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Lagrangian theory for presymplectic systems

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ABSTRACT. — We analyze the lagrangian formalism for singular lagrangian systems in a geometrical way. In particular, the problem of finding a submanifold of the velocity phase space and a tangent vector field which is a solution of the lagrangian equations of motion and a second order differential equation (SODE) is studied. Thus, we develop, in a pure lagrangian context, an algorithm which solves simultaneously the problem of the compatibility of the equations of motion, the consistency of their solutions and the SODE problem. This algorithm allows to construct these solutions and gives all the lagrangian constraints, splitting them into two kinds. In this way, previous works on the same subject are completed and improved.

RÉSUMÉ. — Nous analysons par des méthodes géométriques les systèmes lagrangiens singuliers. Plus concrètement, nous étudions le problème de trouver une sous variété de l'espace des vitesses et un champ de vecteurs tangents qui sont solutions des équations de Lagrange et d'une équation différentielle du second ordre (SODE). C'est-à-dire que nous développons dans un contexte purement lagrangien un algorithme qui résout simultanément le problème de la compatibilité des équations du mouvement, la consistance de leurs solutions et le problème SODE. Cet algorithme permet de construire ces solutions et donne toutes les contraintes lagrangiennes en les séparant en deux catégories. Ce travail complète et améliore des résultats antérieurs sur ce sujet.

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1. INTRODUCTION

Since Dirac and Bergmann ([1], [2]) started the study of the dynamical systems described by singular lagrangians, this theme has been matter of interest for theoretical physicists and mathematicians.

The key of this interest is dual. From the mathematical point of view, it is due to the fact that the differential equations involved are not regular and this leads to the study of their compatibility and consistency of their solutions. From the physical point of view, the fundamental fact is that all the theories exhibiting gauge invariance are necessarily described by singular lagrangians, and these theories are of maximal interest in Modern Physics.

Although in the beginning the study of these systems was done by means of coordinate-dependent descriptions (*see* [3], [4] and references quoted therein), it was early seen that the use of techniques of Differential Geometry [which were successfully applied in order to formulate both the lagrangian and hamiltonian formalisms of Mechanics, as well as other related topics ([5]-[8]) allowed to treat these systems in a very natural way, inside the framework of the Presymplectic Geometry ([9]-[12]).

On the other hand, and thinking in the subsequent quantization of the models, the first descriptions paid special attention to the hamiltonian formalism (*see* [13] and references quoted therein). Later, some authors begin to be interested in the study of the lagrangian formalism and the equivalence between both formulations (*see* for example [14], [15]), and their geometrical description ([16], [17], [18]). In all these works the problem of the compatibility of the equations of motion and consistency of their solutions, as well as the question of the equivalence between the lagrangian and hamiltonian formalisms are studied from different points of view.

But a characteristic feature in the lagrangian formalism is that variational criteria as well as physical motivations demand that the solutions of the presymplectic equations of motion were *second order differential equations* (*SODE*) ([19], [20]). In the case of singular systems this is an additional problem to study, because the solutions of the equations of motion do not, in general, satisfy this condition. One of the more complete analysis of this question is done in [20]; in this work a submanifold of the *velocity phase space* where any SODE solution exists is found, but it is not, in general, the maximal one, as it can be deduced from [21] and [22] (*see* also the examples in the last reference).

In the last years, several works have been devoted to the study of this and other related problems. Thus, in [21] and subsequent papers ([23], [24]), an algorithm is developed for the lagrangian formalism in order to find the maximal submanifold of the velocity phase space where consistent solutions of the *Euler-Lagrange equations* exist. This procedure is equivalent to find a submanifold where there exist some SODE solution of the lagrangian equations of motion [25]. In addition, the complete equivalence between the lagrangian and hamiltonian formalisms for singular systems is shown and a quantitative discussion on the gauge degrees of freedom is performed. In these papers, the treatment is local-coordinate, although the main aspects of the theory have been intrinsically reformulated later ([22], [26], [27], [28]) (*see* also the treatment done in [29]).

Nevertheless, two questions remain unsolved: The above mentioned algorithm is not purely lagrangian in the sense that, in order to find the lagrangian constraints and fix the gauge degrees of freedom, it uses on each step the correspondent hamiltonian algorithm. Moreover, some geometrical aspects (for instance, in relation to the removal of gauge degrees of freedom in the solution) are not clarified.

Thus, in this paper, our aim is to develop a geometric and purely lagrangian algorithm which gives us the maximal submanifold of the velocity phase space where a SODE solution of the lagrangian equations of motion exists and, in addition, allows us to describe how this solution is. In this way, we complete the geometrical description of ([21], [23], [24]) and, as we are going to see, we recover as a particular case the result of [20] and the lagrangian version of the Presymplectic Constraint Algorithm (PCA) [16]. We will follow the same method as in reference [30].

The organization of the paper is the following: In section 2 we establish some notation and terminology and we state the general problem. Sections 3, 4 and 5 are devoted to the study of the *compatibility conditions*, which give rise to the *first generation constraints* and confine the *gauge freedom* of the general solution to its vertical part. Then the splitting into *dynamical* and *non-dynamical constraints* is achieved in an equivalent way as in reference [22]. In section 6 we analyze the *stability condition*, obtaining the general result concerning the new generations of constraints and the removal of gauge degrees of freedom in the general solution. Finally we study the FL-projectability of the solutions in section 7 and we analyze some examples in section 8.

2. STATEMENT OF THE PROBLEM AND NOTATIONS

Let (Q, L) be an almost-regular Lagrangian system [16], where Q is a differentiable manifold, the configuration space of the system, TQ is the coordinate-velocities phase space and L: $\mathrm{TQ} \to \mathbb{R}$ is the Lagrangian function of the system.

