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R. COUTINHO
R. LIMA
R. VILELA MENDES
S. VAÏENTI

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Logarithmic measures, subdimension and Lyapunov exponents of Cantori

by

R. COUTINHO (1), **R. LIMA** (2), (3), **R. VILELA MENDES** (1), (3) **and S. VAÏENTI** (4), (5)

(1) Centro de Física da Matéria Condensada, Av. Gama Pinto, 2, P-1699 Lisboa Codex, Portugal.
(2) Centre de Physique Théorique, C.N.R.S. Luminy, Case 907 13288 Marseille Cedex, France.
(3) Fakültät für Physik, Universität Bielefeld, Postfach 8640, D-4800 Bielefeld 1, R.F.A.
(4) Dept. of Mathematics, University of Southern California, Los Angeles 90034 CA, U.S.A.
(5) Present adress: Centre de Physique Theorique, Luminy, Case 907, 13288 Marseille, France

ABSTRACT. — The characterization of the metric properties of Cantor sets with vanishing fractal measure is discussed. Measures and generalized dimensions are defined using logarithmic functions of the diameters of the cover.

A general result, obtained for Cantor sets with an exponentially decaying gap family, is applied to the cantori in the sawtooth map.

Also proven is the relation between the rate od decay of the gaps in a gap family and the Lyapunov exponent of an hyperbolic cantorus.

1. LOGARITHMIC MEASURES AND SUBDIMENSIONS

Some time ago Li and Bak [1] found, by numerical computation (for the golden mean rotation number in the standard map), that the fractal dimension of the Aubry-Mather set drops abruptly from 1 to 0 at the critical nonlinearity parameter k_c.

At first sight this results seemed counterintuitive insofar as one might expect the fractal dimension to reflect the effectiveness of the cantorus as a leaky diffusion barrier and thus fade away gradually above the critical point. From the point of view of the general properties of the dynamics the results has however a natural interpretation, for it has been proved that hyperbolic cantori have zero dimension [2] and hyperbolicity has been confirmed, at least for sufficiently large k, in the standard map [3]. More recently, Fathi [10] showed that the Hausdorff dimension of the union of the hyperbolic Aubry-Mather sets in twist diffeomorphisms is zero.

A simple calculation shows that, for Cantor sets of zero Lebesgue measure, the fractal dimension vanishes whenever the gaps decay exponentially while the gap multiplicity at all scales is uniformly bounded by a finite number. This implies that, rather then using the direct method of Li and Bak, the vanishing of the fractal dimension may be infered by plotting on a log plot the gap structure (of a rational approximation) and checking whether the gaps lie on straight lines and do not show growth of multiplicity as one moves to smaller scales (see *Fig.* 1).

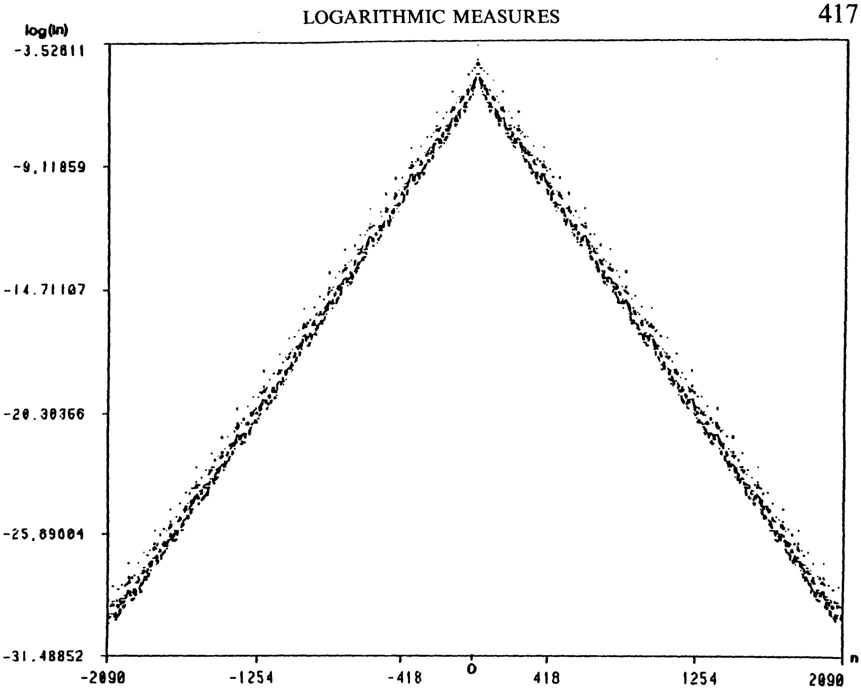
The emergence of a diversity of dynamical behaviors, corresponding to cantorus associated to different values of the non-linearity parameter and rotation number, all having the same dimension, raises the problem of how characterize different sets with vanishing fractal dimension. A possible solution is to use Hausdorff (outer) measures with general (weighing) functions [4].

