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Potential wells in high dimensions I

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ABSTRACT. — Motivated by a problem with a large number of interacting particles with a strong exterior potential, we consider the bottom of the spectrum of the semiclassical Schrödinger operator $-h^2\Delta+V$ in high dimension N. We assume that V satisfies certain conditions uniformly with respect to N and in particular that V has non degenerate local minima. Assuming that $N=\mathcal{O}h^{-N_0})$ for some fixed N_0, we are able to describe a low part of the spectrum. For instance, in the case of one potential well, we get a complete asymptotic expansion in powers of h valid uniformly with respect to N of the lowest eigenvalue and we show that this eigenvalue is simple and separated from the rest of the spectrum by a distance $\geqq h/\text{Const}$.

Key words : WKB, potential wells, high dimension.

RÉSUMÉ. — Motivés par un problème avec un grand nombre de particules qui interagissent mutuellement, avec un potentiel extérieur fort, nous considérons le bas du spectre de l'opérateur de Schrödinger semi-classique $-h^2\Delta+V$ en grande dimension N. On suppose que V vérifie certaines hypothèses uniformément en N, et en particulier que V possède des minimums locaux non dégénérés. Supposant que $N=\mathcal{O}(h^{-N_0})$ pour un N_0 fixé, nous pouvons décrire une partie basse du spectre. Ainsi par exemple dans le cas d'un puits de potentiel, nous obtenons un développement asymptotique complet de la première valeur propre en puissances de h, valable uniformément par rapport à N, et nous montrons que cette valeur propre est simple et séparée du reste du spectre par une distance $\geqq h/\text{Const}$.

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0. INTRODUCTION

The starting point for this work was the thesis of F. Daumer [D] extending earlier results of Albanese ([A1], [A2]), which treated Hartree equations for systems of interacting particles moving in a background potential with potential wells. Naively, we wanted to study the full many body Schrödinger equation and to see how far one can get by WKB-methods and associated *a priori* estimates in the study of the bottom of the spectrum. It turned out that for the semiclassical Schrödinger equation with h denoting Planck's "constant" it is possible under suitable assumptions to make asymptotic expansions of the lowest eigenvalue and of the corresponding eigenfunction when $h \to 0$ uniformly with respect to the number of particles, N, as long as $N = \mathcal{O}(h^{-N_0})$ for some fixed N_0. This type of restriction [which could possibly be replaced by $N = \mathcal{O}(e^{1/Ch})$] is encountered at many places in the argument but we do not know exactly how essential it is for the final result. We hope that the methods developed in the present paper will prove useful in the study of certain problems in solid state physics, statistical mechanics and perhaps quantum field theory (*cf.* [K] and [HeT]), however the motivation for the present paper is a simple model problem, which can be viewed as the semiclassical version for the full Schrödinger operator of the problem considered in [D]:

Let v be a real valued analytic and 2π-periodic function on the real line, $\mathbb{R}$. Assume that $v \geqq 0$ with equality precisely on $2\pi\mathbb{Z}$ and assume further that $v''(0) > 0$. Let B be the number of potential wells, so that the underlying space will be $\mathbb{R}/2\pi B\mathbb{Z}$. On this space we consider N particles which interact by means of a positive potential $w(x_j - x_k)$. More precisely, we are interested in the bottom of the spectrum of the operator

$$\left.\begin{aligned} & -h^2\Delta + V \text{ on } L^2((\mathbb{R}/2\pi B\mathbb{Z})^N), \\ & \text{with} \\ & V = V_0 + W, \qquad V_0(x) = \sum_1^N v(x_j), \qquad W(x) = \sum\sum_{j<k} w(x_j - x_k). \end{aligned}\right\} \tag{0.1}$$

Assume, in order to fix the ideas, that:

$$w(x_1) = \delta \left| B e^{ix_1/B} - B \right|^{-\gamma}, \tag{0.2}$$

that δ, $\gamma > 0$ and that $N \leqq B$. The idea is then that since we have a repulsive interaction, the bottom of the spectrum should be asymptotically determined by the study of the operator near (in some suitable sense) the points $2\pi\alpha = (2\pi\alpha_1, \ldots, 2\pi\alpha_N)$ with $\alpha_j \in \mathbb{Z}/B\mathbb{Z}$, and with $\alpha_j \neq \alpha_k$ when $j \neq k$.

Assume now that $\gamma > 0$ is fixed and that $\delta > 0$ is sufficiently small. Then with α as above, we have

$$|\nabla W(x)|_\infty = \mathcal{O}(\delta), \tag{0.3}$$

for $x \in \mathbb{C}^N$ with $|x - 2\pi\alpha|_\infty \leq C$, provided that $C < \pi$. Here we have written $|z|_\infty = \max |z_j|$ for $z \in \mathbb{C}^N$. *Further, the estimate on* ∇W *is uniform with respect to* N. The object of this paper is to carry out the analysis of the bottom of the spectrum of the operator (0.1) in a box of the form $|x - 2\pi\alpha|_\infty < C$ with suitable boundary conditions, uniformly with respect to N, as long as

$$N = \mathcal{O}(h^{-N_0}), \tag{0.4}$$

for some fixed N_0. We will also give a result involving several potential wells for more general potentials of the form $V_0 + W$, but this result is still somewhat preliminary in that we make the asumption (0.3) globally, which is not realistic for the particular problem explained above, and also in that we do not attempt to analyse the tunnel effect. We intend to discuss these further questions as well as related ones in some future paper (s).

The plan of the present paper is the following: In section 1 we prove some simple (possibly known) estimates for derivatives of holomorphic functions in a polydisc in $\mathbb{C}^N$. These estimates give the key to all the subsequent constructions, and are likely to have other applications.

The rest of the paper is devoted to a general class of potentials of the form $V = V_0 + W$ with W satisfying (0.4).

In section 2, we start the WKB-construction, by solving the eiconal equation $(\varphi_0')^2 = V(x) - V_{min}$ in a complex ball with respect to the l^∞-norm which is centered around a local minimum of V. Here we let V_{min} denote the value of V at the local minimum point [which in the case of the particular problem above will be within a l^∞-distance $\mathcal{O}(\delta)$ from a point $2\pi\alpha$]. The approach is based on the point of view of hyperbolic dynamical systems as in [MS], [HS3].

In section 3, we construct the asymptotic candidates for the first eigenvalue and the corresponding eigenfuction. The eigenfunction is sought of the form $e^{-\varphi(x;h)/h}$ which may be slightly unusual, in the sense that we let the amplitude be one and we let instead the phase be h-dependent: $\varphi(x; h) \sim \varphi_0(x) + \varphi_1(x) h + \ldots$ The crucial problem here is to get a control which is uniform in N over all the terms φ_j, and this is the point where the estimates of section 1 are important. Possibly it is of crucial importance here that we are only looking for the lowest eigenvalue. Indeed an earlier attempt to construct eigenfunctions of the form $a(x; h) e^{-\varphi_0(x)/h}$ did not succeed, maybe because of the fact that this ansatz does not forbid the eigenfunction to have zeros. The asymptotic eigenvalue is of the form $V_{min} + h E(h)$ with $E(h) \sim E_0 + E_1 h + \ldots$ and with $E_j = \mathcal{O}_j(N)$. Without any assumption on the size of N, this asymptotic sum is welldefined only up to $\mathcal{O}(N h^k)$ for every fixed $k \in \mathbb{N}$, and if we want to have $h E(h)$ well defined up to any power of h *uniformly with respect to* N, we are led to introduce the condition (0.4). (Notice that this assumption appears naturally already in the case when $W = 0$, if we do not wish to assume the exact knowledge of the eigenvalue of each one-dimensional Schrödinger operator which enters in the N-dimensional one.)

In section 4, we consider a selfadjoint operator P which equals (0.1) in an l^∞-ball and we show that $\operatorname{dist}(h\,E(h), \sigma(P)) = \mathcal{O}(N\,h^k)$ for every $k \in \mathbb{N}$, where $\sigma(P)$ denotes the spectrum of P.

In section 5 we develop some local *a priori* estimates by making a change of variables which somehow reduces us to the harmonic oscillator. Since we are in high dimensions, the Jacobian plays an important role, and as a matter of fact, the construction of the change of variables requires all the machinery of section 2 and 3. In particular, the simple estimates of section 1 are again needed.

In section 6, the estimates are globalized and we determine asymptotically the low part of the spectrum also for problems with several potential wells. In the case of one well, we get under the assumption (0.4), that the infimum of the spectruçm is given by a simple eigenvalue equal to $V_{\min} + h\,E + \mathcal{O}(h^\infty)$ (uniformly with respect to N) and that this eigenvalue is separated from the remainder of the spectrum by a gap of the order of magnitude h.

The analyticity assumptions on v and w could probably be replaced by suitable estimates on the derivatives of all orders, however this would give rise to long and tedious estimates, so we have preferred (at least to start with) the analytic version. Preliminary results indicate that it is also possible to study the tunnel effect and we hope to treat that question in a future paper. The results of section 1,3 seem to indicate the existence of a general theory of pseudo- and Fourier integral operators in high dimensions, which might be sufficiently interesting to explore further.

We would like to thank B Helffer for stimulating conversations concerning the possible applicability of the results and techniques of this paper to problems of statistical mechanics.

1. ESTIMATES FOR HOLOMORPHIC FUNCTIONS IN MANY VARIABLES

In this section we shall establish the following result:

PROPOSITION 1.1. — *Let* $0 < r_1 < r_0$. *Then there is a constant* $C_0 > 0$ *independent of* N *such that for every holomorphic function on* $B(l^\infty, \mathbb{C}^N; 0, r_0) = \{ x \in \mathbb{C}^N;\ |x|_\infty < r_0 \}$ *satisfying* $|\nabla u(x)|_\infty \leqq 1$, *for* $|x|_\infty < r_0$, *we have for*

$|x|_\infty < r_1$:

$$\left.\begin{array}{l} |\langle \nabla^2 u(x),\ t_1 \otimes t_2 \rangle| \leqq C_0 |t_1|_{p_1} |t_2|_{p_2} \\ \textit{when} \\ t_j \in \mathbb{C}^N, \quad 1 = p_1^{-1} + p_2^{-1}, \quad 1 \leqq p_j \leqq \infty, \end{array}\right\} \tag{1.1}$$

$$\left.\begin{array}{l} |\langle \nabla^3 u(x),\ t_1 \otimes t_2 \otimes t_3 \rangle| \leqq C_0 |t_1|_{p_1} |t_2|_{p_2} |t_3|_{p_3}, \\ \textit{when} \\ t_j \in \mathbb{C}^N, \quad 1 = p_1^{-1} + p_2^{-1} + p_3^{-1}, \quad 1 \leqq p_j \leqq \infty, \end{array}\right\} \tag{1.2}$$

$$|\nabla (\nabla u . \nabla u)|_\infty \leqq C_0 \tag{1.3}$$

$$|\nabla (\Delta u)|_\infty \leqq C_0. \tag{1.4}$$

Here $|\,.\,|_p$ *denotes the* l^p *norm on* $\mathbb{C}^N$.

Proof. – We shall first establish (1.1)-(1.3). Our assumption implies that $|\langle \nabla u(x),\ t_2 \rangle| \leqq |t_2|_1$ for $x \in B(l^\infty,\ \mathbb{C}^N;\ 0,\ r_0) = B(0,\ r_0)$. Let $x \in B(0,\ r_1)$, $t_1 \in l^\infty$ with $|t_1|_\infty \leqq r_0 - r_1$ and consider the holomorphic function of one variable: $D(0,\ 1) \ni z \mapsto \langle \nabla u(x + z t_1),\ t_2 \rangle$ [where in general we denote by $D(z_0,\ r)$ the open disc in $\mathbb{C}$ of center z_0 and radius r], which is of absolute value $\leqq |t_2|_1$ at each point. Applying the Cauchy inequality, we get $|(\partial_z)_{z=0} (< \nabla u(x + z t_1),\ t_2 \rangle)| \leqq |t_2|_1$, or in other words:

$$|\langle \nabla^2 u(x),\ t_1 \otimes t_2 \rangle| \leqq |t_2|_1.$$

For general t_1 we then get

$$|\langle \nabla^2 u(x),\ t_1 \otimes t_2 \rangle| \leqq (r_0 - r_1)^{-1} |t_1|_\infty |t_2|_1,$$

since $\nabla^2 u$ is symmetric, we also have

$$|\langle \nabla^2 u(x),\ t_1 \otimes t_2 \rangle| \leqq (r_0 - r_1)^{-1} |t_1|_1 |t_2|_\infty$$

and by complex interpolation we then get (1.1) with $C_0 = (r_0 - r_1)^{-1}$.

The proof of (1.2) is now a repetition of the same argument: we choose $\rho \in]r_1,\ r_0[$ and start with the fact that

$$|\langle \nabla^2 (x),\ t_2 \otimes t_3 \rangle| \leqq (r_0 - r_1)^{-1} |t_2|_{p_2} |t_3|_{p_3}$$

if

$$p_2^{-1} + p_3^{-1} = 1.$$

Then considering $D(0,\ 1) \ni \mapsto \langle \nabla^2 u(x + z t_1),\ t_2 \otimes t_3 \rangle$ for $|t_1|_\infty \leqq (\rho - r_1)$, $|x|_\infty < r_1$ and applying the Cauchy inequality, we get (1.2) in the special case when $p_1 = \infty$ and with $C_0 = (r_0 - \rho)^{-1} (\rho - r_1)^{-1}$. By the symmetry of $\nabla^3 u$, we then also have the special cases when $p_2 = \infty$ and when $p_3 = \infty$.

The general case follows by complex interpolation (without any increase in the constant).

(1.3) is obtained by a simple computation: $\nabla (\nabla u . \nabla u) = 2 (\nabla^2 u)(\nabla u)$ (viewing $\nabla^2 u$ as a matrix and noticing that (1.1) says that this matrix is of norm $\leqq C_0$ as an operator: $l^p \to l^p$ for every $p \in [1,\ \infty]$) and using (1.1) for $x \in B(0, r_1)$, we get

$$|\nabla(\nabla u . \nabla u)| \leqq 2 \|\nabla^2 u\|_{\mathscr{L}(l^\infty, l^\infty)} |\nabla u|_\infty \leqq 2 C_0.$$

Since we could take $C_0 = (r_0 - r_1)^{-1}$ in (1.1), we can take $C_0 = 2(r_0 - r_1)^{-1}$ in (1.3).

It remains to prove (1.4). In order to estimate $\nabla \Delta u = \Delta \nabla u$ in l^∞, we shall consider $\Delta \langle \nabla u, t \rangle$ for $t \in l^1$, and write it as $\operatorname{tr}(\nabla^2 \langle \nabla u, t \rangle)$. Since $|\langle \nabla u, t \rangle| \leqq |t|_1$ for $|x|_\infty < r_0$, we can use the Cauchy inequality to obtain

$$|\langle \nabla^2 (\langle \nabla u, t \rangle), r \otimes s \rangle| \leqq 4 |t|_1 (r_0 - r_1)^{-2} |r|_\infty |s|_\infty. \tag{1.5}$$

In other words, if $A = \nabla^2 (\langle \nabla u, t \rangle)$ [at some point $x \in B(0, r_1)$], then

$$\|A\|_{\mathscr{L}(l^\infty, l^1)} \leqq 4 (r_0 - r_1)^{-2} |t|_1. \tag{1.6}$$

LEMMA 1.2. — *If* A *is a complex* N × N *matrix, then*

$$|\operatorname{tr}(A)| \leqq \|A\|_{\mathscr{L}(l^\infty, l^1)}. \tag{1.7}$$

Proof of the lemma. — Put $t_j = \omega^j$, where ω is a N:th root of unity. Then $|t|_\infty = 1$ so

$$|\langle A t, \bar{t} \rangle| \leqq \|A\|_{\mathscr{L}(l^\infty, l^1)}. \tag{1.8}$$

On the other hand,

$$\langle A t, \bar{t} \rangle = \sum_0^{N-1} \sum_0^{N-1} a_{j,k} \omega^{j-k} = \sum_0^{N-1} b_\nu \omega^{-\nu}, \tag{1.9}$$

with

$$b_\nu = \sum_{j-k \equiv -\nu \bmod (N)} a_{j,k}. \tag{1.10}$$

Notice that $b_0 = \operatorname{tr}(A)$. Choosing $\omega = \omega_0^k$, where ω_0 is a primitive root of unity, we get from (1.8)–(1.10):

$$\left| \sum_{\nu=0}^{N-1} b_\nu \omega_0^{-k\nu} \right| \leqq \|A\|_{\mathscr{L}(l^\infty, l^1)}, \qquad k = 0, 1, \ldots, N-1, \tag{1.11}$$

so

$$\left| \sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} b_\nu \omega_0^{-k\nu} \right| \leqq N \|A\|_{\mathscr{L}(l^\infty, l^1)}. \tag{1.12}$$

Here we notice that the double sum in (1.11) is equal to Nb(0), since $\sum_{k=0}^{N-1} \omega_0^{-k\nu}$ is equal to zero when $1 \leqq \nu \leqq N-1$ and is equal to N when $\nu = 0$.

