ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 58, nº 1 (1993), p. 55-83

<http://www.numdam.org/item?id=AIHPA_1993__58_1_55_0>

Ann. Inst. Henri Poincaré,

Vol. 58, n° 1, 1993, p. 55-83. *Physique théorique*

Efficient bounds for the spectral shift function

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ABSTRACT. – Let H_0, H be a pair of selfadjoint operators in a separable Hilbert space whose difference $V = H - H_0$ belongs to the trace class and let $\Theta(\lambda) = \Theta(\lambda; H_0, H)$ be the spectral shift function for the pair H_0, H. We obtain the pointwise bounds of $\Theta(\lambda)$ in terms of the compactness properties of the operator $|V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. For example, if it has boundary values in the Neumann-Schatten class $\mathfrak{S}_p$, $p \leqq 1$, as $\Im z \to +0$, then

$$|\Theta(\lambda)| \leqq C \| |V|^{1/2}(H_0 - \lambda - i0)^{-1}|V|^{1/2} \|_p^p.$$

This abstract result allows to obtain the estimates of the scattering phase for the Schrödinger operator which are valid for arbitrary values of the coupling constant and the energy.

RÉSUMÉ. – Soit H_0, H une paire d'opérateurs auto-adjoints dans un espace hilbertien séparable, dont la différence $V = H - H_0$ appartient à la classe des opérateurs à trace et soit $\Theta(\lambda) = \Theta(\lambda; H_0, H)$ la fonction spectrale de perturbation pour la paire H_0, H. On obtient des bornes ponctuelles de $\Theta(\lambda)$ en termes de propriétés de compacité de l'opérateur $|V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. Par exemple, si ce dernier a des valeurs au bord dans la classe de Neumann-Schatten $\mathfrak{S}_p$, $p \leqq 1$, lorsque $\Im z \to +0$, alors

$$|\Theta(\lambda)| \leqq C \| |V|^{1/2}(H_0 - \lambda - i0)^{-1}|V|^{1/2} \|_p^p.$$

Ce résultat abstrait permet d'obtenir des estimations de la phase de diffusion pour l'opérateur de Schrödinger qui sont valables pour n'importe quelle valeur de la constante de couplage et de l'énergie.

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1. INTRODUCTION

Let H_0, H be a pair of selfadjoint operators in a separable Hilbert space whose difference $V = H - H_0$ belongs to the trace class $\mathfrak{S}_1$ and let $\Theta(\lambda) = \Theta(\lambda; H_0, H)$ be the spectral shift function (SSF) for the pair H_0, H. The function $\Theta(\lambda)$ plays an important role in the theory of trace class perturbations. It suffices to mention, for example, the so-called trace formula (*see* [1]-[4]):

$$\operatorname{Tr}[f(H) - f(H_0)] = \int f'(\lambda)\,\Theta(\lambda)\,d\lambda, \tag{1.1}$$

where f is a suitable function such that the l.h.s. of (1.1) is finite. Furthermore, $\Theta(\lambda)$ is related to the scattering matrix $S(\lambda) = S(\lambda; H_0, H)$ for the pair H_0, H. Recall that $S(\lambda)$ is defined for a.a. λ in the absolutely continuous spectrum σ_{ac} of H_0, it is a unitary operator and $S(\lambda) - I$ is trace class. As was proved in [2],

$$\exp(-2i\pi\Theta(\lambda)) = \det S(\lambda), \quad a.a.\, \lambda \in \sigma_{ac}. \tag{1.2}$$

This allows us to interpret $\Theta(\lambda)$ for $\lambda \in \sigma_{ac}$ as the scattering phase. Outside of the essential spectrum of H_0 the function $\Theta(\lambda)$ measures the difference between the discrete spectra of H and H_0 (*see* Section 3 for precise formulation).

We are going to discuss the properties of $\Theta(\lambda)$ in dependence on the size of the perturbation V. The first result in this direction was obtained in [1]. It says that

$$\int |\Theta(\lambda)|\,d\lambda \leqq \|V\|_1. \tag{1.3}$$

In the present paper we obtain the pointwise bounds. Our results are of the conditional character, they are formulated in terms of the boundary values of the operator $B(z) = |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. Suppose that $B(\lambda + i\varepsilon)$ for a.a. λ in some interval $\Lambda \subset R$ has a limit as $\varepsilon \to +0$ in the Neumann-Schatten class $\mathfrak{S}_p$ or in the class Σ_p, $p \geqq 0$ (*see* Section 2 for definition). In such a situation we shall say sometimes that the "limiting absorption principle" in $\mathfrak{S}_p$ or Σ_p holds. Then for $p \leqq 1$

$$|\Theta(\lambda)| \leqq C [B(\lambda + i0)]_p^p, \quad a.a. \lambda \in \Lambda, \tag{1.4}$$

where the constant C does not depend on λ, V, H_0, the bracket $[\ .\ ]_p$ denotes here the quasi-norm in one of the classes $\mathfrak{S}_p$ or Σ_p (Here and below we denote by C and c various constants whose exact values are of no importance.) Obviously, (1.4) differs from (1.3) by the absence of integration. However, on the other hand, the estimate (1.4) contains the resolvent of H_0. At the same time the perturbed operator H does not enter this estimate. In this sense the bound (1.4) is quite efficient. Other possible bounds for $\Theta(\lambda)$ and their discussion are given in Section 3. In particular, we obtain also the estimates similar to (1.4) for the case when only certain powers of the resolvents of H_0 and H differ by a trace class operator.

Roughly speaking the idea of the proof of (1.4) reduces to the following simple observation. The starting point is the definition of $\Theta(\lambda)$ in terms of the perturbation determinant (*see* Section 3), given in [1]:

$$\Theta(\lambda) = \pi^{-1} \lim_{\varepsilon \to +0} \arg \det (I + B(\lambda + i\varepsilon)\Omega), \tag{1.5}$$

where Ω is "the sign" of V, *i.e.* $\Omega = V|V|^{-1}$. First suppose that V is small in a certain sense. Therefore $\det(I + B(\lambda + i\varepsilon)\Omega) \approx 1 + \mathrm{Tr}(B(\lambda + i\varepsilon)\Omega)$, the trace in the r.h.s. being small. Thus (1.5) gives the relation

$$\pi |\Theta(\lambda)| \approx \Im \mathrm{Tr}(B(\lambda + i0)\Omega). \tag{1.6}$$

In the general case we present V in the form $V = V' + V''$, where V' is small and V'' is a finite range operator. The contribution of V' to SSF we estimate by means of (1.6). To take into account V'' it suffices to recall that the SSF for a finite range perturbation is bounded by its dimension.

In Section 4-7 we apply the abstract result (1.4) to the estimates of the SSF for the Schrödinger operator $H_g = H_0 + gV$, $H_0 = -\Delta$, in $L^2(\mathbb{R}^d)$, $d \geqq 2$, where $g \geqq 0$ is the coupling constant. The function V is assumed to decay as $O(|x|^{-\alpha})$, $\alpha > d$ at infinity. The requirement $\alpha > d$ allows to define the SSF for the pair H_0, H_g. We establish the bounds for SSF in dependence on g and the parameter λ which now has the sense of the energy. (Unfortunately we are not able to trace the dependence of SSF on the function V.) We are interested in positive λ, where SSF is interpreted as the scattering phase. This is for the reason that in the case $\lambda < 0$ SSF coincides with the distribution function of the discrete spectrum of H_g and therefore it can be studied in more details by other methods (*see* [20]). Detailed discussion of the results is given in Section 4.

In order to infer the SSF estimates for the Schrödinger operator from (1.4), we have to study the limit as $\varepsilon \to +0$ of the "sandwiched" resolvent

$$(1+|x|)^{-(\alpha/2)}\zeta(H_0\lambda^{-1})(H_0-\lambda-i\varepsilon)^{-1}\zeta(H_0\lambda^{-1})(1+|x|)^{-(\alpha/2)}, \quad (1.7)$$

where the function $\zeta \in C_0^\infty(\mathbb{R})$ equals 1 in a neighbourhood of the point 1. The boundary values of (1.7) are studied by means of a technique which resembles that of Mourre [5] though does not imitate it completely (*see* Section 6).

The paper is organized as follows. The necessary information from the theory of compact operators is collected in Section 2. In Section 3 we present the bounds for SSF in the abstract setting. The estimates of SSF for the Schrödinger operator are stated in Section 4. The auxiliary estimates needed for their proof are established in Section 5. The boundary values of the operator (1.7) are studied in Section 6. Finally, in Section 7 we complete the proof of the estimates of Section 4.

2. AUXILIARY RESULTS

1° – We need some information from the theory of compact operators (*see* [6], [7]). Let $\mathfrak{S}_\infty$ be the class of compact operators, $\lambda_k(A)$ be the eigenvalues of an operator $A \in \mathfrak{S}_\infty$, enumerated in order of decreasing modulus counting multiplicity, Let $s_k(A)=[\lambda_k(A^*A)]^{1/2}=[\lambda_k(AA^*)]^{1/2}$ be singular values (s-values) of $A \in \mathfrak{S}_\infty$. We shall use the following well-known properties of eigenvalues and s-values. For any integer n, m, and any $A, B \in \mathfrak{S}_\infty$

$$s_{n+m-1}(A+B) \leqq s_n(A)+s_m(B), \quad (2.1)$$

$$s_{n+m-1}(AB) \leqq s_n(A)\, s_m(B). \quad (2.2)$$

For any $q>0$ and integer n

$$\sum_{k=1}^{n} |\lambda_k(A)|^q \leqq \sum_{k=1}^{n} s_k^q(A). \quad (2.3)$$

Denote by $\mathfrak{S}_p$, $p>0$, the class of operators with the finite functional

$$\|A\|_p=\left\{\sum_{k=1}^{n} s_k^p(A)\right\}^{1/p}.$$

If $p \geqq 1$ this functional defines a norm. At $p=1\,(p=2)$ the class $\mathfrak{S}_p$ is referred to as to the trace class (Hilbert-Schmidt class). Let $\mathbf{\Sigma}_p$ be the class of operators $A \in \mathfrak{S}_\infty$ with the finite functional

$$\langle A \rangle_p = \sup_k k^{1/p} s_k(A).$$

The classes $\mathbf{\Sigma}_p$ are normable if $p>1$. Further we sometimes denote by $\mathbf{S}_p$ any of the classes $\mathfrak{S}_p$ or $\mathbf{\Sigma}_p$. The notation $[\,.\,]_p$ means either $\|\,.\,\|_p$ or $\langle\,.\,\rangle_p$.

Note the inequality, following from (2.2):

$$[A_1 A_2]_p \leqq C [A_1]_r [A_2]_t, \qquad p^{-1}=r^{-1}+t^{-1}, \quad r>0, \quad t>0, \tag{2.4}$$

where $A_1 \in S_r$, $A_2 \in S_t$ and C depends on p only. One can easily verify that the eigenvalues of $A \in S_p$ obey the bound

$$|\lambda_k(A)| \leqq C k^{-1/p} [A]_p, \tag{2.5}$$

with a constant C not depending on A, C=1 if $S_p = \mathfrak{S}_p$. It is important for us that the operators $A \in S_p$ for $p<1$ satisfy the p-triangle inequality:

$$[A]_p^p \leqq C \sum_{k=1}^{n} [A_k]_p^p, \tag{2.6}$$

where $A = \sum_{k=1}^{n} A_k$ and $C=C(p)$ does not depend on the number of terms in the sum. The following proposition allows to extend (2.6) to all $p<2$ under an additional restriction on A_k.

PROPOSITION 2.1 [8]. – *Let* $A_k \in S_p$, $p<2$, *and*

$$A_j A_k^* = 0, \qquad j \neq k.$$

Then the inequality (2.6) *is valid.*

Note that for $p>1$ (2.6) is stronger than usual triangle inequality.