Let H be the class of functions $g(t)$ defined for all $t>0$ (allowing eventually for the value $+\infty$ for some values of t), monotonic increasing and continuous on the right for $t \geqq 0$ and positive for $t>0$. Let H_0 be the subset of H with $g(0)=0$. Let A be a set in a metric space. The Hausdorff g-measure μ_H^g of A is

$$\mu_H^g = \sup_{\varepsilon>0} \inf_{\substack{d(c_i)\leqq\varepsilon \\ \cup c_i \supset A}} \sum_{i=1}^{\infty} g(d(c_i)) \tag{1.1}$$

where $d(c_i)$ is the diameter of the element c_i of the open covering of A. Because the effect of reducing ε is to reduce the class of covers over which the infimum is taken, the sup may be replaced by $\lim_{\varepsilon \to 0^+}$. When $g(t)=t^d$ we refer to this measure simply as the Hausdorff measure $\mu_H^{(d)}$.

Of particular interest for measuring Cantor sets with exponential gap structure and bounded multiplicity is the class of H_0-functions coinciding



Gap structure in a cantorus of the standard map for $k=0.981{,}635{,}406$, $\rho=(\sqrt{5}-1)/2$.
$\left(\text{Obtained from the rational approximation } \rho'=\dfrac{1{,}597}{4{,}181}\right)$.

in the interval $[0,1-\delta]$ with

$$g_s(t)=\left(\frac{1}{-\log t}\right)^s, \qquad t\in[0,\,1) \tag{1.2}$$

In the space H of functions introduced above there is a partial order relation:

$$g<h \quad \text{if } \frac{h(t)}{g(t)}\to 0 \quad \text{as } t\to 0$$

and one says that g corresponds to a smaller generalized dimension.

For a set A let μ_{H}^{g} be finite for the function of Eq. (1.2) with a certain exponent s_{H}. Then from the comparison theorem [5] for μ_{H}^{g}-measures it follows that

$$\begin{aligned} &\mu_{\mathrm{H}}^{g_s}=0 && \text{for } s>s_{\mathrm{H}} \\ &\mu_{\mathrm{H}}^{g_s}\to\infty && \text{for } s<s_{\mathrm{H}} \end{aligned}$$

To s_{H} we call the *Hausdorff subdimension* of the set A.

If instead of sets of diameter at most ε one considers a covering by a minimum number $n_{\mathrm{A}}(\varepsilon)$ of balls of diameter exactly ε one has, instead of (1.1)

$$\mu_{\mathrm{F}}^{g_s} = \lim_{\varepsilon \to 0} \sup g_s(\varepsilon)\, n_{\mathrm{A}}(\varepsilon) \tag{1.3}$$

