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Quantum mechanics and coherent states on the anti-de Sitter spacetime and their Poincaré contraction

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ABSTRACT. — In this work we show how a Poincaré quantum elementary system arises as the zero curvature limit of an anti-de Sitter (SO (2, 1)) analog. The latter is constructed by applying the method of geometric quantization to the classical motion of a massive freely evolving particle on the anti-de Sitter spacetime. The (unique) invariant polarization that selects the anti-de Sitter quantum elementary system is shown to have as zero curvature limit the Poincaré polarization. In addition, the same limiting process is also applied to a particular family of quantum states, namely the set of SO (2, 1) coherent states, which are optimally localized in phase space. It is shown that their limits are energy eigenstates, and henceforth optimally localized in momentum space.

RÉSUMÉ. — Nous montrons dans ce travail comment un système quantique élémentaire vis-à-vis du groupe de Poincaré s'obtient comme limite de courbure nulle d'un système analogue vis-à-vis du groupe anti-de Sitter (SO (2, 1)). Ce dernier est construit en appliquant la méthode de quantification géométrique au mouvement classique d'une particule massive, évoluant librement sur l'espace-temps anti-de Sitter. Nous montrons

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que l'unique polarisation invariante qui sélectionne le système élémentaire quantique de SO(2, 1), tend dans la limite de courbure nulle vers celle qui sélectionne un système analogue du groupe de Poincaré. De plus, nous appliquons la même procédure de passage à la limite à une famille particulière d'états quantiques, à savoir les états cohérents associés à SO(2, 1); ces derniers sont localisés de manière optimale dans l'espace des phases. Nous montrons alors qu'ils tendent vers des états propres de l'énergie, qui sont de ce fait des états localisés de manière optimale dans l'espace des moments.

1. INTRODUCTION

In this paper we show how to approximate the classical and quantum mechanics of massive particles on Minkowski spacetime by the corresponding theories on an anti-de Sitter spacetime $M_\varkappa$ with small curvature κ.

Physical theories on the anti-de Sitter spacetime have attracted considerable attention (*see* [F], [FF], [GH] and references therein) and have been motivated by a number of different considerations. One of them is the observation that the discreteness of the energy spectrum for massive fields leads to an infra-red cutoff in such theories. This cutoff is considered rather natural, since the anti-de Sitter group is, together with the de Sitter group, the only possible deformation of the Poincaré group [LN] [BLL].

In this paper we start a careful study of the zero-curvature limit of anti-de Sitter spacetime classical and quantum mechanics. We limit the discussion to $1+1$-dimensional spacetimes, in which case the anti-de Sitter group is SO(2, 1). This will allow us to keep the notation relatively simple, while already bringing out many of the essential difficulties of the theory. The theory in $3+1$ dimensions will be elaborated elsewhere [EDB].

Underlying the zero curvature limit considered here is the Inönu-Wigner contraction [IW] [D] of SO(2, 1) with respect to the spacetime isotropy group SO(1, 1), giving in the limit the $1+1$-dimensional Poincaré group. The contraction of Lie algebras is a well defined and much studied notion [IW] [S] [D]. The behaviour of the irreducible representations of the corresponding Lie groups under contraction has been investigated as well in a number of cases (for an overview and further references, *see* [MN] [D]), but generally poses various problems of functional analytic and group theoretic nature. They seem to obstruct the formulation of a general theory, but were overcome for a class of examples (not including the case considered here) in [MN] and [D]. One of the main problems that arises is the need to realize the sequence of Hilbert spaces carrying the representations that are contracted in a way that allows the taking of a meaningful limit (*see* [MN] and [D]). We solve this problem here by identifying among themselves the classical phase spaces (coadjoint orbits of $SO_0(2, 1)$) for different values of κ. This identification is based on the physical interpretation of their points as geodesics on spacetime. It allows us then to realize the above Hilbert spaces as reproducing kernel Hilbert subspaces of the space of L^2-functions on phase space. This formulation permits us to study the limiting behaviour of quantum mechanical states.

We realize in section 2 the classical anti-de Sitter phase spaces in a manner which allows us to easily associate to each of their points a timelike geodesic on the anti-de Sitter spacetime. Using an appropriate coordinate system on spacetime we then identify among themselves the phase spaces for different values of κ. It is then easy to show that the $\kappa \to 0$ limit of the classical theory yields its Poincaré counterpart describing massive test particles on Minkowski spacetime. In sections 3 and 4 we quantize the classical theory. Since canonical quantization can not be applied to find the quantum observables corresponding to the $SO_0(2, 1)$ generators, we use the methods of geometric quantization. The Hilbert space of quantum states will therefore be realized as a reproducing kernel Hilbert space $\mathscr{H}_\kappa^m$ of L^2-functions on phase space, on which $SO_0(2, 1)$ acts with a unitary irreducible representation. This allows us to interpret each state in $\mathscr{H}_\kappa^m$ physically through the phase space probability distribution that it generates. We identify explicitly those states which are optimally localized on phase space. We show that in the zero curvature limit, they contract to eigenstates of the Hamiltonian in the limiting Poincaré invariant theory, thereby becoming completely localized in momentum and completely delocalized in position. We identify the Hilbert space $\mathscr{H}_0^m$ of the limiting theory as a space of functions on the classical phase space, which are however no longer square integrable. In section 5 we show how this space arises naturally if one studies the small κ limit of the $SO_0(2, 1)$-invariant polarization used to define $\mathscr{H}_\kappa^m$ and we establish the link with the geometric quantization theory as applied to the Poincaré group. Section 6 contains our conclusions and a comparison with previous work on the subject [AAG] [BEGG].

2. CLASSICAL MECHANICS OF A MASSIVE TEST PARTICLE

In this section, we describe the classical dynamics and symmetries of a massive test particle of mass m in the $1+1$-dimensional anti-de Sitter spacetime, in a formalism convenient for our purposes, and describe its limit as the curvature tends to zero, recovering the Minkowski theory.

The anti-de Sitter spacetime $M_\varkappa$ is defined as the surface in R^3 $(y=(y^5, y^0, y^1)\in R^3)$ given by

$$y\cdot y \equiv -(y^5)^2-(y^0)^2+(y^1)^2=-\kappa^{-2} \tag{2.1}$$

equipped with the Lorentzian metric $g_\varkappa$, induced by the metric of signature $(-, -, +)$ on R^3. The numbering of the indices is the obvious restriction to the 1+1-dimensional spacetime of the notation of [F], where the 3+1-dimensional spacetime is studied. Here $\kappa>0$ is the curvature of $M_\varkappa$. The isometry group of $M_\varkappa$ is $O(2, 1)$, its component connected to the identity $SO_0(2, 1)$. The generators of $SO_0(2, 1)$, viewed as vector fields on R^3, are

$$X_{\mu\nu}=y_\mu\partial_\nu-y_\nu\partial_\mu; \qquad \mu, \nu=5, 0, 1, \tag{2.2}$$

so that

$$[X_{01}, X_{50}]=-X_{15} \tag{2.3a}$$
$$[X_{01}, X_{15}]=-X_{50} \tag{2.3b}$$
$$[X_{15}, X_{50}]=X_{01} \tag{2.3c}$$

The metric on $M_\varkappa$ has signature $(+, -)$ and we shall take the timelike directions to be those for which the norm of the inner product is negative. This implies that time is compactified on $M_\varkappa$.

As we now first show, the classical dynamics, the classical phase space and its symmetries can be described in an efficient and intrinsic way using constraint Hamiltonian mechanics [Sn] [DB]. Let (y^μ, q_μ) be canonical coordinates on $T^*R^3 \cong R^3\times R^3$. Then the cotangent bundle $T^*M_\varkappa$ to $M_\varkappa$ is identified with the four-dimensional surface

$$y\cdot y \equiv y_\mu y^\mu=-\kappa^{-2} \tag{2.4a}$$
$$y\cdot q \equiv y^\mu q_\mu=0. \tag{2.4b}$$

The evolution space or extended phase space [So] [DB] $E^m_\varkappa$ for a particle of mass m is then

$$y\cdot y=-\kappa^{-2} \tag{2.5a}$$
$$y\cdot q=0 \tag{2.5b}$$
$$q\cdot q=-m^2, \tag{2.5c}$$

with the additional requirement that the timelike vector q is future pointing. A point $(y, q)\in E^m_\varkappa$ represents a particle at the spacetime point y in $M_\varkappa$ with two-momentum q constrained to the forward mass shell. Note that the restriction of the canonical two-form $\omega=dy\wedge dq\equiv dy^\mu\wedge dq_\mu$ on T^*R^3 to (2.4) yields the symplectic two-form $\omega_\varkappa$ on $T^*M_\varkappa$. The restriction of $\omega_\varkappa$ to $E^m_\varkappa$ yields a closed but degenerate two-form $\varepsilon^m_\varkappa$ on $E^m_\varkappa$. A direct calculation or a simple application of the general theory shows that

$$X\equiv m^2 X_{y^2}+\kappa^{-2}X_{q^2} \tag{2.6a}$$

is tangent to $E^m_\varkappa$ and that

$$X \lrcorner\, \varepsilon^m_\varkappa=0, \tag{2.6b}$$

so that X generates the kernel of $\varepsilon_{\varkappa}^{m}$. Here we used the notation X_f for the Hamiltonian vector field corresponding to the function f, *i.e.*

$$X_f \lrcorner\, \omega = df. \tag{2.7}$$

Note that,

$$X = \frac{2}{\kappa^2} X_\pi \tag{2.8a}$$

where

$$\pi = \frac{1}{2} q^2 + \frac{1}{2} \kappa^2 m^2 y^2. \tag{2.8b}$$

The geodesic equations of motion are now obtained as the Hamiltonian equations corresponding to π, *i.e.*

$$\dot{y}^\mu = \{ y^\mu, \pi \} = q^\mu \tag{2.9a}$$
$$\dot{q}_\mu = \{ q_\mu, \pi \} = -\kappa^2 m^2 y_\mu. \tag{2.9b}$$