Now we recall some properties of the trace class operators. If $A \in \mathfrak{S}_1$ then the Lidskii's theorem holds:

$$\sum_{k=1}^{\infty} \lambda_k(A) = \operatorname{Tr} A. \tag{2.7}$$

Further, let $T=(2i)^{-1}(A-A^*)$ be the imaginary part of A. Then

$$\sum_{k=1}^{\infty} |\Im \lambda_k(A)| \leqq \sum_{k=1}^{\infty} s_k(T). \tag{2.8}$$

Let us consider the determinant

$$\det(I+A) = \prod_{k=1}^{\infty} (1+\lambda_k(A)). \tag{2.9}$$

We are going to introduce "the argument" (or "the phase") of $\det(I+A)$. Denote by $\mathscr{M} \subseteqq \mathfrak{S}_1$ the set of those operators which have no eigenvalues on the half-line $(-\infty, -1]$. For $A \in \mathscr{M}$ the equality

$$\frac{1+\lambda_k(A)}{|1+\lambda_k(A)|} = \exp(i\pi\zeta_k(A)) \tag{2.10}$$

and the condition $\zeta_k \in (-1, 1)$ define correctly the numbers $\zeta_k = \zeta_k(A)$, $k = 1, 2, \ldots$ Set

$$\zeta(A) = \sum_{k=1}^{n} \zeta_k(A). \tag{2.11}$$

Clearly, this series converges absolutely and, obviously,

$$\frac{\det(I+A)}{|\det(I+A)|} = \exp(i\pi\zeta(A)).$$

Consequently, it is natural to call $\zeta(A)$ the argument of $\det(I+A)$. Note that for $A, B \in \mathfrak{S}_2$

$$\det(I+AB) = \det(I+BA), \qquad \zeta(AB) = \zeta(BA), \tag{2.12}$$

since non-trivial spectra AB and BA coincide with each other. Now we give upper bounds for $\zeta(A)$.

LEMMA 2.2. – *Let* $A \in \mathcal{M}$. *Then*

$$|\zeta(A)| \leqq C_p \{ \|T\|_1 + [A]_p^p \} \tag{2.13}$$

for any $p \geqq 1$. *If* $A \in S_p$ *with* $p < 1$ *then*

$$|\zeta(A)| \leqq C_p [A]_p^p. \tag{2.14}$$

The constant C_p *does not depend on* A.

Proof. – Let us split the sum (2.11) into two summands:

$$I_1^{(N)} = \sum_{k=1}^{N-1} \zeta_k, \qquad I_2^{(N)} = \sum_{k=N}^{\infty} \zeta_k.$$

Choose the number N in such a way that

$$N - 1 \leqq \gamma [A]_p^p \leqq N \tag{2.15}$$

with some positive constant γ. Since $|\zeta_k| \leqq 1$, it follows from the left inequality (2.15) that

$$|I_1^{(N)}| \leqq \sum_{k=1}^{N-1} 1 = N - 1 \leqq \gamma [A]_p^p. \tag{2.16}$$

To treat $I_2^{(N)}$ we observe that by (2.5) and (2.15) for γ large enough

$$|\lambda_k(A)| \leqq \frac{1}{2}, \qquad k \geqq N.$$

This implies that the numbers $1 + \lambda_k(A)$ at $k \geqq N$ lie in the right half plane. More precisely, $\Re(1 + \lambda_k(A)) \geqq \frac{1}{2}$. Therefore,

$$\zeta_k = \pi^{-1} \arctan \frac{\Im \lambda_k}{1 + \Re \lambda_k}, \qquad k \geqq N. \tag{2.17}$$

Since arctan $x \leqq x$, $x \geqq 0$, then

$$|\zeta_k| \leqq 2\pi^{-1} |\Im \lambda_k|, \qquad k \geqq N.$$

Hence

$$|I_2^{(N)}| \leqq 2\pi^{-1} \sum_{k=N}^{\infty} |\Im \lambda_k|. \tag{2.18}$$

In the case $p \geqq 1$ we estimate the r. h. s. of (2.18) with the help of (2.8):

$$|I_2^{(N)}| \leqq 2\pi^{-1} \sum_{k=1}^{\infty} |\Im \lambda_k| \leqq 2\pi^{-1} \|T\|_1, \qquad p \geqq 1. \tag{2.19}$$

Let $p<1$. Since $|\Im \lambda_k| \leqq |\lambda_k|$ then according to (2.5) the inequality (2.18) yields

$$|I_2^{(N)}| \leqq C \left\{ \sum_{k=N}^{\infty} k^{-1/p} \right\} [A]_p \leqq C N^{-1/p+1} [A]_p.$$

Taking into account the right inequality (2.15) we get

$$|I_2^{(N)}| \leqq C [A]_p^p, \qquad p<1. \tag{2.20}$$

Combining (2.16) with (2.19) or (2.20) we obtain (2.13) or (2.14) respectively. ○

2° – Now we shall discuss the properties of the phase $\zeta(A)$ for operator-valued functions $A=A(z)$. Let $\mathscr{D}$ be an arbitrary open domain in the complex plane and let $\mathscr{M}_c(\mathscr{D})$ be the class of operator-valued functions $\mathfrak{S}_1$-continuous in $z \in \mathscr{D}$, whose values belong to $\mathscr{M}$.

LEMMA 2.3. – *Let* $A(.) \in \mathscr{M}_c(\mathscr{D})$. (i) *Then the function* $\zeta(z)=\zeta(A(z))$ *is continuous in* $z \in \mathscr{D}$.

(ii) *If* $A(.)$ *is* $\mathfrak{S}_1$*-continuous in* $z \in \bar{\mathscr{D}}$, $A(z) \in \mathscr{M}$ *for all* $z \in \mathscr{D}$ *and* (-1) *is not an eigenvalue of* $A(z)$ *for* $z \in \partial\mathscr{D}$ *then the function* $\zeta(.)$ *can be extended by continuity to all* $z \in \bar{\mathscr{D}}$.

Proof. – It suffices to establish (ii). By (2.7) $\operatorname{Tr} T(z) = \sum_{k=1}^{\infty} \Im \lambda_k(z)$, so that

$$\pi\zeta(z) = \operatorname{Tr} T(z) + \sum_{k=1}^{\infty} [\pi\zeta_k(z) - \Im \lambda_k(z)]. \tag{2.21}$$

Since $\operatorname{Tr} T(z) = \Im(\operatorname{Tr} A(z))$ is continuous, it remains to verify the continuity of the sum in (2.21). Let us fix a point $z_0 \in \bar{\mathscr{D}}$ and choose a number $\varepsilon \in (0, 1/2)$ so that the circle $\gamma_\varepsilon = \{\mu \,||\, \mu| = \varepsilon\}$ contains no eigenvalues of $A(z_0)$. Let $N=N(\varepsilon)$ be the number of eigenvalues lying outside of γ_ε, so that

$$|\lambda_k(z_0)| > \varepsilon, \qquad k \leqq N,$$
$$|\lambda_k(z_0)| < \varepsilon, \qquad k \geqq N+1.$$

According to the abstract perturbation theory (*see* [9]) the same estimates hold true (with the same number N) for the eignevalues $\lambda_k(z)$, provided that $z \in \bar{\mathcal{D}}$, $|z-z_0| \leqq \kappa$, and κ is small enough. The numbers $\lambda_k(z)$ with $k \leqq N$ can be renumerated in such a way that they are continuous at the point z_0. If $\Im \lambda_k(z_0) \neq 0$ or $\Im \lambda_k(z_0) = 0$, $\Re \lambda_k(z_0) > -1$, then the phase $\zeta_k(z)$ is trivially continuous by definition (2.10). Let us check the continuity of $\zeta_k(z)$ at a point $z_0 \in \partial\mathcal{D}$ where $\Im \lambda_k(z_0) = 0$ and $\lambda_k(z_0) < -1$. In this case we define $\zeta_k(z_0)$ by the equality

$$\zeta_k(z_0) = \lim_{z \in \mathcal{D},\, z \to z_0} \zeta_k(z).$$

This limit exists since by the condition $A(z) \in \mathcal{M}$, $z \in \mathcal{D}$, the function $\Im \lambda_k(z)$ does not change its sign for z close enough to z_0. More precisely, according to (2.10), $\zeta_k(z_0) = 1$, if $\Im \lambda_k(z) > 0$, and $\zeta_k(z_0) = -1$, if $\Im \lambda_k(z) < 0$ for z near z_0.

Thus we have proved the continuity of the finite sum

$$\sum_{k=1}^{N} [\pi\zeta_k(z) - \Im \lambda_k(z)]$$

in $z \in \bar{\mathcal{D}}$.

It remains to prove that the infinite tail of the sum (2.21) tends to zero as $\varepsilon \to 0$. Since $|\lambda_k| < \varepsilon \leqq 1/2$ for $k \geqq N+1$ then the relation (2.17) holds. Thus

$$\pi\zeta_k - \Im \lambda_k = \left[\arctan \frac{\Im \lambda_k}{1+\Re \lambda_k} - \frac{\Im \lambda_k}{1+\Re \lambda_k}\right] + \left[\frac{\Im \lambda_k}{1+\Re \lambda_k} - \Im \lambda_k\right].$$

Because of the inequality $|\arctan x - x| \leqq x^3/3$ the first summand here does not exceed

$$\frac{1}{3} \frac{|\Im \lambda_k|^3}{(1-|\lambda_k|)^3} \leqq \frac{1}{3(1-\varepsilon)^3} |\lambda_k|^3 \leqq \frac{8}{3} \varepsilon^2 |\lambda_k|.$$

The second summand is bounded by

$$\frac{|\Im \lambda_k| . |\Re \lambda_k|}{1-|\lambda_k|} \leqq \frac{1}{1-\varepsilon} |\lambda_k|^2 \leqq 2\varepsilon |\lambda_k|.$$

Hence

$$\left|\sum_{k=N+1}^{\infty} (\pi\zeta_k - \Im \lambda_k)\right| \leqq C\varepsilon \sum_{k=1}^{\infty} |\lambda_k|.$$

In view of (2.3) for $q=1$ the r. h. s. is bounded by

$$C\varepsilon \sup_{z \in \bar{\mathscr{D}},\, |z-z_0| \leq \varkappa} \| A(z) \|_1 .$$

This is just what we needed. ○

3. SPECTRAL SHIFT FUNCTION ESTIMATES

1° – Here we apply Lemmas 2.1, 2.2 to the study of the spectral shift function. Let H_0, H be a pair of selfadjoint operators in a Hilbert space $\mathfrak{H}$ and let $V=H-H_0 \in \mathfrak{S}_1$. We shall assume that V is represented in the form $V=G^* \Omega G$, where $G \in \mathfrak{S}_2$ is an operator acting from $\mathfrak{H}$ to a certain auxiliary Hilbert space $\mathfrak{N}$ and Ω is a bounded selfadjoint operator in $\mathfrak{N}$, $\|\Omega\|=1$. If $V \geqq 0$ (or $V \leqq 0$) we assume without loss of generality that $\Omega=I$ (or $\Omega=-I$). Denote

$$R_0(z)=(H_0-z)^{-1}, \qquad R(z)=(H-z)^{-1},$$
$$B(z)=GR_0(z)G^*,$$
$$\Gamma(z)=\Im B(z)=(2i)^{-1} G\{R_0(z)-R_0^*(z)\}G^*.$$

These are well known facts (*see*, for example, [4], [10]) that for almost all (a. a.) $\lambda \in \mathbb{R}$ there exists the limit

$$B(\lambda)=\lim_{\varepsilon \to +0} GR_0(\lambda+i\varepsilon)G^* \tag{3.1}$$

in the Hilbert-Schmidt norm and the operator $\Gamma(\lambda)=\Im B(\lambda)$ is trace class.

