_{t_{h-1}} g_{1-\varepsilon(h-1)}, S_{t_h} g_{\varepsilon(h)} \rangle$$

$$\sum_{\substack{1\le x_1,\ldots,x_m\le N \\ |\{x_h\}_{h=1}^m|=m}} \prod_{h=1}^{m} (-1)^{(x_h, N)} \int_{S_{x_h}/z^2}^{T_{x_h}/z^2} \langle S_u f_{x_h}, S_{t_{q_h}} g_{\varepsilon(q_h)} \rangle \, du$$

$$(-1)^{(1/2)m(m-1)} \sum_{\substack{1\le y_1,\ldots,y_m\le N \\ |\{y_h\}_{h=1}^m|=m}} \sum_{h=1}^{m} (-1)^{(y_h, N')}$$

$$\times \int_{S'_{y_h}/z^2}^{T'_{y_h}/z^2} \langle S_{t_{q_{h+1}-1}} g_{\varepsilon(q_{h+1}-1)}, S_v f'_{x_h} \rangle \, dv$$

$$\left\langle \Phi_{N-m}\left(z\int_{S_\alpha/z^2}^{T_\alpha/z^2} S_u f_\alpha \, du,\ \alpha\in\{1,\ldots,N\}\setminus\{x_h\}_{h=1}^m \right), \right.$$

$$\left. \Phi_{N'-m}\left(z\int_{S'_\beta/z^2}^{T'_\beta/z^2} S_u f'_\beta \, du,\ \beta\in\{1,\ldots,N'\}\setminus\{y_h\}_{h=1}^m \right) \right\rangle \quad (3.23)$$

By Theorem (2.1) the last scalar product in (3.23) tends to

$$\langle \Psi_{N-m}(\chi_{[S_\alpha, T_\alpha]}\otimes f_\alpha,\ \alpha\in\{1,\ldots,N\}\setminus\{x_h\}_{h=1}^m),$$
$$\Psi_{N'-m}(\chi_{[S'_\alpha, T'_\alpha]}\otimes f'_\alpha,\ \alpha\in\{1,\ldots,N'\}\setminus\{y_h\}_{h=1}^m)\rangle \quad (3.24)$$

and the same arguments as in the proof of Lemma (3.4) of [1] show that the t-integral term in (3.27) tends to

$$(g_{1-\varepsilon(h-1)} | g_{\varepsilon(h)})$$
$$- \int_{0\le t_{q_m}\le t_{q_{m-1}}\le\ldots\le t_{q_1}\le t} dt_{q_m} \ldots dt_{q_1} \chi_{[S_{x_h}, T_{x_h}]}(t_{q_h}) (f_{x_h} | g_{\varepsilon(q_h)})$$
$$\chi_{[S'_{y_h}, T'_{y_h}]}(t_{q_h}) (g_{1-\varepsilon(q_{h+1}-1)} | f'_{y_h}) \quad (3.25)$$

This proves our result.

4. THE LIMIT OF THE NON-NEGLIGIBLE TERMS

In the previous section we have discussed the limit (1.19) for each fixed n and our main results are Theorems (3.2) and (3.4). The present section is devoted to investigate:

(1) the condition to exchange the limit $z \to 0$ with the sum over $n \in \mathbf{N}$;

(2) the explicit form of the limit.

In the following, we shall use the notation

$$\|g\|_-^2 := \max_{\varepsilon, \sigma \in \{0, 1\}} \int_{-\infty}^{0} |\langle g_\varepsilon, S_t g_\sigma \rangle| \, dt \tag{4.1}$$

THEOREM (4.1). – *For each n, N, $N' \in \mathbf{N}$, $\{f_h\}_{h=1}^{N}$, $\{f'_h\}_{h=1}^{N'} \supset K$,*

$$\sum_{\varepsilon \in \{0, 1\}^n} \int_0^{t/z^2} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}}$$

$$\times dt_n \left| \left\langle \Phi_N \left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \; k=1, \ldots, N \right), \right. \right.$$

$$\left. \left. (I_n(\varepsilon) + II_n(\varepsilon)) \, \Phi_{N'} \left(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \; k=1, \ldots, N' \right) \right\rangle \right|$$

$$\leqq n.16^n . C(N, N', \{f_h\}_{h=1}^{N}, \{f'_h\}_{h=1}^{N'}) \max_{0 \leqq m \leqq n} \left(\frac{t^{n-m}}{(n-m)!} \cdot \|g\|_-^{2m} \right.$$

$$\left. \times \left[\max_{G = g_0, g_1, F \in \{f_h\}_{h=1}^{N} \cup \{f'_h\}_{h=1}^{N'}} \int_{-\infty}^{\infty} |\langle F, S_u G \rangle| \, du \right]^{2(n-m)} \right) \tag{4.2}$$

where,

$$C(N, N', \{f_h\}_{h=1}^{N}, \{f'_h\}_{h=1}^{N'}) := \sup_{z>0} \max_{0 \leqq m \leqq n} \sum_{\substack{1 \leqq x_1, \ldots, x_m \leqq N \\ |\{x_h\}_{h=1}^{m}| = m}} \sum_{\substack{1 \leqq y_1, \ldots, y_m \leqq N \\ |\{y_h\}_{h=1}^{m}| = m}}$$

$$\left| \left\langle \Phi_{N-m} \left(z \int_{S_r/z^2}^{T_r/z^2} S_u f_r \, du, \; r \in \{1, \ldots, N\} \setminus \{x_h\}_{h=1}^{m} \right), \right. \right.$$

$$\left. \left. \Phi_{N'-1} \left(z \int_{S'_r/z^2}^{T'_r/z^2} S_u f'_r \, du, \; r \in \{1, \ldots, N'\} \setminus \{y_h\}_{h=1}^{m} \right) \right\rangle \right| \tag{4.3}$$