To the limit

$$s_{\mathrm{F}}(\mathrm{A}) = \lim_{\varepsilon \to 0} \sup \frac{\log n_{\mathrm{A}}(\varepsilon)}{\log(1/g_1(\varepsilon))} \tag{1.4}$$

we call the *generalized capacity* or *fractal subdimension* of the set A. From the definitions it follows $\mu_{\mathrm{H}}^{g}(\mathrm{A}) \leqq \mu_{\mathrm{F}}^{g}(\mathrm{A})$, therefore

$$s_{\mathrm{H}}(\mathrm{A}) \leqq s_{\mathrm{F}}(\mathrm{A})$$

Fractals are objects such that the number of boxes $\mathrm{N}(\varepsilon)$ needed to cover them has a power law behavior

$$\mathrm{N}(\varepsilon) \sim \varepsilon^{-d}$$

ε being the diameter of the boxes. For the geometrical objects of the type of the Cantor sets in twist maps the corresponding scaling law is

$$\mathrm{N}(\varepsilon) \sim \left(\frac{1}{\log(1/\varepsilon)}\right)^{-s}$$

the exponent being the fractal subdimension defined in Eq. (1.4). To these objects we will call subfractals.

We now come back to our original motivation, namely the metric and dynamical characterization of Cantor sets in two-dimensional twist maps.

2. THE SAWTOOTH MAP

Consider first a piecewise linear map, the sawtooth map, for which many analytical results are known ([6], [7], [11], [12])

$$x_{n+1} = x_n + y_{n+1} \tag{2.1 a}$$

$$y_{n+1} = y_n + k\,\mathrm{S}(x_n) \tag{2.1 b}$$

where $\mathrm{S}(x) = x - \mathrm{Int}(x) - \frac{1}{2}$.

In the sawtooth map there are no KAM circles for $k \neq 0$, all irrational Aubry-Mather sets being cantorus for $k > 0$. The gaps decay exponentially as AB^n, with

$$\mathrm{A}(k) = \frac{1}{2} k \left(k + \frac{1}{4} k^2\right)^{-1/2} \tag{2.2 a}$$

$$\mathrm{B}(k) = 1 + \frac{1}{2} k - \left(k + \frac{1}{4} k^2\right)^{1/2} \tag{2.2 b}$$

and at each scale there are two gaps which may be generated by forward and backward iteration of the main gap of magnitude A.

The Lyapunov numbers are

$$\lambda_{\pm} = 1+\frac{1}{2}k \pm \left(k+\frac{1}{4}k^2\right)^{1/2} \tag{2.3}$$

The rate of decay of the Cantor gaps coincides with the Lyapunov number λ_-. Near the critical point $k_c=0$, the Lyapunov exponent scales as

$$\xi_- = \log \lambda_- \sim -(k-k_c)^{1/2} \tag{2.4}$$

The fractal subdimension s_F of the cantorus is 1 for all k (*see* below). For their metric characterization one requires therefore the calculation of one of the logarithmic measures $\mu_H^{g_1}$ or $\mu_F^{g_1}$. We find that $\mu_F^{g_1}$ [Eq. (3)] is related to the rate of decay of the gaps by

$$\mu_F^{g_1} = -\frac{2}{\log \lambda_-} \tag{2.5}$$

This is a consequence of the following more general result:

THEOREM 1. — *Let* $C \in S^1$ *be a Cantor set with gaps of length* l_n *satisfying:*

(i) $l_n = B_n e^{-\|n\|\xi}$, $\forall n \in \mathbb{Z}$ with $0 < \underline{B} < B_n < \bar{B} < \infty$; $\xi > 0$ (2.6 *a*)

(ii) $\sum_{n \in \mathbb{Z}} l_n = 1$ (2.6 *b*)

Let $N(\varepsilon)$ be the smallest number of open intervals of diameter ε needed to cover C. Then

$$\lim_{\varepsilon \to 0} \frac{N(\varepsilon)}{-\log \varepsilon} = \frac{2}{\xi}. \tag{2.7}$$

Proof. — Let x_k and $\bar{x}_k$ be the right and left ends of the gap l_k. Define the origin of the coordinates in S^1 in such a way that $x_0 = \frac{l_0}{2}$ and $\bar{x}_0 = 1-\frac{l_0}{2}$. Then $l_k = x_k - \bar{x}_k$ $(k \neq 0)$. The proof will be divided in several steps:

(*a*) The distance between any two points in the set $S_N = \{x_k\}_{\|k\| \leqq N} \subset C$ is larger than $\underline{B}\exp(-N\xi)$.