The spectral shift function (SSF) $\Theta(\lambda)$, $\lambda \in \mathbb{R}$, for the pair H_0, H is introduced via the perturbation determinant (*see* [1])

$$\left.\begin{aligned} D(z):=D(z;\, H_0,\, H)=\det(I+VR_0(z))=\det(I+B(z)\Omega),\\ \Im z \neq 0, \end{aligned}\right\} \tag{3.2}$$

where det(.) is defined by (2.9) [the second equality in (3.2) follows from (2.12)]. The function $D(z)$ is analytic in the upper (Π_+) and lower (Π_-) half-planes and $D(z)\neq 0$ as $z \in \Pi_\pm$ because -1 is not an eigenvalue of the operator $B(z)\Omega$, $\Im z \neq 0$. Since $\lim D(z)=1$, $|\Im z| \to \infty$, the condition $\log D(z) \to 0$, $|\Im z| \to \infty$ fixes the branches of the function $\log D(z)$ in Π_+ and Π_-. The next statement gives a precise definition of SSF.

PROPOSITION 3.1 ([1], [3]). – *If* $V \in \mathfrak{S}_1$ *then*

$$\log D(z)=\int_{-\infty}^{\infty} \Theta(\lambda)(\lambda-z)^{-1}\, d\lambda, \qquad \Im z \neq 0,$$

where

$$\Theta(\lambda)=\Theta(\lambda;\, H_0,\, H)=\pi^{-1} \lim_{\varepsilon \to +0} \arg D(\lambda+i\varepsilon;\, H_0,\, H). \tag{3.3}$$

The latter limit exists for a.a. $\lambda \in \mathbb{R}$. *Moreover,* $\Theta(\lambda)$ *satisfies the relation* (1.1) *with* $f(\lambda)=\lambda$ *and the estimate* (1.3). *The function* Θ *is monotonous with respect to the perturbation,* i.e. *if* $H_1 \leqq H_2$ *then* $\Theta(\lambda; H_0, H_1) \leqq \Theta(\lambda; H_0, H_2)$.

Proposition 3.1 gives an upper bound (1.3) for L^1-norm of SSF. Our aim is to get sharp bounds. To that end we make the following assumption. Let $\Lambda \subset \mathbb{R}$ be an open set.

ASSUMPTION 3.2. – The operator $B(z)$, $z \in \Pi_+$, belongs to the class S_p, $0<p<\infty$, and for a.a. $\lambda \in \Lambda$ the limit (3.1) exists in the class S_p.

Note that $B(z)\Omega$ has no real eigenvalues for $\Im z \neq 0$, so surely $B(z)\Omega \in \mathcal{M}$ and the function $\zeta(B(z)\Omega)$ is correctly defined. Now we can state the main result of this section.

THEOREM 3.3. – (i) *The function* $\zeta(B(\lambda+i\varepsilon)\Omega)$ *for a.a.* $\lambda \in \mathbb{R}$ *has a limit as* $\varepsilon \to +0$ *and*

$$\Theta(\lambda) = \lim_{\varepsilon \to +0} \zeta(B(\lambda+i\varepsilon)\Omega), \quad a.a.\,\lambda \in \mathbb{R}. \tag{3.4}$$

(ii) *Let Assumption* 3.2 *be fulfilled for some* $p>0$ *and* $\Lambda \subset \mathbb{R}$. *If* $p \geqq 1$ *then*

$$|\Theta(\lambda)| \leqq C_p \{ \| \Gamma(\lambda) \|_1 + [B(\lambda)]_p^p \}, \quad a, a.\,\lambda \in \Lambda. \tag{3.5}$$

If $p<1$ *then*

$$|\Theta(\lambda)| \leqq C_p [B(\lambda)]_p^p, \quad a.a.\,\lambda \in \Lambda. \tag{3.6}$$

In the particular case $S_1 = \mathfrak{S}_1$ the bound (3.5) yields

$$|\Theta(\lambda)| \leqq C \| B(\lambda) \|_1, \quad a.a.\ \lambda \in \Lambda. \tag{3.7}$$

Let us comment now on Assumption 3.2. As was already mentioned, Assumption 3.2 is fulfilled at least for the class $\mathfrak{S}_2$ when $\Lambda = \mathbb{R}$. Furthermore, as was proved by S. N. Naboko (*see* [11]), the operator $B(z)$ has for a.a. $\lambda \in \mathbb{R}$ non-tangent boundary values in the class $\boldsymbol{\Sigma}_1$ (but not in $\mathfrak{S}_1$!). He also showed in [11] that if $G^* G \in \mathfrak{S}_p$, $p<1$, then $B(\lambda) \in \mathfrak{S}_p$ a.e. Thus we may formulate the following particular case of Theorem 3.3.

COROLLARY 3.4. – *If* $V \in \mathfrak{S}_1$ *then for a.a.* $\lambda \in \mathbb{R}$

$$|\Theta(\lambda)| \leqq C_1 \{ \| \Gamma(\lambda) \|_1 + \langle B(\lambda) \rangle_1 \}. \tag{3.5'}$$

If, in addition, $G^* G \in \mathfrak{S}_p$, $p<1$, *then*

$$|\Theta(\lambda)| \leqq C_p \| B(\lambda) \|_p^p. \tag{3.6'}$$

We shall see in Section 4 that the class $\boldsymbol{\Sigma}_p$, $p<1$, is more natural than $\mathfrak{S}_p$ in the study of the Schrödinger operator. For this reason we would like to have the bound (3.6′) with the quasi-norm $\langle\ .\ \rangle_p$ instead of $\|\ .\ \|_p$. However under the condition $G^* G \in \boldsymbol{\Sigma}_p$, $p<1$, the results of [11] can guarantee only the existence of the limit (3.1) in any class $\mathfrak{S}_q$, $q>p$. As Naboko communicated to the author, the limit (3.1) presumably exists in the class $\boldsymbol{\Sigma}_p$, but a rigourous proof of this fact is still absent.

Proof of Theorem 3.3. – First we check that

$$\pi^{-1} \arg \mathrm{D}(z) = \zeta(\mathrm{B}(z)\Omega) = :\zeta(z), \qquad z \in \Pi_{\pm}. \tag{3.8}$$

Observe that by definition

$$\pi^{-1} \arg \mathrm{D}(z) - \zeta(z) = 2n_{\pm}, \qquad z \in \Pi_{\pm} \tag{3.9}$$

with an integer $n_{\pm} = n_{\pm}(z)$. The operator $\mathrm{VR}_0(z)$ has no real eigenvalues and is analytic in $z \in \Pi_{\pm}$. Therefore it satisfies the conditions of Lemma 2.3 for any open set in $\Pi_{\pm}$ separated from the real axes. So the function $\zeta(z)$ is continuous in $z \in \Pi_{\pm}$, and, consequently, the number $n_{\pm}$ in (3.9) does not depend on z. Since, in addition,

$$\|\mathrm{B}(z)\Omega\|_1 \leqq \|\mathrm{G}\|_2^2 |\Im z|^{-1},$$

we have $\zeta(z) \to 0$, $|\Im z| \to \infty$ by (2.13). Together with the condition $\arg \mathrm{D}(z) \to 0$, $|\Im z| \to \infty$, this gives $n_{\pm} = 0$, which ensures (3.8). Reference to (3.3) completes the proof of (3.4).

In view of monotonicity of $\Theta(\lambda; \mathrm{H}_0, \mathrm{H})$ with respect to V, it is sufficient to establish the bounds (3.5), (3.6) for a perturbation of a fixed sign. To be definite we suppose $\mathrm{V} \geqq 0$, so that $\Omega = \mathrm{I}$. Thus according to Lemma 2.1 for any $z \in \Pi_{\pm}$ we have

$$|\zeta(z)| \leqq \mathrm{C}(\|\Gamma(z)\|_1 + [\mathrm{B}(z))]_p^p)$$

if $p \geqq 1$ and

$$|\zeta(z)| \leqq \mathrm{C}[\mathrm{B}(z)]_p^p$$

if $p<1$. Now, setting $z = \lambda + i\varepsilon$, $\lambda \in \Lambda$, taking into account (3.8) and passing to the limit as $\varepsilon \to 0$, we get from here (3.5) or (3.6). ○

2° – Let us discuss now the relation of SSF with the discrete spectra of H_0 and H. First recall the well known identity for the discrete spectrum of an operator H bounded from below. Let $\mathrm{N}(\lambda; \mathrm{H})$ be the number of eigenvalues of H lying on the left from the point $\lambda < \inf \sigma_{\mathrm{ess}}(\mathrm{H})$. Then (*see* [3])

$$\Theta(\lambda; \mathrm{H}_0, \mathrm{H}) = \mathrm{N}(\lambda; \mathrm{H}_0) - \mathrm{N}(\lambda; \mathrm{H}), \quad \text{a. a. } \lambda < \inf \sigma_{\mathrm{ess}}(\mathrm{H}). \tag{3.10}$$

Now we assume that the spectrum of H_0 (semiboundedness of H_0 is not supposed) has a gap δ and that the perturbation V has a fixed sign. Then the eigenvalues of the operator $\mathrm{H}_g := \mathrm{H}_0 + g\mathrm{V}$ in the gap will be monotonous functions of the parameter $g>0$. Therefore for $\lambda \in \delta$ we can define the so-called counting function $\mathrm{M}(\lambda; \mathrm{H}_0, \mathrm{H})$, which is equal by definition to the number of eigenvalues of H_g crossing the point λ while g varies from 0 to 1. According to the Birman-Schwinger principle (*see*, for example, [12])

$$\left.\begin{array}{ll} M(\lambda;\ H_0,\ H)=n(B(\lambda)), & V\leqq 0; \\ M(\lambda;\ H_0,\ H)=n(-B(\lambda)), & V\geqq 0, \end{array}\right\} \quad (3.11)$$

where $n(A)$ is the number of eigenvalues of the compact selfadjoint operator A on the right from the point 1. We establish the following

THEOREM 3.5. — *If* $\lambda\in\delta$ *is not an eigenvalue of the operator* H *then*

$$\Theta(\lambda;\ H_0,\ H)=\pm M(\lambda;\ H_0,\ H), \quad (3.12)$$

for $\pm V\geqq 0$.

Note that if $\lambda<\inf\sigma_{ess}(H)$ and the perturbation V has a fixed sign then (3.12) gives (3.10).