It is clear that the solutions of (2.9) with initial data on $E_{\varkappa}^{m}$ stay on $E_{\varkappa}^{m}$. Solving (2.9) yields

$$y(\tau) = y(0) \cos(\kappa m \tau) + (\kappa m)^{-1} q(0) \sin(\kappa m \tau), \tag{2.10a}$$
$$q(\tau) = \dot{y}(\tau), \qquad \tau \in \mathbb{R}. \tag{2.10b}$$

Since each solution in (2.10) intersects the surface $y^0 = 0$ in exactly one point, we can identify the symplectic reduction of $E_{\varkappa}^{m}$ with

$$\Sigma_{\varkappa}^{m} = \{ (y, q) \in E_{\varkappa}^{m} \mid y^0 = 0, y^5 > 0 \}. \tag{2.11}$$

$\Sigma_{\varkappa}^{m}$ is the space of motions [So] [DB] or phase space of a test-particle of mass m on $M_{\varkappa}$. To each point in $\Sigma_{\varkappa}^{m}$ corresponds precisely one timelike geodesic on $M_{\varkappa}$, *i.e.* one possible motion of the particle, given by (2.10). The symplectic form $\omega_{\varkappa}^{m}$ on $\Sigma_{\varkappa}^{m}$ is obtained as the restriction of ω to $\Sigma_{\varkappa}^{m}$. To describe the symmetries of this system, we proceed as follows. On $T^* \mathbb{R}^3$, $SO_0(2, 1)$ acts via:

$$(y, q) \xrightarrow{\Lambda \in SO_0(2,1)} (\Lambda y, \Lambda q). \tag{2.12}$$

This action leaves the two-form ω invariant; its generators are therefore Hamiltonian vector fields, generated by the functions

$$L_{\mu\nu} = y_\mu q_\nu - y_\nu q_\mu, \qquad \mu, \nu = 5, 0, 1. \tag{2.13}$$

We remark that indices on y and q are raised and lowered with the metric $\eta_{55} = -1 = \eta_{00}$, $\eta_{11} = 1$ on $\mathbb{R}^3$. Since

$$\{ y^2, L_{\mu\nu} \} = \{ q^2, L_{\mu\nu} \} = \{ y \cdot q, L_{\mu\nu} \} = 0 \tag{1.14}$$

one sees that the action (2.12) of $SO_0(2, 1)$ leaves $E^m_\varkappa$ invariant. In fact, $E^m_\varkappa$ is a homogeneous space for $SO_0(2, 1)$ on which $SO_0(2, 1)$ acts transitively and freely. Fixing a point $(y_{(0)}, q_{(0)})$ on $E^m_\varkappa$ by

$$y^5_{(0)}=\kappa^{-1}, \qquad y^0_{(0)}=0, \qquad y^1_{(0)}=0, \tag{2.15 a}$$
$$q^5_{(0)}=0, \qquad q^0_{(0)}=m, \qquad q^1_{(0)}=0, \tag{2.15 b}$$

one sees that for each $(y, q)\in E^m_\varkappa$, there exists a unique $\Lambda(y, q)\in SO_0(2, 1)$ so that

$$\Lambda(y, q)\, y_{(0)}=y, \tag{2.16 a}$$
$$\Lambda(y, q)\, q_{(0)}=q. \tag{2.16 b}$$

One verifies that $\Lambda(y, q)$ is given by

$$\Lambda(y, q)^\mu_\nu = -q^\mu q_{(0)\,\nu} m^{-2} - y^\mu y_{(0)\,\nu} \kappa^2 + m^{-2}\kappa^2 \varepsilon^{\mu\varkappa\lambda} q_\varkappa y_\lambda \varepsilon_{\nu\rho\sigma} q^\rho_{(0)} y^\sigma_{(0)}, \tag{2.17}$$

where $\varepsilon^{\mu\nu\varkappa}$ is the alternating tensor defined by

$$\varepsilon_{501}=1=\varepsilon^{501}. \tag{2.18}$$

Since the action in (2.12) sends the solution curves of (2.10) into each other, it induces an action on $\Sigma^m_\varkappa$ which leaves $\omega^m_\varkappa$ invariant. Its generators are therefore Hamiltonian vector fields, generated by the restriction of the $L_{\mu\nu}$ in (2.13) to $\Sigma^m_\varkappa$. This last assertion follows again readily from the general theory of Hamiltonian constraints. It is also clear that we can identify $\Sigma^m_\varkappa$ with $SO_0(2, 1)/SO(2)$, where $SO(2)$ is the group of rotations in the y^5-y^0 plane.

We have therefore determined very explicitly the classical phase space $\Sigma^m_\varkappa$ of a particle of mass m on $M_\varkappa$ [*see* (2.11)], together with its symplectic structure, an explicit expression for the generators of the symmetry group $SO_0(2, 1)$ (*i. e.* the moment map [So] [AM], *see* (2.13)), and a clear physical interpretation for its points [*see* (2.9)-(2.10)]. Since in addition $\Sigma^m_\varkappa \cong SO_0(2, 1)/SO(2)$ is a homogeneous space of $SO_0(2, 1)$, we can identify $(\Sigma^m_\varkappa, \omega^m_\varkappa)$ in a natural way with an orbit in the dual of the Lie algebra of $SO_0(2, 1)$.

We are now ready to study the flat spacetime limit $(\kappa \to 0)$ of the classical theory. Notice first that $\Sigma^m_\varkappa$, defined in (2.11), changes as a subset of $T^* R^3 \equiv R^6$ when we vary κ. In addition, the coadjoint orbit associated to $\Sigma^m_\varkappa$ via the moment map changes as a subset of the dual of the Lie algebra of $SO_0(2, 1)$. Indeed, the Casimir operator $-L^2_{50}+L^2_{15}+L^2_{01}$ takes the value $-\left(\frac{m}{\kappa}\right)^2$ on $\Sigma^m_\varkappa$ and hence varies with κ. This shows at once that $\Sigma^m_\varkappa$ and $\Sigma^{m'}_{\varkappa'}$ are identical as symplectic homogeneous spaces of $SO_0(2, 1)$ if and only if $\frac{m}{\kappa}=\frac{m'}{\kappa'}$.

To take a meaningful limit of the group generators in (2.13) restricted to Σ_κ^m as $\kappa \to 0$, it is clearly necessary to identify the Σ_κ^m for different values of κ. Since it follows from the above argument that Σ_κ^m and $\Sigma_{\kappa'}^m$ can not be identical as symplectic homogeneous spaces of $SO_0(2, 1)$ unless $\kappa=\kappa'$, we conclude that this identification can not both intertwine the action of $SO_0(2, 1)$ and preserve the symplectic structure. The identification we construct below preserves the latter and is based on a local identification of the non-isometric spacetimes M_κ and $M_{\kappa'}$ for all $\kappa>0$, as follows.

Consider the system of global coordinates on M_κ, given by

$$y^5 = Y \cos \kappa x^0, \tag{2.19a}$$
$$y^0 = Y \sin \kappa x^0, \tag{2.19b}$$
$$y^1 = x^1, \tag{2.19c}$$

with

$$Y = (\kappa^{-2} + (x^1)^2)^{1/2}. \tag{2.19d}$$

Here $x^1 \in R$, $x^0 \in \left[-\frac{\pi}{\kappa}, \frac{\pi}{\kappa}\right)$. A straightforward calculation yields

$$g_\kappa^{00} = -(\kappa Y)^{-2}, \qquad g_\kappa^{11} = (\kappa Y)^2. \tag{2.20}$$

Note that, uniformly on compact neighbourhoods of $x^0=0$, $x^1=0$, $g_\kappa^{00} \to 1$ and $g_\kappa^{11} \to 1$. In this sense, we can say that the spacetimes M_κ converge to the Minkowski spacetime as $\kappa \to 0$.