Let $x_k > x_{k'}$. Then $\bar{x}_k > x_{k'}$ and

$$\|x_k - x_{k'}\| = x_k - x_{k'} = x_k - \bar{x}_k + \bar{x}_k - x_{k'} > l_k > \underline{B}\, e^{-\|k\|\xi} \geqq \underline{B}\, e^{-N\xi}$$

(*b*) One needs at least $2N+1$ intervals of diameter $\underline{B}\exp(-N\xi)$ to cover C.

By (a), each interval of diameter $\underline{B}\exp(-N\xi)$ contains at most one element of the set $S_N \subset C$. Because the set S_N has $2N+1$ elements the assertion follows.

(c) $\liminf_{\varepsilon \to 0} \dfrac{N(\varepsilon)}{-\log \varepsilon} \geqq \dfrac{2}{\xi}$.

Consider N such that $\underline{B}\exp\{-(N+1)\xi\} < \varepsilon \leqq \underline{B}\exp(-N\xi)$. Then $N(\varepsilon) \geqq N(\underline{B}\, e^{-N\xi})$ and, by (b), $N(\varepsilon) \geqq 2N+1$. Hence

$$\frac{N(\varepsilon)}{-\log \varepsilon} > \frac{2N+1}{-\log(\underline{B}\, e^{-(N+1)\xi})} = \frac{2N+1}{(N+1)\xi - \log \underline{B}}$$

When $\varepsilon \to 0$, $N \to \infty$ and the assertion follows.

(d) The Lebesgue measure of S^1 minus the first $2N+1$ gaps is less than $\dfrac{2\bar{B}}{1-e^{-\xi}} e^{-(N+1)\xi}$. Because $\sum_{n \in \mathbb{Z}} l_n = 1$ one obtains:

$$1 - \sum_{k=-N}^{N} l_k = \sum_{k=-\infty}^{-N-1} l_k + \sum_{k=N+1}^{\infty} l_k \leqq \sum_{k=-\infty}^{-N-1} \bar{B}\, e^{-\|k\|\xi} + \sum_{k=N+1}^{\infty} \bar{B}\, e^{-\|k\|\xi}$$
$$= \frac{2\bar{B}}{1-e^{-\xi}} e^{-(N+1)\xi}$$

(e) $\limsup_{\varepsilon \to 0} \dfrac{N(\varepsilon)}{-\log \varepsilon} \leqq \dfrac{2}{\xi}$.

From (d) it follows that we may cover C with $2N+1$ intervals of diameter $\dfrac{2\bar{B}}{1-e^{-\xi}} e^{-(N+1)\xi}$. Now given $\varepsilon > 0$ we choose N such that

$$\frac{2\bar{B}}{1-e^{-\xi}} e^{-(N+1)\xi} < \varepsilon \leqq \frac{2\bar{B}}{1-e^{-\xi}} e^{-N\xi}. \qquad \text{Then}$$

$$\frac{N(\varepsilon)}{-\log \varepsilon} \leqq \frac{2N+1}{-\log((2\bar{B}/(1-e^{-\xi}))\, e^{-N\xi})} = \frac{2N+1}{N\xi - \log(2\bar{B}/(1-e^{-\xi}))}$$

In the limit $\varepsilon \to 0$, N goes to infinite and the assertion (e) is obtained.

The theorem [Eq. (2.7)] now follows from (c) and (e). □

Remark 1. – The condition $(2.6a)$ on the size of the gaps may be enlarged to $\lim_{\varepsilon \to \pm\infty} \dfrac{1}{\|n\|} \log l_n = -\xi$, which in fact implies that $\forall \delta > 0\ \exists\, \underline{B}$ and $\bar{B} \in \mathbb{R}^+$ such that

$$\underline{B}\, e^{-\|n\|(\xi+\delta)} \leqq l_n \leqq \bar{B}\, e^{-\|n\|(\xi-\delta)}.$$

Remark 2. – The theorem was proved for a Cantor set in S^1. However the result (2.7) still holds for a Cantor set in $S^1 \times \mathbb{R}$, if the projection in S^1 satisfies $(2.6\,a\text{-}b)$ and a Lipschitz condition

$$\exists k > 0, \quad \forall (x, y), (x', y') \in C, \qquad \|y - y'\| \leqq k \|x - x'\|$$

(a Lipschitz condition of this type holds for the Aubry-Mather Cantor sets of twist maps).