Proof. – By formula (3.3) and using the notation (4.3), we know that the left hand side of (4.2) is majorized by

$$\sum_{\varepsilon\in\{0,1\}^n}\sum_{m=0}^{n-1}\sum_{2\leq q_1<\ldots<q_m\leq n}\sum_{\substack{1\leq p_1,\ldots,p_m\leq n-1\\ p_h<q_h,\,h=1,\ldots,m,\,|\{p_h\}_{h=1}^m|=m}} \times z^{2(n-m)}\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n$$

$$\prod_{h=1}^m|\langle S_{t_{p_h}}g_{1-\varepsilon(p_h)},S_{t_{q_h}}g_{\varepsilon(q_h)}\rangle|\,.\,C(N,N',\{f_h\}_{h=1}^N,\{f'_h\}_{h=1}^{N'})\quad(4.4)$$

this is the same, up to a constant, as the right hand side of (4.18) in [1], therefore the application of the same argument as in the proof of Lemma (4.3) of [1] leads to (4.2).

Combining together Theorem (4.1), Theorem (3.1), Theorem (3.2) and Theorem (3.4), one has the following

THEOREM (4.2). – *For each* $N, N'\in\mathbf{N}$, $f_1,\ldots,f_N$, $f'_1,\ldots,f'_{N'}\in K$, $\{S_h,T_h\}_{h=1}^N$, $\{S_h,T_h\}_{h=1}^N\subset\mathbf{R}$, $u,v\in H_0$, $D\in B(H_0)$, *if*

$$\|g\|_-^2<\frac{1}{16\|D\|}\quad(4.5)$$

the limit (1.19) *exists and is equal to*

$$\langle u\otimes\Psi_N(\chi_{[S_\alpha,T_\alpha]}\otimes f_\alpha,\alpha\in\{1,\ldots,N\}),\; v\otimes\Psi_{N'}(\chi_{[S'_\alpha,T'_\alpha]}\otimes f'_\alpha,\alpha\in\{1,\ldots,N'\}\rangle$$

$$+\sum_{n=1}^\infty\sum_{\varepsilon\in\{0,1\}^n}\sum_{m=1}^n\sum_{1=q_1<q_2<\ldots<q_m\leq n}\prod_{r=1}^m\prod_{h=q_r+1}^{q_{r+1}-1}(g_{1-\varepsilon(h-1)}|g_{\varepsilon(h)})$$

$$-\int_{0\leq t_{q_m}\leq t_{q_{m-1}}\leq\ldots\leq t_{q_1}\leq t}dt_{q_m}\ldots dt_{q_1}\langle u,D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\rangle$$

$$\sum_{\substack{1\leq x_1,\ldots,x_m\leq N\\ |\{x_h\}_{h=1}^m|=m}}\prod_{h=1}^m(-1)^{(x_h,N)}\chi_{[S_{x_h},T_{x_h}]}(t_{q_h})(f_{x_h}|g_{\varepsilon(q_h)})$$

$$\sum_{\substack{1\leq y_1,\ldots,y_m\leq N\\ |\{y_h\}_{h=1}^m|=m}}\prod_{h=1}^m(-1)^{(y_h,N')}\chi_{[S'y_h,T'y_h]}(t_{qh})(g_{1-\varepsilon(q_{h+1}-1)}|f'_{y_h})$$

$$\langle\Psi_{N-m}(\chi_{[S_\alpha,T_\alpha]}\otimes f_\alpha,\alpha\in\{1,\ldots,N\}\setminus\{x_h\}_{h=1}^m),\; \Psi_{N'-m}(\chi_{[S'_\alpha,T'_\alpha]}\otimes f'_\alpha,\alpha\in\{1,\ldots,N'\}\setminus\{y_h\}_{h=1}^m)\rangle\quad(4.6)$$

Proof. – By expanding the product

$$(-i)^nV(t_1)\ldots V(t_n)\quad(4.7)$$

to

$$\sum_{\varepsilon\in\{0,1\}^n} D_{\varepsilon(1)}\dots D_{\varepsilon(n)}\otimes A^+(S_{t_1}g_{\varepsilon(1)})$$

$$\times A(S_{t_1}g_{1-\varepsilon(1)})\dots A^+(S_{t_n}g_{\varepsilon(n)})A(S_{t_n}g_{1-\varepsilon(n)}) \quad (4.8)$$

one can write the scalar product in (1.19) in the form

$$\sum_{n=0}^{\infty}\sum_{\varepsilon\in\{0,1\}^n}\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{n-1}}$$

$$\times dt_n\left\langle \Phi_N\left(z\int_{D_k/z^2}^{T_k/z^2}S_u f_k\,du,\ k=1,\ \dots,\ N\right),\right.$$

$$\left.(I_n(\varepsilon)+II_n(\varepsilon))\,\Phi_{N'}\left(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k=1,\ \dots,\ N'\right)\right\rangle$$

$$\times\langle u,\ D_{\varepsilon(1)}\dots D_{\varepsilon(n)}v\rangle \quad (4.9)$$

Applying Theorem (4.1) to (4.9), one knows that if $\|g\|_-^2<1/16\|D\|$, the limit (1.22) is equal to

$$\sum_{n=0}^{\infty}\lim_{z\to 0}\sum_{\varepsilon\in\{0,1\}^n}\int_0^{t/z^2}dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{n-1}}dt_n$$

$$\langle u,\ D_{\varepsilon(1)}\dots D_{\varepsilon(n)}v\rangle\left\langle \Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k\,du,\ k=1,\ \dots,\ N\right),\right.$$

$$\left.(I_n(\varepsilon)+II_n(\varepsilon))\,\Phi_{N'}\left(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k=1,\ \dots,\ N'\right)\right\rangle \quad (4.10)$$

By application of Theorem (3.2) and Theorem (3.3) we finish the proof.