Now we can introduce coordinates (x^0, x^1, p_0, p_1) on $T^* M_\kappa$ by setting

$$q_\mu dy^\mu = p_0 dx^0 + p_1 dx^1 \tag{2.21}$$

or, using (2.19) and $q \cdot y=0$,

$$p_0 = \kappa (q_0 y^5 - q_5 y^0), \tag{2.22a}$$
$$p_1 = q_1 (\kappa Y)^{-2}. \tag{2.22b}$$

Inverting (2.22) yields

$$q_1 = (\kappa Y)^2 p_1 \tag{2.23a}$$
$$q_0 = (\kappa Y^2)^{-1} (y^5 p_0 - \kappa y^0 y^1 q_1) \tag{2.23b}$$
$$q_5 = (\kappa Y^2)^{-1} (-y^0 p_0 - \kappa y^5 y^1 q_1). \tag{2.23c}$$

Now, using (2.20), one sees that equation (2.5c) in the (x,p) coordinates of $T^* M_\kappa$ reads

$$-(\kappa Y)^{-2} p_0^2 + (\kappa Y)^2 p_1^2 = -m^2. \tag{2.24}$$

It follows then from (2.11) that (x^1, p_1) are coordinates on Σ_κ^m so that $(\Sigma_\kappa^m, \omega_\kappa^m)$ is symplectomorphic to $(R^2, dx^1 \wedge dp_1)$. This gives the desired identification between the Σ_κ^m for different values of κ (and of m). Using (2.19) and (2.23) in (2.13) and setting $y^0=0$, one finds, for each value of m and κ,

$$L_{50}(x^1, p_1) = \kappa^{-1} p^0 \tag{2.25a}$$
$$L_{15}(x^1, p_1) = Y p_1 \tag{2.25b}$$
$$L_{01}(x^1, p_1) = (\kappa Y)^{-1} p^0 x^1 \tag{2.25c}$$

where $p^0>0$ is found upon solving (2.24) for p^0 in terms of (x^1, p_1). A direct calculation confirms that one has

$$\{L_{50}, L_{15}\}=L_{01}, \qquad \{L_{50}, L_{01}\}=-L_{15}, \qquad \{L_{15}, L_{01}\}=-L_{50}, \tag{2.26}$$

where the Poisson bracket is taken with respect the canonical pair (x^1, p_1). The geodesics can be recovered by solving

$$\frac{dx^1}{dt}=\{x^1, L_{50}\} \tag{2.27 a}$$

$$\frac{dp_1}{dt}=\{p_1, L_{50}\} \tag{2.27 b}$$

for $(x^1(t), p_1(t))$. A geodesic on $M_\varkappa$ is then given by $x^0=\kappa^{-1}t$, $x^1=x^1(t)$. Giving (2.24), (2.25) and (2.27) is completely equivalent to giving (2.11), (2.13) and (2.9). The latter representation of the massive particle on $M_\varkappa$ is simpler for calculational purposes, but hides the κ dependence of the generators. We therefore need to use the first to study the contraction $\kappa\to 0$.

Recall first that the Poincaré group in 1+1 spacetime dimensions is three-dimensional with generators

$$H(x^1, p_1)=\sqrt{p_1^2+m^2}, \tag{2.28 a}$$

$$P(x^1, p_1)=p_1, \tag{2.28 b}$$

$$K(x^1, p_1)=H x^1, \tag{2.28 c}$$

on the usual phase space $\Sigma_0^m=T^*R\cong R^2$ of a massive test particle of mass m in Minkowski spacetime. One then sees readily that

$$\lim_{\varkappa\to 0}\kappa L_{50}(x^1, p_1)=H(x^1, p_1), \tag{2.29 a}$$

$$\lim_{\varkappa\to 0}\kappa L_{15}(x^1, p_1)=P(x^1, p_1), \tag{2.29 b}$$

$$\lim_{\varkappa\to 0}L_{01}(x^1, p_1)=K(x^1, p_1). \tag{2.29 c}$$

The following proposition implies that the limits in (2.29) are uniform on compacta.

PROPOSITION 2.1. – *Let* $L>0$ *be given. Let*

$$S(L,m)=\{(x^1, p_1)\in\Sigma_0^m \mid |x_1|\sqrt{p_1^2+m^2}\leqq L m\}; \tag{2.30 a}$$

then, $\forall (x^1, p_1)\in S(L,m)$, $\forall\kappa$ *so that* $\kappa L\leqq 1$

$$|\kappa L_{50}-H(x^1, p_1)|\leqq 3\kappa L m \tag{2.30 b}$$

$$|\kappa L_{15}-P(x^1, p_1)|\leqq \kappa L m \tag{2.30 c}$$

$$|L_{01}-K(x^1, p_1)|\leqq(3\kappa L m)|x^1|. \tag{2.30 d}$$

The proof consists of a simple estimate that we omit. Proposition 2.1 can be interpreted as follows. Suppose Poincaré invariance is established experimentally up to a given experimental error, for states of the particle in a compact subset of Σ_0^m. Then the inequalities in (2.30) give an upper bound on κ in terms of the experimental error.

3. PREQUANTIZATION

To quantize the classical theory of section 2 means in the first place to identify the Hilbert space of quantum states, for which we shall write $\mathcal{H}_\varkappa^m$, and to give the physical interpretation of the states in $\mathcal{H}_\varkappa^m$. In addition, we need to quantize the classical observables $L_{\mu\nu}$ in order to obtain (upon integration) a unitary irreducible representation $U_\varkappa^m$ of $SO_0(2, 1)$ on $\mathcal{H}_\varkappa^m$.

One might argue that this is trivial since the unitary irreducible representations of $SO_0(2, 1)$ are well known. This is not true for two reasons. First, one has to decide which irreducible representation is associated to which classical system. Geometric quantization gives an answer to this question and we shall see that quantization of $\Sigma_\varkappa^m$ and of $\Sigma_{\varkappa'}^{m'}$ leads to the same irreducible representation if and only if $\frac{m}{\kappa} = \frac{m'}{\kappa'}$. This follows because the coadjoint orbits associated to $\Sigma_\varkappa^m$ and $\Sigma_{\varkappa'}^{m'}$ are identical as symplectic homogeneous spaces of $SO_0(2, 1)$. Hence, geometric quantization yields the same irreducible representation. Nevertheless, the points of $\Sigma_\varkappa^m$ and of $\Sigma_{\varkappa'}^{m'}$ have different physical meanings, since $M_\varkappa$ and $M_{\varkappa'}$ are different (*i.e.* not isometric) as spacetimes if $\kappa \neq \kappa'$. Similarly, we shall see that states in $\mathcal{H}_\varkappa^m$ and in $\mathcal{H}_{\varkappa'}^{m'}$ have different physical meanings, although $(\mathcal{H}_\varkappa^m, U_\varkappa^m)$ and $(\mathcal{H}_{\varkappa'}^{m'}, U_{\varkappa'}^{m'})$ are unitarily equivalent as irreducible representations of $SO_0(2, 1)$. In order words, we argue that the actual realization of the irreducible representation is of physical importance, not just the representation up to unitary equivalence. This is our second point. This is of course related to the fact that the group generators are not the only physically important observables. Turning now to the explicit quantization of the theory in section 2, we start by remarking that the complicated dependence on (x^1, p_1) of the $L_{\mu\nu}$ in (2.25) makes it impossible to obtain their quantization via canonical quantization. This is why we use the methods of geometric quantization here.

In this section we analyse as a first step the prequantization of the observables $L_{\mu\nu}$ in (2.25) and their contraction. As in (2.29), we make use of the identification of $(\Sigma_\varkappa^m, \omega_\varkappa^m)$ with $(R^2, dx^1 \wedge dp_1)$ given in section 2. This will allow us to study the contraction of the prequantized observables as a strong limit on one fixed Hilbert space (Theorem 3.1). The prequantization map associates with every smooth function f on phase space an operator $\hat{f}$ on $L^2(R^2, dx^1 dp_1)$ as follows [W];

$$\hat{f} = -i\nabla_{X_f} + f \tag{3.1}$$

where, for a vector field X on R^2,

$$\nabla_X = X - iX \lrcorner \theta \tag{3.2}$$

and

$$\theta = p_1 \, dx^1. \tag{3.3}$$

One sees easily that, if the flow of X_f is complete, then $\hat{f}$ is the generator of a one-parameter group of unitaries, and hence self-adjoint on its natural domain. Indeed, let

$$(\hat{U}_t \psi)(x^1, p_1) \equiv \psi(\rho_t(x^1, p_1)) \exp\left(i \int_0^t dt' (-X_f \lrcorner \theta + f)\right) \tag{3.4}$$

where ρ_t is the flow of X_f, *i.e.*

$$\frac{d}{dt} \rho_t(x^1, p_1)|_{t=0} = X_f(x^1, p_1)$$

and the integral is taken along the flow line from (x^1, p_1) to $\rho_t(x^1, p_1)$. Then $\hat{U}_t$ is unitary and

$$\begin{aligned} \frac{d}{dt}(\hat{U}_t \psi)(x^1, p_1)|_{t=0} &= (X_f - iX_f \lrcorner \theta + if)\psi(x^1, p_1) \\ &= i(\hat{f}\psi)(x^1, p_1) \end{aligned} \tag{3.5}$$

The main property of the prequantization map is that

$$-i[\hat{f}, \hat{g}] = \widehat{\{f, g\}}. \tag{3.6}$$

This guarantees in particular that the one-parameter groups corresponding via (3.4) to the generators in (2.25) combine to yield a unitary representation $\hat{U}$ of $SO_0(2, 1)$. We shall refer to $\hat{U}$ as the prequantized representation. It is clear from (3.4) that, in order to write this representation globally, one needs to integrate the flow of the $L_{\mu\nu}$ in (2.25), which is a difficult task in the coordinates used. Since we shall need this representation further on, when dealing with quantization proper, we shall use another method for that purpose below. For now, we note the explicit form of the prequantized operators and study their contractions:

$$\hat{L}_{50} = -Y\frac{m^2}{p_0} + iY(\kappa Y)^2 \frac{p_1}{p_0} \partial_{x^1} - \frac{i}{Y}[p_0^2 + (\kappa Y)^2 p_1^2] \frac{x^1}{p_0} \partial_{p_1} \tag{3.7a}$$

$$\hat{L}_{15} = -iY\partial_{x^1} + \frac{i}{Y} x^1 p_1 \partial_{p_1} \tag{3.7b}$$

$$\hat{L}_{01} = -(\kappa Y)\frac{m^2}{p_0}x^1 + i(\kappa Y)^3 x^1 \partial_{x^1} - i\left[\frac{p_0}{\kappa Y} + \kappa^2(\kappa Y)\frac{p_1^2}{p_0}(x^1)^2\right]\partial_{p_1}. \quad (3.7c)$$

