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## Random Schrödinger operators with a constant electric field

by

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ABSTRACT. — We consider two random one-dimensional Schrödinger operators with a constant electric field:

$$H_{\omega}^F \equiv H_{\omega} - Fx \quad \text{and} \quad \mathcal{H}_{\omega}^{\kappa, F} \equiv \mathcal{H}_{\omega}^{\kappa} - Fx,$$

where  $H_{\omega}$  is the G.M.P. model and  $\mathcal{H}_{\omega}^{\kappa} = -d^2/dx^2 - \kappa W(x)$ ,  $W(x)$  being the Gaussian white noise. We prove that for any  $F > 0$ ,  $H_{\omega}^F$  has purely absolutely continuous spectrum with probability one, whereas  $\mathcal{H}_{\omega}^{\kappa, F}$  exhibits, as we increase  $F > 0$ , a transition from pure point spectrum with power decaying eigenfunctions ( $0 < F < \kappa^2/2$ ) to purely singular continuous spectrum ( $F \geq \kappa^2/2$ ). The proof is based on the combination of a recent result of Gilbert and Pearson and the “Kotani’s trick”.

RÉSUMÉ. — On considère deux opérateurs de Schrödinger aléatoires en dimension un avec un champ électrique constant:

$$H_{\omega}^F \equiv H_{\omega} - Fx \quad \text{et} \quad \mathcal{H}_{\omega}^{\kappa, F} \equiv \mathcal{H}_{\omega}^{\kappa} - Fx,$$

ou  $H_{\omega}$  est le G.M.P.-modèle et  $\mathcal{H}_{\omega}^{\kappa} = -d^2/dx^2 + \kappa W(x)$ ,  $W(x)$  étant le bruit blanc gaussien. On démontre que pour tout  $F > 0$ ,  $H_{\omega}^F$  a un spectre purement absolument continu et que  $\mathcal{H}_{\omega}^{\kappa}$  présente une transition, quand on fait croître  $F > 0$ , d’un spectre purement ponctuel avec fonctions propres, qui décroissent suivant une puissance ( $0 < F < \kappa^2/2$ ), à un spectre purement singulièrement continu ( $F \geq \kappa^2/2$ ). La démonstration est basée

sur la combinaison d'un résultat récent de Gilbert et Pearson avec « Kotani's trick ».

## 1. INTRODUCTION. STATEMENT OF THE MAIN RESULTS

In this paper, we consider two concrete examples of one-dimensional random Schrödinger operators with a constant electric field both of which have, when the electric field is absent, dense pure point spectra with exponentially decaying eigenfunctions.

The first example we consider is the following operator in  $L^2(\mathbb{R}^1)$ :

$$(1.1) \quad H_{\omega}^F = -d^2/dt^2 + G(X_{\omega}(t)) - Ft, \quad -\infty < t < +\infty,$$

where  $F \geq 0$  is the intensity of the electric field,  $\{X_{\omega}(t); -\infty < t < +\infty\}$  is the Brownian motion on the unit circle

$$S^1 = \mathbb{R}^1/Z^1 \quad \text{with} \quad P(X(0) \in dx) = dx,$$

$dx$  being the Lebesgue measure on  $S^1$ , and  $G$  is a non-constant  $C^{\infty}$ -function on  $S^1$ . As our basic probability space, we choose  $\Omega = C(\mathbb{R}^1; S^1)$ , the totality of all continuous functions  $\omega$  from  $\mathbb{R}^1$  to  $S^1$ . As is well known, the measure  $dx$  is the unique invariant measure of the Brownian motion on  $S^1$ , and with this choice of the initial distribution, the above  $\{X(t)\}$  becomes a stationary, ergodic stochastic process. Finally for  $x \in \mathbb{R}^1$ ,  $T_x \omega$  is the "shifted path":  $(T_x \omega)(\cdot) = \omega(\cdot + x)$ .

Since  $H_{\omega}^F = -d^2/dt^2 - Ft$  is self-adjoint on the domain

$$(1.2) \quad D = \left\{ u \in L^2(\mathbb{R}^1) : u(t) \text{ and } u'(t) \text{ are absolutely continuous} \right. \\ \left. \text{and } u''(t) + Ft u(t) \in L^2(\mathbb{R}^1) \right\},$$

and since  $G(X_{\omega}(t))$  is a bounded perturbation,  $H_{\omega}^F$  is also self-adjoint on the domain  $D$  for each  $\omega$  and  $F \geq 0$ .

In the following, the spectrum of a self-adjoint operator  $A$  is generally denoted by  $\Sigma(A)$ .

Now concerning  $H_{\omega}^F$ , we have the following

**THEOREM 1.** — *Let  $F > 0$  be arbitrary. Then with probability one,  $H_{\omega}^F$  has purely absolutely continuous spectrum, and  $\Sigma(H_{\omega}^F) = (-\infty, \infty)$ .*

It is a famous result that  $H_{\omega}^0 = -d^2/dt^2 + G(X_{\omega}(t))$  has dense pure point spectrum with exponentially decaying eigenfunctions. (Exponential localization, see Goldsheid, Molchanov and Pastur [8], Molchanov [17], Car-

mona [3], and Kotani [13] for the proof.) The above theorem says that an arbitrarily weak electric field delocalizes the electron. An analogous result has already been obtained by Bentosela *et al.* [2], who give an example of random Schrödinger operator which shows the transition from exponential localization to purely absolutely continuous spectrum under an arbitrarily weak electric field. But their result is essentially deterministic in the sense that what they actually prove is that a Schrödinger operator  $-d^2/dt^2 + q(t) - Ft$ ,  $F \neq 0$ , has purely absolutely continuous spectrum when  $q(t)$  is bounded up to first and second derivatives. (See Ben-Artzi [1] for generalizations of this result.) In our Theorem 1, on the contrary, the potential  $G(X_\omega(t))$  is not differentiable and the result holds only almost surely (not deterministically).

The second model we are interested in is the Schrödinger operator with white noise potential plus a constant electric field:

$$(1.3) \quad \mathcal{H}_\omega^F = \mathcal{H}_\omega^{F, \kappa, \theta} = -d^2/dt^2 + \kappa B'_\omega(t) - Ft, \quad \kappa > 0, \quad F \geq 0,$$

which we consider in  $L^2([0, \infty))$  under the boundary condition

$$(1.4)_\theta \quad u(0) \cos \theta + u'(0) \sin \theta = 0, \quad \theta \in [0, \pi).$$

Here  $\{B_\omega(t)\}_{t \geq 0}$  is the standard Brownian motion with  $B_\omega(0) = 0$  defined on some probability space  $(\Omega, \mathcal{F}, P)$ , and  $B'_\omega(t)$  is the “derivative” of its sample function, namely the white noise. But since  $B_\omega(\cdot)$  is nowhere differentiable with probability one, the definition of  $\mathcal{H}_\omega$  needs a justification. In fact, we can define the Schrödinger operator

$$\mathcal{H}_Q^\theta = -d^2/dt^2 + Q'(t)$$

with the boundary condition  $(1.4)_\theta$  for any continuous real function  $Q(t)$ ,  $Q(0) = 0$ , in the following manner.

Let  $C_Q^\theta$  be the totality of  $C^1$ -function on  $[0, \infty)$  satisfying  $(1.4)_\theta$  and such that there is a  $v(\cdot) \in L^1_{loc}([0, \infty))$  which satisfies

$$(1.5) \quad u'(t) = u'(0) + Q(t)u(t) - \int_0^t \{Q(s)u'(s) + v(s)\} ds, \quad t \geq 0.$$

This  $v$  is uniquely determined from  $u \in C_Q^\theta$ , and we let  $v \equiv \mathcal{H}_Q^\theta u$ . Finally if we define the domain of  $\mathcal{H}_Q^\theta$  by

$$(1.6) \quad D_Q^\theta = \{u \in C_Q^\theta \cap L^2([0, \infty)); \mathcal{H}_Q^\theta u \in L^2([0, \infty))\},$$

then  $D_Q^\theta$  is dense in  $L^2([0, \infty))$ . For details about  $\mathcal{H}_Q^\theta$ , see Minami [15]. Now we get the definition of the operator  $\mathcal{H}_\omega^F = \mathcal{H}_\omega^{F, \kappa, \theta}$  by letting

$$Q(t) = \kappa B_\omega(t) - \frac{1}{2} Ft^2.$$

The result for  $\mathcal{H}_\omega^\theta$  is the following.