5. THE QUANTUM STOCHASTIC DIFFERENTIAL EQUATION

From the Sections §2, §3 and §4 one has learnt

(1) the limit space on which our limit processes lives;

(2) the conditions allowing to take the limit in (1.19);

(1) the explicit form of the limit (1.19).

Now we want to describe the quantum stochastic process arising in the limit (1.19).

First of all notice that for each $N,\ N'\in\mathbf{N}$, $f_1,\ \dots,f_N,\ f'_1,\ \dots,f'_{N'}\in K$, $\{S_h,\ T_h\}_{h=1}^N$, $\{S_h,\ T_h\}_{h=1}^N\subset\mathbf{R}$, $u,\ v\in H_0$, $D\in B(H_0)$, the scalar product in

(1.19) can be written as

$$\left\langle u, \mathrm{F}_z\left(t; \mathrm{N}, \mathrm{N}'; \begin{pmatrix} f_1, \ldots, f_{\mathrm{N}} \\ f'_1, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle \tag{5.1}$$

and its limit *i.e.* (4.6) can be written in the form

$$\left\langle u, \mathrm{F}\left(t; \mathrm{N}, \mathrm{N}'; \begin{pmatrix} f_1, \ldots, f_{\mathrm{N}} \\ f'_1, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle \tag{5.2}$$

It is clear that both expressions are bounded.

In the following we introduce the following notations: for $\sigma \in \{0, 1\}$

$$\mathrm{D}_g(\sigma) := \sum_{n=0}^{\infty} (g_{1-\sigma} | g_{1-\sigma})_-^n \, (g_\sigma | g_\sigma)_-^n \, (\mathrm{D}_\sigma \mathrm{D}_{1-\sigma})^n \tag{5.3}$$

$$\mathrm{D}_1(\sigma) := \mathrm{D}_g(\sigma)\,\mathrm{D}_\sigma, \qquad \mathrm{D}_2(\sigma) := (g_{1-\sigma} | g_{1-\sigma})_- \, \mathrm{D}_\sigma \mathrm{D}_{1-\sigma} \mathrm{D}_g(\sigma) \tag{5.4}$$

Our first and most important conclusion in this section is

THEOREM (5.1). – *For each* $\mathrm{N}, \mathrm{N}' \in \mathbf{N}$, $f_1, \ldots, f_{\mathrm{N}}, f'_1, \ldots, f'_{\mathrm{N}'} \in \mathrm{K}$, $\{\mathrm{S}_h, \mathrm{T}_h\}_{h=1}^{\mathrm{N}}$, $\{\mathrm{S}_h, \mathrm{T}_h\}_{h=1}^{\mathrm{N}} \subset \mathbf{R}$, $u, v \in \mathrm{H}_0$, $\mathrm{D} \in \mathrm{B}(\mathrm{H}_0)$, *under the conditions* (1.8), (1.9) *and* (4.5), *the expressions* (5.2) *satisfy the system of differential equations*

$$\begin{aligned} &\left\langle u, \mathrm{F}\left(t; \mathrm{N}, \mathrm{N}'; \begin{pmatrix} f_1, \ldots, f_{\mathrm{N}} \\ f'_1, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle \\ &\quad = \langle u \otimes \Psi_{\mathrm{N}} (\chi_{[\mathrm{S}_\alpha, \mathrm{T}_\alpha]} \otimes f_\alpha, \alpha \in \{1, \ldots, \mathrm{N}\}), \\ &\qquad v \otimes \Psi_{\mathrm{N}'} (\chi_{[\mathrm{S}'_\alpha, \mathrm{T}'_\alpha]} \otimes f'_\alpha, \alpha \in \{1, \ldots, \mathrm{N}'\}\rangle \\ &\qquad + \int_0^t ds \sum_{\varepsilon \in \{0, 1\}} \sum_{i=1}^{\mathrm{N}} \sum_{j=1}^{\mathrm{N}'} \chi_{[\mathrm{S}_i, \mathrm{T}_i]}(s) \\ &\qquad \times \chi_{[\mathrm{S}'_j, \mathrm{T}'_j]}(s) (f_i | g_\varepsilon) \cdot (g_{1-\varepsilon} | f'_j) \cdot (-1)^{i+j} \\ &\quad \Big\langle ((g_\varepsilon | f_i) \cdot (f'_j | g_{1-\varepsilon}) \mathrm{D}_1^+(\varepsilon) u + (g_\varepsilon | f_i) \cdot (f'_j | g_\varepsilon) \mathrm{D}_2^+(\varepsilon) u), \\ &\qquad \mathrm{F}\left(s; \mathrm{N}-1, \mathrm{N}'-1; \begin{pmatrix} f_1, \ldots, \hat{f}_i, \ldots, f_{\mathrm{N}} \\ f'_1, \ldots, \hat{f}'_j, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\Big\rangle \end{aligned} \tag{5.5a}$$

$$\begin{aligned} &\left\langle u, \mathrm{F}\left(0; \mathrm{N}, \mathrm{N}'; \begin{pmatrix} f_1, \ldots, f_{\mathrm{N}} \\ f'_1, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle \\ &\quad \langle u \otimes \Psi_{\mathrm{N}} (\chi_{[\mathrm{S}_\alpha, \mathrm{T}_\alpha]} \otimes f_\alpha, \alpha \in \{1, \ldots, \mathrm{N}\}), \\ &\qquad v \otimes \Psi_{\mathrm{N}'} (\chi_{[\mathrm{S}'_\alpha, \mathrm{T}'_\alpha]} \otimes f'_\alpha, \alpha \in \{1, \ldots. \mathrm{N}'\}\rangle \end{aligned} \tag{5.5b}$$

and

$$\langle u, \mathrm{F}(t; 0, 0; 0)\rangle = \langle u, v\rangle \tag{5.5c}$$

Remark. – This Theorem is the analogue of Theorem (5.10) of [1] and the two proofs are also similar. We shall not repeat the details of the proof but only give the main idea and outline the important steps.