We denote by J: $\mathcal{X}(\mathrm{TQ}) \to \mathcal{X}^v(\mathrm{TQ})$ the vertical endomorphism, $\mathcal{X}^v(\mathrm{TQ})$ being the module of vertical fields on TQ. The lagrangian 1-form is defined on TQ by $\theta = d\mathrm{L} \circ \mathrm{J}$ and the lagrangian 2-form is $\omega = -d\theta$. In addition, the function A: TQ → ℝ is given by $\mathrm{A} = \Delta(\mathrm{L})$, where Δ is the Liouville vector field on TQ. Then the energy of the system is $\mathrm{E} = \mathrm{A} - \mathrm{L}$. For more details on notations *see* [7], [19], [31].

In the finite dimensional case, if $\dim \mathrm{Q} = n$ and (U, q^i) is a local chart in Q, then $(\pi^{-1}(\mathrm{U}), q^i, v^i)$ is a local chart in TQ (π is the canonical projection from TQ onto Q). In this chart the expressions of these elements are:

$$\mathrm{A} = \sum (\partial \mathrm{L}/\partial v^i) v^i, \qquad \theta = \sum (\partial \mathrm{L}/\partial v^i) dq^i$$
$$\omega = \sum (\partial^2 \mathrm{L}/\partial q^j \partial v^i) dq^i \wedge dq^j + \sum (\partial^2 \mathrm{L}/\partial v^j \partial v^i) dq^i \wedge dv^j$$

The trajectories of the lagrangian system (Q, L) are the curves $\sigma: [a, b] \to \mathrm{Q}$ such that minimize the variational problem given by the integral:

$$\int_{\dot{\bar{\sigma}}} \mathrm{L}\, dt$$

where $\dot{\sigma}: [a, b] \to \mathrm{TQ}$ is the canonical lift of σ to the tangent bundle. Notice that actually we don't consider TQ but $\mathrm{TQ} \times \mathbb{R}$ with the natural projections on TQ and ℝ, and the curve $\bar{\sigma}: [a, b] \to \mathrm{Q} \times \mathbb{R}$ given by $\bar{\sigma}(t) = (\sigma(t), t)$. Then the curve $\dot{\bar{\sigma}}$ is the natural lift of $\bar{\sigma}$ to $\mathrm{TQ} \times \mathbb{R}$.

It is known that this variational problem is equivalent to the one given by:

$$\int_{\dot{\bar{\sigma}}} \theta + \mathrm{E}\, dt$$

Using the techniques of variatonal calculus (*see* for example [32]), and the presymplectic character of ω, this problem leads to the following: To find a vector field $\mathrm{X} \in \mathscr{X}(\mathrm{TQ})$ and a submanifold $\mathrm{S} \subset \mathrm{TQ}$ such that:

(i) $(i_{\mathrm{X}}\omega - d\mathrm{E})|_{\mathrm{S}} = 0$ (the dynamical equation)

(ii) X is a Second Order Differential Equation (SODE) on the points of S.

(iii) X is tangent to S.

The condition $i_{\mathrm{X}}\omega - d\mathrm{E} = 0$ is the intrinsic version of the Lagrange equations. When the system is degenerate, or singular, then ω is presymplectic and the equation $i_{\mathrm{X}}\omega - d\mathrm{E} = 0$ has not any solution all over TQ, but only in a subset of points of TQ.

The second condition is equivalent to say that the integral curves of the field X are canonical lifts to TQ of curves on Q. Another statement of this condition is $(\mathrm{J}(\mathrm{X}) - \Delta)|_{\mathrm{S}} = 0$.

The third condition is the geometric translation of the fact that the trajectories of the system must remain inside the submanifold S.

Conditions (i) and (iii) are called *compatibility* and *stability* (or *consistency*) *conditions* respectively. If you only consider them, you can solve the problem by means of the known Presymplectic Constraint Algorithm, PCA [9]. In the case that L is regular, the equation $i_X \omega - dE = 0$ has one unique solution on TQ which is automatically a SODE.

Our aim is to obtain an algorithm which gives us the submanifold S and the SODE vector field X on it. To this end we study the constraints induced by every one of the conditions. This algorithm will be a generalization of the before mentioned PCA. At every step we make the following assumption:

GENERAL HYPOTHESIS. – *Every subset of* TQ *obtained by application of the algorithm is a regular closed submanifold and its natural injection is an embedding.*

3. COMPATIBILITY CONDITIONS: DYNAMICAL FIRST GENERATION CONSTRAINTS

In the case we study, the 2-form ω has non trivial kernel, so the equation $i_X \omega - dE$ has no solution everywhere in TQ in general. Hence we have to restrict the problem to the subset

$$\{ x \in TQ ; \, dE(x) \in \operatorname{Im} \omega(x) \}$$

In order to analyze this situation, as usually we assume:

HYPOTHESIS. – $\dim \ker \omega$ *is constant.*

And we have the following known result:

PROPOSITION 1 ([9], [16]). – (i) *The equation* $i_X \omega = dE$ *has solution only in the points of the set:*

$$P_1 = \{ x \in TQ ; \, i_Z dE(x) = 0, \, \forall Z \in \ker \omega \}$$

(ii) *On* P_1, *the solutions of the equations are* $X_0 + Y$, *where* X_0 *is a particular solution and* $Y \in \ker \omega$.

Proof. – (i) ($\Rightarrow$) Let $x \in TQ$ and suppose there exists $u \in T_x TQ$ with $i_u \omega = dE(x)$. If $z \in \ker \omega(x)$ then $i_z dE = \omega(z, u) = 0$.

($\Leftarrow$) Let $x \in P_1$, that is $i_z dE = 0$, for all $z \in \ker \omega(x)$. Then if we prove that $dE(x) \in \operatorname{Img} \omega(x)$, the existence of a solution is assured.