**THEOREM 2.** — Fix  $\theta \in [0, \pi)$ ,  $\kappa > 0$ , and  $F > 0$ .

(i) With probability one,  $\mathcal{H}_\omega^F$  is self-adjoint and  $\Sigma(\mathcal{H}_\omega^F) = (-\infty, \infty)$ .

(ii) If  $0 < F < \kappa^2/2$ , then with probability one,  $\mathcal{H}_\omega^F$  has pure point spectrum and each eigenfunction  $u(t)$  satisfies

$$(1.7) \quad u(t)^2 = O(t^{-\kappa^2/4 F - 1/2 + \delta}), \quad t \rightarrow \infty,$$

and

$$(1.8) \quad \lim_{x \rightarrow +\infty} x^{\kappa^2/4 F + 1/2 + \delta} \int_x^{x+1} |u(t)|^2 dt > 0,$$

for any  $\delta > 0$ .

(iii) If  $F \geq \kappa^2/2$ , then with probability one,  $\mathcal{H}_\omega^F$  has purely singular continuous spectrum.

Just as in the first example, we have exponential localization when  $F = 0$  (see e.g. Minami [16]). For small values of  $F$ , we have power-law localization, and at the critical value  $F_c = \kappa^2/2$ , transition to singular continuous spectrum occurs.

In [5], Delyon, Simon and Souillard analyzed the random Kronig-Penny model with a constant electric field and proved that for sufficiently small  $F > 0$ , one has power-law localization and that for sufficiently large  $F$ , the spectrum is continuous. But they did not determine whether this continuous spectrum is singular or absolutely continuous, and the question of the existence and determination of the critical intensity  $F_c$  of the electric field remained open. For our white noise model, more accurate analysis is possible.

The author originally intended to treat  $\mathcal{H}_\omega^F$  on the whole space  $\mathbb{R}^1$ , rather than on the half-space  $[0, \infty)$  under boundary conditions. But for this purpose, it is necessary to consider  $\mathcal{H}_\omega^{F, \kappa, \theta}$  for  $F < 0$ . Although the author did not succeed to prove it, he believes that when  $F < 0$ ,  $\mathcal{H}_\omega^{F, \kappa, \theta}$  has purely discrete spectrum with probability one, for any  $\theta \in [0, \pi)$ . If it were the case, then  $\mathcal{H}_\omega^{F, \kappa}$ , considered on the whole space, would have exactly the same spectral properties as expressed in Theorem 2. We shall return to the discussion of this problem at the end of this paper.

The proof of Theorems 1 and 2 is based on the combination of recent results of Gilbert and Pearson ([6], [7]) and the so called Kotani's trick. In paragraph 2, we state and prove some general criteria to determine the spectral properties of random Schrödinger operators, and show how these can be used to reduce the proof of Theorems 1 and 2 to the investigation of the asymptotic behavior of the solutions of eigenvalue equations. This asymptotic behavior is studied in detail in paragraph 3.

**2. APPLICATIONS OF GILBERT-PEARSON'S THEORY**

Consider a one-dimensional Schrödinger operator

$$(2.1) \quad L = -d^2/dt^2 + q(t), \quad -\infty < t < +\infty.$$

When we restrict this to the half-space  $[0, \infty)$  under the boundary condition  $(1.4)_\theta$ , we shall denote the corresponding operator by  $L^\theta$ . Here,  $q(t)$  may be any real continuous function, or even a "generalized potential"  $q(t) = Q'(t)$  introduced in paragraph 1. We suppose that  $L$  is in the limit point case at  $\pm\infty$  – or at  $+\infty$  only, when we consider  $L^\theta$ . As is well known, this is equivalent to saying that  $L$  is self-adjoint in  $L^2(\mathbb{R}^1)$  with the domain

$$(2.2) \quad D(L) = \{ u \in L^2(\mathbb{R}^1) : Lu \text{ has a meaning and } u \in L^2(\mathbb{R}^1) \}$$

– or saying that  $L^\theta$  is self-adjoint in  $L^2([0, \infty))$  with the domain

$$(2.2) \quad D^\theta(L) = \{ u|_{[0, \infty)} : u \in D(L) \text{ and } u \text{ satisfies } (1.4)_\theta \}.$$

Following Gilbert and Pearson [6], we say that a non-trivial solution  $u_s$  of the equation  $Lu = \xi u$  is subordinate at  $+\infty$  [at  $-\infty$ ] if for every other linearly independent solution  $v$  of the same equation, one has

$$(2.3) \quad \lim_{T \rightarrow +\infty} \frac{\|u_s\|_T}{\|v\|_T} = 0.$$

where  $\|u\|_T^2 = \int_0^T |u(x)|^2 dx$ . Subordinate solution may not exist, but

when it does, it is unique up to multiplicative constants. We also recall that  $L$  is in the limit point case at  $+\infty$  [ $-\infty$ ] if and only if for some (and hence for all)  $\xi$ ,  $Lu = \xi u$  has at most one linearly independent solution which is square integrable near  $+\infty$  [ $-\infty$ ]. Hence if we are in the limit point case, and if there is a solution which is square integrable near  $+\infty$  [ $-\infty$ ], then it is necessarily the subordinate solution at  $+\infty$  [ $-\infty$ ].

In what follows,  $\mu(d\xi)$  shall stand for the Lebesgue measure on  $\mathbb{R}^1$ . The minimal support of a measure  $\nu$  on  $\mathbb{R}^1$  is, by definition, a Borel set  $S \subset \mathbb{R}$  such that  $\nu(S^c) = 0$  and that  $\mu(A) = 0$  whenever  $A \subset S$  and  $\nu(A) = 0$ . The minimal support  $S$  of a measure  $\nu$  is determined only up to  $(\mu + \nu)$ -null sets, so that in this section, all equalities or inclusion relations concerning minimal supports of measures are understood to hold after modifying suitable null sets.

Let  $\sigma(d\xi)$  [resp.  $\sigma^\theta(d\xi)$ ] be the spectral measure of  $H$  (see [5], [6] or references therein for the definition of the spectral measure). The theorem of Gilbert and Pearson we need in this paper reads as follows.

**THEOREM 3.** – (Theorem 3 of [6] and Theorem 3.6 of [7].) Let  $\sigma = \sigma_{ac} + \sigma_s$  and  $\sigma^\theta = \sigma_{ac}^\theta + \sigma_s^\theta$  be the Lebesgue decomposition of  $\sigma$  and  $\sigma^\theta$  into the absolutely continuous and singular parts, and let  $M_{ac}$ ,  $M_s$ ,  $M_{ac}^\theta$ , and  $M_s^\theta$  be

the minimal supports of  $\sigma_{ac}$ ,  $\sigma_s$ ,  $\sigma_{ac}^0$ , and  $\sigma_s^0$  respectively. Then,  $\theta$  being arbitrary, we have

- (i)  $M_{ac}^0 = \{ \xi \in \mathbb{R}^1; Lu = \xi u \text{ has no subordinate solution at } +\infty \}$ , in particular,  $M_{ac}^0$  is independent of  $\theta$ ;
- (ii)  $M_s^0 = \{ \xi \in \mathbb{R}^1; Lu = \xi u \text{ has a subordinate solution at } +\infty \text{ satisfying the boundary condition } (1.4)_\theta \}$ ;
- (iii)  $M_{ac} = \{ \xi \in \mathbb{R}^1; Lu = \xi u \text{ has no subordinate solution at } +\infty, \text{ or has no subordinate solution at } -\infty \}$ ;
- (iv)  $M_s = \{ \xi \in \mathbb{R}^1; Lu = \xi u \text{ has a solution which is subordinate both at } +\infty \text{ and at } -\infty \}$ .