It is then clear that, formally at least, we have

$$\lim_{\varkappa \to 0} \kappa \hat{L}_{50} = -\frac{m^2}{p_0} + i\frac{p_1}{p_0}\partial_{x^1} = \hat{H} \quad (3.8a)$$

$$\lim_{\varkappa \to 0} \kappa \hat{L}_{15} = -i\partial_{x^1} = \hat{P} \quad (3.8b)$$

$$\lim_{\varkappa \to 0} \hat{L}_{01} = -\frac{m^2}{p_0}x^1 + i\frac{x^1 p_1}{p_0}\partial_{x^1} - ip_0\partial_{p_1} = \hat{K} \quad (3.8c)$$

where now $p^0 = \sqrt{p_1^2 + m^2}$. More precisely, we have the following theorem:

THEOREM 3.1. – $\forall \psi \in L^2(R^2, dx^1 dp_1)$, *we have*, $\forall (a, b, v) \in R^3$

$$\lim_{\varkappa \to 0} \| e^{-ib\varkappa \hat{L}_{50}}\psi - e^{-ib\hat{H}}\psi \| = 0, \quad (3.9a)$$

$$\lim_{\varkappa \to 0} \| e^{-ia\varkappa \hat{L}_{51}}\psi - e^{-ia\hat{P}}\psi \| = 0, \quad (3.9b)$$

$$\lim_{\varkappa \to 0} \| e^{-iv\hat{L}_{01}}\psi - e^{-iv\hat{K}}\psi \| = 0. \quad (3.9c)$$

Proof. – It suffices to show the result on a dense subset of $L^2(R^2, dx^1 dp_1)$. We take $\psi \in C_0^\infty(R^2)$, which is contained in the domain of each generator and left invariant by the corresponding one-parameter groups of unitaries [*see* (3.4)]. We have

$$\| e^{-ib\varkappa \hat{L}_{50}}\psi - e^{-ib\hat{H}}\psi \| = \| \psi - e^{ib\varkappa \hat{L}_{50}} e^{-ib\hat{H}}\psi \|$$
$$= \left\| i\int_b^0 db'\, e^{ib'\varkappa \hat{L}_{50}}(\kappa \hat{L}_{50} - \hat{H})\, e^{-ib' H}\psi \right\|$$
$$\leqq b \sup_{b' \in [0, b]} \| (\kappa \hat{L}_{50} - \hat{H})\, e^{-ib'\hat{H}}\psi \|.$$

Since the one-parameter group $\exp(-ib'\hat{H})$ can be computed explicitly using (3.4), the result follows easily from simple estimates on the coefficients of the first order partial differential operator $\kappa \hat{L}_{50} - \hat{H}$. This ends the proof of (3.9*a*); (3.9*b*) and (3.9*c*) are proven similarly. □

The above result is the prequantum (and global) equivalent of Proposition 2.1, which describes the behaviour of the classical theory under contraction. In sections 4 and 5, we shall study the contraction of the quantized theory as well. For that purpose we shall need an explicit expression for the prequantized unitary representation $\hat{U}$ of $SO_0(2, 1)$. This is most easily found if one works in an intrinsic form, without reference to the (x^1, p_1) coordinates. To do so we remark that the definition of $\hat{f}$ on $L^2(R^2, dx^1\, dp_1)$ is not completely natural in view of the gauge freedom in the choice of θ in (3.3). General theory teaches us $\hat{f}$ is in fact an operator on a space of L^2-sections of a line bundle with connection over $\Sigma^m_\kappa \cong R^2$, associated to the principal bundle [W]

$$SO_0(2, 1) \rightarrow \Sigma^m_\kappa \cong SO_0(2, 1)/SO(2).$$

This means in particular that the natural Hilbert space for prequantization is not $L^2(\Sigma^m_\kappa, dx^1\, dp_1)$, but rather the space of L^2-functions $\hat{\psi}$ on $SO_0(2, 1)$ satisfying

$$\hat{\psi}(\Lambda \exp(\tau e_{50})) = \exp\left(i\frac{m}{\kappa}\tau\right)\hat{\psi}(\Lambda), \qquad (3.10)$$

for which we shall use the notation $\hat{\mathscr{H}}$. Note that this space is non-trivial if and only if $\frac{m}{\kappa}$ is an integer. Here e_{50} is the element of the Lie algebra of $SO_0(2, 1)$ for which

$$\exp(\tau e_{50}) = \begin{pmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.11)$$

The character $\exp\left(i\frac{m}{\kappa}\tau\right)$ of $SO(2)$ is naturally associated to Σ^m_κ as can be seen upon remarking that $L_{50}(0,0) = \frac{m}{\kappa}$, $L_{15}(0,0) = 0 = L_{01}(0, 0)$. To understand the link between $L^2(\Sigma^m_\kappa, dx^1\, dp_1)$ and the Hilbert space $\hat{\mathscr{H}}$ defined in (3.10), we recall from (2.16)-(2.17) that, as homogeneous spaces $E^m_\kappa \cong SO_0(2, 1)$. Hence functions on $SO_0(2, 1)$ are functions on E^m_κ. To understand what (3.10) implies for those functions, we compute the left-invariant vector fields on $SO_0(2, 1)$ as vector fields on E^m_κ:

$$Y_{\mu\nu}(y, q) = \left(\frac{d}{d\alpha}\Lambda(y, q)\exp\alpha\, e_{\mu\nu}\, y_{(0)}\right)^\rho_{\alpha=0}\frac{\partial}{\partial y^\rho} + \left(\frac{d}{d\alpha}\Lambda(y, q)\exp\alpha\, e_{\mu\nu}\, q_{(0)}\right)^\rho_{\alpha=0}\frac{\partial}{\partial q^\rho}. \qquad (3.12a)$$

Hence

$$Y_{50}(y, q) = -\frac{1}{m\kappa}q\cdot\frac{\partial}{\partial y} + m\kappa\, y\cdot\frac{\partial}{\partial q}, \qquad (3.12b)$$

$$Y_{15}(y, q) = -m^{-1}(q \wedge y)\cdot\partial_y, \qquad (3.12c)$$

$$Y_{01}(y, q) = \kappa(q \wedge y)\cdot\partial_q. \qquad (3.12d)$$

We conclude that L^2-functions on $SO_0(2, 1)$ satisfying (3.10) are in 1-1 correspondence with L^2-functions on $E_\varkappa^m$ satisfying

$$(Y_{50}\hat{\psi})(y, q) = i\frac{m}{\kappa}\hat{\psi}(y, q) \qquad (3.13)$$

with Y_{50} given in (3.12 *b*). Remark now that

$$Y_{50} = -(m\kappa)^{-1} X_\pi \qquad (3.14)$$

where X_π is defined in (2.8). It follows then from (3.13)-(3.14) that the functions $\hat{\psi}$ in (3.13) are entirely determined by their restriction to $\Sigma_\varkappa^m$ and, conversely, that to each element ψ in $L^2(\Sigma_\varkappa^m, dx^1\, dp_1)$, we can associate a unique $\hat{\psi}$ on $E_\varkappa^m$ satisfying (3.13); this establishes the link between $\hat{\mathscr{H}}$ and $L^2(\Sigma_\varkappa^m, dx^1\, dp_1)$. Note that $\hat{\mathscr{H}} \subset L^2(SO_0(2, 1), d\mu_\varkappa^m)$, where the invariant measure $d\mu_\varkappa^m$ is determined uniquely (and not just up to a multiplicative factor) by the requirement that the identification of ψ and $\hat{\psi}$ is unitary.

We now turn to the determination of the prequantized representation $\hat{U}$, as acting on $\hat{\mathscr{H}}$. We claim that,

$$\forall \Lambda \in SO_0(2, 1), \qquad (\hat{U}(\Lambda)\hat{\psi})(y, q) = \hat{\psi}(\Lambda^{-1}(y, s)). \qquad (3.15)$$

To prove (3.15), it suffices to computes explicitly the generators of $\hat{U}$ and to compare with the definition (3.1) of $\hat{L}_{\mu\nu}$. We conclude from (3.15) that the prequantized representation is nothing else than the left regular representation on $SO_0(2, 1)$, restricted to those L^2-functions that satisfy (3.13). In other words, it is the representation of $SO_0(2, 1)$ induced from the character $e^{i(m/\varkappa)\tau}$ of SO (2). Note that if $\frac{m}{\kappa}$ is not integer, then we obtain a representation of the universal covering of $SO_0(2, 1)$.

4. QUANTIZATION AND SO(2, 1) COHERENT STATES

To quantize, we use the methods of geometric quantization to select an irreducible subrepresentation of the prequantized representation, using a positive invariant Kähler polarization on $\Sigma_\varkappa^m$. A general discussion of the concept of polarization can be found in [W]. Here we content ourselves with a definition that is correct in the specific example dealt with in this paper.