Proof. – By the change of variable

$$z^2 t_1 \looparrowright t_1 \tag{5.6}$$

in (1.22), one finds that (5.1) is equal to

$$\begin{aligned}
&\Big\langle u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1,\ \dots,\ N\Big),\\
&\qquad v\otimes\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1,\ \dots,\ N'\Big)\Big\rangle\\
&\quad+\Big\langle u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1,\ \dots,\ N\Big),\\
&\qquad z^{-2}\sum_{n=1}^{\infty}\int_0^t dt_1\,(-i)\,V(t_1/z^2)\\
&\qquad \int_0^{t_1/z^2} dt_2\ \dots\int_0^{t_{n-1}} dt_n\,(-i)^{n-1}\,V(t_2)\ \dots\\
&\qquad \times V(t_n)\,\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1,\ \dots,\ N'\Big)\Big\rangle
\end{aligned}\tag{5.7}$$

Using the explicit form (1.6) of the interaction for $V(t_1/z^2)$ and the change of variables

$$m=n-1,\qquad s_1=t_2,\ \dots,\ s_m=t_{n-1} \tag{5.8}$$

(5.7) becomes

$$\begin{aligned}
&\Big\langle u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2} S_{u\ f_k}\,du,\ k=1,\ \dots,\ N\Big),\\
&\qquad v\otimes\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1,\ \dots,\ N'\Big)\Big\rangle\\
&\quad+z^{-2}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\Big\langle D_\varepsilon^+\,u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\,du,\ k=1,\ \dots,\ N\Big),\\
&\qquad 1\otimes A^+(S_{t_1/z^2}g_\varepsilon)\,A(S_{t_1/z^2}g_{1-\varepsilon})\\
&\qquad U_{t_1/z^2}\,\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\,du,\ k=1,\ \dots,\ N'\Big)\Big\rangle
\end{aligned}\tag{5.9}$$

The first term of (5.9) tends, as $z \to 0$, to

$$\langle u \otimes \Psi_N (\chi_{[S_\alpha, T_\alpha]} \otimes f_\alpha, \alpha \in \{1, \ldots, N\}), \quad v \otimes \Psi_{N'} (\chi_{[S'_\alpha, T'_\alpha]} \otimes f'_\alpha, \alpha \in \{1, \ldots, N'\} \rangle \tag{5.10}$$

In the second term of (5.9), the action of the creator $A^+ (S_{t_1/z^2} g_\varepsilon)$ on the $\Phi_{N'}$-number vector gives

$$\sum_{i=1}^{N} \int_{S_i/z^2}^{T_i/z^2} \langle S_{t_1/z^2} g_\varepsilon, S_u f_i \rangle \, du \, (-1)^i \times \Phi_{N-1} \left(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \, k \in \{1, \ldots, N\} \setminus \{i\} \right) \tag{5.11}$$

Therefore the second term of (5.9) is equal to

$$z^{-1} \sum_{i=1}^{N} \sum_{\varepsilon \in \{0, 1\}} \int_0^t dt_1 \int_{S_i/z^2}^{T_i/z^2} \langle S_{t_1/z^2} g_\varepsilon, S_u f_i \rangle \, du \, . \, (-1)^i$$
$$\Big\langle D_\varepsilon^+ u \otimes \Phi_{N-1} \left(z \int_{S_k/z^2}^{T_k/z^2} f_k \, du, \, k \in \{1, \ldots, N\} \setminus \{i\} \right),$$
$$1 \otimes A (S_{t_1/z^2} g_{1-\varepsilon}) \, U_{t_1/z^2} \, \Phi_{N'} \left(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \, k = 1, \ldots, N' \right) \Big\rangle \tag{5.12}$$

The expression $1 \otimes A (S_{t_1/z^2} g_{1-\varepsilon}) \, U_{t_1/z^2}$ is handled with the same techniques as in [1, ..., 6]. Namely: one expands U_{t_1/z^2} using the iterated series and after the change of variable $z^2 . t_2 \rightsquigarrow t_2$, one finds

$$1 \otimes A (S_{t_1/z^2} g_{1-\varepsilon}) + \sum_{n=2}^{\infty} \sum_{\varepsilon' \in \{0, 1\}} \frac{1}{z^2} \times \int_0^{t_1} dt_2 \, D_{\varepsilon'} \otimes A (S_{t_1/z^2} g_{1-\varepsilon}) \, A^+ (S_{t_2/z^2} g_{\varepsilon'}) \, A (S_{t_2/z^2} g_{1-\varepsilon'})$$
$$\int_0^{t_2/z^2} dt_3 \ldots \int_0^{t_{n-1}} dt_n \, (-i)^{n-2} \, V(t_3) \ldots V(t_n) \tag{5.13}$$