Hence (1.10) reduces to $|b_0| \leqq \|A\|_{\mathscr{L}}(l^\infty, l^1)$ which is precisely (1.6). □

End of the proof of the proposition. — Combining (1.7) and (1.6) we get

$$|\Delta \langle \nabla u, t \rangle| = |\operatorname{tr} \nabla^2 (\langle \nabla u, t \rangle)| \leqq 4 (r_0 - r_1)^{-2} |t|_1,$$

and by duality, we conclude that $\| \nabla \Delta u \|_\infty \leqq 4 (r_0 - r_1)^{-2}$. In other words, we have (1.4) with $C = 4 (r_0 - r_1)^{-2}$. □

2. THE EICONAL EQUATION

We shall consider a potential of the form

$$V(x) = V_0(x) + W(x), \tag{2.1}$$

where

$$V_0(x) = \sum_1^N v_j(x_j), \tag{2.2}$$

and $v_j(x_j)$ are real valued potentials extending holomorphically to the complex disc $D(0, r_0) = \{ x_j \in \mathbb{C};\ |x_j| < r_0 \}$, where r_0 is independent of N. (We here spell out the formulas in the case $n = 1$ only, for $n > 1$, we should simply replace the disc in $\mathbb{C}$ by the standard complex ball of radius r_0 in $\mathbb{C}^n$.) We assume further that there exists C_0 independent of N, such that

$$\left. \begin{aligned} &|v_j(x_j)| \leqq C_0, && x_j \in D(0, r_0), \\ &\nabla v_j(0) = 0, && v_j''(0) \geqq 1/C_0. \end{aligned} \right\} \tag{2.3}$$

As for W we assume that W is real valued on the real domain, and that

$$\left. \begin{aligned} &|\nabla W(x)| \leqq \delta \\ &\text{for all} \\ &x \in B(0, r_0) = B(l^\infty, \mathbb{C}^N; 0, r_0) = D(0, r_0)^N. \end{aligned} \right\} \tag{2.4}$$

Here $\delta > 0$ is assumed to be sufficiently small (depending on all the later constructions) but independent of N. According to Proposition 1.1, we may assume after replacing r_0 by any smaller number and replacing δ by Const. $\times\, \delta$, that for all $x \in B(0, r_0)$:

$$\| \nabla^2 W(x) \|_{\mathcal{L}(l^p, l^p)} \leqq \delta, \quad \text{for all } p \in [1, \infty]. \tag{2.5}$$

The object of this section is then to construct solutions of the eiconal equation

$$(\nabla \varphi)^2 = V(x) - V_{\min} \tag{2.6}$$

defined in some complex l^∞-ball. Here $V_{\min}$ denotes the local minimum of V attained at a point within an l^∞-distance $\mathcal{O}(\delta)$ from 0.

Our first task will be to show the existence of this point, and we shall do this by a standard iteration procedure. Consider the real map

$$\kappa:\quad B_{\mathbb{R}}(0, r)=B(l^{\infty}, \mathbb{R}^N; 0, r)\ni x\mapsto \nabla V_0(x)\in\mathbb{R}^N. \tag{2.7}$$

If $r>0$ is small enough, this map is a diffeomorphism onto a set $\Omega_r=\prod_1^N \Omega_{j,r}$, where $\Omega_{j,r}=]-a_{j,r}, b_{j,r}[$, and $a_{j,r}$ and $b_{j,r}$ are uniformly of the same order of magnitude as r. (When $n>1$, the form of $\Omega_{j,r}$ is of course a little more complicated, but the argument below will work without any essential changes.) The inverse, $\rho=\kappa^{-1}$ has the property that $d\rho$ is a diagonal matrix and that every diagonal element is $\mathcal{O}(1)$. Hence

$$\|d\rho(y)\|_{\mathscr{L}(l^p, l^p)}\leqq C_1,\qquad y\in\Omega_r,\quad 1\leqq p\leqq\infty, \tag{2.8}$$

and in particular for $p=\infty$.

We then solve the equation

$$\nabla V_0(x)+\nabla W(x)=0, \tag{2.9}$$

by successive approximations: define the sequence $x^0, x^1, \ldots,$ by $x^0=0$,

$$\nabla V_0(x^{j+1})=-\nabla W(x^j), \tag{2.10}$$

where we make the inductive assumption that $x^0, x^1, \ldots, x^j\in B_{\mathbb{R}}(0, r)$. Since $|\nabla W(x^j)|_\infty\leqq\delta$, we then have $-\nabla W(x^j)\in\Omega_r$ if $\delta>0$ is small enough, and hence $x^{j+1}\in B_{\mathbb{R}}(0, r)$. As for the convergence, we notice that

$$|x^1|_\infty\leqq C_1\,\delta, \tag{2.11}$$

$$|x^{j+1}-x^j|_\infty\leqq C_1\,\delta\,|x^j-x^{j-1}|_\infty, \tag{2.12}$$

hence if $C_1\,\delta<\frac{1}{2}$, x^j converges in l^∞ to a point x^0 with

$$|x^0|_\infty\leqq C_2\,\delta. \tag{2.13}$$

The point x^0 is a local minimum for V, and $V_{\min}$ in (2.6) is by definition equal to $V(x_0)$.

We shall work in a l^∞-neighborhood of x^0, and we translate the coordinates so that x^0 becomes the origin. Moreover we replace V by $V-V_{\min}$ so that $V(0)=0$. Put $A_0=V''(0)\,(=\nabla^2 V(0))$. Then

$$A_0=D_0+B, \tag{2.14}$$

where D_0 is a positive diagonal matrix with diagonal elements in $[C_2^{-1}, C_2]$ for some $C_2>0$ (independant of N), and where

$$\|B\|_{\mathscr{L}(l^p, l^p)}\leqq\delta\qquad\text{for}\quad 1\leqq p\leqq\infty. \tag{2.15}$$

We write

$$A_0^{\pm 1/2}=(2\pi i)^{-1}\int_\Gamma z^{\pm 1/2}(z-A_0)^{-1}\,dz, \tag{2.16}$$

where Γ is the positively oriented boundary of $[C_2^{-1}, C_2]+D(0, 1/2 C_2)$ (assuming C_2 large). If δ is sufficiently small, we can expand $(z-A_0)^{-1}$ in a perturbation series

$$(z-A_0)^{-1}=(z-D_0)^{-1}-(z-D_0)^{-1} B (z-D_0)^{-1} + (z-D_0)^{-1} B (z-D_0)^{-1} B (z-D_0)^{-1} - \dots,$$

and obtain:

$$(z-A_0)^{-1}=(z-D_0)^{-1}+\mathcal{O}(\delta) \quad \text{in } \mathcal{L}(l^p, l^p) \text{ for all } p\in[1, \infty].$$

Hence

$$A_0^{\pm 1/2}=D_0^{\pm 1/2}+C_\pm, \qquad \text{where } \|C_\pm\|_{\mathcal{L}(l^p, l^p)} \leqq C_3 \delta. \tag{2.17}$$

Using Proposition 1.1, we get

$$\langle \nabla V(x)-\nabla^2 V(0) x, t\rangle = \int_0^1 (1-s)\langle (\nabla^3 V)(sx), t\otimes x\otimes x\rangle ds=\mathcal{O}(1)|t|_1 |x|_\infty^2, \tag{2.18}$$

and hence by duality:

$$|\nabla V(x)-\nabla^2 V(0) x|_\infty \leqq C|x|_\infty^2. \tag{2.19}$$

Let $q(x, \xi)=\frac{1}{2}\xi^2-V(x)$, $q_0=\frac{1}{2}\xi^2-\frac{1}{2}V''(0)x.x$, and recall that the Hamilton field H_q of q is given by $H_q=\sum_1^N \partial_{\xi_j} q(x, \xi)\partial_{x_j}-\partial_{x_j} q(x, \xi)\partial_{\xi_j}$ and that H_{q0} is defined similarly. (2.19) implies that:

$$|H_q-H_{q0}|_\infty \leqq C|x|_\infty^2. \tag{2.20}$$

We shall next look at the H_{q0} flow. Put

$$x_\pm=\frac{1}{2}\left(x\pm V''(0)^{-1/2}\xi\right), \qquad \xi_\pm=\frac{1}{2}\left(\pm V''(0)^{1/2}x+\xi\right), \tag{2.21}$$

and notice that

$$x=x_+ + x_-, \qquad \xi=\xi_+ + \xi_-, \tag{2.22}$$

$$\xi_\pm=\pm V''(0)^{1/2} x_\pm. \tag{2.23}$$

Since $V''(0)^{\pm 1/2}$ are uniformly bounded in $\mathcal{L}(l^\infty, l^\infty)$, we also have

$$|x_\pm|_\infty, \qquad |\xi_\pm|_\infty \leqq C|(x, \xi)|_\infty. \tag{2.24}$$

The coordinates $(2^{1/2}x_+, 2^{1/2}\xi_-)$ are symplectic as well as $(2^{1/2}x_-, 2^{1/2}\xi_+)$. The equations for the H_{q0} flow: $\partial_t x(t)=\xi(t)$, $\partial_t \xi(t)=V''(0)x(t)$ give:

$$\partial_t x_\pm(t)=\pm V''(0)^{1/2} x_\pm(t), \qquad \partial_t \xi_\pm(t)=\pm V''(0)^{1/2}\xi_\pm(t). \tag{2.25}$$

In partcular, we have the stable H_{q_0} invariant Lagrangian subspaces $\Lambda^0_\pm$: $\xi=\pm V''(0)^{1/2}x$. The differential equations for the H_q flow are: $\partial_t x(t)=\xi(t)$, $\partial_t\xi(t)=V'(x(t))$. Combining (2.19) (2.24) with the fact that $V''(0)^{\pm 1/2}$ are uniformly bounded in $\mathscr{L}(l^\infty, l^\infty)$, we get the following estimates for the H_q-flow:

$$\left.\begin{aligned}&|\partial_t x_\pm(t)\mp V''(0)^{1/2}x_\pm(t)|_\infty,\\&|\partial_t \xi_\pm(t)\mp V''(0)^{1/2}\xi_\pm(t)|_\infty\end{aligned}\right\}\leqq C|x(t)|^2_\infty. \tag{2.26}$$

Consider the regions

$$\Omega_{\varepsilon_1,\varepsilon_2}=\{(x,\xi)\in\mathbb{C}^{2N};\ |\xi_-|_\infty\leqq\varepsilon_1|x|_\infty,\ |x|_\infty<\varepsilon_2\}.$$

We define

$$\begin{aligned}\partial^-\Omega_{\varepsilon_1,\varepsilon_2}&=\{(x,\xi)\in\partial\Omega_{\varepsilon_1,\varepsilon_2};\ |x|_\infty<\varepsilon_2\},\\ \partial^+\Omega_{\varepsilon_1,\varepsilon_2}&=\{(x,\xi)\in\partial\Omega_{\varepsilon_1,\varepsilon_2};\ |x|_\infty=\varepsilon_2\},\end{aligned}$$

so that $\partial^-\Omega_{\varepsilon_1,\varepsilon_2}$ and $\partial^+\Omega_{\varepsilon_1,\varepsilon_2}$ form a partition of $\partial\Omega_{\varepsilon_1,\varepsilon_2}$.

LEMMA 2.1. – *If $\varepsilon_j>0$, $j=1, 2$ and ε_1 and $\varepsilon_2/\varepsilon_1$ are sufficiently small, then the following holds: Let $t\mapsto(x(t),\xi(t))$ be a H_q-trajectory. If $(x(t_0),\xi(t_0))\in\partial^+\Omega_{\varepsilon_1,\varepsilon_2}$, then $(x(t),\xi(t))\notin\bar\Omega_{\varepsilon_1,\varepsilon_2}$ for $t-t_0>0$ small enough. If $(x(t_0),\xi(t_0))\in\partial^-\Omega_{\varepsilon_1,\varepsilon_2}$ then $(x(t),\xi(t))\in\Omega_{\varepsilon_1,\varepsilon_2}$ for $t-t_0>0$ small enough. In other words, the flow enters through $\partial^-\Omega_{\varepsilon_1,\varepsilon_2}$ and leaves through $\partial^+\Omega_{\varepsilon_1,\varepsilon_2}$.*

Proof (cf. [HS3]). – We shall study the derivatives of the Lipschitz continuous fonctions $t\mapsto|\xi_-(t)|_\infty$ and $t\mapsto|x(t)|_\infty$. [If $f(t)$ is locally Lipschitz on some interval, then $\partial_t f(t)$ exists a.e. and defines an element in L^∞_{loc} with the property that $f(t_2)-f(t_1)=\int_{t_1}^{t_2}\partial_t f(t)\,dt$.] Using (2.26) and the fact that $V''(0)^{1/2}$ is a small perturbation in $\mathscr{L}(l^\infty, l^\infty)$ of a diagonal matrix with diagonal elements > const. > 0, we get:

$$\partial_t|\xi_-(t)|_\infty\leqq -C^{-1}|\xi_-(t)|_\infty+\mathcal{O}(|x(t)|^2_\infty)\ a.e. \tag{2.27}$$

We also have almost everywhere:

$$\partial_t|x(t)|_\infty\geqq C^{-1}|x_+(t)|_\infty-C|x_-(t)|_\infty+\mathcal{O}(|x(t)|^2_\infty). \tag{2.28}$$

In $\bar\Omega_{\varepsilon_1,\varepsilon_2}$ we get from (2.28) [*cf.* also (2.23) and (2.24)]:

$$\partial_t|x(t)|_\infty\geqq C^{-1}|x_+(t)|_\infty-C\varepsilon_1|(x,\xi)|_\infty\geqq(\tilde C^{-1}-\tilde C\varepsilon_1)|(x,\xi)|_\infty, \tag{2.29}$$

and in particular $\partial_t|x(t)|_\infty>0$ for $(x(t),\xi(t))$ near $\partial^+\Omega_{\varepsilon_1,\varepsilon_2}$, which proves the first statement in the lemma.