DEFINITION 4.1. — *A positive invariant Kähler polarization on* $(\Sigma_\varkappa^m, \omega_\varkappa^m)$ *is a complex vector field* $\tilde{Z}_\varkappa^m$ *on* $\Sigma_\varkappa^m$ *satisfying the following properties:*

(i) $\tilde{Z}_\varkappa^m + \bar{\tilde{Z}}_\varkappa^m$ *and* $i(\tilde{Z}_\varkappa^m - \bar{\tilde{Z}}_\varkappa^m)$ span $T_{(x^1, p_1)}\Sigma_\varkappa^m$ *at each point* (x^1, p_1) *of* $\Sigma_\varkappa^m$;

(ii) *there exists a function* $\beta_{\mu\nu}$ *on* $\Sigma^m_\varkappa$ *such that*

$$[\tilde{X}_{L_{\mu\nu}}, \tilde{Z}^m_\varkappa] = \beta_{\mu\nu}\tilde{Z}^m_\varkappa, \qquad \forall\,\mu, \nu = 5, 0, 1; \tag{4.1}$$

(iii) *defining* $J^m_\varkappa : T^C\Sigma^m_\varkappa \to T^C\Sigma^m_\varkappa$ *by*

$$J^m_\varkappa \tilde{Z}^m_\varkappa = i\tilde{Z}^m_\varkappa, \qquad J^m_\varkappa \bar{\tilde{Z}}^m_\varkappa = -i\bar{\tilde{Z}}^m_\varkappa \tag{4.2}$$

and extending linearly, we define

$$g^m_\varkappa(X, Y) = \omega^m_\varkappa(JX, Y) \tag{4.3}$$

We require then that $g^m_\varkappa$ *is a positive non-degenerate metric on* $\Sigma^m_\varkappa$.

Conditions (i), (ii) and (iii) refer respectively to the Kählerian, invariant and positive character of the polarization. Invariant Kähler polarizations can be characterized algebraically [W]. This allows us to prove the following.

LEMMA 4.2. — *There exists on* $\Sigma^m_\varkappa$ *a positive invariant Kähler polarization, unique up to a multiplicative factor, given by*:

$$\tilde{Z}^m_\varkappa = Y_{01} + iY_{15} - i\kappa(q\wedge y)^0(q^0)^{-1}Y_{50} \tag{4.4a}$$
$$= \tilde{Y}_{01} + i\tilde{Y}_{15} \tag{4.4b}$$

where $\tilde{Y}_{01} = Y_{01}$ *and* $\tilde{Y}_{15} = Y_{15} - \kappa(q\wedge y)^0(q^0)^{-1}Y_{50}$ *are both tangent to* $\Sigma^m_\varkappa$.

The proof of Lemma 4.2 is given in Appendix A. Equation (4.2) now implies

$$J^m_\varkappa \tilde{Y}_{01} = -\tilde{Y}_{15}; \qquad J^m_\varkappa \tilde{Y}_{15} = \tilde{Y}_{01} \tag{4.5}$$

so that (4.3) yields

$$g^m_\varkappa(\tilde{Y}_{01}, \tilde{Y}_{01}) = \frac{m}{\kappa} = g^m_\varkappa(\tilde{Y}_{15}, \tilde{Y}_{15}) \tag{4.6}$$

We also introduce

$$Z^m_\varkappa \equiv Y_{01} + iY_{15} \tag{4.7}$$

which is defined on all of $E^m_\varkappa$ [*see* (3.12)].

We now show how the geometric quantization program allows us to use the Kählerian polarization determined above to construct a unitary irreducible subrepresentation of $\hat{U}$. First, one defines the Hilbert space

$$\mathscr{H}^m_\varkappa \equiv \{\psi \in L^2(R^2, dx^1\,dp_1)\,|\,\nabla_{\bar{Z}^m_\varkappa}\psi = 0\}. \tag{4.8}$$

The arguments in Appendix B show that this space is infinite dimensional if $\frac{m}{\kappa} > \frac{1}{2}$ and trivial otherwise. Since we take κ to zero, we can assume to be in the first case. Then, one observes that the prequantized operators $\hat{L}_{\mu\nu}$ in (3.7) leave $\mathscr{H}^m_\varkappa$ invariant, *i.e.*

$$\forall\,\psi\in\mathscr{H}^m_\varkappa, \qquad \hat{L}_{\mu\nu}\psi\in\mathscr{H}^m_\varkappa. \tag{4.9}$$

This follows from the general theory, using (4.1) [W] and can be verified by a direct computation. Equation (4.9) implies that $\hat{U}$ restricts to a representation on $\mathcal{H}^m_\varkappa$ that we shall denote by $U^m_\varkappa$. We write $L^{(m,\varkappa)}_{\mu\nu}$ for its generators, *i.e.* $L^{(m,\varkappa)}_{\mu\nu}=\hat{L}_{\mu\nu}|_{\mathcal{H}^m_\varkappa}$.

We show in Appendix B that the representation $(U^m_\varkappa, \mathcal{H}^m_\varkappa)$ is irreducible and square integrable, *i.e.* that it belongs to the discrete series of representations. $\mathcal{H}^m_\varkappa$ is the Hilbert space of the quantized theory. Each state ψ in $\mathcal{H}^m_\varkappa$ yields a phase space probability distribution $|\psi|^2$. Now, two representations $(\mathcal{H}^m_\varkappa, U^m_\varkappa)$ and $(\mathcal{H}^{m'}_{\varkappa'}, U^{m'}_{\varkappa'})$ are unitarily equivalent if and only if $\frac{m}{\kappa}=\frac{m'}{\kappa'}$ (Appendix B). Nevertheless, the fact that $\mathcal{H}^m_\varkappa$ and $\mathcal{H}^{m'}_{\varkappa'}$ are different as subspaces of $L^2(R^2, dx^1\,dp_1)$ has as a result that the quantum states they contain have different physical interpretations. This is as it should be, since $M_\varkappa$ is not isometric to $M_{\varkappa'}$ if $\kappa\neq\kappa'$ and should be compared to the comments of section 2 [before (2.19)].

In the following, we study those quantum states that are optimally localized, in a sense we now make precise. We shall say the quantum state $\psi\in\mathcal{H}^m_\varkappa$ is localized at the phase space point (x^1, p_1), provided [*see* (2.25)]

$$\langle L^{(\varkappa,m)}_{01}\rangle=L_{01}(x^1, p_1), \qquad \langle L^{(\varkappa,m)}_{15}\rangle=L_{15}(x^1, p_1). \tag{4.10a}$$

We say ψ is optimally localized if, in addition,

$$\langle L^{(\varkappa,m)}_{50}\rangle=L_{50}(x^1, p_1). \tag{4.10b}$$

Here $\langle\,.\,\rangle$ denotes the expectation value in the state ψ. This definition of "localized" is justified by the observation that the values of L_{01} and L_{51} uniquely determine a point (x^1, p_1) in $\Sigma^m_\varkappa$ [*see* (2.25)]. We will justify (4.10 *b*) below.

The state optimally localized at (0, 0) is easily seen to be unique; it is the eigenstate of $L^{(\varkappa,m)}_{50}$ with eigenvalue $\frac{m}{\kappa}$ (*i.e.* the ground state). We denote it by $\varphi_{(0,0)}$. Consequently, all optimally localized states are unique and we denote them by $\varphi_{(x^1,p_1)}$. They can be obtained by applying the unitary operator $U^m_\varkappa(\Lambda(y, q))$ to the state $\varphi_{(0,0)}$. Here (y, q) is related to (x^1, p_1), by (2.19) and (2.23) for $x^0=0$, and $\Lambda(y, q)$ is given in (2.17). In other words

$$\varphi_{(x^1,p_1)}=U^m_\varkappa(\Lambda(y, q))\,\varphi_{(0,0)} \tag{4.11}$$

The states $\varphi_{(x^1,p_1)}$ are also called coherent states [Pe]. Their properties were studied in [Pe] in the context of the Fock-Bargmann realization of the representation $(\mathcal{H}^m_\varkappa, U^m_\varkappa)$, where it is proven in particular that they minimize the "invariant dispersion" of the Casimir operator. This in itself does not justify the term "optimal localization" used above, since the Casimir operator $-L^2_{50}+L^2_{15}+L^2_{01}$ is not positive definite. However, one can verify that $\varphi_{(0,0)}$ minimizes the uncertainty relation

$$(\Delta L_{01}^{(m,\varkappa)})^2 (\Delta L_{15}^{(m,\varkappa)})^2 \geqslant \frac{1}{4} \langle L_{50}^{(m,\varkappa)} \rangle^2 \tag{4.12a}$$

where $\Delta L_{01}^{(m,\varkappa)} = [\langle (L_{01}^{(m,\varkappa)})^2 \rangle - \langle L_{01}^{(m,\varkappa)} \rangle^2]^{1/2}$, etc. For $\varphi_{(0,0)}$ one actually verifies that

$$\Delta L_{01}^{(m,\varkappa)} = \frac{1}{2}\sqrt{\frac{m}{\kappa}} = \Delta L_{51}^{(m,\varkappa)}. \tag{4.13}$$