In this case one replaces

$$\frac{2\bar{B}}{1-e^{-\xi}}e^{-(N+1)\xi} \quad \text{by} \quad \sqrt{1+k^2}\,\frac{2\bar{B}}{1-e^{-\xi}}e^{-(N+1)\xi}$$

in the proof of assertion (*e*), the rest of the proof remaining unchanged.

From Eq. (2.7) it now follows trivially that the subdimension of the Cantor sets in the sawtooh map is one and the logarithmic fractal measure $\mu_F^{g_1}$ is given by (2.5).

It is somewhat unpleasant to have all the cantorus, with exponential gap decay rate (and bounded multiplicity), with the same subdimension and to require, for their metric characterization, the logarithmic fractal measure $\mu_F^{g_1}$, a quantity which is in general much more difficult to compute than the subdimension. One might wonder whether, with the choice of other (weighing) functions $g_s(t)$ instead of those defined by Eq. (2), one might distinguish cantorous with different gap decay rates by its subdimension. It is easy to see that this cannot be the case as long as the gaps decay exponentially and there is an upper bound for the multiplicity at all scales. When computing the subdimension, one compares log (N) with $h(\xi N)=\log g_1(e^{-\xi N})$. For different decay rates to correspond to different subdimensions one should have for large N

$$h(\xi N)\sim f(\xi)\log(N)$$

and this implies $f(\xi)=\text{constant}$, *i.e.* there is no function to distinguish different decay rates by the subdimension.

3. LYAPUNOV EXPONENT AND GAP DECAY RATE

In the sawtooh map, as we saw, it is the Lyapunov exponent, a dynamical quantity, that, coinciding with the gap decay rate, controls the metric properties of the cantorus. It seems to have been assumed by several authors that the Lyapunov exponent coincides in general with the gap decay rate, but, as far as we know, no explicite proof has been given. In fact, and to our surprise, a rigourous proof of this statement turns out to be non-trivial. Below we give an explicite proof for hyperbolic cantorus of C^2 twist maps.

Notation and background

The following notations and general properties of cantorus in twist maps [8] are used:

(1) T^* is an area preserving twist C^2-diffeomorphism of the annulus $(S^1 \times [a, b])$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ a lift of T^* and $DT^N(x)$ the Jacobian matrix of T^N at x.

(2) $C^* \subset S^1 \times [a, b]$ is a T^*-invariant cantorus and $C \in \mathbb{R}^2$ its lift (*i. e.* $(x^1, x^2) \in C^* \Rightarrow (x^1, x^2) \in C$ and $(x^1, x^2) \in C \Rightarrow (x^1 + r, x^2) \in C \; \forall r \in \mathbb{Z}$).

(3) Let $\pi_1(x^1, x^2) = x^1$ and $\pi_2(x^1, x^2) = x^2$.

(4) $\bar{x}_i = T^i \bar{x}_0$ and $x_i = T^i x_0$ are the gap endpoints of a gap orbit, $\bar{x}_0$ and x_0 being the endpoints of the largest gap in the orbit. $\pi_1(\bar{x}_0) < \pi_1(x_0)$. $l_i = x_i - \bar{x}_i \in \mathbb{R}^2$.

(5) There is a $k_1 > 0$ such that $\|\pi_2(x-y)\| \leqq k_1 \|\pi_1(x-y)\| \; \forall x, y \in C$.

(6) $\sum_{n \in \mathbb{Z}} \|l_n\| \leqq \sqrt{1 + k_1^2}$ with $\| \;\; \|$ being the norm in $\mathbb{R}^2$.