If $(x(t_0),\xi(t_0))\in\partial^-\Omega_{\varepsilon_1,\varepsilon_2}\setminus\{(0,0)\}$, then for $t=t_0$: $|\xi_-|_\infty=\varepsilon_1|x|_\infty$ and (2.27) implies that for $t=t_0$:

$$\partial_t|\xi_-(t)|_\infty\leqq -C^{-1}|\xi_-(t)|+\mathcal{O}((\varepsilon_1^{-1}|\xi_-(t)|_\infty)^2), \tag{2.30}$$

which is <0 provided that $\varepsilon_1^{-2}|\xi_-(t)|_\infty$ is sufficiently small, *i.e.* if $\varepsilon_1^{-1}|x(t)|_\infty \leqq \varepsilon_2/\varepsilon_1$ is sufficiently small. In view of (2.29) we also have $\partial_t |x(t)|>0$ at $t=t_0$. Hence at $t=t_0$:

$$\partial_t [|\xi_-(t)|_\infty/|x(t)|_\infty] = (\partial_t |\xi_-(t)|_\infty)/|x(t)|_\infty - |\xi_-(t)|_\infty |\partial_t x(t)|/|x(t)|_\infty^2 < 0,$$

and the second statement follows. □

We shall next analyze the evolution of tangent vectors and tangent space along the H_q flow. From the identity

$$\langle \nabla^2 V(x) - \nabla^2 V(0), t_1 \otimes t_2 \rangle = \int_0^1 \langle \nabla^3 V(sx), x \otimes t_1 \otimes t_2 \rangle ds, \quad (2.31)$$

and from (1.2), we deduce that

$$\| V''(x) - V''(0) \|_{\mathscr{L}(l^p, l^p)} \leqq C |x|_\infty, \qquad 1 \leqq p \leqq \infty. \quad (2.32)$$

The evolution of a tangent vector $t=(t_x, t_\xi)$ under the H_q-flow is given by the system

$$\partial_s \begin{pmatrix} t_x \\ t_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V''(x(s)) & 0 \end{pmatrix} \begin{pmatrix} t_x \\ t_\xi \end{pmatrix} \quad (2.33)$$

As before, we write $t=t^+ + t^-$, $t^\pm = (t_x^\pm, t_\xi^\pm)$, with

$$\left.\begin{aligned} t_x^\pm &= \frac{1}{2}(t_x \pm V''(0)^{-1/2} t_\xi) \\ t_\xi^\pm &= \frac{1}{2}(\pm V''(0)^{1/2} t_x + t_\xi). \end{aligned}\right\} \quad (2.34)$$

If we restrict the attention to integral curves with $|x|_\infty \leqq \varepsilon_2$, we get by combining (2.32)-(2.34):

$$\left.\begin{aligned} \partial_s t_x^+ &= \frac{1}{2}(t_\xi + V''(0)^{-1/2} V''(x(s)) t_x) \\ &= V''(0)^{1/2} t_x^+ + \mathcal{O}(1) \varepsilon_2 |t|_p \quad \text{in } l^p, \\ \partial_s t_x^- &= -V''(0) t_\xi^- + \mathcal{O}(1) \varepsilon_2 |t|_p \quad \text{in } l^p, \\ \partial_s t_\xi^\pm &= \pm V(0)^{1/2} t_\xi^\pm + \mathcal{O}(1) \varepsilon_2 |t|_p \quad \text{in } l^p, \end{aligned}\right\} \quad (2.35)$$

uniformly for $1 \leqq p \leqq \infty$. We take $p=\infty$ and let all the norms be in l^∞ as long as nothing else in indicated. We get from (2.35) and the structure of $V''(0)^{1/2}$:

$$\left.\begin{aligned} \partial_s |t_\xi^+| &\geqq C^{-1} |t_\xi^+| - C\varepsilon_2 |t_\xi^-| \\ \partial_s |t_\xi^-| &\leqq C\varepsilon_2 |t_\xi^+| - C^{-1} |t_\xi^-|, \end{aligned}\right\} \quad (2.36)$$

provided that ε_2 is sufficiently small. (Here we also use that $|t| \sim |t_\xi^+| + |t_\xi^-|$.) It follows that $V_c \underset{\text{def.}}{=} \{ t \in \mathbb{C}^{2N};\ |t_\xi^-| \leqq c |t_\xi^+| \}$ is stable under the differentiated flow as long as $C\varepsilon_2 c < C^{-1}$, $C\varepsilon_2/c < C^{-1}$, that is:

$$C^2\varepsilon_2 < c < 1/(C^2\varepsilon_2). \tag{2.37}$$

Choosing c of the order of magnitude ε_2 we get in particular that there is a constant $C>0$ such that if $\Lambda(0)$ is a Lagrangian subspace of $T_{(x(0),\,\xi(0))}(\mathbb{C}^{2N})$ of the form: $t_\xi = (V''(0)^{1/2} + R(0))\,t_x$ with $R(0)$ symmetric and $\|R(0)\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq \varepsilon_2$, then the image, $\Lambda(s) \subset T_{(x(s),\,\xi(s))}(\mathbb{C}^{2N})$ under the differential of the flow, is of the form: $t_\xi = (V''(0)^{1/2} + R(s))\,t_x$ with $\|R(s)\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq C\varepsilon_2$.

We next consider the evolution of certain Lagrangian manifolds in $\Omega_{\varepsilon_1,\varepsilon_2}$. Let $\Lambda(0)$ be a closed Lagrangian submanifold of $\Omega_{\varepsilon_1,\varepsilon_2}$ of the form $\xi = \varphi'(x)$ such that $\|\varphi''(x) - V''(0)^{1/2}\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq \varepsilon_2$ for all x in the projection of $\Omega_{\varepsilon_1,\varepsilon_2}$. Define

$$\Lambda(s) = \{\exp(s H_q)(\rho);\ \rho \in \Lambda(0), \exp(\sigma H_q)(\rho) \in \Omega_{\varepsilon_1,\varepsilon_2} \text{ for } 0 \leqq \sigma \leqq s\}.$$

Since the points of $\Lambda(s)$ with $|x| \neq 0$ move outwards and cross $\partial^+\Omega_{\varepsilon_1,\varepsilon_2}$, we see that for small $s \geqq 0$, $\Lambda(s)$ is closed and of the form $\xi = \varphi_s'(x)$. The stability remark above for tangent Lagrangian subspaces implies that $\|\varphi_s''(x) - V''(0)^{1/2}\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq C\varepsilon_2$, and we can clearly iterate the argument (without any additional factors C in the last estimate) and conclude that for all $s \geqq 0$, $\Lambda(s)$ is of the form $\xi = \varphi_s'(x)$ with

$$\|\varphi_s''(x) - V''(0)^{1/2}\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq C\varepsilon_2.$$

We now choose $\Lambda(0)$ with the additional property that $\Lambda(0)$ coincides with the stable outgoing manifold for the H_q-flow near (0, 0). (That the stable outgoing manifold is Lagrangian was checked in [HS1].) Since all points in $\Omega_{\varepsilon_1,\varepsilon_2} \setminus \text{neigh}.((0,0))$ are evacuated within some fixed finite time by the flow, we see that for s sufficiently large, $\Lambda(s)$ becomes independent of s and equal to the stable outgoing manifold,

$$\begin{aligned}\Lambda_+ &= \{\rho \in \Omega_{\varepsilon_1,\varepsilon_2};\ \exp(-s H_q(\rho)) \in \Omega_{\varepsilon_1,\varepsilon_2} \text{ for all } s \geqq 0\} \\ &= \{\rho \in \Omega_{\varepsilon_1,\varepsilon_2};\ \exp(-s H_q(\rho)) \in \Omega_{\varepsilon_1,\varepsilon_2} \text{ for all } s \geqq 0 \\ &\qquad \text{and } \exp(-s H_q(\rho)) \to (0,0) \text{ when } s \to +\infty\}.\end{aligned}$$

The corresponding phase $\varphi(x)$ then satisfies the eiconal equation

$$\varphi'^2 = V(x) \qquad \text{for } |x| < \varepsilon_2. \tag{2.38}$$

By construction, we also have

$$|\varphi'(x) - V''(0)^{1/2} x| \leqq \varepsilon_1 |x|, \tag{2.39}$$

$$\|\varphi''(x) - V''(0)^{1/2}\|_{\mathscr{L}(l^\infty,\,l^\infty)} \leqq C\varepsilon_2. \tag{2.40}$$

Actually, (2.40) can be sharpened and generalized, using that $\varphi''(0) = V''(0)^{1/2}$ and that $\|\varphi''(x) - \varphi''(0)\|_{\mathscr{L}(l^p,\,l^p)} \leqq C|x|$:

$$\|\varphi''(x) - V''(0)^{1/2}\| \leqq C|x| \leqq C\varepsilon_2. \tag{2.41}$$

Our estimates give information about the flow $t \mapsto \exp(-t \nabla \varphi(x) . \partial_x)$ which is the x-projection of the flow $t \mapsto \exp(-t H_q)$ restricted to Λ_+. Let $]-\infty, 0] \ni s \mapsto (x(s), \xi(s)) \in \Lambda_+$ be an H_q integral curve. According to (2.29) there is a constant $C>0$ such that $\partial_s |x(s)| \geqq C^{-1} |x(s)|$ and hence:

$$|\exp(-t \nabla \varphi(x) . \partial_x)(x)| \leqq e^{-t/C} |x|, \qquad t \geqq 0. \tag{2.42}$$

If c satisfies (2.37), then $T_{x(s), \xi(s)} \Lambda_+ \subset V_c$ for $-\infty < s \leqq 0$ and (2.36) shows that there is a constant $C>0$ such that $\partial_s |t_\xi^+| \geqq C^{-1} |t_\xi^+|$, if $s \mapsto (t_x(s), t_\xi(s)) \in T \Lambda_+$ is an integral curve of the differentiated flow. Let $c \leqq \frac{1}{2}$. Since $|t_\xi^-| \leqq c |t_\xi^+|$, we have $|t_x| \sim |t_\xi^+|$ and we conclude that $|t_x(s)| \geqq (1/\text{Const.}) e^{(s-\tilde{s})/C} |t_x(\tilde{s})|$ for $\tilde{s} < s$. This can be viewed as an estimate on the differential of the flow of $\exp(-t \nabla \varphi(x) . \partial_x)$ and we get with a new positive constant:

$$\| d(\exp(-t \nabla \varphi(x) . \partial_x) \|_{\mathscr{L}(l^\infty, l^\infty)} \leqq C e^{-t/C}. \tag{2.43}$$

Summing up, we have proved,

PROPOSITION 2.1. – *Let* V *satisfy the assumptions explained in the beginning of this section. Then if* $\delta > 0$ *is sufficiently small, the following holds:*

(A) V *has a non-degenerate local minimum at a point* $x_0 \in \mathbb{R}^N$ *with*

$$|x_0| = \mathcal{O}(\delta) \quad (\textit{and here the norm is the one in } l^\infty).$$

(B) *Translating the coordinates, we may assume that* $x_0 = 0$ *and replacing* V *by* $V - V(0)$, *that* $V(0) = 0$. *Then there exists* $\varepsilon > 0$ *independent of* N *such that* (2.38) *has a solution which is holomorphic in* $\{x \in \mathbb{C}^N; |x| \leqq \varepsilon\}$. *This solution is real-valued on the real domain and satisfies* (2.39), (2.41), (2.42), (2.43) *for some* $C>0$. *The constants* ε_1, ε_2, *can be chosen arbitrarily small if* $\delta > 0$ *and* $\varepsilon > 0$ *are small enough. All constants are independent of* N.

3. ASYMPTOTIC EIGENFUNCTIONS AND EIGENVALUES

We make the same assumptions on V as in the preceding section, and without loss of generality, we may assume that the point of local minimum of V is 0 and that the corresponding value for V is 0.

WKB-constructions for the semiclassical Schrödinger equation can be based on the formula:

$$\left(-\frac{1}{2}h^2\Delta+V(x)-hE\right)(ae^{-\varphi/h})$$
$$=e^{-\varphi/h}\left(\left(V(x)-\frac{1}{2}(\nabla\varphi)^2\right)a\right.$$
$$\left.+h\left(\nabla\varphi(x)\cdot\partial_x+\frac{1}{2}\Delta\varphi(x)-E\right)a-\frac{1}{2}h^2\Delta a(x)\right). \quad (3.1)$$

We may (as for instance in [HS1]) choose φ independent of h, solving the eiconal equation $V(x)-\frac{1}{2}(\nabla\varphi)^2=0$, and then try to find $a(x, h)\sim\sum_0^\infty a_j(x)h^j$ by solving a sequence of transport equations. When we tried this for large N, we were unable to find nice N-independent bounds on the sequence of functions a_j. Instead, we decided to take advantage of the fact that we are only studying the lowest eigenvalue and that the corresponding eigenfunction should be non-vanishing and, say, positive. This led us to take $a=1$ and to make φ dependent of h. According to (3.1), the equation $\left(-\frac{1}{2}h^2\Delta+V(x)-hE\right)(e^{-\varphi/h})=0$ becomes:

$$V(x)-\frac{1}{2}(\nabla\varphi)^2+h\left(\frac{1}{2}\Delta\varphi-E\right)=0. \quad (3.2)$$

We shall solve (3.2) asymptotically by trying solutions φ and E with asymptotic expansions:

$$\varphi(x; h)\sim\varphi_0(x)+\varphi_1(x)h+\dots \quad (3.3)$$
$$E(h)\sim E_0+E_1h+\dots \quad (3.4)$$

and the goal of this section is to obtain estimates for φ_j and E_j which are valid uniformly with respect to N. If we just collect powers of h, (3.3) and (3.4) give us the sequence of equations:

(E) $$V(x)-\frac{1}{2}(\nabla\varphi_0)^2=0,$$

(T 1) $$\nabla\varphi_0(x)\cdot\partial_x\varphi_1(x)=\frac{1}{2}\Delta\varphi_0(x)-E_0,$$

(T 2) $$\nabla\varphi_0(x)\cdot\partial_x\varphi_2(x)=\frac{1}{2}\Delta\varphi_1(x)-\frac{1}{2}(\nabla\varphi_1)^2-E_1,$$

⋮

$$(\mathrm{Tm})\ \nabla \varphi_0 \cdot \partial_x \varphi_m = \frac{1}{2} \Delta \varphi_{m-1} - \frac{1}{2} (\nabla \varphi_1 \cdot \nabla \varphi_{m-1} + \nabla \varphi_2 \cdot \nabla \varphi_{m-2} + \ldots + \nabla \varphi_{m-1} \cdot \varphi_1) - \mathrm{E}_{m-1}$$

⋮

We start by solving these equations in the complex l^∞ ball $\mathrm{B}(0, \varepsilon_0)$ in $\mathbb{C}^\mathrm{N}$, with $\varepsilon_0 > 0$ sufficiently small. For φ_0, we take the function "φ" constructed in section 2. Then (E) holds. The vectorfield $\nabla \varphi_0 \cdot \partial_x$ vanishes at $x=0$, so a necessary condition for solving (T 1) is that

$$\mathrm{E}_0 = \frac{1}{2} \Delta \varphi_0 (0). \tag{3.5}$$

This condition is also sufficient because we have the estimate (2.42), and hence we can solve (T 1) by means of the convergent integral:

$$\varphi_1 (x) = \int_{-\infty}^{0} \left(\frac{1}{2} \Delta \varphi_0 - \mathrm{E}_0 \right) (\exp (t \nabla \varphi_0 \cdot \partial_x) (x))\, dt. \tag{3.6}$$

φ_1 is then holomorphic in $\mathrm{B}(0, \varepsilon_0)$ and it is the unique such solution which vanishes at $x=0$. Once φ_1 has been determined, we can solve (T 2) provided that

$$\mathrm{E}_1 = \frac{1}{2} (\Delta \varphi_1 (0) - (\nabla \varphi_1 (0))^2). \tag{3.7}$$

Continuing in this way, we get a sequence of holomorphic functions $\varphi_j (x)$ defined in the ball $\mathrm{B}(0, \varepsilon_0)$ and a sequence of real numbers $\mathrm{E}_0, \mathrm{E}_1, \ldots$ (the reality of E_j following from that of the $\varphi_{j \mid \mathbb{R}^\mathrm{N}}$) such that (T 1, 2, 3, . . .) are satisfied.

In order to get estimates with N-independent constants, we shall use Proposition 1.1, and we start with the fact that $|\nabla \varphi_0 (x)|_\infty \leqq \mathrm{C}_0$ for $|x|_\infty < \varepsilon_0$, *with* C_0 *independent of N*. Let $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots > \text{const.} > 0$. Applying Proposition 1.1, we see that $\left| \nabla \left(\frac{1}{2} \Delta \varphi_0 - \mathrm{E}_0 \right) (x) \right|_\infty \leqq \tilde{\mathrm{C}}_1$ for $|x|_\infty < \varepsilon_1$ (with $\tilde{\mathrm{C}}_1$ independent of N). On the other hand, we have exponential contractiveness for the differential of $\exp (t \nabla \varphi_0 . \partial_x)$ when $t \leqq 0$ [cf. (2.43)] so from (3.6) we conclude that $|\nabla \varphi_1 (x)|_\infty \leqq \mathrm{C}_1$ for $|x|_\infty < \varepsilon_1$. Assume by induction that

$$|\nabla \varphi_j (x)|_\infty \leqq \mathrm{C}_j \qquad \text{for} \quad |x|_\infty < \varepsilon_j, \tag{3.8}$$

with C_j independent of N, for $j=0, 1, \ldots, m-1$. Let f_m be the right hand side of (Tm). By Proposition 1.1, we have $|\nabla f_m|_\infty \leqq \tilde{\mathrm{C}}_m$ for $|x|_\infty < \varepsilon_m$ and using the formula

$$\varphi_m(x)=\int_{-\infty}^{0} f_m(\exp(t\nabla\varphi_0\,.\,\partial_x)(x))\,dt$$

and the estimate (2.43), we get (3.8) for $j=m$. Hence we have (3.8) for all j.

The values, E_j obtained by imposing the vanishing of the right hand side of (T_j) can be estimated by using (3.8) and Cauchy's inequality:

$$|E_j|\leqq \hat{C}_j N, \qquad j=0,\,1,\,2,\,\ldots \tag{3.9}$$

Summing up, we have proved,

PROPOSITION 3.1. — *We make the same assumptions as in Proposition* 2.1 *and the same reduction as in part* (B) *of that proposition. Let* φ_0 *be the function "*φ*" of Proposition* 2.1. *Then there is an* $\varepsilon>0$ *such that* (3.2) *can be solved asymptotically in* $\{x\in\mathbb{C}^N;\ |x|<\varepsilon\}$ *by* (3.3), (3.4) *with* $\varphi_j(0)=0$, $|\nabla\varphi_j(x)|\leqq C_j$, $|x|<\varepsilon$, $|E_j|\leqq C_j$. *Here* ε *and* C_j *are independent of* N.