Proof of Theorem 3.5. — To be definite we suppose that $V\geqq 0$, *i.e.* $\Omega=1$, and convince ourselves that $\Theta(\lambda)=n(-B(\lambda))$. It is clear that $B(z)$ is continuous in a neighbourhood of λ. Furthermore, (-1) is not an eigenvalue of $B(\lambda)$ (otherwise λ would have been an eigenvalue of H). Thus in view of Lemma 2.3 the function $\zeta(B(z))$ is continuous in z, $\Im z\geqq 0$, near λ. Since $B(\lambda)=(B(\lambda))^*$, by definition (2.10) the partial phases $\zeta_k(B(\lambda))$ equal zero if $\lambda_k(B(\lambda))>-1$. Now let $\lambda_k(B(\lambda))<-1$. Since $\Im B(z)>0$ for $z\in\Pi_+$ then

$$\lim_{\varepsilon\to+0}\zeta_k(B(\lambda+i\varepsilon))=1, \qquad \lambda_k(B(\lambda))<-1.$$

Taking into account the definition (2.11) and the equalities (3.4), (3.11) we arrive at (3.12). ○

3° — It is a well known fact that one can define SSF for two self-adjoint operators H_0, H, bounded from below, satisfying the property (*see* [1]-[4])

$$R^s(z)-R_0^s(z)\in\mathfrak{S}_1 \quad (3.13)$$

for some exponent $s>0$ and some number $z\in\mathbb{C}$ [then (3.13) is automatically satisfied for all z outside of the spectra of H_0 and H]. Choosing the positive number a big enough, define

$$\left.\begin{array}{c} \Theta(\lambda;H_0,H)=-\Theta((\lambda+a)^{-s};\ R_0^s(-a),\ R^s(-a)),\ \lambda>-a; \\ \Theta(\lambda;H_0,H)=0,\ \ \lambda\leqq -a. \end{array}\right\} \quad (3.14)$$

One can prove that the function (3.14) does not depend on the choice of the exponent s, for which (3.13) holds, and the number a. As in the case of trace class perturbations, $\Theta(\lambda;\ H_0,\ H)$ satisfies the trace formula (1.1) (for functions f decreasing sufficiently quickly at infinity) and the Birman-Krein formula (1.2). Assume that $v(a)=R^s(-a)-R_0^s(-a)$ belongs to S_p, $p<1$, or to $\mathfrak{S}_1$. For the sake of brevity we shall always write this condition as $v(a)\in S_p$, $p\leqq 1$, thinking that $S_1=\mathfrak{S}_1$. We are going to obtain bounds for $\Theta(\lambda;\ H_0,\ H)$ by making use of Theorem 3.3 for the unperturbed operator $(H_0+a)^{-s}$ and the perturbation $v(a)$. Now the role of $B(z)$ is played by the operator

$$B_a(z)=|v(a)|^{1/2}((H_0+a)^{-s}-(z+a)^{-s})^{-1}|v(a)|^{1/2}. \qquad (3.15)$$

Let $\zeta\in C^\infty(\mathbb{R})$ be a function such that $\zeta(t)=1$ for $t\leqq 3/2$ and $\zeta(t)=0$ for $t\geqq 2$. Suppose that the operator

$$D_a^0(\lambda,\varepsilon):=|v(a)|^{1/2}\zeta(H_0(a+|\lambda|)^{-1}) \times R_0(\lambda+i\varepsilon)\zeta(H_0(a+|\lambda|)^{-1})|v(a)|^{1/2} \qquad (3.16)$$

has a S_p-limit $D_a^0(\lambda)$ as $\varepsilon\to+0$ for a.a. $\lambda\in\Lambda$, where $\Lambda\subset\mathbb{R}$ is an open set. In other words this means that Assumption 3.2 is fulfilled for the operators H_0 and $G=|v(a)|^{1/2}\zeta(H_0(a+|\lambda|)^{-1})$. Now on the basis of Theorem 3.3 we shall prove

THEOREM 3.6. – *Let $v(a)\in S_p$, $p\leqq 1$, and the operator $D_a^0(\lambda,\varepsilon)$ satisfies Assumption 3.2 for the same p and some set Λ. Then for a.a. $\lambda\in\Lambda$ the bound holds*

$$|\Theta(\lambda;H_0,H)|\leqq C\left\{(a+|\lambda|)^{sp}\left[1+\left(\frac{a+|\lambda|}{a}\right)^{2(s-2)}\right]^p \times[v(a)]_p^p+(a+|\lambda|)^{p(s+1)}[D_a^0(\lambda)]_p^p\right\}, \qquad (3.17)$$

with a constant C independent of a, λ, V, H_0, H.

Proof. – We shall prove that for a.a. $\lambda\in\Lambda$ there exists the S_p-limit $B_a(\lambda)=\lim_{\varepsilon\to+0}B_a(\lambda+i\varepsilon)$ and $B_a(\lambda)$ obeys the bound (3.17). By (3.6) or (3.7) this will ensure (3.17) for $\Theta(\lambda;H_0,H)$. First note the identity

$$[(H_0+a)^{-s}-(z+a)^{-s}]^{-1}=-(z+a)^s-(z+a)^{2s}[(H_0+a)^s-(z+a)^s]^{-1},$$

so that

$$B_a(z)=-(z+a)^s|v(a)|-(z+a)^{2s}|v(a)|^{1/2}\{Y_a(\lambda,\varepsilon)\}|v(a)|^{1/2}, \qquad (3.18)$$

where

$$Y_a(z)=[(H_0+a)^s-(z+a)^s]^{-1}.$$

Let us represent $Y_a(z)$, $z=\lambda+i\varepsilon$, in the form $\sum_{k=0}^{2}Y_a^k(\lambda,\varepsilon)$, where

$$Y_a^0(\lambda,\varepsilon)=s^{-1}(z+a)^{-s+1}\zeta(H_0(a+|\lambda|)^{-1})(H_0-z)^{-1}\zeta(H_0(a+|\lambda|)^{-1}),$$

$$Y_a^1(\lambda,\varepsilon)=\zeta(H_0(a+|\lambda|)^{-1}) \times[Y_a(\lambda,\varepsilon)-s^{-1}(z+a)^{-s+1}(H_0-z)^{-1}]\zeta(H_0(a+|\lambda|)^{-1}),$$

$$Y_a^2(\lambda,\varepsilon)=[1-(\zeta(H_0(a+|\lambda|)^{-1}))^2]Y_a(\lambda,\varepsilon).$$

The operators $Y_a^k(\lambda,\varepsilon)$, $k=1, 2$, are continuous in $\varepsilon\in\mathbb{R}$ and

$$\|Y_a^1(\lambda,0)\|\leqq C(a+|\lambda|)^{s-2}a^{-2s+2},$$

$$\|Y_a^2(\lambda,0)\|\leqq C(a+|\lambda|)^{-s}.$$

Furthermore, by assumption the operator

$$(z+a)^{2s}|v(a)|^{1/2}Y_a^0(\lambda,\varepsilon)|v(a)|^{1/2}=s^{-1}(z+a)^{s+1}D_a^0(\lambda,\varepsilon)$$

has a S_p-limit as $\varepsilon\to+0$. Therefore in view of (3.18), this is true for $B_a(\lambda+i\varepsilon)$ as well and by (2.4), (2.6) $B_a(\lambda)$ obeys the estimate

$$\begin{aligned}[B_a(\lambda)]_p^p &\leq C[v(a)]_p^p(a+|\lambda|)^{sp}\\ &\quad\times\{1+(a+|\lambda|)^s\|Y_a^1(\lambda,0)\|+(a+|\lambda|)^s\|Y_a^2(\lambda,0)\|\}^p\\ &\quad+C(a+|\lambda|)^{p(s+1)}[D_a^0(\lambda)]_p^p\\ &\leq C(a+|\lambda|)^{sp}\left[1+\left(\frac{a+|\lambda|}{a}\right)^{2(s-2)}\right]^p[v(a)]_p^p\\ &\quad+C(a+|\lambda|)^{p(s+1)}[D_a^0(\lambda)]_p^p.\end{aligned}$$

The last inequality coincides with (3.17). ○

Remark 3.7. — Note that we have proved by passing that under the condition $v(a)\in S_p$ the operator

$$B_a(\lambda+i\varepsilon)+s^{-1}(\lambda+i\varepsilon+a)^{s+1}D_a^0(\lambda,\varepsilon)$$

is jointly S_p-continuous in $\lambda\in\mathbb{R}$, $\varepsilon\in\mathbb{R}$.

4. THE SCATTERING PHASE FOR THE SCHRÖDINGER OPERATOR. RESULTS AND DISCUSSION

1° — Let $H_g=H_0+gV$ be the Schrödinger operator in $L^2(\mathbb{R}^d)$, $d\geq 2$, where $H_0=-\Delta$ and V is a bounded real-valued function satisfying the condition

$$|V(x)|\leq C(1+|x|)^{-\alpha},\qquad \alpha>d, \tag{4.1}$$

and $g\geq 0$ is the coupling constant. The absolutely continuous spectrum for H_g coincides with $[0,\infty)$ and has no imbedded eigenvalues. Denote $R_g(z)=(H_g-z)^{-1}$, $\Im z\neq 0$. Let $X_\beta^{(\rho)}$ be a multiplication by the function $(1+\rho|x|)^{-\beta}$, $\beta\in\mathbb{R}$, $X_\beta:=X_\beta^{(1)}$. For all $\lambda>0$, $\beta>1/2$ the operator $X_\beta R_g(\lambda+i\varepsilon)X_\beta$ has a norm limit as $\varepsilon\to+0$, which is continuous in $\lambda>0$.

By Ξ^d we denote the dual space, the operator $\Phi: L^2(\mathbb{R}^d)\to L^2(\Xi^d)$ stands for the Fourier transform:

$$\hat{u}(\xi)=(\Phi u)(\xi)=(2\pi)^{-(d/2)}\int e^{-i\xi x}u(x)\,dx,\qquad \xi\in\Xi^d.$$

Set

$$v_g(a)=[R_g(-a)]^s-[R_0(-a)]^s,$$
$$s>0,\qquad a>g\max_x[-V(x)].$$

It is a well known fact (*see*, for example, [13]) that for s large enough the operator $v_g(a)$ is trace class. Thus we may define for the pair H_0, H_g the SSF $\Theta(\lambda, g)=\Theta(\lambda; H_0, H_g)$, not depending on the particular choice of a and s. Throughout the rest of the paper the number s is assumed to be fixed and the dependence on it is omitted from notations. Furthermore, as a rule we assume that

$$a \geqq a_1 = a_1(g) := 2g \max_x [-V(x)].$$

Under this condition we always have

$$\|(H_g+a)^{-1}(H_0+a)\| \leqq 2, \qquad a \geqq a_1. \tag{4.2}$$

For $V \geqq 0$ (*i.e.* for $a_1=0$) this follows from the obvious inequality $H_g+a \geqq H_0+a$. If $a_1 \neq 0$ then (4.2) is a consequence of the identity $(H_g+a)^{-1}(H_0+a)=I-g(H_g+a)^{-1}V$.

We study the function $\Theta(\lambda, g)$ for $\lambda>0$. Note first its continuity in λ.

LEMMA 4.1. – *The function $\Theta(\lambda, g)$ is continuous in $\lambda>0$.*

The proof of this lemma will follow automatically from the proof of the estimates for $\Theta(\lambda, g)$ formulated below. The constants in Theorems 4.2, 4.4 do not depend on g, λ, but may depend on V.

THEOREM 4.2. – *Let* V *satisfy* (4.1). *Then for all* $\lambda \geqq c$, $g>0$

$$|\Theta(\lambda, g)| \leqq C[g^{(d/2)} + g\lambda^{(d/2)-1}(|\log\lambda|+1)]. \tag{4.3}$$

If, in addition, $V \geqq 0$ *then*

$$|\Theta(\lambda, g)| \leqq C g \lambda^{(d/2)-1}(|\log\lambda|+1). \tag{4.4}$$

Note that for $g \leqq C\lambda$ Theorem 4.2 gives the bound

$$|\Theta(\lambda, g)| \leqq C g \lambda^{(d/2)-1}(|\log\lambda|+1), \qquad g \leqq C\lambda, \tag{4.5}$$

independently of the sign of V. However in the case $g \geqq c\lambda$ the bound (4.4) for $V \geqq 0$ is, obviously, better than (4.3).