Moreover

$$A (S_{t_1/z^2} g_{1-\varepsilon}) \, A^+ (S_{t_2/z^2} g_{\varepsilon'}) \, A (S_{t_2/z^2} g_{1-\varepsilon'}) = \langle S_{t_1/z^2} g_{1-\varepsilon}, S_{t_2/z^2} g_{\varepsilon'} \rangle \, A (S_{t_2/z^2} g_{1-\varepsilon'}) + A^+ (S_{t_2/z^2} g_{\varepsilon'}) \, A (S_{t_2/z^2} g_{1-\varepsilon'}) \, A (S_{t_1/z^2} g_{1-\varepsilon}) \tag{5.14}$$

and by (1.9) the scalar product is not equal to zero only when $\varepsilon'=1-\varepsilon$. Thus (5.12) can be rewritten as

$$z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2}g_\varepsilon, S_u f_i\rangle du.(-1)^i$$

$$\Big\langle D^+_{1-\varepsilon}D^+_\varepsilon u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k du,\ k\in\{1,\ldots,N\}\setminus\{i\}\Big),$$

$$1\otimes A(S_{t_1/z^2}g_{1-\varepsilon})\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k du,\ k=1,\ldots,N'\Big)\Big\rangle$$

$$+z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2}g_\varepsilon, S_u f_i\rangle du.(-1)^i$$

$$\Big\langle D^+_{1-\varepsilon}D^+_\varepsilon u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k du,\ k\in\{1,\ldots,N\}\setminus\{i\}\Big),$$

$$\Big(\frac{1}{z^2}\cdot\int_0^{t_1}dt_2\langle S_{t_1/z^2}g_{1-\varepsilon}, S_{t_2/z^2}g_{1-\varepsilon}\rangle A(S_{t_2/z^2}g_{1-\varepsilon'})U_{t_2/z^2}$$

$$+\frac{1}{z^2}\cdot\int_0^{t_1}dt_2 A^+(S_{t_2/z^2}g_{\varepsilon'})A(S_{t_2/z^2}g_{1-\varepsilon'})A(S_{t_1/z^2}g_{1-\varepsilon})$$

$$\int_0^{t_2/z^2}dt_3\ldots\int_0^{t_{n-1}}dt_n(-i)^{n-2}V(t_3)\ldots V(t_n)\Big)$$

$$\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k du,\ k=1,\ldots,N'\Big)\Big\rangle \tag{5.15}$$

Now let see the third term of (5.15) and try to move the annihilation operator $A(S_{t_1/z^2}g_{1-\varepsilon})$ to the right hand side of product $V(t_3)\ldots V(t_n)$ so that we can let the annihilator act on the $\Phi_{N'}$-number vector. In order to do this, from the formulas (1.13) and (5.14) we know that the annihilator $A(S_{t_1/z^2}g_{1-\varepsilon})$ can appear in two ways:

1. it is used to produce a scalar product with a creator $A^+(S_{t_j}g_{1-\varepsilon})$

$$\langle S_{t_1/z^2}g_{1-\varepsilon}, S_{t_j}g_{1-\varepsilon}\rangle \tag{5.16}$$

where $j=3,4,\ldots,n$;

2. the annihilation operator is simply exchanged with the product $V(t_3)\ldots V(t_n)$.

In the case 1, since $j\geqq 3$, one obtains a term of type II, therefore its limit is zero: all the terms of this type are collected in $o(1)$ below. Thus

(5.16) is equal to

$$z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2}g_\varepsilon, S_u f_i\rangle du.(-1)^i$$

$$\Big\langle D^+_{1-\varepsilon}D^+_\varepsilon u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k\,du,\ k\in\{1,\ldots,N\}\setminus\{i\}\Big),$$

$$1\otimes A(S_{t_1/z^2}g_{1-\varepsilon})\,\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k=1,\ldots,N'\Big)\Big\rangle$$

$$+z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2}g_\varepsilon, S_u f_i\rangle du\,.\,(-1)^i$$

$$\Big\langle D^+_{1-\varepsilon}D^+_\varepsilon u\otimes\Phi_N\Big(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k\,du,\ k\in\{1,\ldots,N\}\setminus\{i\}\Big),$$

$$\Big(\frac{1}{z^2}\cdot\int_0^{t_1}dt_2\Big\langle S_{t_1/z^2}g_{1-\varepsilon}, S_{t_2/z^2}g_{1-\varepsilon}\Big\rangle A(S_{t_2/z^2}g_{1-\varepsilon'})\,U_{t_1/z^2}$$

$$+\frac{1}{z^2}\cdot\int_0^{t_1}dt_2\,A^+(S_{t_2/z^2}g_{\varepsilon'})\,A(S_{t_2/z^2}g_{1-\varepsilon'})$$

$$\int_0^{t_2/z^2}dt_3\ldots\int_0^{t_{n-1}}dt_n(-i)^{n-2}\,V(t_3)\ldots V(t_n)\,A(S_{t_1/z^2}g_{1-\varepsilon})\Big)$$

$$\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k=1,\ldots,N'\Big)\Big\rangle+o(1).\qquad(5.17)$$

By letting the annihilator $A(S_{t_1/z^2}g_{1-\varepsilon})$ act on the number vector $\Phi_{N'}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k=1,\ldots,N'\Big)$ one obtains

$$\sum_{j=1}^{N'}z(-1)^j\int_{S'_j/z^2}^{T'_j/z^2}\langle S_{t_1/z^2}g_{1-\varepsilon}, S_u f'_j\rangle$$

$$\times du\,\Phi_{N'-1}\Big(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\ k\in\{1,\ldots,N'\}\setminus\{j\}\Big)\qquad(5.18)$$