In the next section we develop some easy consequences for the spectrum of our Schrödinger operator, and a more complete analysis will be given in sections 5, 6.

Remark 3.2. — By a scaling argument we may weaken the hypothesies of Proposition 3.1 slightly and assume:

(H) $V(0)=0$, $V'(0)=0$ and there exists $C>0$ such that $|\nabla V(x)|\leqq C$ for $x\in\mathbb{C}^N$, $|x|<1/C$. Moreover there exists a positive diagonal matrix D such that $\|V''(0)-D\|_{\mathscr{L}(l^\infty,\,l^\infty)}\leqq\delta$, $D\geqq 1/C$.

In fact, we have already seen that these properties follow from the hypothesies of Proposition 3.1, and we shall now see that conversely (H) will imply those assumptions, up to a dilation in the x-coordinates, and a corresponding dilation in h. We write $V=V_0+W$, with $V_0=\frac{1}{2}Dx\cdot x$. Then $\|W''(0)\|_{\mathscr{L}(l^\infty,\,l^\infty)}\leqq\delta$. If $W_0=W''(0)x\cdot x$, we get $|\nabla W_0(x)|\leqq\delta$ for $|x|\leqq 1$. The quadratic part of V is $V_2=V_0+W_0$, and we write

$$V=V_2+V_3=V_0+W_0+V_3.$$

Then V_3 vanishes to the third order at $x=0$ and we shall estimate $|\nabla V_3(x)|$. We write

$$\langle\nabla V_3(x),\,t\rangle=\int_0^1(1-s)\,\partial_s^2(\langle\nabla V_3(sx),\,t\rangle)\,ds$$
$$=\int_0^1(1-s)\langle\nabla^3 V_3(sx),\,t\otimes x\otimes x\rangle ds=\mathcal{O}(1)\,|t|_1\,|x|_\infty^2.$$

Hence $|\nabla V_3(x)| \leqq \mathcal{O}(1)|x|^2$, which is small for $|x|$ small. To see that we have the right to restrict the attention to such x, we make the change of variables $x=\lambda y$, which gives:

$$\begin{aligned} -h^2\Delta_x+V(x) &= \lambda^2(-(h/\lambda^2)^2\Delta_y+\lambda^{-2}V(\lambda y)) \\ &= \lambda^2(-(h/\lambda^2)^2\Delta_y+V_0(y)+W_0(y)+\lambda^{-2}V_3(\lambda y)), \end{aligned}$$

and if λ is small but independent of h, then for $|y|<1$:

$$|\nabla_y(\lambda^{-2}V_3(\lambda y))| = |\lambda^{-1}(\nabla V_3)(\lambda y)| = \mathcal{O}(\lambda).$$

Choosing $\lambda \ll \delta$, we then can conclude that

$$|\nabla_y(W_0(y)+\lambda^{-2}V_3(\lambda y))| \leqq 2\delta,$$

so the assumptions of Proposition 3.1 are fulfilled in the y-variables if we replace h by h/λ^2. In conclusion, Proposition 3.1 still holds if we only assume (H).

4. A CONSEQUENCE FOR THE SPECTRUM

We keep the same assumptions as in the preceding two sections with 0 as the point of local minimum of V and with $V(0)=0$. We let φ_j and E_j be the quantities constructed in section 3. In some small complex l^∞-ball, $B(0, \varepsilon_0)$ we then have with N-independent constants C_j:

$$|\nabla\varphi_j(x)|_\infty \leqq C_j, \qquad \varphi_j(0)=0. \tag{4.1}$$

Let $\chi \in C_0^\infty(]-1, 1[; [0, 1])$ be equal to 1 on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. For a suitable sequence $\lambda_j \nearrow +\infty$, we put

$$\varphi(x; h) = \sum_0^\infty \varphi_j(x)\, h^j \chi(\lambda_j h). \tag{4.2}$$

We have: $|h^j\chi(\lambda_j h)\nabla\varphi_j|_\infty \leqq C_j h^j$ and this quantity vanishes when $h \geqq 1/\lambda_j$. Take $\lambda_j = C_j 2^j$ so that $|h^j\chi(\lambda_j h)\nabla\varphi_j|_\infty \leqq 2^{-j}h^{j-1}$. It is then easy to check that (4.2) converges for $x \in B(0, \varepsilon_0)$ and that

$$\left|\nabla\varphi - \sum_0^k h^j\nabla\varphi_j\right|_\infty \leqq C_k h^{k+1}, \qquad \varphi(0; h)=0, \tag{4.3}$$

for some new constants C_k, independent of N.

We next look at the expression in the LHS of (3.2): We write $\varphi=\varphi^{(k)}+r_{k+1}$, with $r_{k+1}=\varphi-\sum_0^k h^j\varphi_j$ so that ∇r_{k+1} is the expression appearing inside the norm in (4.3). We also write $E=E^{(k)}+f_{k+1}$ with

$E^{(k)}=\sum_0^k h^j E_j$. Here E is an asymptotic sum of the series (3.4) and chosen in such a way that $|f_{k+1}|\leqq C_k N h^{k+1}$. We then get

$$V(x)-\frac{1}{2}(\nabla \varphi(x))^2+h\left(\frac{1}{2}\Delta\varphi-E\right)=I+II,$$

with

$$I=V(x)-\frac{1}{2}(\nabla \varphi^{(k)})^2+h\left(\frac{1}{2}\Delta\varphi^{(k-1)}-E^{(k-1)}\right),$$

$$II=-\nabla\varphi^{(k)}.\nabla r_{k+1}-\frac{1}{2}(\nabla r_{k+1})^2+h\left(\frac{1}{2}\Delta r_k-f_k\right).$$

Combining the construction of the φ_j in section 3 and the Proposition 1.1, we see that (after decreasing ε_0):

$$|I(0)|\leqq C_k N h^{k+1}, \qquad |\nabla I|_\infty \leqq C_k h^{k+1}. \tag{4.4}$$

Using the estimates above on r_{k+1}, f_k and Proposition 1.1, we also have

$$|II(0)|\leqq C_k N h^{k+1}, \qquad |\nabla II|_\infty \leqq C_k h^{k+1}. \tag{4.5}$$

Thus if we put $R=V(x)-\frac{1}{2}(\nabla\varphi(x))^2+h\left(\frac{1}{2}\Delta\varphi-E\right)$, we get for $|x|_\infty<\varepsilon_1<\varepsilon_0$:

$$|R(0)|\leqq C_k N h^{k+1}, \qquad |\nabla R(x)|_\infty \leqq C_k h^{k+1}, \qquad k=0, 1, 2, \ldots \tag{4.6}$$

From (4.6) it follows that

$$|R(x)|\leqq C_k N h^{k+1} \tag{4.7}$$

(with a new constant C_k). For simplicity, we assume from now on that ε_0 has been decreased so that (4.6) holds with ε_1 replaced by ε_0, and from now in the section we restrict the attention to the real domain. Let $\psi\in C_0^\infty(]-\varepsilon_0, \varepsilon_0[; [0, 1])$ be equal to 1 on $[-\varepsilon_0/2, \varepsilon_0/2]$, and put

$$\chi(x)=\prod_1^N \psi(x_j), \tag{4.8}$$

$$u(x; h)=e^{-\varphi(x; h)/h}\chi(x). \tag{4.9}$$

We need some preliminary remarks concerning the function $x_j\mapsto\varphi(x; h)$, when $|x'|_\infty<\varepsilon_0$, and $h>0$ is sufficiently small, and where $j\in\{1, 2, \ldots, N\}$ and we write $x'=(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$. We have

$$\|\nabla^2(\varphi(x; h)-\varphi_0(x))\|_{\mathscr{L}(l^\infty, l^\infty)}\leqq Ch,$$

and

$$\|(\nabla^2\varphi_0(x)-\nabla^2\varphi_0(0))\|_{\mathscr{L}(l^\infty, l^\infty)}\leqq C\varepsilon_0.$$

Moreover we recall that $\nabla^2 \varphi_0(0)) = V''(0)^{1/2}$ differs from a positive diagonal matrix (uniformly bounded and with a uniformly bounded inverse) by a matrix which is $\mathcal{O}(\delta)$ in $\mathcal{L}(l^p, l^p)$ for every $1 \leqq p \leqq \infty$. For h, ε_0 and δ small enough, $\nabla^2 \varphi(x; h)$ is then as close as we wish in $\mathcal{L}(l^p, l^p)$, to a constant positive diagonal matrix, and in particular $\partial_{x_j}^2 \varphi(x; h)$ is of the order of magnitude 1. Since $|\nabla \varphi - \nabla \varphi_0|_\infty \leqq Ch$, we have that $|\nabla \varphi(0; h)|_\infty \leqq Ch$ and in particular $|\partial_{x_j} \varphi(0; h)| \leqq Ch$. It follows that $]-\varepsilon_0, \varepsilon_0[\ni x_j \mapsto \varphi(0, \ldots, 0, x_j, 0, \ldots, 0; h)$ has a unique minimum $x_j(0) = \mathcal{O}(h)$. For $|x'|_\infty$ sufficiently small, we can still define $x_j(x')$ as the minimum of $]-\varepsilon_0, \varepsilon_0[\ni x_j \mapsto \varphi(x; h)$. Let $x(x')$ be the point with x'-component equal to x' and with x_j component equal to $x_j(x')$. Applying (2.39), we get $|(V''(0)^{1/2} x)_j| \leqq \varepsilon_1 |x|_\infty$. Using the structure of $V''(0)^{1/2}$, this leads to the estimate:

$$C^{-1} |x_j(x')| \leqq 2\varepsilon_1 |x|_\infty = 2\varepsilon_1 \max(|x'|_\infty, |x_j(x')|),$$

provided that δ is sufficiently small. If ε_1 is small enough, we conclude that

$$|x_j(x')| \leqq 2C\varepsilon_1 |x'|_\infty \ll \varepsilon_0,$$

and we conclude that $x_j(x')$ remains welldefined for $|x'|_\infty < \varepsilon_0$.

After these preparations, we shall compare $\|u\|_{L^2}$ and $\left\|\left(-\frac{1}{2} h^2 \Delta + V - hE\right) u\right\|_{L^2}$. We have

$$\left(-\frac{1}{2} h^2 \Delta + V - hE\right) u = -\frac{1}{2} h^2 [\Delta, \chi] e^{-\varphi/h} + \chi R e^{-\varphi/h}, \tag{4.10}$$

with R defined prior to (4.6). We have:

$$-\frac{1}{2} h^2 [\Delta, \chi] e^{-\varphi/h} = \sum_1^N u_j, \tag{4.11}$$

with:

$$u_j = e^{-\varphi/h} [\prod_{\nu \neq j} \psi(x_\nu)] r_j(x; h), \tag{4.12}$$

and

$$r_j(x; h) = h \partial_{x_j} \varphi(x; h) \partial_{x_j} \psi(x_j) - \frac{1}{2} h^2 \partial_{x_j}^2 \psi(x_j), \tag{4.13}$$

and we shall only use that $r_j = \mathcal{O}(h)$ and that r_j has its support in

$$\left\{x_j \in \mathbb{R}; \frac{1}{2} \varepsilon_0 \leqq |x_j| \leqq \varepsilon_0\right\}.$$

Let $\| \, . \, \|$ denote the L^2-norm if nothing else is specified, and write:

$$\|u_j\|^2 = \int e^{-2\varphi(x;h)/h} [\prod_{\nu \neq j} \psi(x_\nu)]^2 r_j(x;h)^2 \, dx. \tag{4.14}$$

From the properties of r_j and of the function $x_j \mapsto \varphi$, we see that

$$\int e^{-2\varphi(x;h)/h} r_j(x;h)^2 \, dx_j \leq C e^{-1/Ch} \int e^{-2\varphi(x;h)/h} \psi_j(x_j)^2 \, dx_j, \tag{4.15}$$

so from Fubini's theorem, we get:

$$\|u_j\|^2 \leq C e^{-1/Ch} \|u\|^2. \tag{4.16}$$

From (4.7) it is obvious that

$$\|\chi \mathrm{R} e^{-\varphi/h}\| \leq C_k N h^k \|u\|, \tag{4.17}$$

for every $k \in \mathbb{N}$. Combining (4.10), (4.11), (4.17) and Cauchy-Schwarz in order to estimate $(u_j \mid u_k)_{L^2}$ we get for every $k \geq 0$:

$$\left\| \left(-\frac{1}{2} h^2 \Delta + V - h E \right) u \right\| \leq C_k N h^k \|u\|. \tag{4.18}$$

The immediate consequence of this estimate is that if V has some extension to some real domain Ω containing the real l^∞ ball $B_{\mathbb{R}}(0, \varepsilon_0)$ and if P is some corresponding self-adjoint realization in $L^2(\Omega)$ with the property that $Pu = \left(-\frac{1}{2} h^2 \Delta + V - hE \right) u$ for the function u constructed above, then for every $k \in \mathbb{N}$, we have:

$$\operatorname{dist}(hE, \sigma(P)) \leq C_k N h^k. \tag{4.19}$$

5. *A PRIORI* ESTIMATES NEAR THE BOTTOM OF A POTENTIAL WELL

Let $\varphi(x; h)$ and V be as in the preceding section, still with $\nabla V(0) = 0$, $V(0) = 0$. The main step of this section will be to obtain L^2-estimates for the system of 1 : st order differential operators

$$Z_{j,\varphi} = h \partial_{x_j} + \partial_{x_j}(\varphi), \qquad j = 1, 2, \ldots, N, \tag{5.1}$$

and we observe that the left C^∞-module generated by these operators is the set of all operators of the form

$$Z_{\nu,\varphi} = h \nu(x, \partial_x) + \nu(x, \partial_x)(\varphi) = e^{-\varphi/h} h \nu(x, \partial_x) e^{\varphi/h}, \tag{5.2}$$

where ν may be any smooth real vector field.

We shall make a change of variables of the form $x \mapsto y = \nabla f(x)$ for a suitable function f, and then study the new system $h \partial_{y_j} + \partial_{y_j}(\varphi)$.

In view of the identity $\int |u(x)|^2 dx = \int |J^{-1/2} u|^2 dy$ with $J(y) = \det \partial y/\partial x = \det f''(x)$, we are interested in L^2-estimates for the system

$$J^{-1/2}(h\partial_{y_j} + \partial_{y_j}(\varphi))J^{1/2}, \tag{5.3}$$

which therefore should have a simple form. We write $J = e^{\gamma}$, $\gamma = \gamma_f$. Then the operator (5.3) becomes

$$h\partial_{y_j} + \partial_{y_j}(\varphi + h\gamma/2). \tag{5.4}$$

We want to have (approximately) $\partial_{y_j}(\varphi + h\gamma/2) = y_j$, that is:

$$\varphi + h\gamma/2 = \frac{1}{2}\sum_1^N y_j^2 + \text{const.} \tag{5.5}$$

This equation can be expressed in the x-coordinates as:

$$\frac{1}{2}(\nabla f)^2 - \varphi - h\gamma_f/2 = \text{const}, \qquad \gamma_f = \log\det(\nabla^2 f), \tag{5.6}$$

and this has great similarities with the equation (3.2), if we think of φ as the new potential. The only difference is that we have the non-linear term γ_f instead of Δf. We shall treat (5.5) asymptotically similarly to what we did in section 3. First recall that

$$\varphi \sim \varphi_0(x) + \varphi_1(x)h + \ldots, \tag{5.7}$$

where $|\varphi_j(x)|_\infty \leqq C_j$ for all x in a complex l^∞ ball of radius $\varepsilon_1 > 0$ independent of N, and that $\varphi_j(0) = 0$, $\varphi_0''(0) = V''(0)^{1/2}$. Moreover, we recall that $V''(0)^{1/2}$ differs from a positive diagonal matrix $D_0 = \mathcal{O}(1)$ with $D_0^{-1} = \mathcal{O}(1)$ by a matrix which is $\mathcal{O}(\delta)$ in $\mathcal{L}(l^p, l^p)$ for all $1 \leqq p \leqq \infty$, where $\delta > 0$ is the basic perturbation parameter. This means that all the assumptions for V that we have used earlier, are also satisfied by φ_0. (*Cf.* Remark 3.2). The higher order terms in the asymptotic expansion (5.7) give contributions to (5.6) which are easy to handle, and we shall therefore assume for simplicity that $\varphi = \varphi_0$ is independent of h.