In order to prove this inclusion observe that

$$\dim \operatorname{Im} \omega(x) = \dim T_x TQ - \dim \ker \omega(x) = \dim (\ker \omega(x))^{\vdash}$$

and if $\beta \in \operatorname{Im} \omega(x)$, then $\ker \omega(x) \subset \ker \beta$, hence $\beta \in (\ker \omega(x))^{\vdash}$. We have then that $\operatorname{Im} \omega(x) \subset (\ker \omega(x))^{\vdash}$ and both have the same dimension. Consequently $dE(x) \in \operatorname{Im} \omega(x)$. (We denote by $F^{\vdash}$ the annihilator of the subspace $F \subset T_x Q$.)

Note. – This is a finite dimensional proof. You can find another in [9] which is also valid for the infinite dimensional case.

(ii) The equation is linear on X and since P_1 is a closed submanifold of TQ, the result follows. □

DEFINITION 1. – (i) *The submanifold* P_1 *is called the submanifold of dynamical first generation constraints.*

(ii) *If* $Z \in \ker \omega$, *the function* $\zeta^1 = i_Z dE$ *is called dynamical* (or *presymplectic*) *first generation constraint generated by* Z.

This terminology will be justified later.

4. SODE CONDITION: NON-DYNAMICAL FIRST GENERATION CONSTRAINTS

In general, the solutions $\{X_0 + Y;\ Y \in \ker \omega\}$ of the dynamical equation $(i_X \omega - dE)|_{P_1} = 0$ are not SODE on P_1, then we state:

PROPOSITION 2. – *Let* $S_1 = \{x \in P_1;\ \exists Y \in \ker \omega,\ (JX_0 + JY - \Delta)(x) = 0\}$ *where* Δ *is the Liouville vector field on* TQ. *Then*

(i) S_1 *is the maximal set of points of* P_1 *where there exist vector fields* $D = X_0 + Y \in \mathscr{X}(TQ)$ *such that*

(*a*) $(i_D \omega - dE)|_{S_1} = 0.$

(*b*) D *is a SODE on* S_1.

(ii) *Let* D_0 *be a vector field which satisfies conditions* (*a*) *and* (*b*) *and* $W \in \ker \omega \cap \mathscr{X}^v(TQ)$ *then* $D_0 + W$ *satisfies* (*a*) *and* (*b*) *too. Conversely if* D_1 *and* D_2 *verify* (*a*) *and* (*b*) *then* $D_1 - D_2$ *is an element of* $\ker \omega \cap \mathscr{X}^v(TQ)$.

(Remember that $\ker \omega \cap \mathscr{X}^v(TQ) = \ker FL_*$, where $FL : TQ \to T^*Q$ is the Legendre transformation associated to the lagrangian L. *See* [20], [22] for the details.)

Proof. – (i) Evident from the definition of S_1.

(ii) If $W \in \ker \omega \cap \mathscr{X}^v(TQ)$ then $i_W \omega = 0$ and $JW = 0$ therefore $D_0 + W$ satisfies (*a*) and (*b*).

The converse is trivial. □

Now we are going to describe the submanifold S_1 inside P_1 as the zero set of a family of functions.

PROPOSITION 3. – *Let* $Y \in \mathscr{X}(TQ)$ *such that* $X_0 + Y$ *is a SODE, that is* $J(X_0 + Y) = \Delta$. *Then*

$$S_1 = \{x \in P_1;\ (i_Z i_Y \omega)(x) = 0,\ \forall Z \in \mathfrak{M}\}$$

where $\mathfrak{M} = \{Z \in \mathscr{X}(TQ);\ JZ \in \ker FL_*\}$.

Proof. – Consider the set $C=\{x\in P_1;\ (i_Z i_Y \omega)(x)=0,\ \forall Z\in \mathfrak{M}\}$, then we make the proof in three steps:

(*a*) C is independent of the chosen vector field Y.

Take $T=Y_1-Y_2$ where Y_1 and Y_2 verify that X_0+Y_i are SODE. Then T is a vertical vector field, because it is the difference of two SODE. So $T=JU$ for some $U\in\mathcal{X}(TQ)$, then:

$$i_Z i_T \omega = i_Z i_{JU} \omega = -i_{JZ} i_U \omega = 0$$

since $JZ\in\ker FL_*\subset\ker\omega$ and we have used the relation $i_{JU}\omega=-i_U\omega\circ J$ for any $U\in\mathcal{X}(TQ)$ (*see* [19]).

(*b*) $(C\subset S_1)$ Let $x\in C$, then $(i_Z i_Y \omega)(x)=0$ for every $Z\in\mathfrak{M}$. Therefrom $Y_x\in\mathfrak{M}_x^{\perp}$, then there exists any $v\in T_x^v Q$ such that $Y_x+v\in\ker\omega|_x$ (*see* [22]). Hence $(X_0+Y)(x)+v$ verifies the conditions of the above Proposition, so $x\in S_1$.

(*c*) $(S_1\subset C)$ Let $x\in S_1$. There exists $U\in\ker\omega$ with $(X_0+U)(x)$ verifying conditions (*a*) and (*b*) of Proposition 2 (i), therefore $(i_Z i_U \omega)(x)=0$, so $x\in C$. □

Corollary. – *For any SODE* D *we have:*

$$S_1=\{x\in P_1;\ (i_Z i_{(X_0-D)}\omega)(x)=0,\ \forall Z\in\mathfrak{M}\}.$$

Proof. – Since we can take $Y=X_0-D$ the proof is trivial. □

Definition 2. – (i) S_1 *is called the submanifold of first generation constraints.*

(ii) *Given* $Z\in\mathfrak{M}$ *and* $Y\in\mathcal{X}(TQ)$ *such that* X_0+Y *is a SODE, then the function* $\eta^1=i_Z i_Y\omega$ *is called non-dynamical (or non presymplectic) first generation constraint generated by* Z.