We now compute $\varphi_{(0,0)}$. It will be convenient to consider the Hilbert spaces $\mathcal{H}_\varkappa^m$ in (4.8) as subspaces of $\hat{\mathcal{H}}$, the space of L^2-functions on $E_\varkappa^m$ satisfying (3.10). We shall write $\hat{\mathcal{H}}_\varkappa^m$ for these spaces. Recall that we showed in section 3 how to identify $\hat{\mathcal{H}}$ with $L^2(\Sigma_\varkappa^m, dx^1 dp_1)$. We first claim that

$$\hat{\mathcal{H}}_\varkappa^m = \{ \hat{\psi} \in \hat{\mathcal{H}} \mid \bar{Z}_\varkappa^m(\hat{\psi}) = 0 \} \tag{4.14}$$

where $Z_\varkappa^m$ is defined in (4.7). This is easily proven by a direct calculation using (4.4), (4.8), (3.12) and (3.13), or by remarking that $Z_\varkappa^m$ is the horizontal lift of $Z_\varkappa^m$ in (4.4) to the prequantum line bundle $E_\varkappa^m \to \Sigma_\varkappa^m$. The ground state $\hat{\varphi}_{(0,0)} \equiv \hat{\varphi}_0 \in \hat{\mathcal{H}}_\varkappa^m$ of the generator of time translations $\hat{L}_{50}$ is determined uniquely (up to normalization) by the following three equations. First, from (3.13),

$$Y_{50}(\hat{\varphi}_0) = i\frac{m}{\kappa}\hat{\varphi}_0. \tag{4.16}$$

Next, from (4.14),

$$\bar{Z}_\varkappa^m(\hat{\varphi}_0) = 0. \tag{4.17}$$

Finally, remarking that the generators of $\hat{U}$ in (3.15) are the right-invariant vector fields $X_{L_{\mu\nu}}$,

$$X_{L_{50}}(\hat{\varphi}_0) = i\frac{m}{\kappa}\hat{\varphi}_0. \tag{4.18}$$

To solve (4.16-4.18) for $\hat{\varphi}_0$, it is convenient to introduce

$$z = \kappa y - \mathrm{i}m^{-1} q, \qquad z \in \mathbb{C}^3, \tag{4.19}$$

in terms of which

$$Z_\varkappa^m = \frac{\kappa}{m}(q \wedge y)\cdot(m\,\partial_q - i\kappa^{-1}\partial_y) = (z \wedge \bar{z})\cdot\partial_z, \tag{4.20}$$

$$Y_{50} = i(\bar{z}\cdot\partial_{\bar{z}} - z\cdot\partial_z), \tag{4.21}$$

$$X_{L_{50}} = z_5\frac{\partial}{\partial z^0} - z_0\frac{\partial}{\partial z^5} + \bar{z}_5\frac{\partial}{\partial \bar{z}^0} - \bar{z}_0\frac{\partial}{\partial \bar{z}^5}. \tag{4.22}$$

We note that (4.19) identifies $T^* R^3$ with $\mathbb{C}^3$ and that the surface $E_\varkappa^m$ is now given by:

$$z^2 = 0 \tag{4.23a}$$

and

$$\bar{z}\cdot z = -2, \tag{4.23 b}$$

where $z^2 = -z_5^2 - z_0^2 + z_1^2$ and $\bar{z}.z = -\bar{z}_5 z_5 - \bar{z}_0 z_0 + \bar{z}_1 z_1$.

A simple calculation allows us then to solve (4.16)-(4.18) uniquely for

$$\hat{\varphi}_0(y, q) = (-2)^{m/\varkappa}(z_5 + iz_0)^{-m/\varkappa} \tag{4.24}$$

where the normalization is chosen so that $\hat{\varphi}_0(y_{(0)}, q_{(0)}) = 1$. Now, for $z' = \kappa y' - im^{-1} q' \in E^m_\varkappa$, we define

$$\hat{\varphi}_{z'}(y, q) = [\hat{U}(\Lambda(y', q'))\hat{\varphi}_0](y, q) \tag{4.25}$$

which, using (2.17) and (3.15), gives

$$\hat{\varphi}_{z'}(y, q) = (-2)^{m/\varkappa}(\bar{z}'.z)^{-m/\varkappa}. \tag{4.26 a}$$

One can also prove that this implies (*see* Appendix B)

$$\langle \hat{\varphi}_z, \hat{\varphi}_z \rangle = 2\pi \frac{2(m/\kappa)}{2(m/\kappa) - 1}. \tag{4.26 b}$$

Since the prequantized representation $\hat{U}$ leaves $\hat{\mathscr{H}}^m_\varkappa$ invariant, we see that $\hat{\varphi}_{z'} \in \hat{\mathscr{H}}^m_\varkappa$, $\forall z' \in E^m_\varkappa$. The states $\varphi_{(x'^1, p'_1)} \in \mathscr{H}^m_\varkappa$ in (4.11) are obtained from the states $\hat{\varphi}_{z'}$ by restricting both z' and z to $\Sigma^m_\varkappa \subset E^m_\varkappa$. Using the definition of z in (4.19) as well as (2.19) and (2.23), one obtains their explicit expression.

In order to understand the contraction of the quantized theory, we now study the contraction of the states $\varphi_{(x^1, p_1)}$. The result is contained in the following lemma.

LEMMA 4.3. – $\forall z, z' \in \Sigma^m_\varkappa$,

$$\lim_{\varkappa \to 0} \frac{1}{2\pi}\sqrt{\frac{m}{\kappa}}\,\varphi_{(x'^1, p'_1)}(x^1, p_1) = \frac{1}{\sqrt{\pi}} p^0 \delta(p_1 - p'_1)\, e^{i(x^1 - x'^1)p_1}. \tag{4.27}$$

Proof. – First, we recall that the surface $\Sigma^m_\varkappa \subset C^3$ is realized through (4.23) and $y^0 = 0$ so that, using (4.19), (2.19), and (2.23), we can write,

$$z_5 = -\kappa Y + i\frac{\kappa}{m}(\kappa Y)x^1 p_1, \tag{4.28 a}$$

$$z_0 = -i\frac{1}{(\kappa Y)}\frac{p_0}{m} = i\frac{\sqrt{(\kappa Y)^2 p_1^2 + m^2}}{m}, \tag{4.28 b}$$

$$z_1 = \kappa x^1 - i(\kappa Y)^2 \frac{p_1}{m}. \tag{4.28 c}$$

Expanding (4.28) in terms of κ, we obtain after some calculation (noting that from now on, $p^0=\sqrt{p_1^2+m^2}$),

$$\lim_{\varkappa\to 0}\frac{1}{2\pi}\sqrt{\frac{m}{\kappa}}\,\varphi_{(x'^1,\,p'_1)}(x^1,p_1)$$

$$=\frac{1}{2\pi}\lim_{\varkappa\to 0}\sqrt{\frac{m}{\kappa}}\left(\frac{m^2+p'_0p_0-p'_1p_1}{2m^2}\right)^{-m/\varkappa}\times\exp i\left(m^2\frac{(p_1+p'_1)(x^1-x'^1)}{m^2+p'_0p_0-p'_1p_1}\right). \quad (4.29)$$

Setting $\varepsilon=\sqrt{\frac{\kappa}{m}}$ and using

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\sqrt{\pi}}\exp-\frac{x^2}{\varepsilon^2}=\delta(x), \quad (4.30)$$

the result follows after some further computation. □

The result in (4.27) merits some comments. First we note that the contracted states are no longer in $L^2(R^2, dx^1\,dp_1)$. This suggests that the family of Hilbert spaces $\mathcal{H}^m_\varkappa$ does not converge to any limiting Hilbert space contained in $L^2(R^2, dx^1\,dp_1)$. In other words, the limiting Poincaré invariant theory does not arise naturally as a theory formulated on phase space. In section 5 we shall analyse this phenomenon in some detail and relate it to the contraction of the polarization $Z^m_\varkappa$ and in particular to the observation that there is no Poincaré invariant Kähler polarization on $\Sigma^m_0\cong R^2$. In order to identify the quantum Hilbert space of the limiting theory, that we shall denote $\mathcal{H}^m_0$, we remark that the limiting states in (4.27) are (generalized) eigenstates of $\hat{P}$ and $\hat{H}$ [*see* (3.8)] with eigenvalues p'_1 and $\sqrt{p'^2_1+m^2}$, respectively. Hence they span an irreducible unitary representation of the Poincaré group, an observation we shall also make more precise in section 5. Finally, we see from (4.13) that

$$\Delta L_{01}^{(m,\,\varkappa)}\to\infty, \qquad \Delta\kappa\, L_{51}^{(m,\,\varkappa)}\to 0$$

as $\kappa\to 0$, which is also consistent with (4.27) and (2.29*b-c*): the coherent states, when contracted, become exceedingly well localized in momentum, while delocalizing completely in position. In [DBEG], the coherent states are realized as solutions to the Klein-Gordon equation on $M_\varkappa$ and it is shown there how in this picture they contract to plane waves on M_0, which is the spacetime analog of the above phase space picture.

5. CONTRACTION OF THE POLARIZATION

The contraction of the classical and the prequantized theories in sections 2 and 3 was a relatively straightforward matter. This is no longer so at the quantum level, as suggested by Lemma 4.3 and the comments following it. Ideally, we would like to prove an analog of Theorem 3.1 for the operators $L_{\mu\nu}^{(m,\varkappa)}$; we remark however that for different κ, they are defined on different Hilbert spaces $\mathscr{H}_\varkappa^m$ as opposed to the $\hat{L}_{\mu\nu}$ in (3.7), which are for all κ defined on $L^2(R^2, dx^1\, dp_1)$. In other words, we are now dealing with a family of operators $U_\varkappa^m(\Lambda)$, defined on a family of Hilbert spaces $\mathscr{H}_\varkappa^m$, and it is not *a priori* clear how to make sense out of the limit of either as κ tends to zero (*see* [CDB]).