The following properties are a direct consequence of hyperbolicity [9]:

(7) There is a constant $k_2 > 0$ and a contraction constant $0 < \alpha < 1$ such that for each $x \in C$ there is a unique one-dimensional subspace of $\mathbb{R}^2$, E^s_x satisfying

$$\|DT^N(x)|_{E^s_x}\| < k_2 \alpha^N, \qquad \forall N \in \mathbb{Z}.$$

If $v \notin E^s_x$ then $\lim_{N \to \infty} \|DT^N v\| = \infty$.

(8) $v \in E^s_x \Rightarrow DT^N v \in E^s_{T^N x}, \; \forall N \in \mathbb{Z}, \; \forall x \in C$

(9) From hyperbolicity and from $\lim_{i \to \infty} \|l_i\| = 0$ it follows that

$$\exists I \in \mathbb{N} : \quad i \geqq I \quad \Rightarrow \quad \|l_{i+N}\| \leqq k_2 \alpha^N \|l_i\|, \qquad \forall N \geqq 0.$$

We now prove the following

THEOREM 2. — *For an hyperbolic cantorus* C *of a* C^2 *twist map* T *the asymptotic rate of decay of the gaps in a gap family coincides with the negative Lyapunov exponent of* T *on* C, i. e.

$$\lim_{n \to \infty} \frac{1}{n} \log \|l_n\| = \xi \equiv \lim_{N \to \infty} \log \|DT^N(x) v_x\| \qquad \text{with} \quad x \in C, \quad v_x \in E^s_x \tag{3.1}$$

Remark. — Since by (9) the endpoints of l_n belong to the same stable manifolds for n large, wa can conclude that $\limsup_{n \to \infty} \frac{1}{n} \log \|l_n\| \leqq \xi$ (*see* Th. (6.3) in [14]). The lower bound will be proved in the following; for completeness we also give the upper bound.

The proof is divided into several steps:

(*a*) Let x_{i_n} be a sequence of gap endpoints (as defined above). If $x_{i_n} \to \mathrm{x} \in \mathrm{C}$ then

$$\frac{l_{i_n}}{\| l_{i_n} \|} \to v_x^s \in \mathrm{E}_x^s \qquad \text{with} \quad \pi_1(v_x^s) > 0. \ (\| v_x^s \| = 1)$$

Proof. – The proof simply follows observing that any convergent subsequence of $\frac{l_{i_n}}{\| l_{i_n} \|}$ contains a subsequence converging to a vector $v \in \mathrm{E}_x^s$ and this by the hyperbolic properties (7) and (9) defining the stable manifolds.

(*b*) The function $x \to v_x^s$ (where $v_x^s \in \mathrm{E}_x^s$, $\| v_x^s \| = 1$, $\pi_1(v_x^s) > 0$) is a continuous function on C.

Proof. – Let $x \in \mathrm{C}$ and $x(k) \in \mathrm{C}$ such that $\lim\limits_{k \to \infty} x(k) = x$. For each k let $x_{i_n^{(k)}}$ be a sequence of gap endpoints satisfying

$$\| x_{i_n^{(k)}} - x(k) \| < \frac{1}{n}$$

$$\left\| \frac{l_{i_n^{(k)}}}{\| l_{i_n^{(k)}} \|} - v_{x(k)}^s \right\| < \frac{1}{n} \qquad \text{[possible by } (a)\text{]}$$

For the diagonal sequence $i_k^{(k)}$

$$\| x_{i_k^{(k)}} - x \| \leqq \| x_{i_k^{(k)}} - x(k) \| + \| x(k) - x \| \leqq \frac{1}{k} + \| x(k) - x \|.$$

Hence $\lim\limits_{k \to \infty} x_{i_k^{(k)}} = x$ and by (*a*) $\lim\limits_{k \to \infty} \frac{l_{i_k^{(k)}}}{\| l_{i_k^{(k)}} \|} = v_x^s$. Then

$$\| v_x^s - v_{x(k)}^s \| \leqq \left\| v_x^s - \frac{l_{i_k^{(k)}}}{\| l_{i_k^{(k)}} \|} \right\| + \frac{1}{k} \quad \text{implying} \quad \lim_{k \to \infty} v_{x(k)}^s = v_x^s.$$

(*c*) Let, as before, x_i be a sequence of gap endpoints and consider $v_{x_i}^s \in \mathrm{E}_{x_i}^s$, $\| v_{x_i}^s \| = 1$, $\pi_1(v_{x_i}^s) > 0$. Then $\lim\limits_{i \to \infty} \left\| v_{x_i}^s - \frac{l_i}{\| l_i \|} \right\| = 0$.