5. GENERAL EXPRESSION OF FIRST GENERATION CONSTRAINTS

Proposition 4. – *For any SODE* D *we have:*

$$S_1=\{x\in TQ;\ (i_Z(i_D\omega-dE))(x)=0,\ \forall Z\in\mathfrak{M}\}.$$

Proof. – Consider the set

$$C=\{x\in TQ;\ (i_Z(i_D\omega-dE))(x)=0,\ \forall Z\in\mathfrak{M}\}$$

where D is a SODE.

Taking into account that $\ker\omega\subseteq\mathfrak{M}$ (*see* [22] for the proof), we have:

If $Z\in\ker\omega\subseteq\mathfrak{M}$ then $i_Z(i_D\omega-dE)=i_Z dE$, which are the dynamical constraints.

On the other hand, taking $Z \in \mathfrak{M}$, but $Z \notin \ker \omega$, then if $D = X_0 + Y$ is a SODE for some Y we have:

$$i_Z (i_D \omega - dE)|_{P_1} = -i_Z i_Y \omega$$

which are the non-dynamical constraints. □

Summarizing, we have arrived to a submanifold S_1 of TQ and to a family of vector fields

$$\{X_0 + Y_0 + V;\ V \in \ker FL_*\} \tag{1}$$

such that

(*a*) $(i_{(X_0 - Y_0 + V)} \omega - dE)|_{S_1} = 0$

(*b*) $X_0 + Y_0 + V$ are SODE

Observe that X_0 and Y_0 are fixed vector fields, meanwhile V is arbitrary with the only condition that it belongs to $\ker FL_*$.

The submanifold S_1 is defined as:

$$S_1 = \{x \in TQ;\ (i_Z (i_D \omega - dE))(x) = 0,\ \forall Z \in \mathfrak{M}\}$$

for any SODE D. The constraints that define S_1 are of two different kinds:

Dynamical constraints: $\zeta^1 = i_Z dE$, $Z \in \ker \omega$.

Non-dynamical constraints: $\eta^1 = i_Z i_Y \omega = 0$, $Z \in \mathfrak{M}$, $Z \notin \ker \omega$, $Y \in \mathcal{X}(TQ)$ with $X_0 + Y$ a SODE.

It is known that the dynamical constraints can be expressed as FL-projectable functions, whose counterparts in T^*Q are all the secondary constraints in the hamiltonian formalism ([16], [33]). But this is not the situation for the non-dynamical ones, as we are going to prove.

PROPOSITION 5. – *If* η *is a non-dynamical first generation constraint, then it cannot be expressed as a* FL-*projectable function.*

We need the following results:

LEMMA 1. – *For every* $V \in \operatorname{Ker} FL_*$, *there exists* $Z \in \mathfrak{M}$, *such that* $JZ = V$ *and* Z *is* FL-*projectable.*

Proof. – It is evident that if $V \in \mathcal{X}^v(TQ)$ is FL-projectable, then we can choose an FL-projectable vector field $X \in \mathcal{X}(TQ)$ with $JX = V$ (in fact, the only problem is to take the vertical part which is arbitrary). In particular, $\forall V \in \ker FL_*$, we have $FL_* V = 0$, this implies they are FL-projectable, hence there exists some $Z \in \mathfrak{M}$ which is FL-projectable and $JZ = V$. □

LEMMA 2. – *If* $D \in \mathcal{X}(TQ)$ *is a SODE, then* $J[V, D] = V$ *for every* $V \in \mathcal{X}^v(TQ)$.

Proof. – Let $X \in \mathcal{X}(TQ)$ be a vector field such that $JX = V$, then

$$\begin{aligned} J[V, D] &= J[JX, D] = [JX, JD] - J[X, JD] \\ &= [JX, \Delta] - J[X, \Delta] = -(L_\Delta J)X = JX = V \end{aligned}$$

where use is made of the properties (*see* [19]):

$$[JX, JY] = J[JX, Y] + J[X, JY]$$
$$-JX = (L_\Delta J)X = [\Delta, JX] - J[\Delta, X]. \quad \square$$

LEMMA 3. — *If* $D \in \mathscr{X}(TQ)$ *is a SODE, then* $[V, D] \in \mathfrak{M}$, *for every* $V \in \operatorname{Ker} FL_*$.

(*Consequently, if* D *is a SODE, it is not* FL-*projectable.*)

Proof. — It is a straightforward consequence of the above lemma. □

Comment. — Observe that if D is a SODE then

$$[\ker FL_*, D] + \mathscr{X}^v(TQ) = \mathfrak{M}$$

because for all $X \in \mathfrak{M}$, we have that $JX = J[JX, D] \in \ker FL_*$, then $X - [JX, D]$ is a vertical vector field.

Proof of the proposition. — If η is a non-dynamical first generation constraint then, according to the proposition 4, $\exists Z \in \mathfrak{M}$, $Z \notin \ker \omega$, such that

$$\eta = i_Z(i_D \omega - dE)$$

(and, moreover, Z cannot be a vertical vector field since the 1-form $i_D \omega - dE$ is horizontal and then $i_Z(i_D \omega - dE) = 0$). Now, suppose that η is FL-projectable, then $\forall V \in \ker FL_*$ we have

$$0 = V(\eta) = V(i_Z(i_D \omega - dE)) = -V(i_D i_Z \omega + i_Z dE)$$

but, by the lemma 1, Z can be chosen FL-projectable and, since dE is also FL-projectable, so is $i_Z dE$; therefore $V(i_Z dE) = 0$ and

$$0 = V(\eta) = -V(i_D i_Z \omega) = -i_{[V, D]} i_Z \omega - i_D L_V i_Z \omega$$