Theorem 4.2 states a universal result, which does not depend on the rate of fall-off of V. Now we shall make an additional assumption which will allow us to distinguish the functions V with different exponents α in (4.1). It is stated in terms of the "sandwiched" free resolvent. Let $\zeta \in C^\infty(\mathbb{R})$ be the function which enters the definition (3.16) of the operator $D_a^0(\lambda, \varepsilon)$. Denote

$$\left.\begin{aligned} T_\beta^{(\rho)}(\lambda, \varepsilon) &= X_\beta^{(\rho)}\zeta(H_0\lambda^{-1})R_0(\lambda+i\varepsilon)\zeta(H_0\lambda^{-1})X_\beta^{(\rho)}, \\ T_\beta(\lambda, \varepsilon) &:= T_\beta^{(1)}(\lambda, \varepsilon). \end{aligned}\right\} \tag{4.6}$$

By $T_\beta^{(\rho)}(\lambda)$, $\beta>1/2$, we denote the norm limit of $T_\beta^{(\rho)}(\lambda, \varepsilon)$ as $\varepsilon \to +0$. As we shall see later (*see* Remark 5.5) the operator $X_\beta\zeta(H_0\lambda^{-1})$ for $\beta>0$ belongs to $\Sigma_{d\beta^{-1}}$, so that $T_\beta(\lambda, \varepsilon) \in \Sigma_p$, $p=d(2\beta)^{-1}$ by (2.2). The following assumption requires that $T_\beta(\lambda)$ should belong to the same class.

ASSUMPTION 4.3. — For a fixed $\lambda>0$ and $\beta>d/2$ the operator $T_\beta(\lambda, \varepsilon)$ has a limit $T_\beta(\lambda)$ in the class Σ_p, $p=d(2\beta)^{-1}$.

Note that the existence of the limit $T_\beta(\lambda)$ at some fixed point $\lambda>0$ provides that for all $\lambda>0$. Indeed, let U_r, $r>0$, be the unitary dilation operator: $(U_r u)(x)=r^{d/2} u(rx)$. Then, obviously,

$$U_r \zeta(H_0) U_r^* = \zeta(H_0 r^{-2}),$$
$$U_r R_0(z) U_r^* = r^2 R_0(zr^2), \qquad U_r X_\beta^{(\rho)} U_r^* = X_\beta^{(\rho r)}.$$

Thus

$$U_r T_\beta^{(\rho)}(\lambda, \varepsilon) U_r^* = r^2 T_\beta^{(r\rho)}(\lambda r^2, \varepsilon r^2). \tag{4.7}$$

Since $T_\beta^{(r)}(\lambda, \varepsilon)=\{X_\beta^{(r)} X_{-\beta}\} T_\beta(\lambda, \varepsilon)\{X_{-\beta} X_\beta^{(r)}\}$, the relation (4.7) with $\rho=1$, $r=\lambda^{-1/2}$ yields

$$[T_\beta(\lambda)]_p \leqq C\lambda^{\beta-1}[T_\beta(1)]_p, \qquad \lambda \geqq c, \tag{4.8}$$

for any $p>0$.

One should mention that Assumption 4.3 is equivalent to Assumption 3.2 for the operator $B(z)=G(H_0-z)^{-1}G^*$ where $H_0=-\Delta$, $G=X_\beta \zeta(H_0\lambda^{-1})$, $\beta>d/2$, on the interval $\Lambda=(0, \infty)$ in the class Σ_p, $p=d(2\beta)^{-1}$.

THEOREM 4.4. — *Let* V *satisfy* (4.1) *and Assumption* 4.3 *be fulfilled for* $\beta=\alpha/2$.
Then for $\lambda \geqq c$

$$|\Theta(\lambda, g)| \leqq C[(g\lambda^{-1})^{d/\alpha}\lambda^{d/2}+g^{d/2}]. \tag{4.9}$$

If, in addition, $V \geqq 0$, *then*

$$|\Theta(\lambda, g)| \leqq C(g\lambda^{-1})^{d/\alpha}\lambda^{d/2}. \tag{4.10}$$

Note that for $g \leqq C\lambda$ the estimates (4.9) and (4.10) ensure that

$$|\Theta(\lambda, g)| \leqq C(g\lambda^{-1})^{d/\alpha}\lambda^{d/2}$$

for arbitrary V. Since $d\alpha^{-1}<1$ this bound is worse than (4.5) (up to $\log \lambda$). Comparing Theorems 4.4 and 4.2, in the case $g \geqq c\lambda$, we see that the estimate (4.9) is essentially the same as (4.3). On the contrary, (4.10) prescribes more moderate growth of $\Theta(\lambda, g)$ than (4.4). At the same time, the r.h.s. of (4.10) transforms into that of (4.4) in the limit $\alpha \to d$.

2° — It is natural to compare the results obtained with the quasi-classical formula for the scattering phase:

$$\Theta(\lambda, g) \sim -\frac{\omega_d}{(2\pi)^d}\int \{(\lambda-gV)_+^{d/2}-\lambda^{d/2}\}\,dx. \tag{4.11}$$

(here ω_d is the volume of the unit ball in $\mathbb{R}^d$) not caring of the precise conditions of its validity. On a heuristic level this formula can be derived on the basis of analogy between SSF and the distribution function of discrete spectrum. We mention that in the rigorous sense the asymptotics (4.11) was proven in [14]-[17] for $\lambda \to \infty$, $g=\text{const.}$ (high energy asymtotics) and in [18] for $g=c\lambda \to \infty$ (quasi-classical limit).

Note first that for $g \leqq C\lambda$ it follows from (4.11) that

$$|\Theta(\lambda, g)| \leqq C g \lambda^{(d/2)-1},$$

which is in agreement with (4.5) (up to log λ). In the case $g \geqq c\lambda$ (4.11) predicts different behaviour of $\Theta(\lambda, g)$ for non-negative potentials and potentials with no restriction on the sign. Let $g\lambda^{-1} \to \infty$. Suppose that $V \leqq 0$ and V has a compact support. Then it follows from (4.11) that

$$\Theta(\lambda, g) \sim -\frac{\omega_d}{(2\pi)^d} g^{d/2} \int |V(x)|^{d/2} dx. \tag{4.12}$$

Clearly, up to log λ, the bound (4.3) [or (4.9)] is precise if compared with (4.12). Now let $V \geqq 0$. In this case $\Theta(\lambda, g)$ depends explicitly on the behaviour of V at infinity. Suppose for simplicity that

$$V(x) = \begin{cases} |x|^{-\alpha}, & |x| \geqq 1; \\ 1, & |x| < 1. \end{cases}$$

Hence

$$\begin{aligned}\Theta(\lambda, g) &\sim \frac{\omega_d}{(2\pi)^d} \int_{|\psi|=1} d\psi \int \{(\lambda - g|x|^{-\alpha})_+^{d/2} - \lambda^{d/2}\} |x|^{d-1} dx \\ &= \left\{-\frac{\omega_d}{(2\pi)^d} \int_{|\psi|=1} d\psi \int_0^\infty [(1-t^{-\alpha})_+^{d/2} - 1] t^{d-1} dt\right\} (g\lambda^{-1})^{d/\alpha} \lambda^{d/2}. \end{aligned} \tag{4.13}$$

Obviously, (4.10) and (4.13) fully agree with each other.

5. SINGULAR VALUES ESTIMATES FOR AUXILIARY INTEGRAL OPERATORS

1° – In this section we prepare various estimates to be used in the proof of Theorems 4.2, 4.4. First we list a number of results on the s-values estimates for integral operators, which we shall rely upon.

PROPOSITION 5.1 [13]. – *Let*

$$T = f\,\Phi^*\,h,$$

(i) *If* $f \in L^p(\mathbb{R}^d)$ *and* $h \in L^p(\Xi^d)$, $p>2$, *then* $T \in \mathfrak{S}_p$ *and*

$$\|T\|_p \leqq C_p \|f\|_{L^p} \|h\|_{L^p}. \tag{5.1}$$

(ii) *If* $f \in L^p_w(\mathbb{R}^d)$ *and* $h \in L^p(\Xi^d)$, $p>2$, *then* $T \in \Sigma_p$ *and*

$$\langle T \rangle_p \leqq C_p \|f\|_{L^p_w} \|h\|_{L^p}. \tag{5.2}$$

The constant C_p *depends only on d and p.*

From (i) with $p=2$ we obtain immediately the following simple test for nuclearity.

PROPOSITION 5.2. – *Let*

$$T=f_1\Phi^* h\Phi f_2, \tag{5.3}$$

where $f_i\in L^2(\mathbb{R}^d)$, $i=1, 2$, *and* $h\in L^1(\Xi^d)$. *Then* $T\in\mathfrak{S}_1$ *and*

$$\|T\|_1\leqq\|f_1\|_{L^2}\|f_2\|_{L^2}\|h\|_{L^1}.$$

The part (ii) of Proposition 5.1 is nothing but the famous Cwickel's estimate (*see* [19]). It allows us to study the operators whose s-values decay slower than $n^{-1/2}$. To treat the operators with better decay properties we shall use the following result from [8], which provides the s-values estimates in dependence of the smoothness of the kernel. Below $H_m(.)$ denotes the Sobolev space.

PROPOSITION 5.3. – *Let* $Q\subset\mathbb{R}^d$, $d\geqq 1$, *be a unit cube and* $T:L^2(Q)\to L^2(\mathbb{R}^d)$ *be the integral operator with the kernel*

$$t(x, y)\,b(y),$$

where $t(x, .)\in H_m(Q)$, $2m>d$ *for a.a.* $x\in\mathbb{R}^d$. *Then*

$$s_n(T)\leqq C\,n^{-(1/2)-(m/d)}\|b\|_{L^2}\left\{\int dx\,\|t(x, .)\|^2_{H_m(Q)}\right\}^{1/2},$$

if the r.h.s. is finite. The constant C *depends only on d, m.*

2° – Now we apply Propositions 5.1-5.3 to the operator

$$K_{\beta, l, t}=K_{\beta, l, t}(a)=X_\beta\Phi^* Y^{(a)}_{l, t},\qquad \beta>0,\quad l>0,\quad t>0,\quad a>0, \tag{5.4}$$

where $Y^{(a)}_{l,t}$ is the multiplication in $L^2(\Xi^d)$ by the function $Y^{(a)}_{l,t}(\xi)=(\xi^2+a)^{-l}(\xi^2+1)^{-t}$.

THEOREM 5.4. – *If* $l+t>\beta/2$ *then* $K_{\beta, l, t}$ *belongs to the class* Σ_p, $p=d/\beta$. *If, in addition,* $t<\beta/2$ *then*

$$\langle K_{\beta, l,t}(a)\rangle_p\leqq C\,a^{-l-t+(\beta/2)} \tag{5.5}$$

and for any $q\in(p, d(2t)^{-1})$,

$$\|K_{\beta, l, t}(a)\|_q\leqq C\,a^{-l-t+(d/2q)}. \tag{5.6}$$

Proof. – We shall treat separately three cases: $\beta<d/2$, $\beta>d/2$, $\beta=d/2$.

(i) Let $\beta<d/2$. Since $X_\beta\in L^p_w(\mathbb{R}^d)$ with $p=d/\beta>2$ and $X_\beta\in L^q(\mathbb{R}^d)$, $q>p$; $Y^{(a)}_{l,t}\in L^q(\Xi^d)$, $q\geqq p$, we may apply Proposition 5.1. Direct calculation showhs that

$$\|Y^{(a)}_{l, t}\|_{L^q}\leqq C\,a^{-l-t+(d/2q)},\qquad p\leqq q<\frac{d}{2t}. \tag{5.7}$$

Substituting (5.7) with $q=p$ into (5.2) and with $q>p$ into (5.1), we get (5.5) or (5.6) respectively.