Let us now turn to the consideration of random Schrödinger operator

$$(2.4) \quad L_\omega = -d^2/dt^2 + q_\omega(t), \quad -\infty < t < +\infty,$$

or  $L_\omega^0$  considered on the half-space  $[0, \infty)$  under the boundary condition  $(1.4)_\theta$ . Here  $\{q_\omega(t); -\infty < t < +\infty\}$  is a random function on a probability space  $(\Omega, \mathcal{F}, P)$  with continuous sample functions, or a random generalized function  $q_\omega(t) = Q'_\omega(t)$ , where  $\{Q_\omega(t); -\infty < t < +\infty\}$  is a random function with continuous sample functions such that  $Q_\omega(0) = 0$ . We assume that  $L_\omega$  is in the limit point case at  $\pm\infty$  for  $P$ -a.a.  $\omega$ , hence  $L_\omega$  and  $L_\omega^0$  are self-adjoint with probability one. For such  $\omega$ 's, let  $\sigma_\omega(d\xi)$  [resp.  $\sigma_\omega^0(d\xi)$ ] be the spectral measure of  $L_\omega$  [resp.  $L_\omega^0$ ].

Now we make the following assumption, which is technical but fundamental in this section. Let  $\mathcal{G} = \sigma[q_\omega(t); t \notin [0, 1]]$  be the  $\sigma$ -field generated by the random potential outside the interval  $[0, 1]$ , and let  $\mathcal{B}(\mathbb{R}^1)$  be the Borel field over  $\mathbb{R}^1$ . Then we assume that the following condition holds:

*Condition K.* — For any  $B \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^1)$  such that  $(P \times \mu)(B) = 0$ , one has

$$\iint_B P(d\omega) \sigma_\omega(d\xi) = \iint_B P(d\omega) \sigma_\omega^0(d\xi) = 0.$$

*Remark.* — Kotani [13] showed, for the first time, that an analogue of Condition K holds in the model of Goldsheid, Molchanov and Pastur ([8], [17]), and that by means of it, Carmona's proof of exponential localization can be further simplified. Kotani's argument was later reformulated and extended by several authors (see Kotani-Simon [14] and references therein), and is often called "Kotani's trick".

**THEOREM 4.** — *Let Condition K hold and let  $J$  be an interval. Then the following assertions hold.*

- (i) *If for  $\mu$ -almost all  $\xi \in J$ , we have*

$$P \left( \begin{array}{l} L_\omega u = \xi u \text{ has solutions } u_+ \text{ and } u_- \text{ such that} \\ u_+ \in L^2((0, \infty)), u_- \in L^2((-\infty, 0)) \end{array} \right) = 1,$$

*then for  $P$ -almost all  $\omega$ ,  $\sigma_\omega(d\xi)$  is pure point in  $J$ .*

(ii) If for  $\mu$ -almost all  $\xi \in J$ , we have

$$P\left(L_\omega u = \xi u \text{ has a solution in } L^2([0, \infty))\right) = 1,$$

then for any  $\theta \in [0, \pi)$ ,  $\sigma_\omega^\theta(d\xi)$  is pure point in  $J$  with probability one.

(iii) If for  $\mu$ -almost all  $\xi \in J$ , we have

$$P\left(\begin{array}{l} L_\omega u = \xi u \text{ has no solution subordinate at } +\infty, \\ \text{or has no solution subordinated at } -\infty \end{array}\right) = 1,$$

then for  $P$ -almost all  $\omega$ ,  $\sigma_\omega(d\xi)$  is purely absolutely continuous in  $J$ , and  $\sigma_\omega(d\xi)/d\xi > 0$  almost everywhere in  $J$ .

(iv) If for  $\mu$ -almost all  $\xi \in J$ , we have

$$P(L_\omega u = \xi u \text{ has no subordinate solution at } +\infty) = 1,$$

then for any  $\theta \in [0, \pi)$ , and for  $P$ -almost all  $\omega$ ,  $\sigma_\omega^\theta(d\xi)$  is absolutely continuous in  $J$ , and  $\sigma_\omega^\theta(d\xi)/d\xi > 0$  almost everywhere in  $J$ .

(v) If for  $\mu$ -almost all  $\xi \in J$ , we have

$$P\left(\begin{array}{l} L_\omega u = \xi u \text{ has two solutions } u_+ \text{ and } u_- \text{ which are} \\ \text{subordinate respectively at } +\infty \text{ and } -\infty, \text{ but such that} \\ u_+ \notin L^2((0, \infty)) \text{ or } u_- \notin L^2((-\infty, 0)) \end{array}\right) = 1,$$

then for  $P$ -almost all  $\omega$ ,  $\sigma_\omega(d\xi)$  is purely singular continuous in  $J$ .

(vi) If for  $\mu$ -almost all  $\xi \in J$ , we have

$$P\left(\begin{array}{l} L_\omega u = \xi u \text{ has a subordinate solution at } +\infty \\ \text{which is not in } L^2([0, \infty)) \end{array}\right) = 1,$$

then for any  $\theta \in [0, \pi)$ ,  $\sigma_\omega^\theta(d\xi)$  is purely singular continuous with probability one.

We need the following lemma, of which we postpone the proof until Appendix A.

LEMMA 1. — Let  $A_1^{+[\text{resp. } -]} \equiv \{(\omega, \xi) : \xi \in \mathbb{R}, L_\omega u = \xi u \text{ has a solution which is square integrable near } +\infty [\text{resp. } -\infty]\}$ ; and

$$A_2^{+[\text{resp. } -]} \equiv \{(\omega, \xi) : \xi \in \mathbb{R}, L_\omega u = \xi u$$

has a subordinate solution at  $+\infty$  [resp.  $-\infty$ ]\}.

Then we have  $A_i^\pm \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^1)$ ,  $i = 1, 2$ . In particular the events inside  $P(\cdot)$ 's of Theorem 4 are all measurable.

Proof of Theorem 4. — We shall prove only the assertion (i), (iii) and (v). The other assertions can be proved similarly.

(i) From the condition,  $A_1^+ \cap A_1^-$  has full  $P \times \mu$ -measure in  $\Omega \times J$ . Hence from Condition K and Lemma 1,  $\sigma_\omega(d\xi)|_J$  is concentrated on  $A_1^+(\omega) \cap A_1^-(\omega)$  with probability one. (Here we have set

$$A_i^j(\omega) = \{\xi \mid (\omega, \xi) \in A_i^j\}$$



for  $i=1,2$  and  $j=+, -$ .) Since a square integrable solution is necessarily subordinate, we see from (iii) of Theorem 3 that  $\sigma_\omega(d\xi)|_J$  is singular with probability one. Hence by (iv) of Theorem 3 and the uniqueness of subordinate solution,  $u_+$  and  $u_-$  must be linearly dependent for  $\sigma_\omega$ -almost all  $\xi \in J$ . Therefore for P-almost all  $\omega$ , and for  $\sigma_\omega$ -almost all  $\xi \in J$ ,  $L_\omega u = \xi u$  has a solution in  $L^2(\mathbb{R}^1)$ , and hence  $\sigma_\omega(d\xi)|_J$  is pure point.

(iii) In the same way as the proof of (i), we see that

$$\sigma_\omega(A_2^+(\omega) \cap A_2^-(\omega) \cap J) = 0$$

holds with probability one. But  $M_s(\omega) \subset A_2^+(\omega) \cap A_2^-(\omega)$ , so that by (iv) of Theorem 3,  $\sigma_\omega(d\xi)|_J$  does not have singular part with probability one. The last assertion follows from (ii) of Theorem 3 and Fubini's Theorem.

(v) As before, we see that with probability one,  $\sigma_\omega(d\xi)|_J$  is singular and is concentrated on  $J \setminus [A_1^+(\omega) \cap A_1^-(\omega)]$ , hence cannot have any point mass in  $J$ .  $\square$

*Remark.* - The proof of (i) and (ii) actually shows how the eigenfunctions are obtained. Namely let a functions  $u_\pm(t, \xi, \omega)$  of  $(t, \xi, \omega) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \Omega$  be such that for each  $(\xi, \omega)$ ,  $u_+(\cdot, \xi, \omega)$  [resp.  $u_-(\cdot, \xi, \omega)$ ] is a non trivial solution of  $L_\omega u = \xi u$  which is in  $L^2([0, \infty))$  [resp.  $L^2((-\infty, 0])$ ], whenever such a solution exists, and is  $\equiv 0$ , say, otherwise. Then under the condition of (i), for P-a.a.  $\omega$  and for  $\sigma_\omega$ -a.a.  $\xi \in J$ ,  $u_+(\cdot, \xi, \omega)$  and  $u_-(\cdot, \xi, \omega)$  are linearly dependent,  $\neq 0$ , and hence give an eigenfunction of  $L_\omega$ . Similarly under the condition of (ii), it is seen that for any  $\theta \in [0, \pi)$ , for P-a.a.  $\omega$ , and for  $\sigma_\omega^\theta$ -a.a.  $\xi \in J$ ,  $u_+(\cdot, \xi, \omega) \neq 0$  satisfies the boundary condition (1.4) $_\theta$ , and hence is an eigenfunction of  $L_\omega^\theta$ .