We look for f of the form

$$f \sim f_0 + f_1 h + f_2 h^2 + \ldots, \tag{5.8}$$

and we first get the characteristic equation

$$\frac{1}{2}(\nabla f_0)^2 - \varphi = 0, \tag{5.9}$$

which can be solved in some small l^∞-ball centered at 0, precisely as in section 2. f_0 will then have the same properties as φ. f_1 should fulfill the

1st transport equation:

$$\nabla f_0 . \partial_x f_1 = \frac{1}{2} \gamma_{f_0} + \text{const.}, \tag{5.10}$$

so we have to investigate $\gamma_{f_0} = \log \det \nabla^2 f_0$. We have

$$\partial_{x_j} \gamma_{f_0} = \operatorname{tr} (\nabla^2 f_0)^{-1} (\partial_{x_j} \nabla^2 f_0). \tag{5.11}$$

Here we recall that if e_j is the jth unit vector, then

$$|\langle \partial_{x_j} \nabla^2 f_0, t \otimes s \rangle| = |\langle \nabla^3 f_0, e_j \otimes t \otimes s \rangle| \leqq \mathcal{O}(1) |t|_\infty |s|_\infty$$

since $|e_j|_1 = 1$, and hence $\partial_{x_j} \nabla^2 f_0 : l^\infty \to l^1$ is $\mathcal{O}(1)$ in norm, uniformly w.r.t. j, N. The same thing then holds for $(\nabla^2 f_0)^{-1} \partial_{x_j} \nabla^2 f_0$ [cf. (2.42)] so (5.11) and Lemma 1.2 give:

$$\partial_{x_j} \gamma_{f_0} = \mathcal{O}(1) \quad \text{uniformly in } j, \text{N}, \tag{5.12}$$

that is:

$$|\nabla \gamma_{f_0}|_\infty = \mathcal{O}(1) \text{ in } l^\infty, \tag{5.13}$$

for all x in a complex l^∞-ball of slightly smaller radius than the radius of the ball, where f_0 is defined. (In the following we will have a slight decrease of the radius in each step, exactly as in section 3, and we shall not mention this explicitly each time.) We now choose the constant in (5.10) so that the RHS of that equation vanishes for $x=0$, and we get a solution of (5.10) which is uniquely determined by the requirement that $f_1(0)=0$. We also see that

$$|\nabla f_1|_\infty = \mathcal{O}(1). \tag{5.14}$$

Here we make a general observation. Assume $f = f_0 + f_1 h + \ldots + f_k h^k$, with $\nabla_x f_j = \mathcal{O}_j(1)$ in l^∞. Put $\gamma_f(x; h) = \log \det f''$, so that $\partial_{x_\nu} \gamma_f = \operatorname{tr} (\nabla^2 f)^{-1} \partial_{x_\nu} f''$. Then $\partial_h^m \partial_{x_\nu} \gamma_f$ is a finite linear combination of terms of the type

$$\operatorname{tr} [(f'')^{-1} (\partial_h^{m_1} f)'' (f'')^{-1} (\partial_h^{m_2} f)'' \circ \ldots \circ (f'')^{-1} (\partial_h^{m_p} f)'' (f'')^{-1} (\partial_{x_\nu} \partial_h^{m_{p+1}} f'')]$$

with $m_1 + m_2 + \ldots + m_{p+1} = m$ and with coefficients independent of N. As for γ_{f_0} it follows that

$$|\nabla_x \partial_h^m \gamma_f|_\infty = \mathcal{O}_m(1). \tag{5.15}$$

By Taylor's formula we get for every m:

$$\gamma_f(x; h) = (\gamma_f)_0(x) + (\gamma_f)_1(x) h + \ldots + (\gamma_f)_m(x) h^m + r_{f, m+1}(x; h) h^{m+1}, \tag{5.16}$$

with

$$\left.\begin{aligned} &\nabla_x (\gamma_f)_j = \mathcal{O}_j(1), \\ \text{and}& \\ &\nabla_x (r_{f,\,m+1}) = \mathcal{O}_m(1) \quad \text{in } l^\infty\text{-norm.}\end{aligned}\right\} \tag{5.17}$$

Assume by induction that we have already constructed $f_0, f_1, \ldots, f_k$ with

$$\nabla_x f_j = \mathcal{O}_j(1) \quad \text{in } l^\infty, \qquad \text{for } j=1, 2, \ldots, k., \tag{5.18}$$

and with f_0 and f_1 as above, so that if $f^{(k)} = f_0 + f_1 h + \ldots f_k h^k$, then:

$$\frac{1}{2}(\nabla f^{(k)})^2 - \varphi - h\gamma_{f^{(k)}}/2 = \text{Const.} + \mathcal{O}_{k,\,\mathrm{N}}(h^{k+1}). \tag{5.19}$$

According to the estimates just made the right hand side of (5.19) is of the form

$$\sim \mathrm{C}_0 + \mathrm{C}_1 h + \ldots + \mathrm{C}_k h^k + g_{k,\,k+1}(x) h^{k+1} + g_{k,\,k+2}(x) h^{k+2} + \ldots,$$

with

$$\nabla g_{k,\,j}(x) = \mathcal{O}_j(1) \quad \text{in } l^\infty. \tag{5.20}$$

Let f_{k+1} be the solution to

$$\nabla f_0 \,.\, \partial_x f_{k+1} = -g_{k,\,k+1}(x) + \text{const.}, \qquad f_{k+1}(0) = 0. \tag{5.21}$$

Then

$$\nabla f_{k+1} = \mathcal{O}_{k+1}(1) \quad \text{in } l^\infty.$$

Put

$$f^{(k+1)} = f_0 + f_1(x) h + \ldots + f_k h^k + f_{k+1} h^{k+1}.$$

Then $\gamma_{f^{(k)}} - \gamma_{f^{(k+1)}} = \mathcal{O}_{k,\,\mathrm{N}}(h^{k+1})$ so it is clear that

$$\frac{1}{2}(\nabla f^{(k+1)})^2 - \varphi - h\gamma_{f^{(k+1)}}/2 = \text{const.} + \mathcal{O}_{k+1,\,\mathrm{N}}(h^{k+2}). \tag{5.22}$$

By iteration we get an infinite sequence with the properties above. If we use (5.16) and (5.17), we see that (5.19) can be made more precise:

$$\frac{1}{2}(\nabla f^{(k)})^2 - \varphi - h\gamma_{f^{(k)}}/2 = \text{Const.} + r_{k+1}(x;\, h) h^{k+1}, \tag{5.23}$$

where

$$\nabla r_{k+1} = \mathcal{O}_{k+1}(1) \text{ in } l^\infty. \tag{5.24}$$

We now return to the problem in the beginning of this section and take $f = f^{(k)}$ for some sufficiently large k to be chosen later. Since $\nabla^2 f(x) - \nabla^2 f(0)$ is small in $\mathcal{L}(l^\infty, l^\infty)$ when x belongs to a real small l^∞-ball, centered at 0, and since $\nabla^2 f(0)$ has a uniformly bounded inverse in $\mathcal{L}(l^\infty, l^\infty)$, it is clear that the map $x \mapsto \nabla f(x)$ is a diffeomorphism from a small l^∞-ball centered at 0 onto a domain, Ω in $\mathbb{R}^N$ which can be sandwiched between two other such small balls. Moreover, the inverse map has a differential which is $\mathcal{O}(1)$ in $\mathscr{L}(l^\infty, l^\infty)$. The operator (5.3), (5.4) takes the form:

$$h\,\partial_{y_j}+y_j+h^{k+1}\,r_{j,k}(y;\,h), \tag{5.25}$$

with

$$r_{j,k}=\mathcal{O}_k(1), \qquad j=1,2,\ldots,\mathrm{N}. \tag{5.26}$$

We now use that the first eigenvalue of $\sum_1^{\mathrm{N}}(h\,\partial_{y_j}+y_j)^*(h\,\partial_{y_j}+y_j)$ acting in L^2 is simple $=0$ with the eigenfunction $u_0=\mathrm{C}(h,\mathrm{N})\,e^{-y^2/2h}$ and that the second eigenvalue is $2h$. If $u\in\mathrm{C}_0^\infty(\mathbb{R}^N)$, we then have:

$$2h\,\|u\|^2\leqq\sum_1^{\mathrm{N}}\|(h\,\partial_{y_j}+y_j)\,u\|^2+2h\,|(u\,|\,u_0)|^2, \tag{5.27}$$

where the norms are the L^2-ones if nothing else is specified. Using (5.25), (5.26), we get:

$$2h\,\|u\|^2\leqq\sum_1^{\mathrm{N}}[(1+\varepsilon^2)\,\|(h\,\partial_{y_j}+y_j+h^{k+1}\,r_{j,k}(y;\,h))\,u\|^2 \\ +(1+\varepsilon^{-2})\,\mathrm{C}_k\,h^{2(k+1)}\,\|u\|^2]+2h\,|(u\,|\,u_0)|^2, \tag{5.28}$$

for $u\in\mathrm{C}_0^\infty(\Omega)$, where $\varepsilon>0$ is arbitrary.

We now introduce our fundamental assumption:

(A) There exists $\mathrm{N}_0\in\mathbb{N}$ such that $\mathrm{N}=\mathcal{O}(h^{-\mathrm{N}_0})$.

Let $\mathrm{M}\in\mathbb{N}$. If we choose k and ε suitably, we get from (5.28):

$$2h(1-\mathrm{C_M}\,h^{\mathrm{M}})\,\|u\|^2\leqq\sum_1^{\mathrm{N}}\|(h\,\partial_{y_j}+y_j+h^{k+1}\,r_{j,k}(y;\,h))\,u\|^2 \\ +2h(1+\mathrm{C_M}\,h^{\mathrm{M}})\,|(u\,|\,u_0)|^2, \tag{5.29}$$

$u\in\mathrm{C}_0^\infty(\Omega)$. We can write this as:

$$2h(1-\mathrm{C_M}\,h^{\mathrm{M}})\,\|\mathrm{J}^{-1/2}\,u\|^2\leqq\sum_1^{\mathrm{N}}\|\mathrm{J}^{-1/2}\,(h\,\partial_{y_j}+\partial_{y_j}(\varphi))\,u\|^2 \\ +2h(1+\mathrm{C_M}\,h^{\mathrm{M}})\,|(\mathrm{J}^{-1/2}\,u\,|\,u_0)|^2, \tag{5.30}$$

where we still use the norm of $\mathrm{L}^2(dy)$. Going back to the x-variables, we get:

$$2h(1-\mathrm{C_M}\,h^{\mathrm{M}})\,\|u\|^2 \\ \leqq\sum_1^{\mathrm{N}}\|(h\,\partial_{y_j}+\partial_{y_j}(\varphi))\,u\|^2+2h(1+\mathrm{C_M}\,h^{\mathrm{M}})\,|(u\,|\,\mathrm{J}^{1/2}\,u_0)|^2, \tag{5.31}$$

now for u in C_0^∞ with support in a small l^∞ ball centered at 0 and with the $L^2(dx)$-norms. The L^2-norm of $J^{1/2}u_0$ over such a ball is $1+\mathcal{O}(e^{-1/Ch})$, and we have:

$$J^{1/2}u_0 = C(N;h)\exp\left[-\frac{1}{2}\sum_1^N y_j^2 + h\frac{1}{2}\gamma_{f^{(k)}}\right]/h$$
$$= C(N;h)\exp h^{-1}[-\varphi - \text{Const.}(N;h) - r_{k+1}(x;h)h^{k+1}]$$
$$= \hat{C}(N;h)\exp h^{-1}[-\varphi - r_{k+1}(x;h)h^{k+1}]$$
$$= \hat{C}(N;h)(1+\mathcal{O}(Nh^k))e^{-\varphi/h}. \quad (5.32)$$

We could also L^2-normalize $e^{-\varphi/h}$ directly over a small real l^∞-ball $B(0,\varepsilon)$ and the corresponding normalization factor would be unique up to a factor $1+\mathcal{O}(h^\infty)$, if we consider variations of the ball. Let

$$e_0 = \tilde{C}(N;h)e^{-\varphi/h} \quad (5.33)$$

be such a normalized function. It is then clear that

$$\tilde{C}(N;h) = \hat{C}(N;h)(1+\mathcal{O}(Nh^k)),$$

and

$$J^{1/2}u_0 - e_0 = \mathcal{O}(Nh^k) \quad \text{in } L^2(B(0,\varepsilon)). \quad (5.34)$$

From now on we fix a small ε and take u with support in $B(0,\varepsilon)$. Assuming k sufficiently large (depending on M) and using the assumption (A), we get

$$2h(1-C_M h^M)\|u\|^2$$
$$\leq \sum_1^N \|(h\partial_{y_j} + \partial_{y_j}(\varphi))u\|^2 + 2h(1+C_M h^M)|(u|e_0)|^2, \quad (5.35)$$

with $L^2(dx)$ norms. It only remains to replace the operators $h\partial_{y_j} + \partial_{y_j}(\varphi)$ by $h\partial_{x_j} + \partial_{x_j}(\varphi)$. We have

$$h\partial_{x_j} + \partial_{x_j}(\varphi) = \sum_k (\partial_{x_j}\partial_{x_k} f(x))(h\partial_{y_k} + \partial_{y_k}(\varphi))$$

or shorter: $(h\partial_x + \partial_x(\varphi))u = f''(x)(h\partial_y + \partial_y(\varphi))u$. Here $f''(x)$ is bounded and has a bounded inverse in $\mathcal{L}(l^2, l^2)$. Hence:

$$\sum_1^N |(h\partial_{x_j} + \partial_{x_j}(\varphi))u(x)|^2$$
$$= |(h\partial_x + \partial_x(\varphi))u(x)|_2^2 = |f''(x)(h\partial_y + \partial_y(\varphi))u(x)|_2^2$$
$$\geq \lambda_{\min}(x)^2 |(h\partial_y + \partial_y(\varphi))u|_2^2, \quad (5.36)$$

where $\lambda_{\min}(x)$ is the smallest eigenvalue of $f''(x)$. As we have already observed:

$$\lambda_{\min}(x) \geq \lambda_0 > 0, \quad (5.37)$$

where λ_0 is independent of the dimension, and if we combine this with (5.35) and (5.36), we get:

$$\sum_1^N \| (h\partial_{x_j} + \partial_{x_j}(\varphi)) u \|^2 \geqq \lambda_0^2 \sum_1^N \| (h\partial_{y_j} + \partial_{y_j}(\varphi)) u \|^2$$
$$\geqq 2h\lambda_0^2 h (1 - C_M h^M) \| u \|^2 - 2h\lambda_0^2 (1 + C_M h^M) | (u | e_0) |^2,$$

or in other words:

$$2\lambda_0^2 h (1 - C_M h^M) \| u \|^2 \leqq \sum_1^N \| (h\partial_{x_j} + \partial_{x_j}(\varphi)) u \|^2 + 2\lambda_0^2 h (1 + C_M h^M) | (u | e_0) |^2, \quad (5.38)$$

for $u \in C_0^\infty(B(0, \varepsilon))$ with $\varepsilon > 0$ small enough.

Using that φ is an approximate solution of (3.2) [*cf.* (4.7)]:

$$V(x) - \frac{1}{2}(\nabla\varphi(x))^2 + h\left(\frac{1}{2}\Delta\varphi(x) - E\right) = \mathcal{O}(h^M), \quad (5.39)$$

for all M [where we also use the assumption (A)], we get by a simple computation:

$$\frac{1}{2}\sum_1^N (h\partial_{x_j} + \partial_{x_j}(\varphi))^* \circ (h\partial_{x_j} + \partial_{x_j}(\varphi))$$
$$= -\frac{1}{2}h^2\Delta + V(x) - hE(h) + \mathcal{O}(h^M). \quad (5.40)$$

Using this in (5.36) gives after an integration by parts:

$$2\lambda_0^2 h (1 - C_M h^M) \| u \|^2 \leqq \left(\left(-\frac{1}{2}h^2\Delta + V - hE\right) u \,\middle|\, u\right) + 2\lambda_0^2 h (1 + C_M h^M) (u | e_0) |^2, \quad (5.41)$$

for all $u \in C_0^\infty(B(0, \varepsilon))$. Here ε is assumed to be sufficiently small. Later we shall use (5.41) with $M = 2$.

(5.41) will be our basic estimate near a local minimum, of the potential, but in order to obtain global estimates in the next section it will also be necessary to extend (5.41) to the case when we put certain exponential weights. For M arbitrarily large, we observe that

$$\tilde{V} = V_M = V - \sum x_j^{2M} \quad (5.42)$$

satisfies the same general assumptions as V. Let $\tilde{\varphi}$, $h\tilde{E}$ be the corresponding phase and eigenvalue respectively.