(ii) Let $\beta>d/2$, so that $p=d/\beta<2$. We may assume $q<2$ as well, since the bounds for $q\geqq 2$ can be obtained as in (i) by Proposition 5.1. Let us divide the space Ξ^d into a lattice of disjoint unit cubes Q_k, $k=1, 2, .,$ so that $\bigcup_k Q_k=\Xi^d$, and consider separately the operators

$$K^{(k)}=K^{(k)}_{\beta, l, t}:=K_{\beta, l, t}\chi_k,$$

χ_k being a characteristic function of Q_k. Further, we split every $K^{(k)}$ in its turn in the sum

$$K^{(k)}=Z_{R, 1}+Z_{R, 2},$$

where

$$Z_{R, 1}=\eta_R K^{(k)}, \qquad Z_{R, 2}=(1-\eta_R)K^{(k)}, \qquad Z_{R, 2}=(1-\eta_R)K^{(k)},$$

η_R being a characteristic function of the ball $\{x\in\mathbb{R}^d \| x|\leqq R\}$. According to Proposition 5.3, for any $m>d/2$ we have

$$s_n(Z_{R, 1})\leqq C_m n^{-(1/2)-(m/d)}\left\{\int_{|x|\leqq R} dx(1+|x|)^{-2\beta} \times\left[\sum_{|j|\leqq m}\int_{Q_k} d\xi\,|D^j_\xi e^{i\xi x}|^2\right]\right\}^{1/2} J^{(k)}_{l, t}(a), \quad (5.8)$$

where we denoted

$$J^{(k)}_{l, t}(a)=\left[\int_{Q_k} d\xi\,(a+\xi^2)^{-2l}(1+\xi^2)^{-2t}\right]^{1/2}.$$

The integrand in (5.8) clearly does not exceed

$$C(1+|x|)^{-2\beta}(|x|^{2m}+C).$$

Thus, choosing $m>\beta-d/2$, we get from (5.8)

$$s_n(Z_{R, 1})\leqq C n^{-(1/2)-(m/d)} R^{-\beta+m+(d/2)} J^{(k)}_{l, t}(a) \quad (5.9)$$

with a constant C depending on m only.

For the operator $Z_{R, 2}$ it suffices to estimate its Hilbert-Schmidt norm:

$$\|Z_{R, 2}\|_2\leqq CR^{-\beta+(d/2)} J^{(k)}_{l, t}(a),$$

so that

$$s_n(Z_{R, 2})\leqq C n^{-(1/2)} R^{-\beta+(d/2)} J^{(k)}_{l, t}(a). \quad (5.10)$$

By (2.1), combining (5.9) and (5.10), we arrive at the bound

$$s_{2n}(K^{(k)})\leqq s_{2n-1}(K^{(k)})\leqq CR^{-\beta}(n^{-(1/2)-(m/d)} R^{(d/2)+m}+n^{-(1/2)} R^{d/2}) J^{(k)}_{l, t}(a).$$

Setting $R=n^{1/d}$, we obtain from here

$$s_n\,(K^{(k)})\leqq C n^{-(\beta/d)} J^{(k)}_{l, t}(a), \quad (5.11)$$

which is equivalent to

$$\langle K^{(k)} \rangle_p \leqq C J_{l,t}^{(k)}(a), \qquad p=\frac{d}{\beta}. \tag{5.12}$$

It follows also from (5.11) that

$$\| K^{(k)} \|_q \leqq C J_{l,t}^{(k)}(a), \tag{5.13}$$

for all $q>p$. Since $K^{(k)}(K^{(j)})^*=0$, $j\neq k$, and $p<2$, $q<2$, we are able to apply Proposition 2.1 to the operator $K_{\beta,l,t}=\sum K^{(k)}$. It ensures us that

$$\langle K_{\beta,l,t} \rangle_p \leqq C [\sum_k [J_{l,t}^{(k)}(a)]^p]^{1/p}, \tag{5.14}$$

$$\| K_{\beta,l,t} \|_q \leqq C_q [\sum_k [J_{l,t}^{(k)}(a)]^q]^{1/q}, \qquad p<q<2. \tag{5.15}$$

The r. h. s. of (5.14) and (5.15) does not exceed the integral

$$C\left[\int d\xi\,(a+\xi^2)^{-lq}(1+\xi^2)^{-tq}\right]^{1/q},$$

where $q=p$ for (5.14) and $p<q<2$ for (5.15). Since $2(t+l)q\geqq 2(t+l)d\beta^{-1}>d$, this integral is finite. If, in addition, $q<\min(d(2t)^{-1}, 2)$, it satisfies the bound (5.7), which yields (5.5) and (5.6). Note in the parentheses that the condition $p<2$ was used only once, when passing from (5.12), (5.13) to (5.14), (5.15).

(iii) Let $\beta=d/2$. Since $[A]_q^2=[A^*A]_{q/2}^2$ for any $A\in S_q$, $q>0$ and

$$K_{\beta,l,t}^* K_{\beta,l,t}=K_{\beta_1,l,t}^* K_{\beta_2,l,t}, \qquad \beta_1+\beta_2=2\beta, \quad \beta_1<\frac{d}{2}<\beta_2,$$

it remains to use the bounds (5.5), (5.6) for $\beta=\beta_1$ and $\beta=\beta_2$. ○

Remark 5.5. – If follows directly from Theorem 5.4 that $X_\beta\zeta(H_0\lambda^{-1})\in\Sigma_p$, $p=d/\beta$, for any function $\zeta\in C_0^\infty(\mathbb{R})$. Consequently, $T_\beta(\lambda, \varepsilon)\in\Sigma_{p/2}$.

3° Let us consider the operator

$$\left.\begin{aligned} M_\beta(g, a)=(H_g+a)X_\beta(H_g+a)^{-1}X_{-\beta}, \\ a>g\max_x[-V(x)]. \end{aligned}\right\} \tag{5.16}$$

LEMMA 5.6. – *Let* $\beta\geqq 0$. *If* $a\geqq\max\{a_1(g), a_2\}$ *with some constant* $a_2=a_2(\beta)>0$, *then* $\| M_\beta(g, a)\|\leqq 2$.

Proof. – The statement of the Lemma will follow immediately if we prove the bound

$$\left.\begin{aligned} \|(H_g+a)u\|\leqq 2\|X_\beta(H_g+a)X_{-\beta}u\|, \\ u\in C_0^\infty(\mathbb{R}^d). \end{aligned}\right\} \tag{5.17}$$

Direct calculation shows that

$$(H_g+a)u=X_\beta(H_g+a)X_{-\beta}u-\{X_\beta(-\Delta X_{-\beta})u-2X_\beta\nabla X_{-\beta}\nabla u\}. \quad (5.18)$$

Let us look at the summand in the curly brackets. For any $\varepsilon>0$ we have

$$\|X_\beta(-\Delta X_{-\beta})u-2X_\beta\nabla X_{-\beta}\nabla u\|^2$$
$$\leqq C_\beta(\|u\|^2+\|\nabla u\|^2)=C_\beta\int(1+\xi^2)|\hat{u}(\xi)|^2 d\xi$$
$$\leqq\varepsilon\int(\xi^4+C_{\varepsilon,\beta})|\hat{u}(\xi)|^2 d\xi\leqq\varepsilon\|(H_0+C_{\varepsilon,\beta}^{1/2})u\|^2, \quad (5.19)$$

where $C_{\varepsilon,\beta}=C_\beta\varepsilon^{-1}[C_\beta(4\varepsilon)^{-1}+1]$. Thus, choosing $a_2=a_2(\varepsilon,\beta)=C_{\varepsilon,\beta}^{1/2}$, we see that the r. h. s. of (5.19) is bounded by

$$\varepsilon\|(H_0+a)u\|^2, \qquad a\geqq a_2.$$

Taking into account (4.2) we get

$$\|X_\beta(-\Delta X_{-\beta})u-2X_\beta\nabla X_{-\beta}\nabla u\|^2\leqq 4\varepsilon\|(H_g+a)u\|^2,$$
$$a\geqq\max\{a_1(g), a_2\}.$$

Now it follows from (5.18) that

$$(1-8\varepsilon)\|(H_g+a)u\|^2\leqq 2\|X_\beta(H_g+a)X_{-\beta}u\|^2.$$

For $\varepsilon=1/16$ this yields (5.17). ○

Relying on Theorem 5.4 for $t=0$ and Lemma 5.6 we are able to prove

THEOREM 5.7. – *Let* $0<\beta<2l$. *Then for all* $g\geqq 0$ *and* $a\geqq a_0(\beta,g):=\max\{a_1(g), a_3\}$ *with some constant* $a_3=a_3(\beta)>0$, *the operator* $X_\beta(H_g+a)^{-l}$ *belongs to* Σ_p, $p=d/\beta$, *and*

$$\langle X_\beta(H_g+a)^{-l}\rangle_p\leqq C a^{-l+(\beta/2)}. \quad (5.20)$$

Moreover, for any $q>p$

$$\|X_\beta(H_g+a)^{-l}\|_q\leqq C a^{-l+(d/2q)}. \quad (5.21)$$

The constant C *does not depend on* a, g.

Proof. – Since $X_\beta(H_0+a)^{-l}=K_{\beta,l,0}\Phi$ [*see* (5.4)], the bounds (5.20) and (5.21) for $g=0$ follow immediately from Theorem 5.4. By (4.2) this implies, in particular, that Theorem 5.7 is valid for $l=1$. The desired result for $l\geqq 2$ we shall obtain by induction. Namely, assuming the bounds (5.20), (5.21) for some l_0, β_0, $2l_0>\beta_0$, we shall prove them for $l=l_0+1$, $\beta=\beta_0+\beta_1$, $\beta_1<2$. To that end we present $X_\beta(H_g+a)^{-l}$ in the form

$$[X_{\beta_1}(H_g+a)^{-1}]M_{\beta_0}(g,a)[X_{\beta_0}(H_g+a)^{-l_0}]. \quad (5.22)$$

According to Lemma 5.6, $\|M_{\beta_0}(g,a)\|\leqq 2$. By assumption the first and the third factors in (5.22) belong to Σ_{p_1} and Σ_{p_0} with $p_1=d/\beta_1$ and $p_0=d/\beta_0$ respectively. Using the bound (5.20) for them and taking into account the inequality (2.4), we get

$$\langle X_\beta (H_g+a)^{-l}\rangle_p \leqq C \langle X_{\beta_1}(H_g+a)^{-1}\rangle_{p_1} \langle X_{\beta_0}(H_g+a)^{-l_0}\rangle_{p_0} \leqq C a^{-l+(\beta/2)}.$$

Thus the bound (5.20) is proved for $\beta=\beta_0+\beta_1$, $l=l_0+1$, and, consequently, for all l, β, $\beta<2l$. The estimate (5.21) is justified analogously. ○

6. LIMITING ABSORPTION PRINCIPLE IN THE TRACE CLASS

1° – As was mentioned in Section 1, the principal role in the proof of Theorem 4.2 is played by the properties of the operator $T_\beta^{(\rho)}(\lambda, \varepsilon)$, defined by (4.6). Namely, we prove for it a kind of a limiting absorption principle in the class $\mathfrak{S}_1$.

THEOREM 6.1. – *If* $\beta>d/2$ *then for all* $\lambda>0$, $\rho>0$ *the operator* $T_\beta^{(\rho)}(\lambda, \varepsilon)$ *is* $\mathfrak{S}_1$*-continuous in* (λ, ε) *as* $\lambda>0$, $\varepsilon\geqq 0$ *and*

$$\| T_\beta^{(\rho)}(\lambda)\|_1 \leqq C \lambda^{(d/2)-1} \rho^{-d} \log(\lambda\rho^{-2}+2) \tag{6.1}$$

with a constant C *independent of* λ, ρ.