Let us return to the proof of Theorems 1 and 2.

If we set

$K = \{ \omega \in \Omega; H_\omega^F \text{ has purely absolutely continuous spectrum and}$

$$\Sigma(H_\omega^F) = (-\infty, \infty) \},$$

then it is measurable and shift-invariant (*i.e.*  $T_x K = K$  for all  $x \in \mathbb{R}^1$ ), as is clear from the relation

$$H_{T_x \omega}^F = (-d^2/dt^2 + G(X_\omega(t+x)) - F(x+t)) + Fx.$$

On the other hand, the process  $\{X_\omega(t)\}$  is ergodic under the flow  $\{T_x\}$ . Hence in order to prove Theorem 1, it suffices to prove that  $P(K) > 0$ . This, in turn, is an immediate consequence of Lemma 2 and Theorem 5 below. Indeed, Lemma 2 means that the Condition  $K$  holds for our random operator  $\{H_\omega^F\}_\omega$  considered on the restricted probability space  $(C, P(\cdot | C))$ , where  $P(\cdot | C)$  is the conditional probability given the event  $C$ , and Theorem 5 means that the condition of Theorem 4 (iii) is satisfied.

LEMMA 2. — Take an interval  $[a, b]$  in  $S^1$ , on which  $G'(x) > 0$  (this is possible since  $G(x) \neq \text{constant}$ ). Fix a sufficiently small  $\delta > 0$  and set

$$C = \{ \omega \in \Omega; X_t(\omega) \in [a + \delta, b - \delta], \text{ for } t \in [0, 1] \}.$$

Then  $P(C) > 0$  and for all  $B \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^1)$  with  $(P \times \mu)(B) = 0$ , we have

$$\iint_{(C \times \mathbb{R}^1) \cap B} P(d\omega) \sigma_\omega(d\xi) = 0,$$

where  $\mathcal{G} = \sigma[X_t(\omega); t \leq 0 \text{ or } t \geq 1]$  and  $\sigma_\omega(d\xi)$  is the spectral measure of  $H_\omega^F$ .

The proof of this lemma can be carried out in the same spirit as Kotani-Simon [14]. We shall give its outline in Appendix B.

THEOREM 5. — Fix a  $\xi \in \mathbb{R}^1$ . Then with probability one, every non-trivial solution  $u$  of  $H_\omega^F u = \xi u$  is such that the finite limit

$$\lim_{T \rightarrow +\infty} T^{-1/2} \int_0^T |u(t)|^2 dt$$

exists and is strictly positive. In particular, we have

$$\begin{aligned} \lim_{T \rightarrow +\infty} \int_0^T |u(t)|^2 dt / \int_0^T |v(t)|^2 dt \\ = \lim_{T \rightarrow +\infty} T^{-1/2} \int_0^T |u(t)|^2 dt / T^{-1/2} \int_0^T |v(t)|^2 dt > 0 \end{aligned}$$

for all pair  $\{u, v\}$  of non-trivial solutions, which means the absence of subordinate solution at  $+\infty$  of the equation  $H_\omega^F u = \xi u$ .

This theorem is proved in paragraph 3.2.

The proof of Theorem 2 proceeds in the following manner. First, we need the following theorem, which we shall prove in paragraph 3.3.

THEOREM 6. — Fix  $F > 0, \kappa > 0, \theta \in [0, \pi]$  and  $\xi \in \mathbb{R}^1$  arbitrarily.

(i) The solution of  $\mathcal{H}_\omega^F u = \xi u$  satisfying the boundary condition (1.4)<sub>0</sub> does not belong to  $L^2([0, \infty))$  with probability one.

(ii) With probability one,  $\mathcal{H}_\omega^F u = \xi u$  has a solution  $u_0(t)$  which satisfies

$$u_0(t) = 0 (t^{-1/4 - \kappa^2/8 F + \delta}), \quad \text{as } t \rightarrow \infty,$$

for any  $\delta > 0$ .

(iii) With probability one,  $u_0(t)$  in (ii) satisfies

$$\lim_{x \rightarrow +\infty} x^{1/2 + \kappa^2/4 F + \delta} \int_x^{x+1} |u_0(t)|^2 dt > 0$$

for all  $\delta > 0$ .

(iv) *With probability one, any solution  $v(t)$  of  $\mathcal{H}_\omega^F u = \xi u$  which is linearly independent of  $u_0(t)$  satisfies*

$$\lim_{x \rightarrow +\infty} x^{1/2 - \kappa^2/4 F + \delta} \int_x^{x+1} |v(t)|^2 dt > 0$$

for all  $\delta > 0$ .

(v) *When  $F = \kappa^2/2$ ,  $u_0(t)$  in (ii) does not belong to  $L^2([0, \infty))$  with probability one.*

From (i),  $\mathcal{H}_\omega^F u = \xi u$  has, with probability one, a solution which is not square integrable near  $+\infty$ , hence  $\mathcal{H}_\omega^F$  is in the limit point case at  $+\infty$ , and so with probability one,  $\mathcal{H}_\omega^F = \mathcal{H}_\omega^{F, \theta}$  is a self-adjoint operator for all  $\theta \in [0, \pi)$ . Let  $\sigma_\omega^\theta(d\xi)$  be its spectral measure. The following lemma can be proved in the same way as Lemma 2 (see Appendix B).

LEMMA 3. — *Let  $\mathcal{G} = \sigma[B_\omega(t) - B_\omega(1); t \geq 1]$ . Then for any  $\theta \in [0, \pi)$  and for any  $B \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^1)$  with  $(P \times \mu)(B) = 0$ , one has*

$$\iint_B P(d\omega) \sigma_\omega^\theta(d\xi) = 0.$$

By this lemma, our  $\mathcal{H}_\omega^F = \mathcal{H}_\omega^{F, \kappa, \theta}$  satisfies Condition K for any  $\theta \in [0, \pi)$ . Suppose  $0 < F < \kappa^2/2$ . Then we see from (ii) of the above theorem that  $u_0(t) \in L^2([0, \infty))$ , so that the condition of Theorem 4 (ii) is satisfied. Hence  $\mathcal{H}_\omega^F$  has pure point spectrum with probability one. According to the remark following Theorem 4, (ii) and (iii) of Theorem 6 give the desired estimates for the eigenfunctions.

On the other hand, for any value of  $F > 0$ , (ii) and (iv) show that  $u_0(t)$  is the subordinate solution at  $+\infty$  of  $\mathcal{H}_\omega^F u = \xi u$ . From (iii) and (v), we see that  $u_0(t) \notin L^2([0, \infty))$  whenever  $F \geq \kappa^2/2$ . Hence the condition of Theorem 4 (vi) is satisfied, and we obtain the assertion (iii) of Theorem 2. It remains to prove  $\Sigma(\mathcal{H}_\omega^F) = (-\infty, \infty)$ . This is easy if we use the following non-random result of Hartman [10]. Let  $L^\theta$  be self-adjoint on the domain  $D^\theta(L)$ , and let  $u_{\theta, \xi}(t)$  be a non-trivial solution of  $L^\theta u = \xi u$  satisfying the boundary condition (1.4) $_\theta$ . We denote by  $N(T; \xi)$  the number of zeros of  $u_{\theta, \xi}(\cdot)$  in the interval  $[0, T]$ .