To shed light on this question, we shall make use of the observation that the Hilbert spaces $\mathscr{H}_\varkappa^m$ are all subspaces of $L^2(R^2, dx^1\, dp_1)$. Introducing the corresponding the corresponding projectors

$$\Pi_\varkappa^m:\quad L^2(R^2, dx^1\, dp_1) \to \mathscr{H}_\varkappa^m \subset L^2(R^2, dx^1\, dp_1), \qquad (5.1\,a)$$

we study their limit as κ tends to zero. We start by showing that the $\Pi_\varkappa^m$ do not have a non-trivial limit in the weak operator topology by studying the contraction of the polarization $\tilde{Z}_\varkappa^m$ as $\kappa \to 0$ (Lemma 5.2).

PROPOSITION 5.1. – *If for some* $\psi \in L^2(R^2, dx^1\, dp_1)$ *there exists a* $\psi_0 \in L^2(R^2, dx^1\, dp_1)$ *so that*

$$w-\lim_{\varkappa \to 0} \Pi_\varkappa^m \psi = \psi_0$$

then $\psi_0 = 0$.

This makes the comment following Lemma 4.3 precise *i.e.*, if $w-\lim_{\varkappa \to 0} \Pi_\varkappa^m$ exists, it is identically equal to zero.

To prove Proposition 5.1 we start by recalling from (4.4) and (3.12) that

$$\tilde{Z}_\varkappa^m = \frac{\kappa}{m}(q \wedge y)(m\,\partial_q - i\kappa^{-1}\,\partial_y) - i\kappa (q \wedge y)^0 (q^0)^{-1}\, Y_{50} \qquad (5.2\,a)$$

with

$$Y_{50} = -\frac{1}{m\kappa} q \cdot \frac{\partial}{\partial y} + m\kappa y \cdot \frac{\partial}{\partial q}. \qquad (5.2\,b)$$

Since $\tilde{Z}_\varkappa^m$ is tangent to $\Sigma_\varkappa^m$, it is possible to expand it on the basis of $T\Sigma_\varkappa^m$ given by $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial p_1}$. The lengthy but straightforward calculation is based on (2.19), (2.5) and (2.25) and yields:

$$\tilde{Z}_\varkappa^m(x^1, p_1) = (\kappa Y)^{-2} p_0 \frac{\partial}{\partial p_1} - i\left[(\kappa m)\frac{x^1 p_1}{p_0}\frac{\partial}{\partial p_1} - \frac{m}{\kappa}\frac{(\kappa Y)^2}{\rho_0}\frac{\partial}{\partial x^1}\right] \qquad (5.3)$$

Formally, we have then

$$\lim_{\varkappa \to 0} \kappa \tilde{Z}_\varkappa^m = \tilde{Z}_0^m \qquad (5.4)$$

where

$$\tilde{Z}_0^m = -i \frac{m}{\sqrt{p_1^2+m^2}} \frac{\partial}{\partial x^1}. \tag{5.5}$$

Applying the analysis of Appendix A to the Poincaré group, it is not hard to show that $\tilde{Z}_0^m$ is the unique Poincaré-invariant polarization on phase space. This explains its appearance as the zero curvature limit of the $\tilde{Z}_\varkappa^m$. Note nevertheless that $\tilde{Z}_0^m$ is no longer positive or Kählerian, which explains why the Hilbert space of the limiting theory is not a subspace of $L^2(R^2, dx^1\, dp_1)$ [*see* (5.9)].

The following Lemma gives a precise meaning to (5.4).

LEMMA 5.2. – *Let* $\psi \in \mathscr{S}(R^2)$ (*Schwartz space*). *Then*

$$s-\lim_{\varkappa \to 0} \| (\kappa \tilde{Z}_\varkappa^m - \tilde{Z}_0^m)\psi \| = 0, \tag{5.6 a}$$

$$s-\lim_{\varkappa \to 0} \| (\kappa \nabla_{\bar{\tilde{Z}}_\varkappa^m})^* - \nabla_{\bar{\tilde{Z}}_0^m}] \psi \| = 0. \tag{5.6 b}$$

The proof consists of a simple estimate that we omit. The functions $\psi(x^1, p_1)$, covariantly constant along $\tilde{Z}_0^m$, *i. e.*

$$\nabla_{\bar{\tilde{Z}}_0^m} \psi = 0, \tag{5.7 a}$$

are of the form

$$\psi(x^1, p_1) = \phi(p_1) e^{ip_1 x^1}. \tag{5.7 b}$$

Such functions can never be in $L^2(R^2, dx^1\, dp_1)$. We use this observation together with Lemma 5.2 to prove Proposition 5.1.

Proof of Proposition 5.1. – For $\phi \in \mathscr{S}(R^2)$, we have

$$\begin{aligned} |\langle \nabla_{\bar{\tilde{Z}}_0^m} \phi, \psi_0 \rangle| &= |\lim_{\varkappa \to 0} \langle \nabla_{\bar{\tilde{Z}}_0^m} \phi, \Pi_\varkappa^m \psi \rangle| \\ &= |\lim_{\varkappa \to 0} \langle \nabla_{\bar{\tilde{Z}}_0^m} \phi - \kappa (\nabla_{\bar{\tilde{Z}}_0^m})^* \phi, \Pi_\varkappa^m \psi \rangle| \\ &\leqq \lim_{\varkappa \to 0} \| \nabla_{\bar{\tilde{Z}}_0^m} - \kappa (\nabla_{\bar{\tilde{Z}}_0^m})^* \phi \| \, \| \psi \| = 0 \end{aligned}$$

where we used Lemma 5.2 and that $\nabla_{\bar{\tilde{Z}}_0^m} \Pi_\varkappa^m = 0$. So

$$\langle \nabla_{\bar{\tilde{Z}}_0^m} \phi, \psi_0 \rangle = 0, \qquad \forall \phi \in \mathscr{S}(R^2).$$

Hence $\psi_0 \in \mathscr{D}(\nabla_{\bar{\tilde{Z}}_0^m})$ and $\nabla_{\bar{\tilde{Z}}_0^m} \psi_0 = 0$. As a result of (5.7), we conclude that $\psi_0 = 0$. □

Proposition 5.1 explains why the Hilbert space $\mathscr{H}_0^m$ of the limiting theory can not be a subspace of $L^2(R^2, dx^1\, dp_1)$. To correctly identify $\mathscr{H}_0^m$, we remark that the solutions of (5.7 *a*) are determined completely by their restriction to $x^1 = 0$ [*see* (5.7 *b*)]. But the line $x^1 = 0$ has a group theoretic and κ-independent meaning. It is the orbit of (0, 0) under the SO(1, 1) subgroup of $SO_0(2, 1)$ generated by L_{01}, *i.e.* of the subgroup with respect to which we contract. We have

$$[X_{L_{01}}]_{x^1=0}=\frac{1}{\sqrt{p_1^2+m^2}}\partial_{p_1}. \tag{5.8 a}$$

The unique κ-independent SO (1, 1) invariant measure on $x^1=0$ is

$$\frac{dp_1}{\sqrt{p_1^2+m^2}}. \tag{5.8 b}$$

Hence we define

$$\mathcal{H}_0^m=\Big\{\psi\,|\,\nabla_{\bar{Z}_0^m}\psi=0 \text{ and } \int|\psi(0, p_1)|^2\frac{dp_1}{p^0}<\infty\Big\}\cong L^2\Big(R, \frac{dp_1}{p^0}\Big). \tag{5.9}$$

It is now easy to verify that $\mathcal{H}_0^m$ is invariant under the action of $\hat{H}$, $\hat{P}$ and $\hat{K}$ in (3.8). One has

$$(\hat{H}\psi)(x^1, p_1)=(p^0\phi)(p_1)e^{ip_1x^1}, \tag{5.10 a}$$

$$(\hat{P}\psi)(x^1, p_1)=(p_1\phi)(p_1)e^{ip_1x^1}, \tag{5.10 b}$$

$$(\hat{K}\psi)(x^1, p_1)=ip^0\partial_{p_1}\phi(p_1)e^{ip_1x^1}, \tag{5.10 c}$$

so that one can see that the prequantized representation of the Poincaré group generated by $\hat{H}$, $\hat{P}$ and $\hat{K}$ restricts to a unitary irreducible representation on $\mathcal{H}_0^m$, equivalent to the usual Wigner representation. To conclude we remark that the limiting states in (4.27) are of the form (5.7 *b*) and are indeed (generalized) eigenstates of $\hat{H}$ in $\mathcal{H}_0^m$.

6. CONCLUSION

We have shown in which sense physics on the anti-de Sitter spacetime can be viewed as a perturbation of the Minkowski theory for small curvature. Our treatment of the classical (and prequantized) theory can be considered complete. In the quantum theory, we identified a particular family of states (the coherent states) and showed they have a correct zero curvature limit, tending to eigenstates of the Hamiltonian in the flat spacetime limit. In addition, one would like to prove an analog of Theorem 3.1 for the quantum theory. This was achieved after the completion of this work in [CDB].