Proof. – Note that by (4.7) it suffices to check the statement of Theorem for $\lambda=1$. Then the estimate (6.1) takes the form

$$\| T_\beta^{(\rho)}(1)\|_1 \leqq C \rho^{-d} \log(\rho^{-2}+2), \qquad \rho>0. \tag{6.2}$$

Let $K_\varepsilon^{(\rho)}$ be a multiplication by the function $K_\varepsilon^{(\rho)}(x)=X_\beta^{(\rho)}(x)\varphi(\varepsilon|x|)$, where $\varphi\in C^\infty(\mathbb{R}_+)$, $\varphi(t)=1$ for $t\leqq 1$ and $\varphi(t)=0$ for $t\geqq 2$. Denote

$$G_\varepsilon=(H_0-1-i\varepsilon)^{-1}, \qquad F_\varepsilon^{(\rho)}=K_\varepsilon^{(\rho)}\zeta(H_0)G_\varepsilon\zeta(H_0)K_\varepsilon^{(\rho)}.$$

Sometimes we denote by G_ε a multiplication in $L^2(\Xi^d)$ by the function $(\xi^2-1-i\varepsilon)^{-1}$. Note the following obvious estimates:

$$\int |K_\varepsilon^{(\rho)}(x)|^2\,dx \leqq C\rho^{-d}, \qquad \varepsilon\geqq 0, \tag{6.3}$$

$$\int |K_\varepsilon^{(\rho)}(x)-K_0^{(\rho)}(x)|^2\,dx \leqq C\rho^{-2\beta}\varepsilon^{2\beta-d}, \qquad \varepsilon\geqq 0,$$

$$\int [\zeta(\xi^2)]^2\,|G_\varepsilon(\xi)|\,d\xi \leqq C(|\log\varepsilon|+1), \qquad \varepsilon>0, \tag{6.4}$$

$$\int \left|\frac{dK_\varepsilon^{(\rho)}(x)}{d\varepsilon}\right|^2 dx \leqq C\rho^{-2\beta}\varepsilon^{2(\beta-(d/2))-2}. \tag{6.5}$$

Without loss of generality we may assume that $2\beta<d+2$. Under this condition

$$\int |K_\varepsilon^{(\rho)}(x)|^2\,|x|^2\,dx \leqq C\rho^{-2\beta}\varepsilon^{2(\beta-(d/2))-2}. \tag{6.6}$$

To verify the nuclearity we use Proposition 5.2. For example, the operator $F_\varepsilon^{(\rho)}$ has the form (5.3) with $f_1 = f_2 = K_\varepsilon^{(\rho)}$ and $h = G_\varepsilon$. Thus

$$\| F_\varepsilon^{(\rho)} \|_1 \leqq C \rho^{-d} (|\log \varepsilon| + 1), \qquad \varepsilon > 0. \tag{6.7}$$

Similarly,

$$\| F_\varepsilon^{(\rho)} - T_\beta^{(\rho)}(1, \varepsilon) \|_1 \leqq C (\rho^{-(d/2)-\beta} \varepsilon^{\beta-(d/2)} + \rho^{-2\beta} \varepsilon^{2\beta-d}) (|\log \varepsilon| + 1).$$

Since the r. h. s. of this inequality tends to zero as $\varepsilon \to 0$, it ensures us that it suffices to check (6.2) for $F_\varepsilon^{(\rho)}$. To that end we use a technique similar to that of Mourre [5]. For brevity from now on we omit ρ from the notation of $F_\varepsilon^{(\rho)}$. We claim that

$$\left\| \frac{dF_\varepsilon}{d\varepsilon} \right\|_1 \leqq C \{ \rho^{-(d/2)-\beta} \varepsilon^{(\beta-(d/2)-1} + \rho^{-d} \} (|\log \varepsilon| + 1), \qquad \varepsilon > 0. \tag{6.8}$$

By Proposition 5.2 and (6.3)-(6.5) the first two summands in the r. h. s. of the identity

$$\frac{dF_\varepsilon}{d\varepsilon} = \frac{dK_\varepsilon}{d\varepsilon} \zeta G_\varepsilon \zeta K_\varepsilon + K_\varepsilon \zeta G_\varepsilon \zeta \frac{dK_\varepsilon}{d\varepsilon} + K_\varepsilon \zeta \frac{dG_\varepsilon}{d\varepsilon} \zeta K_\varepsilon$$

satisfy the estimate (6.8). Let us look at the third term:

$$D_\varepsilon = K_\varepsilon \zeta \frac{dG_\varepsilon}{d\varepsilon} \zeta K_\varepsilon.$$

If we applied to D_ε Proposition 5.2 directly, the L^1-norm of $dG_\varepsilon/d\varepsilon$ would give us the factor ε^{-1}. It does not suit us. Instead of this we represent D_ε in a more convenient form. Set $A = (2i)^{-1}(x\nabla + \nabla x)$. Using the well known relation $i[H_0, A] = 2H_0$, we find that

$$\begin{aligned} i(1+i\varepsilon) \frac{dG_\varepsilon}{d\varepsilon} &= (H_0 - 1 - i\varepsilon)^{-1}(-1 - i\varepsilon) \\ &= (H_0 - 1 - i\varepsilon)^{-1} - (H_0 - 1 - i\varepsilon)^{-1} H_0 (H_0 - 1 - i\varepsilon)^{-1} \\ &= (H_0 - 1 - i\varepsilon)^{-1} - \frac{i}{2} (H_0 - 1 - i\varepsilon)^{-1} [H_0 - 1 - i\varepsilon, A] (H_0 - 1 - i\varepsilon)^{-1}. \end{aligned}$$

Hence

$$i(1+i\varepsilon) D_\varepsilon = D_\varepsilon^{(1)} + D_\varepsilon^{(2)} + D_\varepsilon^{(3)},$$

where

$$\begin{aligned} D_\varepsilon^{(1)} &= F_\varepsilon = K_\varepsilon \zeta G_\varepsilon \zeta K_\varepsilon, \\ D_\varepsilon^{(2)} &= -\frac{i}{2} K_\varepsilon \zeta A G_\varepsilon \zeta K_\varepsilon, \\ D_\varepsilon^{(3)} &= \frac{i}{2} K_\varepsilon \zeta G_\varepsilon A \zeta K_\varepsilon. \end{aligned}$$

We have already the bound (6.7) for F_ε. Let us look, for example, at $D_\varepsilon^{(3)}$. Since $A = i^{-1}(\nabla x - d/2)$ then

$$D_\varepsilon^{(3)} = \frac{1}{2} K_\varepsilon \zeta G_\varepsilon \nabla x \zeta K_\varepsilon - \frac{d}{4} D_\varepsilon^{(1)}. \tag{6.9}$$

To treat the first summand we "transfer" the multiplication by x through ζ:

$$\frac{\partial}{\partial x_j} x_j \zeta = -\Phi^* \xi_j \frac{\partial}{\partial \xi_j} \zeta(\xi^2)\Phi$$
$$= -\Phi^* \xi_j \zeta(\xi^2) \frac{\partial}{\partial \xi_j} \Phi - 2\Phi^* \xi_j^2 \zeta'(\xi^2)\Phi = \tilde{\zeta}_j(H_0) x_j + \tilde{\tilde{\zeta}}_j(H_0)$$

with suitable functions $\tilde{\zeta}_j$, $\tilde{\tilde{\zeta}}_j \in C_0^\infty(\mathbb{R})$. Thus the firt summand in the r. h. s. of (6.9) equals

$$\frac{1}{2} K_\varepsilon \zeta G_\varepsilon \tilde{\zeta}_j x_j K_\varepsilon + \frac{1}{2} K_\varepsilon \zeta G_\varepsilon \tilde{\tilde{\zeta}}_j K_\varepsilon.$$

The second operator here can be estimated as $D_\varepsilon^{(1)}$ and therefore satisfies (6.7). By virtue of Proposition 5.2 and estimates (6.3), (6.4), (6.6) the trace norm of the first operator does not exceed

$$C\rho^{-\beta-(d/2)}(|\log \varepsilon| + 1)\varepsilon^{(\beta-(d/2))-1}.$$

Putting together the inequalities for $D_\varepsilon^{(1)}, D_\varepsilon^{(2)}$ and $D_\varepsilon^{(3)}$, we obtain that D_ε, and consequently, $dF_\varepsilon/d\varepsilon$ satisfy (6.8).

Since $\beta > d/2$, the r. h. s. of (6.8) is integrable near $\varepsilon = 0$. Thus the operator F_ε is $\mathfrak{S}_1$- continuous in $\varepsilon \geqq 0$. Moreover, in view of (6.7)

$$\|F_\varepsilon\|_1 \leqq \int_\varepsilon^\mu \left\| \frac{dF_\varepsilon}{d\varepsilon} \right\|_1 d\varepsilon + \|F_\mu\|_1$$
$$\leqq C\rho^{-\beta-(d/2)}\mu^{\beta-(d/2)}(|\log \mu| + 1) + C\rho^{-d}(|\log \mu| + 1)$$

for $0 < \varepsilon \leqq \mu$. This leads to (6.2) if we set $\mu = \min(\rho, 1)$.

To prove the joint $\mathfrak{S}_1$-continuity of $T_\beta^{(\rho)}(\lambda, \varepsilon)$ in λ, ε, it suffices to establich the $\mathfrak{S}_1$-continuity in $\lambda > 0$ uniform in $\varepsilon \geqq 0$. According to (4.7) it is equivalent to the $\mathfrak{S}_1$-continuity of the operator

$$U_r^* T_\beta^{(r)}(1, \varepsilon r^2) U_r = U_r^* [X_\beta^{(r)} X_{-\beta}] T_\beta(1, \varepsilon r^2) [X_{-\beta} X_\beta^{(r)}] U_r$$

with respect to $r > 0$ uniform in ε. Since this is true for the operator $T_\beta(1, \varepsilon r^2)$, the desired result follows from the norm-continuity of $X_\beta^{(r)} X_{-\beta}$ and strong continuity of U_r. The proof is complete. ○

2° Now we prove an auxilliary result for an operator similar to $T_\beta(\lambda, \varepsilon)$. Namely, set for $a > 0$

$$\left.\begin{aligned} F_{\beta,a}(\lambda, \varepsilon) &= X_\beta \zeta(H_0(a+\lambda)^{-1}) R_0(\lambda + i\varepsilon) \zeta(H_0(a+\lambda)^{-1}) X_\beta, \\ F_{\beta,a}(\lambda) &= \lim_{\varepsilon \to +0} F_{\beta,a}(\lambda, \varepsilon). \end{aligned}\right\} \tag{6.10}$$

The latter limit is taken in the class $\mathfrak{S}_1$. The following statement is a consequence of Theorems 6.1 and 5.4.

THEOREM 6.2. – *If* $\beta > d/2$ *then for all* $\lambda > 0$ *the bound*

$$\| F_{\beta, a}(\lambda) \|_1 \leqq C \left[\frac{\lambda+1}{\lambda} (a+\lambda)^{(d/2)-1} + \lambda^{(d/2)-1} \log(\lambda+2) \right] \quad (6.11)$$

holds. If Assumption 4.3 *is fulfilled with* $p = d(2\beta)^{-1}$ *then for all* $\lambda > c$ *the bound*

$$\langle F_{\beta, a}(\lambda) \rangle_p \leqq C (a+\lambda)^{\beta-1} \quad (6.12)$$

holds. The constant C *does not depend on* λ, a.

Proof. – Clearly, $F_{\beta, a}(\lambda) = T_\beta(\lambda) + Z_{\beta, a}(\lambda)$, where

$$Z_{\beta, a}(\lambda) = X_\beta \{ [\zeta^2 (H_0 (a+\lambda)^{-1}) - \zeta^2 (H_0 \lambda^{-1})] R_0(\lambda) \} X_\beta. \quad (6.13)$$

The operator $T_\beta(\lambda)$ obeys the estimate (6.11) by Theorem 6.1. Under Assumption 4.3 $T_\beta(\lambda)$ satisfies (6.12) in view of (4.8). Thus it remains to treat $Z_{\beta, a}(\lambda)$.