THEOREM 7. — (Hartmann [10]) *If for  $\lambda < \mu$ , one has*

$$(2.5) \quad \lim_{T \rightarrow +\infty} \{N(T; \mu) - N(T; \lambda)\} = \infty,$$

then

$$(2.6) \quad \# \{ \Sigma(L^\theta) \cap (\lambda, \mu) \} = \infty.$$

In case  $L^\theta = \mathcal{H}_\omega^{F, \theta}$ , the following assertion holds as we shall prove in paragraph 3.4.

LEMMA 4. — Let  $L^0 = \mathcal{H}_\omega^{\mathbb{F}, 0}$ . Then for any pair  $\lambda < \mu$  of real numbers, (2.5) holds with probability one.

From this, we see that for P-a.a.  $\omega \in \Omega$ , (2.5) holds for any pair  $\lambda < \mu$  of rational numbers. Hence for such an  $\omega$ , we have  $\Sigma(\mathcal{H}_\omega^{\mathbb{F}}) = (-\infty, \infty)$  by (2.6).

### 3. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE EIGENVALUE EQUATIONS

#### 3.1. Preliminaries

The purpose of this section is to prove Theorems 5 and 6, and Lemma 4. This is done through a transformation of the eigenvalue equations  $H_\omega^{\mathbb{F}} u = \xi u$  and  $\mathcal{H}_\omega^{\mathbb{F}} u = \xi u$  defined below. Let  $H_0^{\mathbb{F}} = -d^2/dt^2 - \mathbb{F}t$ ,  $\mathbb{F} > 0$ .

LEMMA 5. — For any  $\xi \in \mathbb{R}^1$ , there exist two real solutions  $\varphi_+(t)$  and  $\varphi_-(t)$  of  $H_0^{\mathbb{F}} u = \xi u$  with the following properties:

- (i)  $[\varphi_+, \varphi_-](t) \equiv \varphi_+(t) \varphi'_-(t) - \varphi'_+(t) \varphi_-(t) = 1$ ; and
- (ii) as  $t \rightarrow +\infty$ ,

$$(3.1) \quad \varphi_\pm(t) = \mathbb{F}^{-1/4} (t + \xi/\mathbb{F})^{-1/4} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left( \frac{2}{3} \mathbb{F}^{1/2} (t + \xi/\mathbb{F})^{3/2} \right) + (t^{-7/4}),$$

$$(3.2) \quad \varphi'_\pm(t) = \mathbb{F}^{1/4} (t + \xi/\mathbb{F})^{1/4} \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} \left( \frac{2}{3} \mathbb{F}^{1/2} (t + \xi/\mathbb{F})^{3/2} \right) + (t^{-5/4}).$$

This lemma is derived without difficulty from the asymptotic expansion of Airy function (see e. g. [4]). However, for reader's convenience, we sketch the proof of Lemma 5 in Appendix C.

In the following, we frequently use the following asymptotic estimates for

$$(3.3) \quad \rho(t) \equiv \{ \varphi_+(t)^2 + \varphi_-(t)^2 \}^{1/2},$$

which are easily derived from Lemma 5: First of all, we have

$$(3.4) \quad \rho(t)^2 = \mathbb{F}^{-1/2} t^{-1/2} + O(t^{-3/2}),$$

in particular,

$$(3.5) \quad \rho(t) = O(t^{-1/4})$$

and

$$(3.6) \quad \rho(t)^4 = \mathbb{F}^{-1} t^{-1} + O(t^{-2}); \quad \text{Next we obtain}$$

$$(3.7) \quad \frac{d}{dt}(\rho(t)^2) = 2 \{ \varphi_+(t) \varphi'_+(t) + \varphi_-(t) \varphi'_-(t) \} \\ = O(t^{-3/2}),$$

or more generally

$$(3.8) \quad \frac{d}{dt}(\rho(t)^k) = O(t^{-k/4-1}), \quad \text{for any } k.$$

Now let  $\psi_{\pm}(t) \equiv \varphi_{\pm}(t)/\rho(t)$ , and let us perform the following transformation for solution  $u(t)$  of  $\mathbf{H}_{\omega}^F u = \xi u$  or  $\mathcal{H}_{\omega}^F u = \xi u$ :

$$(3.9) \quad \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} \varphi_+(t) & \varphi_-(t) \\ \varphi'_+(t) & \varphi'_-(t) \end{pmatrix} \begin{pmatrix} \psi_-(t) & \psi_+(t) \\ -\psi_+(t) & \psi_-(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

so that in particular,

$$(3.10) \quad u(t) = \rho(t)y(t).$$

By direct calculation, it is seen that  $x(t)$  and  $y(t)$  thus introduced satisfy the ordinary differential equation

$$(3.11) \quad \begin{cases} x'(t) = \rho(t)^{-2}y(t) - \rho^2(t)G(X_{\omega}(t))y(t) \\ y'(t) = -\rho(t)^{-2}x(t) \end{cases}$$

in the case of  $\mathbf{H}_{\omega}^F$ , or the stochastic differential equation

$$(3.12) \quad \begin{cases} dx(t) = \rho(t)^{-2}y(t)dt - \kappa\rho(t)^2y(t)d\mathbf{B}(t) \\ dy(t) = -\rho(t)^{-2}x(t)dt \end{cases}$$

in the case of  $\mathcal{H}_{\omega}^F$ .

If we further introduce  $r(t)$  and  $\theta(t)$  by

$$(3.13) \quad x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t),$$

then the equations satisfied by these are

$$(3.14) \quad \begin{cases} r'(t) = -\rho(t)^2r(t)G(X_{\omega}(t))\sin\theta(t)\cos\theta(t) \\ \theta'(t) = -\rho(t)^{-2} + \rho(t)^2G(X_{\omega}(t))\sin^2\theta(t) \end{cases}$$

in the case of  $\mathbf{H}_{\omega}^F$ , or

$$(3.15) \quad \left\{ \begin{array}{l} dr(t) = -\kappa\rho(t)^2r(t)\sin\theta(t)\cos\theta(t)d\mathbf{B}(t) \\ \quad \quad \quad + \frac{1}{2}\kappa^2\rho(t)^4r(t)\sin^4\theta(t)dt \\ d\theta(t) = \kappa\rho(t)^2\sin^2\theta(t)d\mathbf{B}(t) - \rho(t)^{-2}dt \\ \quad \quad \quad + \kappa^2\rho(t)^4\sin^3\theta(t)\cos\theta(t)dt \end{array} \right.$$

in the case of  $\mathcal{H}_{\omega}^F$ . The extra terms on the right hand sides of (3.15) are due to Itô's formula (see Ikeda-Watanabe [11]): Indeed by Itô's formula and (3.13), we have

$$\begin{aligned} dx(t) &= \cos\theta(t)dr(t) - r(t)\sin\theta(t)d\theta(t) \\ &\quad - \frac{1}{2}r(t)\cos\theta(t)(d\theta(t))^2 - \sin\theta(t)dr(t)d\theta(t) \end{aligned}$$

and

$$dy(t) = \sin \theta(t) dr(t) + r(t) \cos \theta(t) d\theta(t) - \frac{1}{2} r(t) \sin \theta(t) (d\theta(t))^2 + \cos \theta(t) dr(t) d\theta(t).$$

Hence by (3.12),

$$r(t) d\theta(t) + dr(t) d\theta(t) = \kappa \rho(t)^2 r(t) \sin^2 \theta(t) dB(t) - \rho(t)^{-2} r(t) dt$$

and

$$dr(t) - \frac{1}{2} r(t) (d\theta(t))^2 = -\kappa \rho(t)^2 r(t) \sin \theta(t) \cos \theta(t) dB(t).$$

But noting  $(dB(t))^2 = dt$ ,  $dB(t) dt = (dt)^2 = 0$ , we see that

$$dr(t) d\theta(t) = (-\kappa \rho(t)^2 r(t) \sin \theta(t) \cos \theta(t)) (\kappa \rho(t)^2 \sin^2 \theta(t)) (dB(t))^2$$

and

$$(d\theta(t))^2 = \kappa^2 \rho(t)^4 \sin^4 \theta(t) dt.$$

Inserting these into the above equations, we obtain (3.15).