Finally recall that as $z\to 0$ one has

$$\left.\begin{array}{c}\displaystyle\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2}g_\sigma, S_u f_i\rangle du\to\chi_{[S_i,T_i]}(t_1)(g_\sigma\,|\,f_i);\\ \sigma=\varepsilon,\ 1-\varepsilon\end{array}\right\}\qquad(5.19a)$$

and

$$z^{-2}\int_0^{t_{n-1}} dt_n \langle S_{t_{n-1}/z^2} g_\sigma, S_{t_n/z^2} g_\sigma \rangle \to (g_\sigma | g_\sigma)_-; \tag{5.19b}$$

$$\sigma=\varepsilon, \qquad 1-\varepsilon, n\in\mathbf{N}$$

From the above, in the notations (5.1), (5.2), we deduce:

$$\left\langle u, F_z\left(t; N, N'; \begin{pmatrix} f_1, \ldots, f_N \\ f'_1, \ldots, f'_{N'} \end{pmatrix}\right)\right\rangle$$
$$= \langle u\otimes\Psi_N(\chi_{[S_\alpha, T_\alpha]}\otimes f_\alpha, \alpha\in\{1, \ldots, N\}),$$
$$v\otimes\Psi_{N'}(\chi_{[S'_\alpha, T'_\alpha]}\otimes f'_\alpha, \alpha\in\{1, \ldots, N'\}\rangle$$
$$+\sum_{i=1}^{N}\sum_{j=1}^{N'}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1 \chi_{[S_i, T_i]}(t_1)(f_i|g_\varepsilon)$$
$$\times\chi_{[S'_j, T'_j]}(t_1)(g_{1-\varepsilon}|f'_j)(-1)^{i+j}$$

$$\left\langle D_\varepsilon^+ u, F\left(s; N-1, N'-1; \begin{pmatrix} f_1, \ldots, \hat{f}_i, \ldots, f_N \\ f'_1, \ldots, \hat{f}'_j, \ldots, f'_{N'} \end{pmatrix}\right)\right\rangle$$
$$+z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\int_{S_i/z^2}^{T_i/z^2}\langle S_{t_1/z^2} g_\varepsilon, S_u f_i\rangle du\,.\,(-1)^i$$

$$\left\langle D_{1-\varepsilon}^+ D_\varepsilon^+ u\otimes\Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2} S_u f_k\, du, k\in\{1, \ldots, N\}\setminus\{i\}\right),\right.$$
$$\frac{1}{z^2}\,.\int_0^{t_1} dt^2 \langle S_{t_1/z^2} g_{1-\varepsilon}, S_{t_2/z^2} g_{1-\varepsilon}\rangle A(S_{t_2/z^2} g_{1-\varepsilon'}) U_{t_2/z^2}$$
$$\left.+\Phi_{N'}\left(z\int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k\, du, k=1, \ldots, N'\right)\right\rangle + o(1) \tag{5.20}$$

Notice that in (5.20) the last term is similar to (5.12), therefore by repeating the discussion from (5.12) to (5.20), we have

$$\left\langle u, F_z\left(t; N, N'; \begin{pmatrix} f_1, \ldots, f_N \\ f'_1, \ldots, f'_{N'} \end{pmatrix}\right)\right\rangle$$
$$= \langle u\otimes\Psi_N(\chi_{[S_\alpha, T_\alpha]}\otimes f_\alpha, \alpha\in\{1, \ldots, N\}),$$
$$v\otimes\Psi_{N'}(\chi_{[S'_\alpha, T'_\alpha]}\otimes f'_\alpha, \alpha\in\{1, \ldots, N'\}\rangle$$
$$+\sum_{i=1}^{N}\sum_{j=1}^{N'}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1 \chi_{[S_i, T_i]}(t_1)(f_i|g_\varepsilon)$$
$$\times\chi_{[S'_j, T'_j]}(t_1)(g_{1-\varepsilon}|f'_j)\,.\,(-1)^{i+j}$$

$$\left\langle D_\varepsilon^+ u, F\left(t_1; N-1, N'-1; \begin{pmatrix} f_1, \ldots, \hat{f}_i, \ldots, f_N \\ f'_1, \ldots, \hat{f}'_j, \ldots, f'_{N'} \end{pmatrix}\right)\right\rangle$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N'}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\,\chi_{[S_i,T_i]}(t_1)(f_i\,|\,g_\varepsilon)$$
$$\times(g_{1-\varepsilon}\,|\,g_{1-\varepsilon})-\cdot\,\chi_{[S_j,T_j]}(t_1)(g_\varepsilon\,|\,f'_j)\,.\,(-1)^{i+j}$$
$$\left\langle D^+_{1-\varepsilon}D^+_\varepsilon\,u\,F\left(t_1;\,N-1,\,N'-1:\begin{pmatrix}f_1,\,\ldots,\hat{f}_i,\,\ldots,f_N\\ f'_1,\,\ldots,\hat{f}'_j,\,\ldots,f'_{N'}\end{pmatrix}\right)\right\rangle$$
$$+z^{-1}\sum_{i=1}^{N}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\,\chi_{[S_i,T_i]}(t_1)(f_i\,|\,g_\varepsilon)$$
$$\times z^{-2}\int_0^{t_1}dt_2\,\langle S_{t_1/z^2}g_{1-\varepsilon},\,S_{t_2/z^2}g_{1-\varepsilon}\rangle$$
$$\times z^{-2}\int_0^{t_2}dt_3\,\langle S_{t_2/z^2}g_\varepsilon,\,S_{t_3/z^2}g_\varepsilon\rangle$$
$$\left\langle D^+_\varepsilon\,D^+_{1-\varepsilon}D^+_\varepsilon\,u\otimes\Phi_N\left(z\int_{S_k/z^2}^{T_k/z^2}S_u f_k\,du,\,k\in\{1,\,\ldots,\,N\}\setminus\{i\}\right),\right.$$
$$1\otimes A(S_{t_3/z^2}g_\varepsilon)\,U_{t_3/z^2}\,\Phi_{N'}$$
$$\left.\times\left(z\int_{S'_k/z^2}^{T'_k/z^2}S_u f'_k\,du,\,k=1,\,\ldots,\,N'\right)\right\rangle+o(1)\qquad(5.21)$$