Proof. – Suppose the assertion to be false. Then there is $\delta > 0$ and a subsequence i_n such that $\left\| v_{x_{i_n}}^s - \frac{l_{i_n}}{\| l_{i_n} \|} \right\| > \delta$, $\forall n \in \mathbb{N}$. The associated sequence x_{i_n} contains a convergent $x_{i_{n'}}$ such that $x_{i_{n'}} \to x \pmod 1$ with $x \in \mathrm{C}$.

Then by (a) $\frac{l_{i_{n'}}}{\| l_{i_{n'}} \|} \to v_x^s$ $(n' \to \infty)$ and by (b) $v_{x_{i_{n'}}}^s \to v_x^s$ $(n' \to \infty)$, implying

$$\lim_{n' \to \infty} \left\| v_{x_{i_{n'}}}^s - \frac{l_{i_{n'}}}{\| l_{i_{n'}} \|} \right\| = 0$$

(d) The following inequality will be used: If $\frac{\| B \|}{\| A \|} < \frac{1}{2}$ then

$$\log \| A \| - 2 \frac{\| B \|}{\| A \|} \leqq \log \| A + B \| \leqq \log \| A \| + \frac{\| B \|}{\| A \|} \tag{3.2}$$

(e) $\lim_{n \to \infty} \frac{1}{n} \log \| l_n \| = \xi$ where $\xi < 0$ is the negative Lyapunov exponent of T on C.

Proof. — T is uniquely ergodic since the motion is semi-conjugate to an irrational rotation.

This and the continuity of the function $\log \| DT(x) v_x^s \|$ following from (b), insures us that ξ is the same *for all* $x \in C$. In fact we can write:

$$\xi = \lim_{N \to \infty} \frac{1}{N} \log \| DT^N(x) v_x^s \| = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \log \| DT(T^k x) v_x^s \| \tag{3.3}$$

and the convergence of this limit is uniform for $x \in C$ (*cf.*, e.g., [13]).

For all $x \in C$ and all $u \in \mathbb{R}^2$ there is a number $R > 1$ such that

$$\| u \| R^{-1} \leqq \| DT(x) u \| \leqq R \| u \|;$$

and, by Tailor's formula since T is C^2:

$$l_{i+1} = DT(x_i) l_i + r_i \qquad \text{with} \quad \| r_i \| < C \| l_i \|^2.$$

Let $\delta > 0$. Since $\| l_i \| \to 0 \mid (i \to \infty) \exists i_1$ such that $i \geqq i_1 \Rightarrow \| l_i \| \leqq \frac{\delta}{4CR}$.

By (c) $\exists i_2$ such that $i \geqq i_2 \Rightarrow \left\| v_{x_i}^s - \frac{l_i}{\| l_i \|} \right\| < \frac{\delta}{4R^2}$.

Let now $i > \max(i_1, i_2)$ and compute

$$\frac{1}{N} \log \frac{\| l_{i+N} \|}{\| l_i \|} = \frac{1}{N} \sum_{k=0}^{N-1} \log \frac{\| l_{i+k+1} \|}{\| l_{i+k} \|}$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} \log \left\| DT(x_{i+k}) \frac{l_{i+k}}{\| l_{i+k} \|} + \frac{r_{i+k}}{\| l_{i+k} \|} \right\|$$