LEMMA 5.1. — *We have the estimates:*

$$\varphi - \tilde{\varphi} = \mathcal{O}(1)(|x|_{2M}^{2M} + h|x|_{2(M-1)}^{2(M-1)} + \ldots + h^{M-1}|x|_2^2 + h^M N) \quad (5.43)$$
$$E - \tilde{E} = \mathcal{O}(1) N h^{M-1}. \quad (5.44)$$

Proof. – We assume that M is fixed and put $V_t = V - t\sum_1^N x_j^{2M}$, and let $\varphi_t \sim \varphi_{0,t} + h\varphi_{1,t} + \dots$ and $E_t \sim E_{0,t} + hE_{1,t} + \dots$ be the corresponding quantities. The characteristic equation for $\varphi_{0,t}$ is $V_y = \frac{1}{2}(\nabla\varphi_{0,t})^2$ and if we take the derivative w.r.t. t we get: $(\nabla\varphi_{0,t})\cdot\partial_x(\partial_t\varphi_{0,t}) = \partial_t V_t$. Hence: $\partial_t\varphi_{0,t} = \mathcal{O}_N(|x|^{2M})$ (where for the moment we do not specify which norm on $\mathbb{C}^N$ to use). We then look at (T1):

$$(\nabla_x\varphi_{0,t})\cdot\partial_x(\varphi_{1,t}) = \frac{1}{2}\Delta\varphi_{0,t} - E_{0,t}.$$

Taking the t-derivative we get:

$$\begin{aligned}(\nabla_x\varphi_{0,t})\cdot\partial_x(\partial_t\varphi_{1,t}) &= \frac{1}{2}\Delta\partial_t\varphi_{0,t} - (\nabla_x\partial_t\varphi_{0,t})\cdot\partial_x\varphi_{1,t} - \partial_t E_{0,t} \\ &= \mathcal{O}(|x|^{2M-2} + |x|^{2M-1}) - \partial_t E_{0,t}.\end{aligned}$$

If $M \geqq 2$, we must have $\partial_t E_{0,t} = 0$, $\partial_t\varphi_{1,t} = \mathcal{O}_N(|x|^{2(M-1)})$.

We now introduce the inductive assumption

$$(I_m)\quad \begin{cases}\partial_t\varphi_{k,t} = \mathcal{O}_N(|x|^{2(M-k)}), & 0 \leqq k \leqq m, \\ \partial_t E_{k,t} = 0, & 0 \leqq k \leqq m-1,\end{cases}$$

for $m < M$. We have proved I_0 and in the case when $M \geqq 2$ we also have I_1.

Assume now that $2 \leqq m < M$ and that we have shown I_{m-1}. Write (Tm):

$$\begin{aligned}\nabla\varphi_{0,t}\cdot\partial_x\varphi_{m,t} = \frac{1}{2}\Delta\varphi_{m-1,t} - \frac{1}{2}(&\nabla\varphi_{1,t}\cdot\nabla\varphi_{m-1,t} \\ &+ \nabla\varphi_{2,t}\cdot\nabla\varphi_{m-2,t} + \dots + \nabla\varphi_{m-1,t}\cdot\nabla\varphi_{1,t}) - E_{m-1,t}.\end{aligned}$$

We differentiate this w.r.t. t and see that

$$\nabla\varphi_{0,t}\cdot\partial_x\partial_t\varphi_{m,t} = \mathcal{O}_N(|x|^{2(M-m)}) - \partial_t E_{m-1,t}.$$

We then see that $\partial_t E_{m-1,t} = 0$ and that $\partial_t\varphi_{m,t} = \mathcal{O}_N(|x|^{2(M-m)})$. In other words, we have shown I_m. It is then clear that I_m holds for all $m < M$, and we have then shown that

$$\left.\begin{aligned}\varphi_0 - \tilde\varphi_0 = \mathcal{O}_N(|x|^{2M}), \quad \varphi_1 - \tilde\varphi_1 = \mathcal{O}_N(|x|^{2M-1}), \quad \dots, \\ \varphi_{M-1} - \tilde\varphi_{M-1} = \mathcal{O}_N(|x|^2),\end{aligned}\right\} \tag{5.45}$$

$$(E_0 - \tilde E_0) = \dots = (E_{M-2} - \tilde E_{M-2}) = 0. \tag{5.46}$$

Since $E_j, \tilde E_j = \mathcal{O}_j(N)$, we conclude that $E - \tilde E = \mathcal{O}(N h^{M-1})$. We also recall that $|\nabla_x\varphi_j|_\infty = \mathcal{O}_j(1)$ in a complex l^∞-ball, and as we have seen, this implies that

$$|\langle \nabla_x^2 \varphi_j(x), t_1 \otimes t_2 \rangle| = \mathcal{O}_j(1) |t_1|_{p_1} |t_2|_{p_2}$$

if $1 \leqq p_j \leqq \infty$, $p_1^{-1} + p_2^{-2} = 1$, for all x in a concentric l^∞-ball with a slightly smaller radius. More generally, the Cauchy inequalities imply that for x in a slightly decreased l^∞-ball:

$$|\langle \nabla_x^k \varphi_j(x), t_1 \otimes t_2 \otimes \ldots \otimes t_k \rangle| = \mathcal{O}_j(1) |t_1|_1 |t_2|_\infty \cdot \ \ldots \ \cdot |t_k|_\infty,$$

and by symmetry, we may permute the indices $1, 2, \ldots, k$ in the right hand side. By complex interpolation we get the more general estimate:

$$|\langle \nabla_x^k \varphi_j(x), t_1 \otimes t_2 \otimes \ldots \otimes t_k \rangle| = \mathcal{O}_j(1) |t_1|_{p_1} |t_2|_{p_2} \cdot \ \ldots \ \cdot |t_k|_{p_k},$$

with $1 \leqq p_k \leqq \infty$, $p_1^{-1} + p_2^{-1} + \ldots + p_k^{-1} = 1$, and with the constant $\mathcal{O}_j(1)$ independent of the choice of $p_1, p_2, \ldots, p_k$. These estimates are equally valid for $\tilde{\varphi}_j$ so (5.45) and Taylor's formula give:

$$\varphi_j - \tilde{\varphi}_j = \mathcal{O}_j(1) |x|_{2\,(\mathrm{M}-j)}^{2\,(\mathrm{M}-j)}, \qquad 0 \leqq j \leqq \mathrm{M} - 1. \tag{5.47}$$

From this we get the estimate on φ-$\tilde{\varphi}$ in the lemma. □

We shall use the lemma in order to compare $e^{-\varphi/h}$ and $e^{-\tilde{\varphi}/h}$ in $\mathrm{L}^2(\mathrm{B}(0, \varepsilon))$ where $\mathrm{B}(0, \varepsilon)$ denotes the real l^∞-ball. Put $\varphi_t = (1-t)\varphi + t\tilde{\varphi}$ and consider

$$g(t) = \| e^{-\varphi_t/h} \|^2, \tag{5.48}$$

where $\| \cdot \|$ is the L^2-norm over $\mathrm{B}(0, \varepsilon)$. Let x_t denote the critical point of φ_t, and notice that we have $|x_t|_\infty = \mathcal{O}(h)$. From this we see that we can replace x by $x - x_t$ in the RHS of (5.43). Then [with all integrals taken over $\mathrm{B}(0, \varepsilon)$ unless otherwise is specified]:

$$g'(t) = \int 2(\varphi - \tilde{\varphi}) h^{-1} e^{-2\varphi_t/h} dx = \mathcal{O}(1) h^{(\mathrm{M}-1)\mathrm{N}} \int e^{-2\varphi_t/h} dx + \sum_0^{\mathrm{M}-1} \int h^{j-1} |x - x_t|_{2\,(\mathrm{M}-j)}^{2\,(\mathrm{M}-j)} e^{-2\varphi_t/h} dx. \tag{5.49}$$

Here we use that $|x - x_t|_{2p} \sim |\nabla \varphi_t|_{2p}$ and get

$$\int |x - x_t|_{2p}^{2p} e^{2\varphi_t/h} dx \leqq \mathrm{C}_p \int |\nabla \varphi_t|_{2p}^{2p} e^{-2\varphi_t/h} dx = \mathrm{C}_p \sum_1^{\mathrm{N}} \int (\partial_{x_j} \varphi_t)^{2p} e^{-2\varphi_t/h} dx.$$

As in section 4, we see that

$$\int_{-\varepsilon}^{\varepsilon} (\partial_{x_j} \varphi_t)^{2p} e^{-2\varphi_t/h} dx_j = \mathcal{O}(1) h^p \int_{-\varepsilon}^{\varepsilon} e^{-2\varphi_t/h} dx_j,$$

so by Fubini's theorem, we get:

$$\int |\nabla \varphi_t|_{2p}^{2p} e^{-2\varphi_t/h} dx \leqq \mathrm{C N} h^p \int e^{2\varphi_t/h} dx.$$

Using this in (5.49), we get

$$g'(t) = \mathcal{O}(\mathrm{N} h^{\mathrm{M}-1}) g(t). \tag{5.50}$$

Let N_0 be as in the assumption (A). Then

$$g'(t) = \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1}) g(t), \tag{5.51}$$

and if we choose $\mathrm{M} > \mathrm{N}_0 + 1$, we conclude that $g(t)$ is of constant order of magnitude:

$$g(t)/g(s) \leqq \mathrm{C}, \qquad 0 \leqq t, s \leqq 1, \tag{5.52}$$

with C independent of h, t, s. Then (5.51) implies that

$$g(t) = (1 + \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1})) g(s). \tag{5.53}$$

Put $u_t = e^{-\varphi_t/h}$ so that

$$\| u_t \| = (1 + \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1})) \| u_s \|. \tag{5.54}$$

Then

$$|\partial_t (\| u_t - u_0 \|^2)| = 2 |(u_t - u_0 | \partial_t u_t)| \leqq 2 |(u_t | \partial_t u_t)| + 2 |(u_0 | \partial_t u_t)|. \tag{5.55}$$

In the same way as above, we get:

$$\begin{aligned}\partial_t (\| u_t - u_0 \|^2) &= \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1}) (\| u_t \|^2 + (u_0 | u_t)) \\ &= \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1}) \| u_t \|^2 = \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1}) \| u_0 \|^2.\end{aligned} \tag{5.56}$$

Integrating this relation, we get

$$\| u_1 - u_0 \|^2 = \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1}) \| u_0 \|^2. \tag{5.57}$$

We can recapitulate our estimates in:

LEMMA 5.2. – *Choose first* $\mathrm{M} > 0$ *sufficiently large and fixed. Let then* $\varepsilon > 0$ *be sufficiently small and fixed, and take all* L^2*-norms over the real* l^∞*-ball* $\mathrm{B}(0, \varepsilon)$. *Then:*

$$\| e^{-\tilde{\varphi}/h} - e^{-\varphi/h} \| = \mathcal{O}(h^{(\mathrm{M}-\mathrm{N}_0-1)/2}) \| e^{-\varphi/h} \| \tag{5.58}$$

$$\| e^{-\tilde{\varphi}/h} \| = (1 + \mathcal{O}(h^{\mathrm{M}-\mathrm{N}_0-1})) \| e^{-\varphi/h} \|. \tag{5.59}$$

If we let $e_0 = u_0 / \| u_0 \|$, $\tilde{e}_0 = u_1 / \| u_1 \|$, it follows that

$$\| e_0 - \tilde{e}_0 \| = \mathcal{O}(h^{(\mathrm{M}-\mathrm{N}_0-1)/2}). \tag{5.60}$$

We rewrite (5.41) in the form:

$$\begin{aligned}&(h\mathrm{E} + 2\lambda_0^2 h - \mathcal{O}(h^{\tilde{\mathrm{M}}})) \| u \|^2 \\ &\qquad \leqq \left(\left(-\frac{1}{2} h^2 \Delta + \mathrm{V}\right) u \,\middle|\, u\right) + 2\lambda_0^2 h |(u | e_0)|^2\end{aligned} \tag{5.61}$$

for $u \in C_0^\infty(B(0, \varepsilon))$ for some sufficiently small $\varepsilon > 0$ which may depend on $\tilde{M}$. In this inequality, we may replace λ_0 by any smaller positive constant. The corresponding inequality holds after the substitution: $(V, e_0, E, \lambda_0) \mapsto (\tilde{V}, \tilde{e}_0, \tilde{E}, \tilde{\lambda}_0)$, and it is also easy to see that we can take $\tilde{\lambda}_0 \geqq (1 - o(1)) \lambda_0$ when $\varepsilon \to 0$. Let λ_0 from now on denote the infimum of the earlier λ_0 and of $\tilde{\lambda}_0$. Also choose $\tilde{M} = 2$ from now on. We then have

$$(h E + 2 \lambda_0^2 h - \mathcal{O}(h^2)) \| u \|^2 \leqq \left(\left(-\frac{1}{2} h^2 \Delta + \tilde{V} \right) u \,|\, u \right) + 2 \lambda_0^2 h \,|\, (u \,|\, e_0) \,|^2, \quad (5.62)$$

which can be viewed as a sharpening of (5.61) (in the case $\tilde{M} = 2$).

Let $\psi \in C^\infty$ be realvalued and defined in a neighborhood of $\overline{B(0, \varepsilon)}$. Since:

$$e^{\psi/h} \circ \left(-\frac{1}{2} u^2 \Delta \right) \circ e^{-\psi/h} = -\frac{1}{2} ((h \partial_x)^2 - (h \partial_x \circ \psi'_x + \psi'_x \circ h \partial_x) + (\psi'_x)^2),$$

and the middle term of the RHS is formally anti-selfadjoint, we get for $u \in C_0^\infty(B(0, \varepsilon))$:

$$\operatorname{Re} \left(e^{\psi/h} \left(-\frac{1}{2} h^2 \Delta + V \right) e^{-\psi/h} u \,|\, u \right) = \left(\left(-\frac{1}{2} h^2 \Delta + V - \frac{1}{2} (\psi'_x)^2 \right) u \,|\, u \right). \quad (5.63)$$

(As in [HS1] we may observe that this identity remains valid in the case when ψ is merely Lipschitz continuous.) If we now assume that

$$V - \frac{1}{2} (\psi'_x)^2 \geqq \tilde{V}, \quad (5.64)$$

or in other words that $\frac{1}{2} (\psi'_x)^2 \leqq \sum_1^N x_j^{2M}$, where M now is fixed and sufficiently large, then (5.62) and (5.63) imply that

$$(h E + 2 \lambda_0^2 h - \mathcal{O}(h^2)) \| u \|^2 \leqq \operatorname{Re} \left(e^{\psi/h} \left(-\frac{1}{2} h^2 \Delta + V \right) e^{-\psi/h} u \,|\, u \right) + 2 \lambda_0^2 h \,|\, (u \,|\, e_0) \,|^2. \quad (5.65)$$

Put $v = e^{-\psi/h} u$. We then get for $z \in \mathbb{C}$:

$$(h E + 2 \lambda_0^2 h - \operatorname{Re} z - \mathcal{O}(h^2)) \| e^{\psi/h} v \|^2 \leqq \operatorname{Re} \left(e^{\psi/h} \left(-\frac{1}{2} h^2 \Delta + V - z \right) v \,|\, e^{\psi/h} v \right) + 2 \lambda_0^2 h \,|\, (e^{\psi/h} v \,|\, e_0) \,|^2. \quad (5.66)$$

6. A GLOBAL PROBLEM WITH SEVERAL POTENTIAL WELLS

We shall now consider a Schrödinger operator on $\mathbb{R}^{nN}$. Everything would work equally well on domains of the form $I_1 \times I_2 \times \ldots \times I_N$, where I_j could be either a large torus, or a suitable subset of $\mathbb{R}^n$. The unperturbed potential will be of the form

$$V_0(x) = \sum_1^N v_j(x_j), \tag{6.1}$$

where $v_j \in C^\infty(\mathbb{R}^n; [0, +\infty[)$ has the following properties:

(B) – The set of points where $v_j(x)=0$ is discrete and non-empty. We denote this set by $\{x_{j,k};\ k \in \mathscr{A}_j\}$. It may be finite or infinite.

– There exists a constant $C>0$ such that for every k, v_j has a holomorphic extension to $\{x_j \in \mathbb{C}^n;\ |x_j - x_{j,k}| < 1/C\}$, satisfying in this set: $|\nabla v_j(x_j)|_\infty \leqq C$. Moreover $\nabla^2 v_j(x_{j,k}) \geqq 1/C$ in the sense of symmetric matrices, and for real x_j in the same set, we have $v_j(x_j) \geqq C^{-1}|x_j - x_{j,k}|^2$.

We will also need an assumption about the behaviour far away from the union of the $x_{j,k}$:

(C) With C as in assumption (B), we assume that there exists a constant $\tilde{C}$ such that for $x_j \in \mathbb{R}^n \setminus \bigcup_k B(x_{j,k}, 1/C)$, we have:

$$v_j(x_j) \geqq \tilde{C}^{-1} \inf_k |x_j - x_{j,k}|.$$

Notice that the statement in (C) remains valid if we increase "C" in (B), possibly after increasing "$\tilde{C}$".