Using again the lemma 1 we have that $i_Z \omega$ is FL-projectable and then $L_V i_Z \omega = 0$. Therefore

$$0 = V(\eta) = -i_{[V, D]} i_Z \omega$$

But, for all X in $\mathfrak{M}$, taking into account the above comment, $X = [V, D] + V'$, for some $V \in \ker FL_*$ and $V' \in \mathscr{X}^v(TQ)$, so we have

$$i_X i_Z \omega = i_{[V, D]} i_Z \omega + i_{V'} i_Z \omega = i_{[V, D]} i_Z \omega = 0$$

since $i_{V'} i_Z \omega = i_{JY} i_Z \omega = -i_Y i_{JZ} \omega = 0$, for some $Y \in \mathscr{X}(TQ)$, because $JZ \in \ker FL_* \subset \ker \omega$. Hence we conclude that $Z \in \mathfrak{M}^\perp$ and therefore $Z \in \mathfrak{M}^\perp \cap \mathfrak{M}$. But $\mathfrak{M}^\perp \cap \mathfrak{M}$ and $\ker \omega \cap \mathfrak{M}$ defer only on their vertical part, hence, since $Z \notin \mathscr{X}^v(TQ)$, we have $Z \in \ker \omega \cap \mathfrak{M}$ which is absurd because $Z \notin \ker \omega$ by hypothesis. □

A consequence of the non FL-projectability of the non dynamical constraints is that these constraints remove degrees of freedom on the leaves of the foliation defined by $\ker FL_*$ [33].

6. STABILITY CONDITIONS: NEW GENERATIONS OF CONSTRAINTS

In general none of the fields of the family (1) of solutions on S_1 is tangent to the submanifold S_1. So we must search for the points of S_1 where there exists any vector field $V \in \ker FL_*$ such that X_0+Y_0+V is tangent to S_1. In order to solve this problem we define:

$$S_2=\{x \in S_1;\ \exists V \in \ker FL_* \text{ with } (X_0+Y_0+V)(x) \in T_x S_1\}$$

But S_1 is the zero set of a family of functions. So a vector field D is tangent to S_1 if $(Df)(x)=0$ for every f in the family and $x \in S_1$. Hence we have:

$$\begin{aligned}S_2=\{x \in S_1;\ \exists V \in \ker FL_* \text{ such that}\\ &(a)\ ((X_0+Y_0+V)(i_Z dE))(x)=0,\ \forall Z \in \ker \omega\\ &(b)\ ((X_0+Y_0+V)(i_Z i_Y \omega))(x)=0,\\ &\qquad \forall Z \in M,\ Z \notin \ker \omega,\ Y \in \mathcal{X}(TQ),\ X_0+Y \text{ SODE}\}\end{aligned}$$

Where we have used the above expressions of the constraints.

But the dynamical constraints can be expressed as FL-projectable functions, so we have $V(i_Z dE)=0$ for every $V \in \ker FL_*$. Hence condition (*a*) is

$$((X_0+Y_0)(i_Z dE))(x)=0, \qquad \forall Z \in \ker \omega$$

which, in general, gives us new constraints.

On the other hand, for the non-dynamical constraints, we have

$$((X_0+Y_0+V)(i_Z i_Y \omega))(x) = ((X_0+Y_0)\,\omega(Z, Y))(x)+(V\,\omega(Z, Y))(x)=0 \qquad (2)$$

This is a system of linear equations for V, and we have:

LEMMA. — *The system* (2) *is compatible on all the points of* S_1.

Proof. — Locally you can take a finite set of non-dynamical independent constraints, $\eta_1, \ldots, \eta_h$, from the expresssions $\omega(Z, Y)$ for $Z \in \mathfrak{M}$. As we said above, these constraints remove h degrees of freedom on the leaves of the foliation defined by $\ker FL_*$. Then the matrix of this linear system for V has maximal rank h, hence the system is compatible at least locally. (*See* another different proof of this result in [21].)

For any collection of local solutions we can construct a global one on S_1 using a partition of unity on this manifold, hence the system is compatible on all the points of S_1. □

So the stability of these constraints does not give new constraints but eliminates gauge degrees of freedom of the solution.

Now the dynamical vector fields which are solution of the problem can be written in the form $X_0+Y_0+V_0+V'$ where $X_0+Y_0+V_0$ is tangent to

S_1 on the points of S_2, V_0 is a solution of the above system and $V' \in \ker FL_*$ is tangent to S_1 on the points of S_2 and contains all the gauge freedom. In addition V' is any solution of the system $V\,\omega(Z, Y)=0$ for all $Z \in \mathfrak{M}$.

All this discussion can be summarized as:

PROPOSITION 6. — *(a) Let*

$$S_2 = \{x \in S_1;\ ((X_0+Y_0)(i_Z dE))(x)=0,\ \forall Z \in \ker \omega\}.$$

Then S_2 *is the maximal subset of* S_1 *such that there exists* $V \in \ker FL_*$ *verifying:*

(i) $(i_{(X_0+Y_0+V)}\,\omega - dE)\,|_{S_2}=0$

(ii) X_0+Y_0+V *is a SODE*

(iii) X_0+Y_0+V *is tangent to* S_1

(b) The set

$$\mathscr{V} = \{V \in \ker FL_*;\ (X_0+Y_0)\,\omega(Z, Y)+V\,\omega(Z, Y)=0,\ \forall Z \in \mathfrak{M}\}$$

contains all the vector fields belonging to $\ker FL_*$ *such that* X_0+Y_0+V *verifies those conditions.*

(c) If $V_0 \in \mathscr{V}$ *then:*

$$\mathscr{V} = V_0 + \{V' \in \ker FL_*;\ V'\,\omega(Z, Y)=0,\ \forall Z \in \mathfrak{M}\}$$

(Note that these last vector fields V' *are tangent to* S_1.)