The contraction of the generators in the quantum theory (but not of the coherent states) was also studied in [AAG]. There, the irreducible representation of $SO_0(2, 1)$ is taken in its Fock-Bargmann realization (*see* Appendix B), as a space of holomorphic functions on a classical domain, to be thought of as the classical phase space (the case $SO_0(3, 2)$ is treated in [BEGG]). The contraction is performed using a κ-dependent real coordinatization of the phase space, given by

$$z=\frac{m\kappa q+ip}{\pi_0+m},\qquad \pi_0=[m^2(1+\kappa^2 q^2)+p^2]^{1/2},\qquad (q,p)\in\mathrm{R}^2. \quad (7.1)$$

This parametrization is not given a physical interpretation, however. Two problems are then encountered. First, divergent terms in κ^{-1} appear and are eliminated upon requiring that the states of the limiting theory satisfy a polarization condition emerging from the calculation. Second, the generators obtained in the limit do not satisfy the commutation relations of the Poincaré group, but a spurious factor $\frac{1}{2}$ appears which is eliminated by an ad hoc scaling argument. The work in this paper shows that the polarization that appears is in fact the unique Poincaré-invariant polarization on phase space and can be obtained as the limit of $SO_0(2, 1)$-invariant Kählerian polarizations. It can then also be seen that the factor $\frac{1}{2}$ appears because what is contracted in [AAG] [BEGG] is actually the holomorphic part of the prequantized representation. Adding back in the "forgotten" anti-holomorphic part of the prequantized representation eliminates both the terms in κ^{-1} and the bothersome $\frac{1}{2}$.

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APPENDIX A

An algebraic characterization of a positive invariant polarization on Σ_κ^m is possible, since the latter is a homogeneous symplectic space for $SO_0(2, 1)$, namely $\Sigma_\kappa^m \cong SO_0(2, 1)/SO(2)$ [W] [R]. To find such a polarization, we first need to look for a complex subalgebra h of $so^c(2, 1)$, the complexified Lie algebra of $SO_0(2, 1)$, satisfying

(i) $\{e_{50}\}\subset h,$

(ii) $\dim_c h=2,$

(iii) $B(e_{50},[h,h])=0,$

(iv) h *is* $\mathrm{Ad}_{\exp \tau e_{50}}$-invariant,

(v) $iB(e_{50},[\bar{h},h])\geqq 0,$

here $\dim_c h$ is the complex dimension of h, $B(\ ,\ )$ is the Killing form for $SO_0(2, 1)$ and Ad is the adjoint representation of $SO_0(2, 1)$. Conditions (iv) and (v) are the algebraic counterparts of the equations (4.1) and (4.3), respectively. Once h is obtained, we construct for each one of its elements the corresponding complex left invariant vector field on $SO_0(2, 1) \cong E^m_\varkappa$. Their projection onto $T^C{}^m_\varkappa$ yields the positive invariant polarization on $\Sigma^m_\varkappa$.

Using (i) to (v) above, one easily finds h to be the algebra generated by e_{50} and $e_{01}+ie_{15}$. The algebra Y_h is then generated by Y_{50} and $Z^m_\varkappa \equiv Y_{01}+iY_{15}$, where Y_{50}, Y_{01} and Y_{15} are given in (3.12). Therefore the unique (up to a multiplicative factor) positive invariant polarization $\tilde{Z}^m_\varkappa$ on $\Sigma^m_\varkappa$ is obtained by projecting Y_h onto $T^C\Sigma^m_\varkappa$, which gives

$$\tilde{Z}^m_\varkappa = \tilde{Y}_{01} + i\tilde{Y}_{15}. \tag{A.1}$$

To project $Z^m_\varkappa$ on $T^C\Sigma^m_\varkappa$ one first looks for the unique functions α_{01} and α_{15} satisfying

$$[Y_{01}+\alpha_{01} Y_{50}](y^0)=0, \tag{A.2a}$$

and

$$[Y_{15}+\alpha_{15} Y_{50}](y^0)=0. \tag{A.2b}$$

Then the projected vector fields $\tilde{Y}_{01}$ and $\tilde{Y}_{15}$ are obtained by taking $y^0=0$ in $Y_{01}+\alpha_{01} Y_{50}$ and $Y_{15}+\alpha_{15} Y_{50}$, respectively. One easily finds

$$\alpha_{01}=0 \quad \text{and} \quad \alpha_{15}=-\kappa(q \wedge y)^0 (q^0)^{-1}. \tag{A.3}$$

This proves Lemma 4.2.

APPENDIX B

We display here explicitly the unitary transformation intertwining the representation $(U^m_\varkappa, \mathscr{H}^m_\varkappa)$ of section 4 and the discrete series Fock-Bargmann representation of $SO_0(2, 1)$ that we now describe [Pe]. This will both establish the irreducibility of $(U^m_\varkappa, \mathscr{H}^m_\varkappa)$ and allow us to verify (4.26b). Let $\mathscr{H}^{E_0}_{FB}$ be the Hilbert space of analytic functions $f(z)$ on the unit disc $D \subset C$ with norm

$$\| f \|_{E_0} \equiv \frac{2E_0-1}{\pi} \int_D |f(z)|^2 (1-|z|^2)^{2E_0-2} d^2 z, \tag{B.1a}$$

where

$$d^2 z = \frac{i}{2} dz \wedge d\bar{z}. \tag{B.1b}$$

For $E_0 > \frac{1}{2}$, $\mathscr{H}^{E_0}_{FB}$ carries a unitary irreducible representation of the covering of $SO_0(2, 1)$ which, for E_0 integer, yields a unitary irreducible and square integrable representation of $SO_0(2, 1)$ itself [Pe]. In our notation, its generators are

$$K_{01} = -\left(\frac{z^2+1}{2}\frac{d}{dz} + E_0 z\right), \qquad (B.2a)$$

$$K_{15} = -i\left(\frac{z^2-1}{2}\frac{d}{dz} + E_0 z\right), \qquad (B.2b)$$

$$K_{50} = z\frac{d}{dz} + E_0. \qquad (B.2c)$$

To find the unitary mapping from $\mathcal{H}_\varkappa^m$ onto $\mathcal{H}_{FB}^{E_0}$, with

$$E_0 = \frac{m}{\kappa}, \qquad (B.3)$$

we proceed as follows. First, we find the complex coordinate function $\zeta(x^1, p_1)$ on $\Sigma_\varkappa^m$ associated to the Kähler polarization $\tilde{Z}_\varkappa^m$ determined in section 4; $\zeta(x^1, p_1)$ is determined uniquely, up to a conformal transformation, by

$$\tilde{Z}_\varkappa^m(\zeta) = 0. \qquad (B.4)$$

A solution of (B.4) is

$$\zeta(x^1, p_1) = \tau((\kappa Y) p_1) + i\rho(x^1). \qquad (B.5)$$

Here

$$\tau(\eta) = \cot^{-1}\left(-\frac{\eta}{m}\right), \qquad \eta \in R, \qquad (B.6a)$$

and

$$\rho(x) = sh^{-1}(\kappa x). \qquad (B.6b)$$

Note that $\rho \in R$, $\tau \in (0, \pi)$. It is then not hard to show, using the definition (4.8), that any $\psi \in \mathcal{H}_\varkappa^m$ can be written as

$$\psi(x^1, p_1) = (\sin\tau)^{-m/\varkappa} f(\bar{\zeta}) \qquad (B.7)$$

where f is an analytic function on the strip $R + i(0, \pi)$. Finally, introduce

$$z = \frac{e^{i\zeta} - i}{e^{i\zeta} + i} \in C, \qquad |z| < 1. \qquad (B.8)$$

Lengthy, but straightforward calculations then establish

$$\omega = dx^1 \wedge dp_1 = 2i\frac{m}{\kappa}\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} \qquad (B.9a)$$

and

$$(\sin\tau)^2 = \frac{(1-|z|^2)^2}{|1-z^2|^2}. \qquad (B.9b)$$

Now define

$$\mathcal{H}_\varkappa^m \ni \psi \overset{U}{\mapsto} (U\psi) \in \mathcal{H}_{FB}^{E_0} \qquad (B.10a)$$

by

$$(\mathrm{U}\psi)(z)=\left(2\pi\frac{2\mathrm{E}_0}{2\mathrm{E}_0-1}\right)^{1/2}(1-z^2)^{-\mathrm{E}_0}f(\zeta(z)) \qquad (\mathrm{B}.10\,b)$$

where f is defined in (B.7). Then

$$\begin{aligned}\|\psi\|^2&=\int|\psi|^2(x^1,p_1)\,dx^1\,dp_1\\&=\int|(1-z^2)^{-\mathrm{E}_0}f(\zeta(z))|^2\,2i\mathrm{E}_0\frac{dz\wedge d\bar{z}}{(1-|z|^2)^{2-2\mathrm{E}_0}}\\&=\frac{2\mathrm{E}_0-1}{\pi}\int|(\mathrm{U}\psi)(z)|^2\frac{d^2z}{(1-|z|^2)^{2-2\mathrm{E}_0}},\end{aligned}$$

so that

$$\|\psi\|^2=\|\mathrm{U}\psi\|^2_{\mathrm{E}_0}. \qquad (\mathrm{B}.10\,c)$$

This establishes the unitary of U. Furthermore, using (B.10), one establishes that

$$\mathrm{U}\hat{\mathrm{L}}_{\mu\nu}\mathrm{U}^{-1}=\mathrm{K}_{\mu\nu}, \qquad (\mathrm{B}.11)$$

so that we can conclude that U intertwines the Fock-Bargmann representation and $(\mathrm{U}^m_\varkappa, \mathscr{H}^m_\varkappa)$. Note that the spectrum of K_{50} and hence of $\mathrm{L}^{(m,\varkappa)}_{50}$ is $\frac{m}{\kappa}, \frac{m}{\kappa}+1, \ldots, \frac{m}{\kappa}+n, \ldots$

It is now easy to establish (4.26 *b*) by computing the norm of $(\mathrm{U}_{\varphi(0,0)})(z)$.

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