Since $\dfrac{\|r_{i+k}\|}{\|l_{i+k}\|}\ \dfrac{\|l_{i+k}\|}{\|\mathrm{DT}(x_{i+k})\,l_{i+k}\|} \leqq \mathrm{CR}\,\|l_{i+k}\| < \dfrac{\delta}{4}$ one has, using (d)

$$\frac{1}{N}\sum_{k=0}^{N-1}\log\left\|\mathrm{DT}(x_{i+k})\frac{l_{i+k}}{\|l_{i+k}\|}\right\| - \frac{\delta}{2} \leqq \frac{1}{N}\log\frac{\|l_{i+N}\|}{\|l_i\|}$$
$$\leqq \frac{1}{N}\sum_{k=0}^{N-1}\log\left\|\mathrm{DT}(x_{i+k})\frac{l_{i+k}}{\|l_{i+k}\|}\right\| + \frac{\delta}{2} \qquad (3.4)$$

Now from

$$\left\|\mathrm{DT}(x_{i+k})\frac{l_{i+k}}{\|l_{i+k}\|}\right\| = \left\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}} + \mathrm{DT}(x_{i+k})\left(\frac{l_{i+k}}{\|l_{i+k}\|} - v^s_{x_{i+k}}\right)\right\|$$

and

$$\frac{\|\mathrm{DT}(x_{i+k})\,((l_{i+k}/\|l_{i+k}\|) - v^s_{x_{i+k}})\|}{\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}}\|} \leqq \mathrm{R}^2\left\|\frac{l_{i+k}}{\|l_{i+k}\|} - v^s_{x_{i+k}}\right\| < \frac{\delta}{4}$$

one obtains, using (d)

$$\frac{1}{N}\sum_{k=0}^{N-1}\log\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}}\| - \frac{\delta}{2} \leqq \frac{1}{N}\sum_{k=0}^{N-1}\log\left\|\mathrm{DT}(x_{i+k})\frac{l_{i+k}}{\|l_{i+k}\|}\right\|$$
$$\leqq \frac{1}{N}\sum_{k=0}^{N-1}\log\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}}\| + \frac{\delta}{2} \qquad (3.5)$$

From (3.4) and (3.5) it now follows that $\forall\,\delta>0\ \exists\, i_0$ such that $i>i_0$ implies.

$$\frac{1}{N}\sum_{k=0}^{N-1}\log\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}}\| - \delta \leqq \frac{1}{N}\log\frac{\|l_{i+k}\|}{\|l_i\|}$$
$$\leqq \frac{1}{N}\sum_{k=0}^{N-1}\log\|\mathrm{DT}(x_{i+k})\,v^s_{x_{i+k}}\| + \delta$$

Therefore

$$\xi - \delta \leqq \limsup_{N\to\infty}\frac{1}{N}\log\frac{\|l_{i+N}\|}{\|l_i\|} \leqq \xi + \delta$$

But

$$\limsup_{N\to\infty}\frac{1}{N}\log\frac{\|l_{i+N}\|}{\|l_i\|}$$
$$= \limsup_{N\to\infty}\frac{i+N}{N}\,\frac{1}{i+N}\log\|l_{i+N}\| = \limsup_{N\to\infty}\frac{1}{N}\log\|l_N\|$$

Then

$$\xi-\delta \leqq \lim_{N\to\infty} \sup \frac{1}{N} \log \| l_N \| \leqq \xi+\delta$$

and, since this holds for every $\delta > 0$,

$$\lim_{N\to\infty} \sup \frac{1}{N} \log \| l_N \| = \xi.$$

The same argument leads to

$$\lim_{N\to\infty} \inf \frac{1}{N} \log \| l_N \| = \xi$$

Hence

$$\lim_{N\to\infty} \frac{1}{N} \log \| l_N \| = \xi$$

Remark 1. – A a similar argument may be applied to the homoclinic intersections of stable and unstable manifolds of an hyperbolic fixed point. The rate of approach of the intersections to the fixed point would therefore also be controlled by the Lyapunov exponent at the hyperbolic fixed point.

Remark 2. – The rate of decay of the gaps of the cantorus is a quantity which is relatively easy to obtain, with precision, from the numerical study of a rational approximation. This result provides therefore an accurate way to measure the Lyapunov exponent.

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