In this section, we shall use the following notation for an economy of language: For a function  $f(t)$  of a real variable  $t \geq 0$ , we shall write

$$f(t) = O(t^{a+0}), \quad t \rightarrow +\infty$$

when we have

$$f(t) = O(t^{a+\delta}), \quad t \rightarrow +\infty$$

for every  $\delta > 0$ . In the same way, we write

$$f(t) = O(t^{a-0}), \quad t \rightarrow 0$$

when we have

$$f(t) = O(t^{a-\delta}), \quad t \rightarrow 0$$

for any  $\delta > 0$ . Also we shall freely use the notation like  $O(h(t)^{a+0})$  or  $O(\eta(x)^{a-0})$ . Their meaning should be obvious from the context. On the other hand, if we have  $\lim_{t \rightarrow +\infty} t^{-a} f(t) > 0$ , then we shall write

$$f(t) \geq O(t^a),$$

and if we have  $f(t) \geq 0(t^{a-\delta})$  for any  $\delta > 0$ , then we write

$$f(t) \geq O(t^{a-0}).$$

Before proceeding further, we prepare a lemma concerning continuous martingales, which we use repeatedly in this section.

Let  $\mathcal{F}_t$  be the sub  $\sigma$ -field of  $\mathcal{F}$  generated by  $\{B_\omega(s); 0 \leq s \leq t\}$ . For any continuous  $(\mathcal{F}_t)$ -martingale  $\{M(t)\}$  which is square integrable (namely  $E[M(t)^2] < \infty$ ), there is a unique increasing continuous process  $\{\langle M \rangle(t)\}$ ,

called the quadratic variational process of  $\{M(t)\}$ , such that the process

$$\{M(t)^2 - \langle M \rangle(t)\}_{t \geq 0}$$

becomes again a martingale (see [11]). Virtually every martingale appearing in this work is given by stochastic integral:

$$M(t) = \int_0^t a_\omega(s) dB_\omega(s),$$

where  $\{a_\omega(t)\}$  is a continuous  $(\mathcal{F}_t)$ -adapted process with  $E[a(t)^2] < \infty$ . In such a case, we have

$$\langle M \rangle(t) = \int_0^t a_\omega(s)^2 ds.$$

LEMMA 6. — Let  $\{M(t)\}_{t \geq 0}$  be a real valued, square integrable martingale with continuous sample paths.

(i) On the set  $\{\langle M \rangle(\infty) < \infty\}$ , the finite limit

$$(3.16) \quad M(\infty) = \lim_{t \rightarrow +\infty} M(t)$$

exists almost surely, and one has

$$(3.17) \quad |M(t) - M(\infty)| = O(\{\langle M \rangle(\infty) - \langle M \rangle(t)\}^{1/2-0}), \quad t \rightarrow +\infty,$$

and

$$(3.18) \quad |M(t) - M(s)| = O(|\langle M \rangle(t) - \langle M \rangle(s)|^{1/2-0}), \quad |t-s| \rightarrow 0.$$

(ii) On the set  $\{\langle M \rangle(\infty) = \infty\}$ , one has almost surely

$$(3.19) \quad M(t) = O(\langle M \rangle(t)^{1/2+0}), \quad t \rightarrow +\infty.$$

*Proof.* — By the martingale representation theorem (see [11], Ch. II, Theorem 7.2'), there exists a Brownian motion  $\{\tilde{B}(t)\}_{t \geq 0}$  (on an enlarged probability space) such that

$$M(t) = \tilde{B}(\langle M \rangle(t)).$$

Hence (3.16) to (3.19) follow at once from the continuity of sample paths of the Brownian motion, local and global laws of the iterated logarithm respectively (see Ito-McKean [12] § 1.8).

3.2. *Proof of Theorem 5.* — Let  $r(t) = r_\xi(t; \omega, r, \theta)$  and  $\theta(t) = \theta_\xi(t; \omega, \theta)$  be the solutions of (3.14) with  $\theta(0) = \theta$  and  $r(0) = r$ .

LEMMA 7. — For any  $\theta \in [0, \pi]$  and real  $\beta \neq 0$ , the finite limit

$$\lim_{t \rightarrow +\infty} \int_0^t \rho(s)^2 e^{i\beta\theta(s)} G(X(s)) ds$$

exists with probability one.

*Proof.* — Define

$$\alpha(t) \equiv - \int_0^t \rho(s)^{-2} ds$$

and  $\tilde{\theta}(t) \equiv \theta(t) - \alpha(t)$ . Then  $\tilde{\theta}(t)$  satisfies

$$\frac{d}{dt} \tilde{\theta}(t) = \rho(t)^2 G(X(t)) \sin^2 \theta(t).$$

On the other hand, if we set

$$(3.20) \quad m(t) = \int_0^t dG(X(s)) - \int_0^t AG(X(s)) ds,$$

where  $A$  is the infinitesimal generator of  $\{X(t)\}$ , then  $\{m(t)\}$  is a square integrable martingale with continuous sample paths such that

$$(3.21) \quad d\langle m \rangle(t) \leq \text{constant} \cdot dt.$$

By integrating by parts,

$$\begin{aligned} & \int_0^t \rho(s)^2 e^{i\beta\theta(s)} G(X(s)) ds \\ &= - \frac{1}{i\beta} \int_0^t \rho(s)^4 e^{i\beta\tilde{\theta}(s)} \frac{d}{ds} (e^{i\beta\alpha(s)} G(X(s))) ds \\ &= - \frac{1}{i\beta} [\rho(s)^4 e^{i\beta\theta(s)} G(X(s))]_0^t \\ &+ \frac{1}{i\beta} \int_0^t e^{i\beta\theta(s)} G(X(s)) \frac{d}{ds} (\rho(s)^4) ds \\ &+ \int_0^t e^{i\beta\theta(s)} \rho(s)^6 G(X(s))^2 \sin^2 \theta(s) ds \\ &+ \frac{1}{i\beta} \int_0^t e^{i\beta\theta(s)} \rho(s)^4 \{ dm(s) + AG(X(s)) ds \}. \end{aligned}$$

From (3.5) and (3.8), it is clear that the first three terms on the right hand side converge to finite limits as  $t \rightarrow +\infty$ . The fourth term

$$Y(t) \equiv \frac{1}{i\beta} \int_0^t e^{i\beta\theta(s)} \rho(s)^4 dm(s)$$

is a complex valued, square integrable martingale with continuous paths. By (3.5) and (3.21), its real and imaginary parts satisfy

$$\langle \text{Re } Y \rangle(\infty) + \langle \text{Im } Y \rangle(\infty) < \infty$$



almost surely. Hence  $Y(\infty)$  exists almost surely by Lemma 6. As for the last term, we see by the same integration by parts as before,

$$\begin{aligned}
 (3.22) \quad & \frac{1}{i\beta} \int_0^t e^{i\beta\theta(s)} \rho(s)^4 \text{AG}(X(s)) ds \\
 &= \beta^{-2} [\rho(s)^6 e^{i\beta\theta(s)} \text{AG}(X(s))]_0^t \\
 &\quad - \beta^{-2} \int_0^t e^{i\beta\theta(s)} \text{AG}(X(s)) \frac{d}{ds} (\rho(s)^6) ds \\
 &\quad - \frac{i}{\beta} \int_0^t e^{i\beta\theta(s)} \rho(s)^8 G(X(s)) \text{AG}(X(s)) \sin^2 \theta(s) ds \\
 &\quad - \beta^{-2} \int_0^t e^{i\beta\theta(s)} \rho(s)^6 \{ d\tilde{m}(s) + A^2 G(X(s)) ds \}.
 \end{aligned}$$

Here  $\{\tilde{m}(t)\}$  is again a square integrable martingale which satisfies (3.20) and (3.21) with  $G$  replaced by  $\text{AG}$ . Again using (3.5), (3.8) and Lemma 6, we see that every term on the right hand side of (3.22) converges to a finite limit as  $t \rightarrow \infty$ . This completes the proof of Lemma 7.

LEMMA 8. — For any  $r > 0$ , the finite limit

$$r(\infty) = \lim_{t \rightarrow +\infty} r(t)$$

exists and is strictly positive with probability one. Moreover if we set

$$\bar{\theta}(t) \equiv \bar{\theta}(t) - \frac{1}{2} \int_0^t \rho(s)^2 G(X(s)) ds,$$

then the finite limit

$$\bar{\theta}(\infty) \equiv \lim_{t \rightarrow +\infty} \bar{\theta}(t)$$

exists with probability one.