We shall write $x_\alpha = (x_{1,\alpha_1}, x_{2,\alpha_2}, \ldots, x_{N,\alpha_N})$,where α varies in the set $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_N$.

The perturbed potential V is supposed to be of the form $V = V_0(x) + W(x)$, where W is realvalued and smooth and satisfies:

(D) W has a holomorphic extension to $\bigcup_{\alpha \in \mathscr{A}} B(x_\alpha, 1/C)$ (complex l^∞-balls), and satisfies $|\nabla W(x)|_\infty \leqq \delta$, for all x in $\mathbb{R}^{nN} \cup (\bigcup_{\alpha \in \mathscr{A}} B(x_\alpha, 1/C))$.

Here $\delta > 0$ is the basic perturbation parameter and our main result concerning the bottom of the spectrum of our Schrödinger operator will be valid, provided that δ is small enough, depending on the constants appearing in (B), (C) but independently of N and h.

Our analysis of the situation with one potential well, shows that V has local minimum points, $\tilde{x}_\alpha$, $\alpha \in \mathscr{A}$ with the property that $|\tilde{x}_\alpha - x_\alpha|_\infty = \mathcal{O}(\delta)$. The following assumption will be satisfied in the cases when $W \geqq 0$ and when $\mathscr{A}$ is finite.

(E) $$\inf_{\alpha \in \mathscr{A}} V(\tilde{x}_\alpha) > -\infty.$$

We shall cover $\mathbb{R}^{nN}$ by boxes of the type $B_c = B_{c_1} \times B_{c_2} \times \ldots \times B_{c_N}$, where $c = (c_1, \ldots, c_N) \in \mathbb{R}^{nN}$ is the center of the box. Here we fix some large constant $C > 0$, and we let $\varepsilon > 0$ be sufficiently small and fixed. Then c_j is either equal to $\tilde{x}_{j,k}$ for some $k \in \mathscr{A}_k$ or $\inf_{k \in \mathscr{A}} |c_j - \tilde{x}_{j,k}| \geqq (C-1)\varepsilon$. We require that $B_{c_j} = B(c_j, C\varepsilon)$ in the first case and that B_{c_j} is equal to $B(c_j, \varepsilon)$ in the second case. Here $B(c, r)$ will denote the real l^∞-ball of center c and of radius r, either in $\mathbb{R}^n$ or in $\mathbb{R}^{nN}$, depending on the context.

If c is of the form $\tilde{x}_\alpha$ for some $\alpha \in \mathscr{A}$, then we shall apply the estimate (5.66) for v with support in B_c. If c is not of the form $\tilde{x}_\alpha$ for any $\alpha \in \mathscr{A}$, then we let $\alpha(c) \in \mathscr{A}$ be an index for which $|c - \tilde{x}_\alpha|_1$ is minimal, and we shall compare $V(x)$ with $V(x - (c - \tilde{x}_\alpha))$, $x \in B_c$, when $\alpha = \alpha(c)$. Assuming that $\delta > 0$ is sufficiently small depending on ε, we then have with a constant $\tilde{C}$, which is independent of δ:

$$V_0(x) \geqq V_0(x - (c - \tilde{x}_\alpha)) + \tilde{C}^{-1} |c - \tilde{x}_\alpha|_1, \qquad x \in B_c. \tag{6.1}$$

Also notice that $x \in B_c \Rightarrow x - (c - \tilde{x}_\alpha) \in B_{\tilde{x}_\alpha}$. (It is at this point, that we need to introduce the constant C in the definition of the boxes: just think of the case of a quadratic potential in one variable.) If we use the asumption (D) we get from (6.1), that for $\delta > 0$ sufficiently small, we have with a new constant $\tilde{C}$:

$$V(x) \geqq V(x - (c - \tilde{x}_\alpha)) + \tilde{C}^{-1} |c - \tilde{x}_\alpha|_1, \qquad x \in B_c. \tag{6.2}$$

Let us now recapitulate (5.66) in the case when $c = \tilde{x}_\alpha$ for some $\alpha \in \mathscr{A}$: Let $M > 0$ be fixed as prior to (5.65) and assume that ψ is a real-valued smooth function satisfying $\frac{1}{2}(\psi')^2 \leqq |x - c|_{2M}^{2M}$. Then for every $u \in C_0^\infty(B_c)$ and every $z \in \mathbb{C}$, we have

$$(\mu_c + 2\lambda_0^2 h - \operatorname{Re} z - \mathcal{O}(h^2)) \| e^{\psi/h} u \|^2 \leqq \operatorname{Re}\left(e^{\psi/h} \left(\frac{1}{2} h^2 \Delta + V - z \right) u \,|\, e^{\psi/h} u \right) + 2\lambda_0^2 h |(e^{\psi/h} u \,|\, e_c)|^2, \tag{6.3}$$

where we have put $\mu_c = V(c) + h E_c$, provided that we also assume (A) of section 5, as we shall do from now on.

Here we have to introduce the value $V(c)$ since we do not want to assume that this value vanishes. Moreover $h E_c$ is the asymptotic eigenvalue associated to c obtained as in section 3, and e_c is the corresponding asymptotic normalized eigenfunction. Finally λ_0 now denotes the infimum of the various λ_0's that we get in (5.53) for the various c's. If $\operatorname{Re} z \leqq \mu_c + 2\lambda_0^2 h - h/C$ for some arbitrary but fixed C, then there is a new constant C such that

$$C^{-1} h \| e^{\psi/h} u \|^2 \leqq \operatorname{Re}\left(e^{\psi/h} \left(-\frac{1}{2} h^2 \Delta + V - z \right) u \,|\, e^{\psi/h} u \right) + Ch |(u \,|\, e^{\psi/h} e_c)|^2. \tag{6.4}$$

Later, we shall choose ψ to be constant in some small l^∞-ball of some fixed radius, centered at c. In that case, we may replace $e^{\psi/h} e_c$, by $e^{\psi(c)/h} e_c$. Actually, since we are not trying to obtain global weigthed L^2-estimates in this paper, we shall choose $\psi(c)=0$, in which case the last term in (6.4) can be replaced by: $\mathrm{C}h|(u|e_c)|^2$, If $\mathrm{Re}\, z \leqq \mu_c - h/\mathrm{C}$, for an arbitrary, but fixed C, then there is a new constant C such that:

$$\mathrm{C}^{-1} h \| e^{\psi/h} u \|^2 \leqq \mathrm{Re}\left(e^{\psi/h}\left(-\frac{1}{2} h^2 \Delta + \mathrm{V} - z \right) u \,|\, e^{\psi/h} u \right). \tag{6.5}$$

From now on we restrict z to the half plane

$$\left.\begin{aligned} &\mathrm{Re}\, z \leqq \mu_{\inf} + 2\lambda_0^2 h - h/\mathrm{C}_0, \\ &\text{where} \\ &\qquad \mu_{\inf} = \inf_{\alpha \in \mathscr{A}} \mu_{\tilde{x}_\alpha}, \end{aligned}\right\} \tag{6.6}$$

for some fixed $\mathrm{C}_0 > 0$. (For simplicity, we shall sometimes write μ_α instead of $\mu_{\tilde{x}_\alpha}$.) We next consider the case, when c is not of the form $\tilde{x}_\alpha$ for any α, and let $c_0 = \tilde{x}_{\alpha(c)}$ be a point which minimizes $\alpha \mapsto |c - \tilde{x}_\alpha|_1$. Then according to (6.2), we have for $x \in \mathrm{B}_c : \mathrm{V}(x) \geqq \mathrm{V}(x - (c - c_0)) + 2\lambda_0^2 h$, provided that $h > 0$ is sufficiently small, and since we now have (6.5) with z replaced by $z - 2\lambda_0^2 h$, it follows that:

$$\mathrm{C}^{-1} h \| e^{\psi/h} u \|^2 \leqq \mathrm{Re}\left(e^{\psi/h}\left(\frac{1}{2} h^2 \Delta + \mathrm{V} - z \right) u \,|\, e^{\psi/h} u \right), \tag{6.7}$$

for all $u \in \mathrm{C}_0^\infty(\mathrm{B}_c)$.

The next task will be to patch together the estimates (6.4) and (6.7) and it appears that an ordinary partition of unity will not be very convenient, since it gives rise to exponentially large cutoff errors after summation. Instead we shall construct a resolution of the identity. In order to do so, we start with the case of one of the variables, say $x = x_1$, and we denote for simplicity the points $\tilde{x}_{1,k}$ simply by x_k. Let $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ be positive and have the properties:

$f = 0$ in $\mathrm{B}(0, \varepsilon/2)$, $f > 0$ outside $\mathrm{B}(0, 3\varepsilon/4)$
and is of the order of magnitude $|x|$ near infinity. (6.8)

$$|\nabla f|_\infty \leqq \tilde{\varepsilon}. \tag{6.9}$$

Here ε is the same as in our box construction, and we shall choose $\tilde{\varepsilon} > 0$ sufficiently small depending on ε. With $\tilde{\varepsilon}$ sufficiently small, we have

$$(\nabla f(x))^2 \leqq |x|^{2\mathrm{M}}, \tag{6.10}$$

where M is the number that we fixed in the end of section 5. Let $\mathrm{F} \in \mathrm{C}^\infty(\mathbb{R}^n; [0, 1])$ have its support in $\mathbb{R}^n \setminus \bigcup_k \mathrm{B}(x_k, (\mathrm{C}-1)\varepsilon)$ and be equal to 1 near $\mathbb{R}^n \setminus \bigcup_k B\left(x_k, \left(C-\frac{1}{2}\right)\varepsilon\right)$. We also assume that the gradient of F is uniformly bounded. Put $G=1-F$, so that G has its support in $\bigcup_k B\left(x_k, \left(C-\frac{1}{2}\right)\varepsilon\right)$. We define f_0 by means of:

$$e^{-f_0(x)/h}=1-\int e^{-f(x-\alpha)/h} C(h) F(\alpha)\, d\alpha=\int e^{-f(x-\alpha)/h} C(h) G(\alpha)\, d\alpha, \quad (6.11)$$

where the constant $C(h)$ is determined by the requirement that $C(h)\int e^{-f(x)/h}\, dx=1$. We shall estimate the gradient of f_0, and by differentiating (6.11) with respect to x, we first write two different formulas for this gradient, by noticing that the last member of (6.11) is a convolution:

$$\nabla f_0(x)=\int e^{-f(x-\alpha)/h}(\nabla f)(x-\alpha) G(\alpha)\, d\alpha \Big/ \int e^{-f(x-\alpha)/h} G(\alpha)\, d\alpha, \quad (6.12)$$

$$\nabla f_0(x)=-h\int e^{-f(x-\alpha)/h}(\nabla G)(\alpha)\, d\alpha \Big/ \int e^{-f(x-\alpha)/h} G(\alpha)\, d\alpha. \quad (6.13)$$

From the first formula, we see that

$$|\nabla f_0|_\infty \leqq \| |\nabla f|_\infty \|_\infty \leqq \tilde{\varepsilon}, \quad (6.14)$$

and from the second formula, we see that ∇f_0 is uniformly exponentially small (w.r.t. h) in $\bigcup_k B(x_k, (C-1)\varepsilon)$. The same fact concerning f_0 holds also, and follows directly from (6.11), which also implies that f_0 is bounded from below by a positive constant outside $\bigcup_k B\left(x_k, \left(C-\frac{1}{4}\right)\varepsilon\right)$. After modifying f_0 by an exponentially small function, we may assume that f_0 vanishes in $\bigcup_k B(x_k, (C-1)\varepsilon)$, that (6.14) still holds, and that instead of (6.11), we have:

$$e^{-f_0(x)/h}+\int e^{-f(x-\alpha)/h} C(h) F(\alpha)\, d\alpha=1+\mathcal{O}(e^{-1/Ch}). \quad (6.15)$$

Committing another exponentially small error, we may replace $e^{-f_0(x)/h}$ in (6.15) by $\sum_k e^{-f_k(x)/h}$, where f_k is equal to f_0 in $B(x_k, 1/C)$, and where the properties of f_k outside this ball are uninteresting since we are also going to throw in cutoff functions. Put $f(x, \alpha; h)=f_k(x)$ when $\alpha=x_k$, and when α is different from all the x_k's, put $f(x, \alpha; h)=f(x-\alpha)$. Let $\mu(d\alpha)=\sum_k \delta(\alpha-x_k)+C(h) F(\alpha)\, d\alpha$, so that we can write the just modified form of (6.15) as

$$\int e^{-f(x,\alpha;h)/h}\mu(d\alpha)=1+\mathcal{O}(e^{-1/\mathrm{C}h}). \tag{6.16}$$

Let $\chi(x,\alpha)=\tilde{\chi}(x-\alpha)$ when α is of the form x_k for some k, and let $\chi(x,\alpha)=\hat{\chi}(x-\alpha)$ otherwise. Here we choose $\tilde{\chi}\in\mathrm{C}_0^\infty(\mathrm{B}(0,\mathrm{C}\varepsilon))$ equal to 1 on $\mathrm{B}\left(0,\left(\mathrm{C}-\frac{1}{4}\right)\varepsilon\right)$, and we let $\hat{\chi}\in\mathrm{C}_0^\infty(\mathrm{B}(0,\varepsilon))$ be equal to 1 on $\mathrm{C}_0^\infty\left(\mathrm{B}\left(0,\frac{1}{2}\varepsilon\right)\right)$. Then we still have (6.14) after introduction of $|\chi(x,\alpha)|^2$ into the integral.

We next return to the case of N copies of $\mathbb{R}^n$. Let f_j, μ_j, χ_j denote the quantities that we have constructed for the j: th copy. Put

$$\varphi(x,c;h)=\sum_j f_j(x_j,c_j;h),$$
$$\mu(dc)=\mu_1(dc_1)\otimes\ldots\otimes\mu_\mathrm{N}(dc_\mathrm{N}),$$
$$\chi(x,c)=\chi_1(x_1,c_1).\ \ldots\ .\chi_\mathrm{N}(x_\mathrm{N},c_\mathrm{N}).$$

Then if we make use of the assumption (A) in section 5, we get

$$\int e^{-\varphi(x,c;h)/h}|\chi(x,c)|^2\mu(dc)=1+\mathcal{O}(e^{-1/\mathrm{C}h}). \tag{6.17}$$

Let $u\in\mathrm{C}_0^\infty(\mathbb{R}^{n\mathrm{N}})$. We then apply (6.4)/(6.5) or (6.7), depending on wheter c is of the form $\tilde{x}_\alpha$ or not, to the function $\chi(.,c)u$, and with ψ chosen so that $2\psi=-\varphi(.,c;h)$. $\left(\text{If }\tilde{\varepsilon}\text{ is small enough this function will satisfy the assumption: }\frac{1}{2}(\psi')^2\leqq|x-c|_{2\mathrm{M}}^{2\mathrm{M}}\text{ for }x\in\mathrm{B}_c.\right)$ When c is not of the form $\tilde{x}_\alpha$ for any α, we get the estimate:

$$\begin{aligned}\mathrm{C}^{-1}h\int e^{-\varphi(x,c;h)/h}|\chi(x,c)|^2|u(x)|^2\,dx\\ \leqq 2\,\mathrm{Re}\int e^{-\varphi(x,c;h)/h}|\chi(x,c)|^2\left[\left(-\frac{1}{2}h^2\Delta+\mathrm{V}-z\right)u\right](x)\bar{u}(x)\,dx\\ +\mathrm{Re}\int e^{-\varphi(x,c;h)/h}\chi(x,c)\left(\left[-\frac{1}{2}h^2\Delta,\chi(.,c)\right]u\right)(x)\bar{u}(x).\end{aligned} \tag{6.18}$$

When c is of the form $\tilde{x}_\alpha$ for some α we have the same estimate provided that we add the term $\mathrm{C}h|(u|\chi(.,c)e_c)|^2$ to the right hand side if $\mathrm{Re}\,z>\mu_c-h/\mathrm{Const.}$ We shall then integrate these estimate with respect to $\mu(dc)$ and we start by estimating the contribution from the last term in

(6.18). Here we use that

$$\int e^{-\varphi(x, c; h)/h} |\nabla_x \chi(x, c)|_1 \mu(dc)$$

and

$$\int e^{-\varphi(x, c; h)/h} |\Delta_x \chi(x, c)| \mu(dc)$$

are both $\mathcal{O}(e^{-1/\mathrm{Ch}})$. If we also use (6.17), we get with a new constant C:

$$\begin{aligned} \mathrm{C}^{-1} h \int |u(x)|^2 dx & \\ \leqq \mathrm{Re} \int (1+\mathcal{O}(e^{-1/\mathrm{Ch}})) & \left(\left(-\frac{1}{2} h^2 \Delta + \mathrm{V} - z\right) u\right) \bar{u}\, dx \\ & + \int \mathcal{O}(e^{-1/\mathrm{Ch}}) |\nabla u| \,.\, |u|\, dx \\ & + \sum_{\alpha \in \mathscr{B}} \mathrm{Ch} |(u | \chi(., \tilde{x}_\alpha) e_{\tilde{x}_\alpha})|^2, \quad (6.19) \end{aligned}$$

where $\mathscr{B}$ is the set of β in $\mathscr{A}$ for which $\mu_\beta < \mu_{\mathrm{inf}} + 2\lambda_0^2 h - h/\mathrm{C}_0 + h/\mathrm{C}_1$ and where C_1 may be any fixed positive constant.