Iterating n times the above procedure one finds that the scalar product

$$\left\langle u,\,F_z\left(t;\,N,\,N';\begin{pmatrix}f_1,\,\ldots,f_N\\ f'_1,\,\ldots,f'_{N'}\end{pmatrix}\right)\right\rangle$$

is expressed as a sum of several terms. Denoting by T_k the sum of all the terms obtained in the k-th step with the exception of the first and the last summands, one has

$$T_n=T_{n+1}+\sum_{i=1}^{N}\sum_{j=1}^{N'}\sum_{\varepsilon\in\{0,1\}}\int_0^t dt_1\,\chi_{[S_i,T_i]}(t_1)(f_i\,|\,g_\varepsilon)$$
$$\times(g_{1-\varepsilon}\,|\,g_{1-\varepsilon})-(g_\varepsilon\,|\,g_\varepsilon)-\,\ldots\,(g_{\varepsilon_n}\,|\,g_{1-\varepsilon_n})$$
$$-\cdot\,\chi_{[S_j,T_j]}(t_1)(g_{\varepsilon_n}\,|\,f'_j)\,.\,(-1)^{i+j}$$
$$\left\langle D^+_{1-\varepsilon_n}\ldots D^+_{1-\varepsilon}D^+_\varepsilon\,u\,F\left(t_1;\,N-1,\,N'-1;\right.\right.$$
$$\left.\left.\begin{pmatrix}f_1,\,\ldots,\hat{f}_i,\,\ldots,f_N\\ f'_1,\,\ldots,\hat{f}'_j,\,\ldots,f'_{N'}\end{pmatrix}\right)\right\rangle\qquad(5.22)$$

where

$$\varepsilon_n := \begin{cases} 1-\varepsilon & \text{if } n \text{ is odd} \\ \varepsilon, & \text{if } n \text{ is even} \end{cases} \tag{5.23}$$

Moreover the last summand in the n-th step is

$$\begin{aligned} z^{-1} \sum_{i=1}^{N} \sum_{\varepsilon \in \{0,1\}} \int_0^t & dt_1 \, \chi_{[S_i, T_i]}(t_1) (f_i | g_\varepsilon) \\ & \times z^{-2} \int_0^{t_1} dt_2 \langle S_{t_1/z^2} g_{1-\varepsilon}, S_{t_2/z^2} g_{1-\varepsilon} \rangle \\ & \times z^{-2} \int_0^{t_2} dt_3 \langle S_{t_2/z^2} g_\varepsilon, S_{t_3/z^2} g_\varepsilon \rangle \\ & \times \ldots \times z^{-2} \int_0^{t_{n-1}} dt_n \langle S_{t_{n-1}/z^2} g_{\varepsilon_n}, S_{t_n/z^2} g_{\varepsilon_n} \rangle \\ & \times \Big\langle D^+_{1-\varepsilon_n} \ldots D^+_\varepsilon D^+_{1-\varepsilon} D^+_\varepsilon u \\ & \otimes \Phi_N \Big(z \int_{S_k/z^2}^{T_k/z^2} S_u f_k \, du, \, k \in \{1, \ldots, N\} \setminus \{i\} \Big), \\ & 1 \otimes A(S_{t_n/z^2} g_{\varepsilon_n}) U_{t_n/z^2} \Phi_{N'} \\ & \Big(z \int_{S'_k/z^2}^{T'_k/z^2} S_u f'_k \, du, \, k = 1, \ldots, N' \Big) \Big\rangle + o(1) \end{aligned} \tag{5.24}$$

Notice that the term (5.24) differs from the corresponding one in the (n-1)-st step in that the operator $A(S_{t_{n-1}/z^2} g_{\varepsilon_{n-1}})$ has been replaced by

$$z^{-2} \int_0^{t_{n-1}} dt_n \langle S_{t_{n-1}/z^2} g_{\varepsilon_n}, S_{t_n/z^2} g_{\varepsilon_n} \rangle \,.\, A(S_{t_n/z^2} g_{\varepsilon_n})$$

and the operator

$$D^+_{1-\varepsilon_{n-1}} \ldots D^+_\varepsilon D^+_{1-\varepsilon} D^+_\varepsilon$$

has been replaced by

$$D^+_{1-\varepsilon_n} \ldots D^+_\varepsilon D^+_{1-\varepsilon} D^+_\varepsilon .$$