*Proof.* — It is easy to derive from (3.14)

$$r(t)^{-1} \frac{d}{ds} r(t) + i \frac{d}{ds} \bar{\theta}(t) = -\frac{i}{2} \rho(t)^2 G(X(t)) e^{2i\theta(t)},$$

so that

$$-\bar{\theta}(t) + i \log r(t) = -\theta + i \log r + \frac{1}{2} \int_0^t \rho(s)^2 e^{2i\theta(s)} G(X(s)) ds.$$

Hence setting  $\beta=2$  in Lemma 7, the conclusion of Lemma 8 follows at once.

Now we are ready to finish the proof of Theorem 5. Let  $u_1, u_2$  be the solutions of  $H_\omega^F u = \xi u$  satisfying  $u_1(0) = u_2'(0) = 1$ ,  $u_1'(0) = u_2(0) = 0$ , and let  $(x_j(t), y_j(t))$ ,  $(r_j(t), \theta_j(t))$ ,  $j=1, 2$ , be obtained from  $u_j(t)$ ,  $j=1, 2$ , by the transformations (3.9) and (3.13). In exactly the same way as Lemma 7,

we see that the finite limits

$$(3.23) \quad \lim_{T \rightarrow +\infty} \int_0^T \rho(t)^2 e^{i(\theta_j(t) + \theta_k(t))} dt, \quad j, k = 1, 2$$

exist almost surely. Also, from Lemma 8,

$$(3.24) \quad \lim_{T \rightarrow +\infty} r_j(t) = r_j(\infty) \in (0, \infty), \quad j = 1, 2$$

and

$$(3.25) \quad \lim_{T \rightarrow +\infty} \bar{\theta}_j(t) = \bar{\theta}_j(\infty), \quad j = 1, 2$$

exist almost surely.

Now let us fix an  $\omega$  for which the limits (3.23), (3.24) and (3.25) exist, and let  $u(t) = au_1(t) + bu_2(t)$ ,  $(a, b) \neq (0, 0)$ , be any non-trivial solution of  $H_\omega^F u = \xi u$ . For such  $\omega$  and  $u$ , we shall prove

$$\lim_{T \rightarrow +\infty} \frac{1}{R(T)} \int_0^T u(t)^2 dt > 0,$$

where

$$R(T) \equiv \int_0^T \rho(t)^2 dt \sim 2F^{-1/2} T^{1/2}, \quad T \rightarrow +\infty.$$

But from (3.23)-(3.25),

$$\begin{aligned} \int_0^T u_j(t) u_k(t) dt &= \int_0^T \rho(t)^2 r_j(t) r_k(t) \sin \theta_j(t) \sin \theta_k(t) dt \\ &= \frac{1}{2} \int_0^T \rho(t)^2 r_j(t) r_k(t) \{ \cos(\bar{\theta}_j(t) - \bar{\theta}_k(t)) - \cos(\theta_j(t) + \theta_k(t)) \} \\ &= \frac{1}{2} r_j(\infty) r_k(\infty) \cos(\bar{\theta}_j(\infty) - \bar{\theta}_k(\infty)) R(T) + o(R(T)), \end{aligned}$$

so that

$$\begin{aligned} \lim_{T \rightarrow +\infty} \frac{1}{R(T)} \int_0^T u(t)^2 dt &= \lim_{T \rightarrow +\infty} \frac{1}{R(T)} \int_0^T \{ au_1(t) + bu_2(t) \}^2 dt \\ &= \frac{1}{2} \left\| a \begin{pmatrix} r_1(\infty) \cos \bar{\theta}_1(\infty) \\ r_1(\infty) \sin \bar{\theta}_1(\infty) \end{pmatrix} + b \begin{pmatrix} r_2(\infty) \cos \bar{\theta}_2(\infty) \\ r_2(\infty) \sin \bar{\theta}_2(\infty) \end{pmatrix} \right\|^2. \end{aligned}$$

To see that this is strictly positive, it suffices to note that two vectors,  $(r_j(\infty) \cos \bar{\theta}_j(\infty), r_j(\infty) \sin \bar{\theta}_j(\infty))$ ,  $j = 1, 2$ , are linearly independent. But

this is immediate from

$$\begin{aligned} \det \begin{pmatrix} r_1(\infty) \cos \bar{\theta}_1(\infty) & r_2(\infty) \cos \bar{\theta}_2(\infty) \\ r_1(\infty) \sin \bar{\theta}_1(\infty) & r_2(\infty) \sin \bar{\theta}_2(\infty) \end{pmatrix} \\ = \lim_{t \rightarrow +\infty} \det \begin{pmatrix} x_1(t) x_2(t) \\ y_1(t) y_2(t) \end{pmatrix} = \lim_{t \rightarrow +\infty} \det \begin{pmatrix} u_1(t) u_2(t) \\ u'_1(t) u'_2(t) \end{pmatrix} = 1. \end{aligned}$$

3.3. *Proof of Theorem 6.* — Let  $r(t) = r_\xi(t; \omega, r, \theta)$  and  $\theta(t) = \theta_\xi(t; \omega, \theta)$  be the solutions of (3.15) with non-random initial data  $r(0) = r$  and  $\theta(0) = \theta$ .

LEMMA 9. — For any real numbers  $k > -2$  and  $\beta \neq 0$ , the finite limit

$$\lim_{t \rightarrow +\infty} \int_0^t \rho(s)^k e^{i\beta\theta(s)} ds$$

exists almost surely.

*Proof.* — Consider  $\tilde{\theta}(t) \equiv \theta(t) - \alpha(t)$  as before, where  $\alpha(t)$  was defined in the proof of Lemma 7. Then  $\tilde{\theta}(t)$  satisfies

$$(3.26) \quad d\tilde{\theta}(t) = \kappa \rho(t)^2 \sin^2 \theta(t) dB(t) + \kappa^2 \rho(t)^4 \sin^3 \theta(t) \cos \theta(t) dt.$$

By integrating by parts,

$$\begin{aligned} \int_0^t \rho(s)^k e^{i\beta\theta(s)} ds &= \frac{i}{\beta} \int_0^t \rho(s)^{k+2} \left( \frac{d}{ds} e^{i\beta\alpha(s)} \right) e^{i\beta\tilde{\theta}(s)} ds \\ &= \frac{i}{\beta} [\rho(s)^{k+2} e^{i\beta\theta(s)}]_0^t \\ &\quad - \frac{i}{\beta} \int_0^t e^{i\beta\theta(s)} \frac{d}{ds} \rho(s)^{k+2} ds \\ &\quad + \kappa \int_0^t \rho(s)^{k+4} \sin^2 \theta(s) e^{i\beta\theta(s)} dB(s) \\ &\quad + \kappa^2 \int_0^t \rho(s)^{k+6} \sin^3 \theta(s) \cos \theta(s) e^{i\beta\theta(s)} ds \\ &\quad + \frac{1}{2} \beta \kappa^2 \int_0^t \rho(s)^{k+6} \sin^4 \theta(s) e^{i\beta\theta(s)} ds. \end{aligned}$$

Noting (3.5), (3.8), and Lemma 6, it is easy to see, in the same way as in the proof of Lemma 7, that each term on the right hand side converges to a finite limit as  $t \rightarrow +\infty$ .