After the Fourier transform the term in the curly brackets in (6.13) acts as a multiplication by the function

$$[\zeta^2 (\xi^2 (a+\lambda)^{-1}) - \zeta^2 (\xi^2 \lambda^{-1})] (\xi^2 - \lambda)^{-1}.$$

This function does not exceed

$$C \frac{\lambda+1}{\lambda} (\xi^2+1)^{-1} \zeta^2 (\xi^2 (a+\lambda)^{-1}).$$

Further, since $\zeta(\xi^2 (a+\lambda)^{-1}) \leqq C (a+\lambda)^l (\xi^2 + a + \lambda)^{-l}$ for arbitrary l, the s-values of $Z_{\beta, a}$ are estimated by those of the operator

$$(a+\lambda)^{2l} \frac{\lambda+1}{\lambda} K_{\beta, l, 1/2}(a) K^*_{\beta, l, 1/2}(a),$$

where $K_{\beta, l, t}(a)$ is defined in (5.4). For $l > \beta/2 - 1/2$ the operator $K_{\beta, l, 1/2}(a)$ satisfies the conditions of Theorem 5.4. In particular, the requirement $t = 1/2 < \beta/2$ is fulfilled. Thus we obtain from (5.5) and (5.6) that

$$\| K_{\beta, l, 1/2}(a) \|_2^2 \leqq C (a+\lambda)^{-2l + (d/2) - 1}$$
$$\langle K_{\beta, l, 1/2}(a) \rangle_{2p}^2 \leqq C (a+\lambda)^{-2l + \beta - 1}.$$

By (2.4) this immediately yields

$$\| Z_{\beta, a}(\lambda) \|_1 \leqq C \frac{\lambda+1}{\lambda} (a+\lambda)^{(d/2)-1}, \quad (6.14)$$

$$\langle Z_{\beta, a}(\lambda) \rangle_p \leqq C \frac{\lambda+1}{\lambda} (a+\lambda)^{\beta-1}. \quad (6.15)$$

Combining (6.14) or (6.15) with the inequalities (6.1) or (4.8) for $T_\beta(\lambda)$, we arrive at (6.11) or (6.12). ○

7. PROOF OF THEOREMS 4.2, 4.4

1° – The proof of Theorems 4.2, 4.4 reduces to the application of abstract Theorem 3.6. To that end we shall obtain appropriate estimates for (quasi-)norms of the operators $v_g(a)$ and (3.16) in the classes $\mathfrak{S}_1$ or Σ_p, $p=d\alpha^{-1}$, $\alpha>d$. In what follows we always assume that $a\geqq a_0(\alpha/2, g)$, where a_0 is the constant introduced in Theorem 5.7, and the number s is integer. First we establish some properties of the perturbation $v_g(a)$.

LEMMA 7.1. – *If* $s>\alpha/2-1$ *then* $v_g(a)\in\Sigma_p$ *and*

$$\langle v_g(a)\rangle_p\leqq C g a^{-1-s+(\alpha/2)}, \tag{7.1}$$

$$\|v_g(a)\|_1\leqq C g a^{-1-s+(d/2)}, \tag{7.2}$$

$$\|X_{-(\alpha/2)}|v_g(a)|^{1/2}\|+\||v_g(a)|^{1/2}X_{-(\alpha/2)}\|\leqq C g^{1/2} a^{-((1+s)/2)}. \tag{7.3}$$

with a constant C *not depending on a, g.*

Proof. – Direct calculation shows that

$$v_g(a)=-g\sum_{k=1}^{s}(H_g+a)^{-k}V(H_0+a)^{-s+k-1}=-g\sum_{k=1}^{s}S_k. \tag{7.4}$$

Since $s>\alpha/2-1$ for a fixed k one can find such $\beta_1>0$, $\beta_2>0$ that $\beta_1+\beta_2=\alpha$ and $2(s-k+1)>\beta_2$, $2k>\beta_1$. Thus the k-th term of the sum (7.4) can be represented as

$$S_k=\{(H_g+a)^{-k}X_{\beta_1}\}\{X_{-\beta_1}VX_{-\beta_2}\}\{X_{\beta_2}(H_0+a)^{-s+k-1}\}.$$

Obviously, the factor in the middle is bounded. Using the bound (5.20) for the first and the third factors and taking into account (2.4), we get

$$\langle S_k\rangle_p\leqq C\langle(H_g+a)^{-k}X_{\beta_1}\rangle_{d/\beta_1}\langle X_{\beta_2}(H_0+a)^{-s+k-1}\rangle_{d/\beta_2}$$
$$\leqq C a^{-k+(\beta_1/2)}a^{-s+k-1+(\beta_2/2)}=C a^{-s-1+(\alpha/2)}.$$

Applying the p-triangle inequality (2.6) to the sum (7.4), we arrive at (7.1). The estimate (7.2) follows analogously from (5.21).

To check (7.3) we establish the bound

$$\|X_{-(\alpha/2)}|v_g(a)|X_{-(\alpha/2)}\|\leqq C g a^{-s-1}. \tag{7.5}$$

In view of monotonicity of $v_g(a)$ with respect to V, it is sufficient to prove (7.5) separately for $V=V_+$ and $V=-V_-$. Under any of these conditions the perturbation $v_g(a)$ has a fixed sign, so (7.5) reduces to

$$\|X_{-(\alpha/2)}v_g(a)X_{-(\alpha/2)}\|\leqq C g a^{-s-1}. \tag{7.5'}$$

The operator $X_{-(\alpha/2)} S_k X_{-(\alpha/2)}$ can be rewritten in the form $\{X_{-(\alpha/2)} (H_g + a)^{-1} X_{\alpha/2}\}^k \{X_{-(\alpha/2)} V X_{-(\alpha/2)}\} \{X_{\alpha/2} (H_0 + a)^{-1} X_{-(\alpha/2)}\}^{s-k+1}$. Note that

$$X_{\alpha/2} (H_g + a)^{-1} X_{-(\alpha/2)} = (H_g + a)^{-1} M_{\alpha/2} (g, a),$$

where $M_{\alpha/2}(g, a)$ is defined in (5.16). Thus by Lemma 5.6

$$\| X_{-(\alpha/2)} (H_g + a)^{-1} X_{\alpha/2} \| \leqq 2 \| (H_g + a)^{-1} \| \leqq C a^{-1}.$$

This gives immediately the estimate $\| X_{-(\alpha/2)} S_k X_{-(\alpha/2)} \| \leqq C a^{-s-1}$, which leads to (7.5′) and consequently to (7.3). ○

Let us proceed to the study of the operator (3.16) with $v(a) = v_g(a)$:

$$D^0_{g,a}(\lambda, \varepsilon) = | v_g(a) |^{1/2} \zeta (H_0 (a + |\lambda|)^{-1}) R_0 (\lambda + i\varepsilon) \times \zeta (H_0 (a + |\lambda|)^{-1}) | v_g(a) |^{1/2}.$$

LEMMA 7.2. — *The operator* $D^0_{g,a}(\lambda, \varepsilon)$ *is jointly* $\mathfrak{S}_1$*-continuous in* $\lambda > 0$, $\varepsilon \geqq 0$ *and*

$$\| D^0_{g,a}(\lambda) \|_1 \leqq C g a^{-1-s} \left[\frac{\lambda + 1}{\lambda} (a + \lambda)^{(d/2)-1} + \lambda^{(d/2)-1} \log (\lambda + 2) \right]. \quad (7.6)$$

If in addition, Assumption 4.3 *is fulfilled for* $\beta = \alpha/2$ *then* $D^0_{g,a}(\lambda, \varepsilon)$ *for all* $\lambda > 0$ *has a* Σ_p*-limit as* $\varepsilon \to +0$ *and for* $\lambda \geqq c$

$$\langle D^0_{g,a}(\lambda) \rangle_p \leqq C g a^{-1-s} (a + \lambda)^{(\alpha/2)-1}. \quad (7.7)$$

The constant C *does not depend on* g, a, λ.

Proof. — Let $F_{\beta,a}$ be the operator defined by (6.10). Then obviously

$$D^0_{g,a}(\lambda, \varepsilon) = \{ | v_g(a) |^{1/2} X_{-(\alpha/2)} \} F_{\alpha/2,a}(\lambda, \varepsilon) \{ X_{-(\alpha/2)} | v_g(a) |^{1/2} \},$$

By lemma 7.1 the operators in curly brackers are bounded by $g^{1/2} a^{-(1+s)/2}$. It remains to apply Theorem 6.2. ○

COROLLARY 7.3. — *Lemma* 4.1 *follows directly from Lemma* 7.2. *Indeed, according to Lemma* 7.2 *and Remark* 3.7 *the operator-valued function* $B_{g,a}(\lambda, \varepsilon)$ *defined by* (3.15) *with* $v(a) = v_g(a)$, *is jointly* $\mathfrak{S}_1$*-continuous in* $\lambda > 0$, $\varepsilon \geqq 0$. *Furthermore,* (-1) *is not an eigenvalue of* $B_{g,a}(\lambda) \Omega_g$, $\Omega_g = v_g(a) | v_g(a) |^{-1}$ *for* $\lambda > 0$ (*otherwise the operator* $H_g = H_0 + gV$ *would have a positive eigenvalue*). *The desired continuity of* $\Theta(\lambda)$ *follows from Lemma* 2.3 *and equality* (3.4).

2° — To complete the proof of Theorems 4.2, 4.4 we choose the constant a in a convenient way. First we assume that $c \leqq \lambda \leqq a$. Then (7.6), (7.7) reduce to

$$\| D^0_{g,a}(\lambda) \|_1 \leqq C g a^{-1-s} [a^{(d/2)-1} + \lambda^{(d/2)-1} \log (\lambda + 2)], \quad (7.6')$$

$$\langle D^0_{g,a}(\lambda) \rangle_p \leqq C g a^{-2-s+(\alpha/2)}. \quad (7.7')$$

Therefore the inequality (3.17) for the class $\mathfrak{S}_1$ and (7.2), (7.6′) give the bound

$$|\Theta(\lambda, g)| \leqq C g (a^{(d/2)-1} + \lambda^{(d/2)-1} \log(\lambda+2)). \qquad (7.8)$$

Similarly, (3.17) for Σ_p, $p = d\alpha^{-1}$, and (7.1), (7.7′) lead to the estimate

$$|\Theta(\lambda, g)| \leqq C g^{d/\alpha} (a^{\alpha/2-1})^{d/\alpha} \leqq C \left(\frac{g}{a}\right)^{d/\alpha} a^{d/2}. \qquad (7.9)$$

Now set

$$a = \lambda + a_1(g) + a_3(\alpha/2) = \lambda + 2g \max_x [-V(x)] + a_3$$

(*see* Theorem 5.7 for definition of a_3). Then according to (7.8)

$$|\Theta(\lambda, g)| \leqq C (g^{d/2} + g \lambda^{(d/2)-1} \log(\lambda+2)),$$

which coincides with (4.3). If $V \geqq 0$ then $a = \lambda + a_3$ and consequently

$$|\Theta(\lambda, g)| \leqq C g \lambda^{(d/2)-1} \log(\lambda+2),$$

which proves (4.4). Similarly, from (7.9) we get (4.9) and (4.10). This completes the proof of Theorems 4.2, 4.4.

ACKNOWLEDGEMENTS

This paper was written during my stay at the University Paris-Nord. It is a pleasure for me to thank Professor A. Martinez and the staff of Département de Mathématiques et Informatique for their hospitality. I am indebted to C.I.E.S. for financial support due to which my stay in France became possible.

I thank the referee for his remarks.

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(*Manuscript received May* 13, 1991;
revised version received July 27, 1991.)