DEFINITION 3. — (i) *The functions* $(X_0+Y_0)(i_Z dE)$, *for* $Z \in \ker \omega$, *are called second generation constraints.*

(ii) S_2 *is the submanifold of second generation constraints.*

The situation is the same that at the end of section 5: We have a submanifold S_2 where the solution is $X_0+Y_0+V_0+V'$ and $V_0+V' \in \mathscr{V}$. This solution is a SODE and is tangent to S_1 at the points of S_2 but in general it is not tangent to S_2.

Now we must look for the points of S_2 where some of those solutions are tangent to S_2. In order to solve this problem we need to split the second generation constraints in two different kinds, dynamical and non-dynamical and stabilize all of them. This procedure is the same at each step of the algorithm.

Then the general situation is the following: In order to obtain S_{i+1}, $i \geqq 2$, from S_i, we have to

(i) classify the constraints defining S_i and

(ii) stabilize them.

The dynamical constraints will originate new constraints, but the non-dynamical ones will remove gauge degrees of freedom, as we are going to prove.

THEOREM. — *Let* S_i $(i>1)$ *be the closed regular submanifold obtained at the i-th step of the above procedure.*

(*a*) *The following subsets of* S_i *are the same:*

(i) $A=\{x\in S_i;\ X_0\, i_T\, dE(x)=0,\ \forall T\in \mathscr{X}(S_{i-1})^{\perp},\ Y_0\, i_T\, dE|_{S_i}=0\}$

(ii) $B=\{x\in S_i;\ i_T\, dE(x)=0,\ T\in \mathscr{X}(S_i)^{\perp}\}$

(iii) $C=$ *The submanifold of* S_i *of zeros of* FL-*projectable* $(i+1)$-*th generation constraints.*

The functions ζ^{i+1} *defining this subset will be called dynamical* (*or presymplectic*) $(i+1)$-*th generation constraints.*

(*b*) *Let* $T\in \mathscr{X}(S_{i-1})^{\perp}$ *be such that* $Y_0\, i_T\, dE|_{S_i}\neq 0$, *then the function* $\eta^{i+1}=(X_0+Y_0)\, i_T\, dE$ *can not be expressed as a* FL-*projectable function. All these functions will be called non-dynamical* (*or non-presymplectic*) $(i+1)$-*th generation constraints.*

(*c*) *The stability condition for the dynamical* $(i+1)$-*th generation constraints can be written as:*

$$(X_0+Y_0)(X_0(i_T\, dE))=0$$

for $T\in \mathscr{X}(S_{i-1})^{\perp}$ *such that* $Y_0\, i_T\, dE|_{S_i}=0$, *and these functions are a new generation of constraints.*

(*d*) *The stability condition for the non-dynamical* $(i+1)$-*th generation constraints does not give any new constraints but remove gauge degrees of freedom from the solution.*

In order to prove it we need the following results:

LEMMA 1. — *If* $P\subset TQ$ *is a submanifold defined by* FL-*projectable constraints and* $S\subset P$ *is a submanifold defined in* P *by non* FL-*projectable functions, then* $\mathscr{X}(P)^{\perp}=\mathscr{X}(S)^{\perp}$.

Proof. — It is a consequence of proposition 3.1 of [33]. □

LEMMA 2. — *There exists one* FL-*projectable solution* X_0 *of the equation* $(i_X\,\omega-dE)|_{P_1}=0$.

Proof. — Similar to the proof of the lemma in section IV of [20]. □

Proof of the theorem. — The proof of these statements is the same for every generation of constraints. For simplicity we make it for $i=1$. In this case we have that $\mathscr{X}(S_{i-1})^{\perp}=\ker\omega$.

(*a*) (C=B) By lemma 1, the constraints defining B are the lagrangian PCA constraints (*see* [16]). But it is known that every lagrangian PCA constraint can be expressed as a FL-projectable function ([16], [33]). On the other hand, according to the theorem 1 of [24], the only FL-projectable constraints are the PCA constraints. Hence the result follows.

(C⊂A) By lemma 2, the constraints defining A can be expressed as FL-projectable functions, then the results is trivial.

($A \subset B$) The second generation lagrangian PCA constraints are consequence of the stability condition of the first generation ones. These are $\zeta^1 = i_T dE$, $T \in \ker\omega$, and the set of solutions along P_1 is $X_0 + Y$, $Y \in \ker\omega$. Therefore the stability condition is $(X_0 + Y) i_T dE|_{S_1} = 0$.

We obtain new constraints only if $T \in \ker\omega$ verifies that $Y(i_T dE)|_{S_1} = 0$ for every $Y \in \ker\omega$. Hence the second generation lagrangian PCA constraints are

$$\zeta^2 = X_0 (i_T dE), \quad T \in \ker\omega, \qquad Y(i_T dE)|_{S_1} = 0, \quad \forall Y \in \ker\omega$$

Then these constraints define B. But all these constraints vanish on the points of the set A.

(*b*) Immediate from part (*a*).

(*c*) The expression of these constraints is $\zeta^2 = X_0 i_T dE$ for $T \in \ker\omega$ and $Y_0 i_T dE|_{S_1} = 0$ hence the stability condition is:

$$(X_0 + Y_0 + V_0 + V')(X_0 i_T dE)|_S = 0$$

but ζ^2 are FL-projectable, then $(V_0 + V')\zeta^2 = 0$ and we obtain the desired result.

(*d*) The expression of these constraints is $\eta^2 = (X_0 + Y_0) i_T dE$ for $T \in \ker\omega$ and $Y_0 i_T dE|_{S_1} \neq 0$, so the stability condition is

$$(X_0 + Y_0 + V_0 + V')(X_0 + Y_0) i_T dE = 0$$

that is:

$$V'(X_0 + Y_0) i_T dE = -(X_0 + Y_0 + V_0)(X_0 + Y_0) i_T dE$$

This system is compatible for the same reason that in the lemma before proposition 6, and then the result follows immediately. □

Observe that the expression of ζ^{i+1} only uses the particular solution X_0 of the dynamical equation $i_X \omega = dE$. Meanwhile the expression of η^{i+1} contains the vector field Y_0, coming from the SODE condition. This comment justifies the above terminology.