Clearly $\chi(., \tilde{x}_\alpha) e_{\tilde{x}_\alpha}$, has the same properties as $e_{\tilde{x}_\alpha}$, and we modify and simplify the notation, by naming this function e_α. We may assume that the e_α form and orthonormal system in $\mathrm{L}^2(\mathbb{R}^{n\mathrm{N}})$. The operator $\mathrm{R}_+ : \mathrm{L}^2(\mathbb{R}^{n\mathrm{N}}) \to l^2(\mathscr{A})$, defined by:

$$\mathrm{R}_+ u(\alpha) = (u | e_\alpha), \qquad \alpha \in \mathscr{B}, \quad (6.20)$$

is of norm 1, and the last term in (6.19) can be written $\mathrm{Ch} \| \mathrm{R}_+ u \|^2$.

Estimating the two first terms of the RHS in (6.19) by means of Cauchy-Schwarz, we get

$$\mathrm{C}^{-1} h \| u \| \leqq \left\| \left(-\frac{1}{2} h^2 \Delta + \mathrm{V} - z\right) u \right\| + \mathcal{O}(e^{-1/\mathrm{Ch}}) \| \nabla u \| + \mathrm{Ch} \| \mathrm{R}_+ u \|. \quad (6.21)$$

In order to get rid of the gradient term to the right we shall make a rough estimate, starting with:

$$\mathrm{Re}\left(\left(-\frac{1}{2} h^2 \Delta + \mathrm{V} - z\right) u \,\middle|\, u\right) = ((\mathrm{V} - \mathrm{Re}\, z) u | u) + \frac{1}{2} h^2 \| \nabla u \|^2.$$

Combining the assumptions (B)-(D) as before, we notice that

$$\mathrm{V}(x) \geqq \inf_{\alpha \in \mathscr{A}} \mathrm{V}(\tilde{x}_\alpha), \quad (6.22)$$

and if we recall the assumption (6.6) about z and that $E_{\tilde{x}_\alpha}$ is $\mathcal{O}(N)$, we get:

$$\operatorname{Re}\left(\left(-\frac{1}{2}h^2\Delta+V-z\right)u\,|\,u\right)\geqq -\operatorname{Ch}N\,\|u\|^2+\frac{1}{2}h^2\,\|\nabla u\|^2,$$

and if we combine this with (6.21), we get:

$$h^2\,\|\nabla u\|^2\leqq\mathcal{O}(1)\left((N/h)\left\|\left(-\frac{1}{2}h^2\Delta+V-z\right)u\right\|^2\right.$$
$$\left.+e^{-1/\mathrm{Ch}}\,\|\nabla u\|^2+Nh\,\|R_+u\|^2\right).$$

We multiply this by h/N and get after taking square roots:

$$C^{-1}(h/N)^{1/2}\,\|h\nabla u\|\leqq\left\|\left(-\frac{1}{2}h^2\Delta+V-z\right)u\right\|$$
$$+\mathcal{O}(e^{-1/\mathrm{Ch}})\,\|\nabla u\|+\operatorname{Ch}\|R_+u\|. \quad (6.23)$$

Combining this with (6.21), we can absorb the gradient terms of the RHS in both estimates, and we get:

$$C^{-1}(h\,\|u\|+(h/N)^{1/2}\,\|h\nabla u\|)$$
$$\leqq\left\|\left(-\frac{1}{2}h^2\Delta+V-z\right)u\right\|+\operatorname{Ch}\|R_+u\|, \quad (6.24)$$

where it may be time to recall that $u\in C_0^\infty(\mathbb{R}^{nN})$ and that z should satisfy (6.6).

From (E) and (6.22) we see that $-\frac{1}{2}h^2\Delta+V$ is semibounded from below and symmetric with domain $C_0^\infty(\mathbb{R}^{nN})$. The corresponding Friedrichs extension will then be denoted by $-\frac{1}{2}h^2\Delta+V$ or by P. If instead of (6.6), we make the stronger assumption:

$$\operatorname{Re}z\leqq\inf_{\alpha\in\mathscr{A}}V(\tilde{x}_\alpha)+h\,E_{\tilde{x}_\alpha}-h/C_0=\mu_{\inf}-h/C_0 \quad (6.25)$$

then the proof of (6.24) will give the stronger estimate

$$C^{-1}(h\,\|u\|+(h/N)^{1/2}\,\|h\nabla u\|)\leqq\left\|\left(-\frac{1}{2}h^2\Delta+V-z\right)u\right\|,$$

and from this we see that P has no spectrum in $]-\infty,\mu_0-h/C_0]$. Notice that (6.26) remains valid for general u in the domain of P, and that the same remark applies to (6.24).

Now let z satisfy (6.6) and in addition that

$$|z-\mu_{\inf}|\leqq\operatorname{Ch}\quad\text{for some constant } C>0. \quad (6.27)$$

Let $R_- : l^2(\mathscr{A}) \to L^2(\mathbb{R}^{nN})$ be the adjoint of R_+, so that $R_- u^- = \sum u_\alpha^- e_\alpha$. If $\mathscr{D}$ is the domain of P, we then consider the operator

$$\mathscr{P} = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathscr{D} \times l^2 \to L^2 \times l^2. \tag{6.28}$$

When z is real, we can also consider $\mathscr{P} = \mathscr{P}(z)$ as an unbounded self-adjoint operator in $L^2 \times l^2$ with domain $\mathscr{D} \times l^2$. We shall show that $\mathscr{P}(z)$ is bijective, and to do so we consider the corresponding system of equations:

$$\left.\begin{aligned} &(P - z)u + \sum_\alpha u_\alpha^- e_\alpha = v && (a) \\ &(u \mid e_\alpha) = v_\alpha^+ && (b) \end{aligned}\right\} \tag{6.29}$$

for $u \in \mathscr{D}$, $v \in L^2$, u^-, $v^+ \in l^2$. We shall first derive an *a priori* estimate for this system. Taking the scalar product of the equation (a) with e_β gives:

$$(u \mid (P - \bar{z}) e_\beta) + u_\beta^- = (v \mid e_\beta), \tag{6.30}$$

and we use that

$$(P - \mu_\alpha) e_\alpha = r_\alpha \qquad \text{with} \quad \| r_\alpha \| = \mathcal{O}(h^\infty), \tag{6.31}$$

and $(r_\alpha \mid r_\beta) = 0$ for $\alpha \neq \beta$. This gives

$$(\mu_\beta - z) v_\beta^+ + \mu_\beta^- = (v \mid e_\beta) - (u \mid r_\beta), \tag{6.32}$$

and since $\mu_\beta - z = \mathcal{O}(h)$, we get:

$$\| u^- \| \leqq C(\| v \| + h \| v^+ \| + \mathcal{O}(h^\infty) \| u \|). \tag{6.33}$$

We then have the same estimate for $\| \sum u_\alpha^- e_\alpha \|$, and if we write (6.29) (a): $(P - z) u = v - \sum u_\alpha^- e_\alpha$, and use (6.24) and (6.29) (b) we get:

$$h \| u \| + (h/N)^{1/2} \| h \nabla u \| + \| u^- \| \leqq C(\| v \| + h \| v^+ \|). \tag{6.34}$$

This shows that $\mathscr{P}(z)$ is injective $\mathscr{D} \times l^2 \mapsto L^2 \times l^2$ and has a closed image. (The control of the $\mathscr{D}$-norm is easily obtained from the control of the L^2-norm of u and the l^2-norm of u^-.) For real z we can use the selfadjointness of $\mathscr{P}$, to infer that $\mathscr{P}$ is bijective. For complex z, we then conclude that $\mathscr{P}(z)$ is bijective by a continuity argument.

Let

$$\mathscr{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \tag{6.35}$$

be the inverse of $\mathscr{P}(z)$. We then know that z belongs to the spectrum of P iff 0 belongs to the spectrum of $E_{-+}(z)$, and we shall now determine $E_{-+}(z)$ up to an operator of norm $\mathcal{O}(h^\infty)$. Put $v = 0$ and define the following approximate solution of (6.29):

$$\left.\begin{aligned} \tilde{u} &= \sum v_\alpha^+ e_\alpha \underset{\text{def.}}{=} \tilde{E}_+ v^+, \\ \tilde{u}_\alpha^- &= (z - \mu_\alpha) v_\alpha^+ \underset{\text{def.}}{=} (\tilde{E}_{-+}(z) v^+)_\alpha, \end{aligned}\right\} \tag{6.36}$$

so that $\tilde{E}_{-+}(z)$ is the diagonal matrix with diagonal element $(z-\mu_\alpha)$. Then we get $(P-z)\tilde{u}+\sum u_\alpha^- e_\alpha=\sum v_\alpha^+ r_\alpha=\mathcal{O}(h^\infty)\|v^+\|$ in L^2. Moreover $(\tilde{u}|e_\alpha)=v_\alpha^+$. If we now let (u, u^-) be the exact solution to (6.29) in the case $v=0$, we get thanks to (6.34):

$$\|u-\tilde{u}\|+\|u^- -\tilde{u}^-\|=\mathcal{O}(h^\infty)\|v^+\|. \tag{6.37}$$

This means that

$$\|E_\pm-\tilde{E}_\pm\|=\mathcal{O}(h^\infty), \qquad \|E_{-+}-\tilde{E}_{-+}\|=\mathcal{O}(h^\infty), \tag{6.38}$$

where we take the operator norms between the various L^2 and l^2 spaces. We have already seen in section 4 that the values μ_α are within a distance $\mathcal{O}(h^\infty)$ from the spectrum of P. On the other hand if $z\in\mathbb{R}$ satisfies (6.6) and is in the spectrum of P, then in view of the second estimate in (6.38) we see that $z-\mu_\alpha=\mathcal{O}(h^\infty)$ for some α which depends on z and h. In other words we have localized modulo $\mathcal{O}(h^\infty)$ the spectrum of P in $]-\infty, \mu_0+2h\lambda_0^2-h/C_0]$, where C_0 is an arbitrary fixed constant, and h is supposed to be sufficiently small.

If we make a gap-assumption, we can be more precise: Assume that I is an interval of length $\mathcal{O}(h)$ in $]-\infty, \mu_{\text{inf}}+2\lambda_0^2-h/C_0-2h^{M_0}]$ for some fixed M_0, such that $(I+[-2h^{M_0}, 2h^{M_0}])\setminus I$ does not intersect the set of values $\{\mu_\alpha;\ \alpha\in\mathcal{A}\}$ (or equivalently the set $\{\mu_\alpha;\ \alpha\in\mathcal{B}\}$). Let $\mathcal{T}$ be the set of $\alpha\in\mathcal{A}$ with $\mu_\alpha\in I$. Let Π be the spectral projection associated with P and $I+[-h^{M_0}, h^{M_0}]$. Then we claim that

$$\operatorname{rank}(\Pi)=\text{the number of elements of } \mathcal{T}. \tag{6.39}$$

To show (6.37), we let Γ be the boundary of the rectangle defined by: $\operatorname{Re} z\in I+[-h^{M_0}, h^{M_0}]$, $|\operatorname{Im} z|\leq h$. For $z\in\Gamma$, we know that $E_{-+}(z)^{-1}$ exists and is equal to $\tilde{E}_{-+}(z)^{-1}+\mathcal{O}(h^\infty)$, in $\mathcal{L}(l^2, l^2)$. On the other hand, $(z-P)^{-1}=-E(z)+E_+(z)E_{-+}(z)^{-1}E_-(z)$, so

$$\Pi=(2\pi i)^{-1}\int_\Gamma E_+(z)E_{-+}(z)^{-1}E_-(z)\,dz, \tag{6.40}$$

so modulo an operator of norm $\mathcal{O}(h^\infty)$ we have $\Pi\equiv\tilde{\Pi}$, where

$$\tilde{\Pi}=(2\pi i)^{-1}\int_\Gamma \tilde{E}_+(z)\tilde{E}_{-+}(z)^{-1}\tilde{E}_-(z)\,dz.$$

An easy computation shows that $\tilde{\Pi}$ is the orthogonal projection onto the space spanned by the orthonormal family $\{e_\alpha\}_{\alpha\in\mathcal{T}}$. It follows that Π and $\tilde{\Pi}$ have the same rank (finite or infinite), so we obtain (6.39).

Let us sum up the discussion into the following.

THEOREM 6.1. – *Let* $V=V_0(x)+W(x)$ *be a potential on* $\mathbb{R}^{nN}$ *which satisfies the assumptions* (B)-(E) *of the beginning of this section. Let* $\tilde{x}_\alpha$, $\alpha\in\mathcal{A}$ *be the set of local minima of* V, *introduced prior to* (E) *and let*

$\mu_\alpha = V(\tilde{x}_\alpha) + h\,E_{\tilde{x}_\alpha}$ *be the corresponding* WKB-*eigenvalues constructed in section* 3. *Put* $\mu_{\inf} = \inf_{\alpha \in \mathscr{A}} \mu_\alpha$, *and let* P *be the Friedrichs extension of* $-\frac{1}{2}h^2\Delta + V$. *Then there is a constant* $\lambda_0 > 0$ *such that the following holds when* $\delta > 0$ *is sufficiently small: Fix some* $N_0 > 0$ *and assume that* N *satisfies* (0.4). *Then for* $h > 0$ *sufficiently small we have: For every* μ_α *in* $[\mu_{\inf}, \mu_{\inf} + 2\lambda_0^2 h[$ *we have* $\operatorname{dist}(\mu_\alpha, \sigma(P)) = \mathcal{O}(h^k))$ *for every* $k \in \mathbb{N}$, *uniformly with respect to* α *and* N. *For every* $\mu \in \sigma(P) \cap]-\infty, \mu_{\inf} + 2\lambda_0^2 h]$ *we have* $\operatorname{dist}(\mu, \{\mu_\alpha; \alpha \in \mathscr{A}\}) = \mathcal{O}(h^k)$ *for every* $k \in \mathbb{N}$, *uniformly with respect to* N. *If we further assume that* $I \subset [\mu_{\inf}, \mu_{\inf} + 2\lambda_0^2 h[$ *is a closed interval with* $(I + [-2h^M, 2h^M]) \setminus I$ *disjoint from the set of all* μ_α *for some fixed* $M > 0$, *then for* $h > 0$ *sufficiently small, the dimension of the spectral subspace associated to* P *and the interval* $I + [-h^M, h^M]$ *is equal to the number of* $\alpha \in \mathscr{A}$ *for which* $\mu_\alpha \in I$.

As mentioned in the beginning of this section, our results remain valid for potentials in certain product domains. Let us formulate one such result in the case of a single well, probably a starting point for the study of tunneling. Let I_i be the l^∞-ball of radius r_j in $\mathbb{R}^n$ with $C^{-1} \leqq r_j \leqq C$. Let $v_j \in C^\infty(\bar{I}_j)$ satisfy (B), (C) with $\mathscr{A}_j = \{0\}$, $x_{j,k} = 0$. Let W be realvalued and smooth on $\prod_1^N \bar{I}_j$ satisfying (D) with $\mathscr{A}$ reduced to $\{0\}$, and with $x_0 = 0$. Let $\mu_0 = V(\tilde{x}_0) + h\,E_{\tilde{x}_0}$ be the corresponding WKB-eigenvalue, and let P denote the Dirichlet realization of $-\frac{1}{2}h^2\Delta + V(x)$ on $L^2\left(\prod_1^N I_j\right)$. Then we have:

THEOREM 6.2. — *There is a constant* $\lambda_0 > 0$ *such that the following holds when* $\delta > 0$ *is sufficiently small: Fix some* $N_0 > 0$ *and assume that* N *satisfies* (0.4). *Then for* $h > 0$ *sufficiently small we have: The lowest eigenvalue of* P *is simple and of the form* $\mu_0 + \mathcal{O}(h^\infty)$. *The distance from this eigenvalue to the rest of the spectrum is* $\geqq 2\lambda_0^2 h$.

We also notice that λ_0 can be estimated in terms of the hessian of V by examining the argument of section 5.

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