Therefore it follows, from the induction argument and formula (5.19 *b*) that

$$\left\langle u, \mathrm{F}_z\left(t; \mathrm{N}, \mathrm{N}'; \begin{pmatrix} f_1, \ldots, f_\mathrm{N} \\ f'_1, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle$$

$$= \langle u \otimes \Psi_\mathrm{N}(\chi_{[\mathrm{S}_\alpha, \mathrm{T}_\alpha]} \otimes f_\alpha, \alpha \in \{1, \ldots, \mathrm{N}\}),$$

$$v \otimes \Psi_{\mathrm{N}'}(\chi_{[\mathrm{S}'_\alpha, \mathrm{T}'_\alpha]} \otimes f'_\alpha, \alpha \in \{1, \ldots, \mathrm{N}'\}\rangle$$

$$+ \sum_{n=1}^{\infty} \sum_{i=1}^{\mathrm{N}} \sum_{j=1}^{\mathrm{N}'} \sum_{\varepsilon \in \{0, 1\}} \int_0^t dt_1\, \chi_{[\mathrm{S}_i, \mathrm{T}_i]}(t_1)(f_i | g_\varepsilon)$$

$$\times (g_{1-\varepsilon} | g_{1-\varepsilon}) - (g_\varepsilon | g_\varepsilon) - (g_{1-\varepsilon} | g_{1-\varepsilon}) - \cdot \chi_{[\mathrm{S}_j, \mathrm{T}_j]}(t_1)(g_\varepsilon | f'_j) \cdot (-1)^{i+j}$$

$$\left\langle \mathrm{D}^+_{1-\varepsilon} \mathrm{D}^+_\varepsilon \mathrm{D}^+_{1-\varepsilon} \mathrm{D}^+_\varepsilon u \mathrm{F}\left(t_1; \mathrm{N}-1, \mathrm{N}'-1; \begin{pmatrix} f_1, \ldots, \hat{f}_i, \ldots, f_\mathrm{N} \\ f'_1, \ldots, \hat{f}'_j, \ldots, f'_{\mathrm{N}'} \end{pmatrix}\right)\right\rangle + o(1) \quad (5.25)$$

Finally by rewriting the right hand side of (5.25) as the sum of two terms corresponding to n odd or even and letting z tend to zero, we obtain (5.5 *a*). It is easy to check (5.5 *b*) and (5.5 *c*).

Now let us introduce some notations on the Fock space $\Gamma(\mathrm{L}^2(\mathbf{R} \otimes (\mathrm{K}, (.\,|\,.)))$. For each $\xi \in \mathrm{B}(\mathrm{L}^2(\mathbf{R}))$, $\mathrm{T} \in \mathrm{B}(\mathrm{K})$, denote $\mathrm{N}(\xi \otimes \mathrm{T})$ the number operator, characterized by the property

$$\langle \Psi_\mathrm{N}(\eta_r \otimes f_r, r=1, \ldots, \mathrm{N}), \mathrm{N}(\xi \otimes \mathrm{T}) \Psi_{\mathrm{N}'}(\eta'_r \otimes f'_r, r=1, \ldots, \mathrm{N}')\rangle$$

$$= \sum_{j=1}^{\mathrm{N}} \sum_{k=1}^{\mathrm{N}'} (-1)^{j+k} \langle \eta_j, \xi \eta'_k \rangle \cdot \langle f_j, \mathrm{T} f'_k \rangle$$

$$\langle \Psi_{\mathrm{N}-1}(\eta_r \otimes f_r, r \in \{1, \ldots, \mathrm{N}\} \setminus \{j\}),$$

$$\mathrm{N}(\xi \otimes \mathrm{T}) \Psi_{\mathrm{N}'-1}(\eta'_r \otimes f'_r, r \in \{1, \ldots, \mathrm{N}'\} \setminus \{k\})\rangle \quad (5.26)$$

For each $f, g \in \mathrm{K}$, $s \geqq 0$, define the number process $\mathrm{N}_s(f, g)$ by $\mathrm{N}(\chi_{[0, s]} \otimes | f \rangle \langle g |)$. Consider the quantum stochastic differential equation

$$\mathrm{U}(t) = 1 + \int_0^t \sum_{\sigma \in \{0, 1\}} (\mathrm{D}_1(\sigma) \otimes d\mathrm{N}_s(\sigma, 1-\sigma) + \mathrm{D}_2(\sigma) \otimes d\mathrm{N}_s(\sigma, \sigma))\, \mathrm{U}(s) \quad (5.27)$$

where,

$$\mathrm{N}_s(\sigma, \varepsilon) := \mathrm{N}_s(g_\sigma, g_\varepsilon) \quad (5.28)$$

THEOREM (5.3). — *The quantum stochastic differential equation* (5.27) *has a unique and unitary solution.*

Proof. – The existence and uniqueness of the solution of q.s.d.e. (5.27) follows form the fact that $D_1(\sigma)$, $D_2(\sigma)$ are bounded operators. The proof of unitarity is the same as the one of Theorem (6.3) of [1].

Now our last assertion can be stated and proved as following:

THEOREM (5.4). – Under the conditions (1.8), (1.9) and (4.5), the limit (1.19) is of form

$$\langle u\otimes\Psi_N(\chi_{[S_r,T_r]}\otimes f_r\; r=1,\ldots,N),\; U(t)\,v\otimes\Psi_{N'}(\chi_{[S'_r,T'_r]}\otimes f'_r,\; r=1,\ldots,N') \tag{5.29}$$

and where $U(t)$ is the solution of the quantum stochastic differential equation (5.27).

Proof. – Clearly (5.29) can be written in the form:

$$\left\langle u,\, G\left(t;\, N,\, N';\, \begin{pmatrix} f_1, \ldots, f_N \\ f'_1, \ldots, f'_{N'} \end{pmatrix}\right)\right\rangle \tag{5.30}$$

Using the QSDE (5.27) it is easy to show that (5.30) satisfies the system of differential equations (5.5 *a*, *b*, *c*).

Since $D_1(\sigma)$, $D_2(\sigma)$ are the bounded operators, one knows that the differential equation has a unique solution. This allows to identify (5.29) with (5.2) and therefore, by Theorem (5.1), with the limit (1.19). This completes the proof.

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(*Manuscript received January* 18, 1991;
revised version received November 10, 1991.)