The physical interesting case arises when there exists a final constraint submanifold S_f, that is, an integer k such that $S_{k+1} = S_k = S_f$ with $\dim S_f > 0$. Then, on this submanifold, there exist SODE solutions D which are tangent to S_f. They have the form $X_0 + Y_0 + V_\delta + V$, where V_δ, $V \in \ker FL_*$ and V_δ is completely determined, meanwhile V, which is also tangent to S_f, denotes the remaining arbitrariness or gauge freedom.

7. ON THE FL-PROJECTABILITY OF THE SOLUTIONS

As we have said in the introduction, the problem of finding a SODE solution to the presymplectic lagrangian equations has been studied in reference [20]. There, the obtained final constraint submanifold where this

SODE exists only contains one point in every fibre of the foliation defined by $\ker FL_*$ and, on this submanifold the SODE solution is FL-projectable.

But we are going to prove that this is also a necessary condition for a SODE solution to be *weakly FL-projectable* (that is, FL-projectable on the points of the final constraint submanifold S_f) (*see* [33] for the terminology). In fact, suppose S_f contains two different points in the same fibre of that foliation, denoted $A=(q_a, v_a)$ and $B=(q_b, v_b)$ in a local chart of TQ. Then $q_a=q_b$ but $v_a \neq v_b$, since the fibres of the foliation are contained in the fibres of TQ. Now, if D is a SODE solution, then

$$D(A)=(v_a\,\partial/\partial q+f\,\partial/\partial v)\,|_A, \qquad D(B)=(v_b\,\partial/\partial q+f\,\partial/\partial v)\,|_B$$

and $FL(A)=FL(B)$ but $FL_*(D(A)) \neq FL_*(D(B))$ and the result follows.

Taking into account this result together with the above mentioned of reference [20], we can state the following:

PROPOSITION 7. — *The necessary and sufficient condition for a SODE solution of the presymplectic equations of motion to be weakly* FL-*projectable on the final constraint submanifold* S_f, *is that this submanifold only contains one point of every fibre of the foliation defined by* $\ker FL_*$.

As a consequence, this weakly FL-*projectable SODE solution does not contain any gauge degree of freedom, that is, it is unique.*

8. EXAMPLES

(*a*) Consider the dynamical system defined by the lagrangian

$$L=v_1^2+q_3 v_4+q_2^2+q_4^2$$

partially studied in reference [26]. We have

$$\omega=-2\,dv_1\wedge dq_1-dq_3\wedge dq_4, \qquad E=-q_2^2-q_4^2+v_1^2$$

then, a local base for $\ker\omega$ is

$$\ker\omega=\{Z_1=\partial/\partial q_2;\ V_1=\partial/\partial v_2,\ V_2=\partial/\partial v_3,\ V_3=\partial/\partial v_4\}$$

and for the non-vertical part of $\mathfrak{M}$ we take

$$\{Z_1=\partial/\partial q_2,\ Z_2=\partial/\partial q_3,\ Z_3=\partial/\partial q_4\}$$

The compatibility condition of the presymplectic equation $i\omega=dE$ leads to the dynamical constraint

$$\zeta_1^1=q_2=\langle Z_1\,|\,dE\rangle$$

and the solution on P_1 is

$$X=v_1\,\partial/\partial q_1+b\,\partial/\partial q_2-2\,q_4\,\partial/\partial q_3+A\,\partial/\partial v_2+B\,\partial/\partial v_3+C\,\partial/\partial v_4$$

Now, if we search for a SODE solution D of this equation, we obtain, in addition, two non-dynamical constraints:

$$\eta_1^1 = v_3 + 2q_4 = \langle Z_3 \mid i_{D_0}\omega - dE \rangle$$
$$\eta_2^1 = v_4 = \langle Z_2 \mid i_{D_0}\omega - dE \rangle$$

(D_0 is an arbitrary SODE), and this solution, on S_1 is

$$D = v_1\,\partial/\partial q_1 + v_2\,\partial/\partial q_2 + v_3\,\partial/\partial q_3 + A\,\partial/\partial v_2 + B\,\partial/\partial v_3 + C\,\partial/\partial v_4$$
$$= X_0 + Y_0 + V_1 + V_2 + V_3$$

with $Y_0 = v_2\,\partial/\partial q_2$ and $X_0 = v_1\,\partial/\partial q_1 + v_3\,\partial/\partial q_3$ (on S_1).

The stability condition originates, in a first step, a new non-dynamical constraint and fixes two arbitrary functions

$$\eta_1^2 = v_2 = Y_0(\zeta_1^1)$$
$$D(\eta_1^1) = B = 0, \qquad D(\eta_2^1) = C = 0$$

Finally, in a second step a new arbitrary function is determined

$$D(\eta_1^2) = A = 0$$

and the final SODE solution on S_2 is

$$D = v_1\,\partial/\partial q_1 + v_3\,\partial/\partial q_3$$

The Legendre transformation is given by

$$p_1 = v_1, \qquad p_2 = 0, \qquad p_3 = 0, \qquad p_4 = q_3$$

Using the constraint η_1^1, one can see that the SODE solution D is weakly FL-projectable on S_2, since we are in the conditions stated in proposition 7.

(*b*) Another example is given by the lagrangian

$$L = 1/2\,(v_1^2 - q_1^2 - 2q_1q_3 + (v_4 - q_2)^2)$$

which is a modified version of the one given in reference [9]. We have

$$\omega = -dv_1 \wedge dq_1 - dv_4 \wedge dq_4 + dq_2 \wedge dq_4$$
$$E = 1/2\,(v_1^2 + v_4^2 + q_1^2 + 2q_1q_3 - q_2^2)$$

Local basis for $\ker\omega$ and the non-vertical part of $\mathfrak{M}$ are now

$$\ker\omega = \{Z_1 = \partial/\partial q_2 + \partial/\partial v_4,\ Z_2 = \partial/\partial q_3;\ V_1 = \partial/\partial v_2,\ V_2 = \partial/\partial v_3\}$$
$$\{Z_1 = \partial/\partial q_2 + \partial/\partial v_4,\ Z_2 = \partial/\partial q_3\}.$$

The compatibility condition of the presymplectic equation $i_X\omega = dE$ leads to the dynamical constraints

$$\zeta_1^1 = v_4 - q_2 = \langle Z_1 \mid dE \rangle$$
$$\zeta_2^1 = q_1 = \langle Z_2 \mid dE \rangle$$

The solution on P_1 is

$$X = v_1\,\partial/\partial q_1 + A\,\partial/\partial q_2 + c\,\partial/\partial q_3 + v_4\,\partial/\partial q_4$$
$$-(q_1 + q_3)\,\partial/\partial v_1 + B\,\partial/\partial v_2 + C\,\partial/\partial v_3 + A\,\partial/\partial v_4$$

A SODE solution D is obtained from this vector field by taking the particular element in the non-vertical part of $\ker \omega$,

$$Y_0 = v_3\, \partial/\partial q_3 + v_2\, (\partial/\partial q_2 + \partial/\partial v_4)$$

(that is, $c = v_3$ and $A = v_2$), whereby first generation non-dynamical constraints do not appear. Hence, on P_1 we have

$$D = X_0 + Y_0 + V_1 + V_2 = v_1\, \partial/\partial q_1 + v_2\, \partial/\partial q_2 + v_3\, \partial/\partial q_3 + v_4\, \partial/\partial q_4 \\ - (q_1 + q_3)\, \partial/\partial v_1 + B\, \partial/\partial v_2 + C\, \partial/\partial v_3 + v_2\, \partial/\partial v_4.$$

At the successive steps, the stability condition only originates two new dynamical constraints and a non-dynamical one

$$\zeta_1^2 = v_1 = X_0(\zeta_2^1), \qquad \zeta_1^3 = q_1 + q_3 = X_0(\zeta_1^2)$$
$$\eta_1^4 = v_1 + v_3 = Y_0(\zeta_1^2)$$

Finally, one arbitrary function is determined $D(\eta_1^4) = C = 0$, and the final SODE solution on S_4 is

$$D = v_2\, \partial/\partial q_2 + v_3\, \partial/\partial q_3 + v_4\, \partial/\partial q_4 + B\, \partial/\partial v_2$$

The Legendre transformation is given by

$$p_1 = v_1, \qquad p_2 = 0, \qquad p_3 = 0, \qquad p_4 = v_4 - q_2$$

hence, the vector field D is not weakly FL-projectable on S_4 due to the component $v_2\, \partial/\partial q_2$. Then in order to obtain a weakly FL-projectable SODE solution we have to restrict the final constraint submanifold by introducing a new (non-dynamical) constraint of the form

$$\eta_1^5 = v_2 - f(q_i, v_1, v_4) \qquad (i = 1, \ldots, 4)$$

(notice that f is a FL-projectable function). Its stabilization determines the remaining arbitrary function B in D

$$D(\eta_1^5) = f\, \partial f/\partial q_2 + v_3\, \partial f/\partial q_3 + v_4\, \partial f/\partial q_4 - B = 0$$

and so the final SODE solution on S_5 is

$$D = v_2\, \partial/\partial q_2 + v_3\, \partial/\partial q_3 + v_4\, \partial/\partial q_4 + (f\, \partial f/\partial q_2 + v_1\, \partial f/\partial q_3 + v_4\, \partial f/\partial q_4)\, \partial/\partial v_2$$

which is, in fact, weakly FL-projectable on S_5 (using the constraint η_1^4). For example, the simplest solution is to take $f = 0$, and then $B = 0$.

CONCLUSIONS

We have done a complete study of the lagrangian equations of motion of dynamical systems described by singular lagrangians. Our goal has been dual. On one hand, we have found the maximal submanifold of the velocity phase space of the system where these equations are compatible and there exists some tangent vector field solution of the equations of movement, with the condition that it is a SODE. On the other hand, we give the maximal information about the structure of such a vector field.

The procedure has been algorithmic. First of all we have determined the submanifold where the lagrangian equations are compatible and have a solution which is a SODE. This manifold is usually defined in TQ by constraints of two different kinds: The so-called *dynamical* (*or presymplectic*) *constraints* and the *non-dynamical* (*or non-presymplectic*) *ones*. The first ones arise from the compatibility condition, meanwhile the second ones born from the SODE condition (all these results were already known. We include them here in order to give a complete unified exposition). In addition, the SODE condition compels that the arbitrariness of the solution (*gauge freedom*) lies in its vertical part, which is in ker FL_*.

Next, the stability procedure (*tangency condition*) generally leads to new generations of constraints and removes some degrees of the gauge freedom of the solution. Thus, on each step, the stability of the non-dynamical constraints eliminates the same number of gauge degrees of freedom as the number of independent constraints of this kind there are at the previous step; meanwhile, the stability of the dynamical constraints can originate new constraints. Some of them are equivalent to the presymplectic constraints obtained in the PCA (which is an algorithm that does not consider the SODE condition) and are also called *dynamical* (*or presymplectic*) *constraints*. The others are new constraints which are related to the SODE character of the solution, and they are also called *non-dynamical* (*or non-presymplectic*) *constraints*. As a consequence of the analysis, we also prove that only the dynamical constraints can be expressed as FL-projectable functions.

Finally we prove that, in the final submanifold, the SODE solution is FL-projectable if, and only if, this submanifold contains just one point on each leaf of the foliation induced by ker FL_*. So the result of [20] on this problem is recovered and reinterpreted.

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