LEMMA 10. — Let  $r(t)$  be as above. Then with probability one,

$$\lim_{t \rightarrow +\infty} \frac{\log r(t)}{\log t} = \kappa^2/8 F.$$

*Proof.* — From equation (3.15), one has

$$(3.27) \quad \log r(t) = \log r(0) - \kappa \int_0^t \rho(s)^2 \sin \theta(s) \cos \theta(s) dB(s) \\ + \frac{\kappa^2}{2} \int_0^t \rho(s)^4 \sin^2 \theta(s) (\sin^2 \theta(s) - \cos^2 \theta(s)) ds.$$

For a while, let us denote the second and third terms on the right hand side by  $Y(t)$  and  $Z(t)$  respectively. Then  $\{Y(t)\}$  is a square integrable martingale such that

$$\langle Y \rangle(t) = O(\log t), \quad t \rightarrow +\infty.$$

Hence by Lemma 6,

$$Y(t) = o(\log t), \quad t \rightarrow +\infty.$$

On the other hand, since we have

$$\sin^4 \theta = \frac{3}{8} + \frac{1}{16}(e^{4i\theta} + e^{-4i\theta} - 2e^{2i\theta} - 2e^{-2i\theta})$$

and

$$\sin^2 \theta \cos^2 \theta = \frac{1}{8} - \frac{1}{16}(e^{4i\theta} + e^{-4i\theta}),$$

it is seen from Lemma 9 that

$$Z(t) = \frac{\kappa^2}{2} \int_0^t \rho(s)^4 \left( \frac{3}{8} - \frac{1}{8} \right) ds + O(1) \sim \frac{\kappa^2}{8F} \log t, \quad t \rightarrow +\infty,$$

with probability one. This completes the proof of Lemma 10.  $\square$

This lemma says, in particular, that for any  $r > 0$  and  $\theta \in [0, \pi)$ ,  $r_\xi(t; \omega, r, \theta)$  grows up to infinity as  $t \rightarrow +\infty$  with probability one. From this, assertion (i) of Theorem 6 follows. Indeed, on the event

$$A \equiv \left\{ \omega; \int_0^\infty u(t)^2 dt = \int_0^\infty \rho(t)^2 r(t)^2 \sin^2 \theta(t) dt < \infty \right\},$$

one has

$$\left( \int_0^\infty \rho(t)^4 r(t) \sin^4 \theta(t) dt \right)^2 \\ = \left( \int_0^\infty \rho(t) r(t) \sin \theta(t) \cdot \rho(t)^3 \sin^3 \theta(t) dt \right) \\ \leq \int_0^\infty \rho(t)^2 r(t)^2 \sin^2 \theta(t) dt \cdot \int_0^\infty \rho(t)^6 \sin^6 \theta(t) dt < \infty,$$

and

$$\int_0^\infty \rho(t)^4 r(t)^2 \sin^2 \theta(t) \cos^2 \theta(t) dt \leq \text{const.} \int_0^\infty \rho(t)^2 r(t)^2 \sin^2 \theta(t) dt < \infty.$$

Hence by (3.15) and Lemma 6, the finite limit

$$\lim_{t \rightarrow +\infty} r(t)$$

exists almost surely on A, and so  $P(A) = 0$  as required.  $\square$

To continue, let  $u_1$  and  $u_2$  be the solutions of  $\mathcal{H}_\omega^F u = \xi u$  such that their transforms  $(x_j(t), y_j(t)), j = 1, 2$ , according to (3.9) satisfy

$$x_1(0) = y_2(0) = 1, \quad x_2(0) = y_1(0) = 0,$$

and let  $(r_j(t), \theta_j(t)), j = 1, 2$ , be defined by (3.13) from  $(x_j, y_j)$ . Consider the matrix-valued random function

$$V(t, \omega) = \begin{pmatrix} x_1(t) x_2(t) & \\ y_1(t) y_2(t) & \end{pmatrix} = \begin{pmatrix} r_1(t) \cos \theta_1(t) & r_2(t) \cos \theta_2(t) \\ r_1(t) \sin \theta_1(t) & r_2(t) \sin \theta_2(t) \end{pmatrix}.$$

Then clearly

$$(3.28) \quad \det V(t, \omega) = 1,$$

and by Lemma 10, we have with probability one,

$$(3.29) \quad \lim_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t, \omega)\| = \kappa^2 / 8 F,$$

where  $\|V\|$  is the operator norm of the matrix  $V$ .

The first step toward the proof of (ii) to (v) of Theorem 6 is to obtain an analogue of Oseledec's theorem for the random matrix  $\{V(t)\}_{t \geq 0}$ . To this end, note first that the eigenvalues of the matrix  $\{V(t)^* V(t)\}^{1/2}$  are  $\|V(t)\|$  and  $\|V(t)\|^{-1}$ , and that the projection  $P(t) = P(t, \omega)$  onto the eigenspace belonging to the eigenvalue  $\|V(t)\|^{-1}$  is given by

$$(3.30) \quad P(t) = \frac{1}{1 + \beta^2} \begin{pmatrix} 1 & \beta \\ \beta & \beta^2 \end{pmatrix},$$

where  $\beta = \beta_\omega(t)$  is defined by

$$(3.31) \quad \beta = \frac{1}{2 \cos(\theta_1 - \theta_2)} \left[ \begin{pmatrix} r_2 & -r_1 \\ r_1 & r_2 \end{pmatrix} - \left\{ \begin{pmatrix} r_2 & r_1 \\ r_1 & r_2 \end{pmatrix}^2 - \frac{4}{r_1^2 r_2^2} \right\}^{1/2} \right].$$

LEMMA 11. — *With probability one,*

$$\frac{r_2(t)}{r_1(t)}, \quad \frac{r_1(t)}{r_2(t)}, \quad \text{and} \quad \theta_1(t) - \theta_2(t)$$

converge to finite limits

$$\frac{r_2}{r_1}(\infty), \quad \frac{r_1}{r_2}(\infty), \quad \text{and} \quad (\theta_1 - \theta_2)(\infty)$$

as  $t \rightarrow +\infty$  respectively. Moreover the speeds of convergence are estimated as

$$(3.32) \quad \max \left\{ \left| \frac{r_2(t)}{r_1(t)} - \frac{r_2}{r_1}(\infty) \right|, \left| \frac{r_1(t)}{r_2(t)} - \frac{r_1}{r_2}(\infty) \right|, \left| (\theta_1(t) - \theta_2(t)) - (\theta_1 - \theta_2)(\infty) \right| \right\} = O(t^{-\kappa^2/4 F + 0})$$

In particular, the finite limit

$$(3.33) \quad b = \lim_{t \rightarrow +\infty} \beta_\omega(t)$$

exists, and we have

$$(3.34) \quad |\beta(t) - b| = O(t^{-\kappa^2/4 F + 0}).$$

*Proof.* — It suffices to investigate the asymptotic behavior of

$$\log \left( \frac{r_1(t)}{r_2(t)} \right) + i(\theta_1(t) - \theta_2(t)) = \log(x_1(t) + iy_1(t)) - \log(x_2(t) + iy_2(t)),$$

as  $t \rightarrow +\infty$ . But from (3.12) and Itô's formula, we can deduce

$$(3.35) \quad \begin{aligned} & \log(x_1(t) + iy_1(t)) - \log(x_2(t) + iy_2(t)) \\ &= \kappa \int_0^t \rho(s)^2 \frac{1}{(x_1(s) + iy_1(s))(x_2(s) + iy_2(s))} dB(s) \\ & \quad + \frac{\kappa^2}{2} \int_0^t \rho(s)^4 \frac{x_1(s)y_2(s) + x_2(s)y_1(s) + 2iy_1(s)y_2(s)}{(x_1(s) + iy_1(s))^2(x_2(s) + iy_2(s))^2} ds, \end{aligned}$$

where we have used the fact

$$x_1(t)y_2(t) - x_2(t)y_1(t) = \det V(t) = 1.$$

Let us denote the first and the second terms on the right hand side of (3.35) by  $M(t)$  and  $A(t)$  respectively.  $\{M(t)\}$  is a complex valued square integrable martingale, and we have almost surely

$$\langle \text{Re } M \rangle(\infty) + \langle \text{Im } M \rangle(\infty) = \int_0^\infty \rho(s)^4 \frac{1}{r_1(s)^2 r_2(s)^2} ds < \infty$$

because the integrand is of  $O(s^{-1-\alpha})$  with  $\alpha > 0$  by (3.5) and Lemma 10. Hence by Lemma 6, the finite limit

$$M(\infty) = \lim_{t \rightarrow +\infty} M(t)$$

exists almost surely. Also  $A(\infty)$  exists, since it is the indefinite integral of a function which is of  $O(s^{-1-\alpha})$ ,  $\alpha > 0$ . The first assertion of the lemma has been thus proved.

In order to prove (3.33), it is sufficient to estimate the rate of convergence of  $M(t)$  and  $A(t)$ , to  $M(\infty)$  and  $A(\infty)$  respectively. As for  $A(t)$ , it is easily seen that

$$|A(\infty) - A(t)| = O\left(\int_t^\infty \frac{\rho(s)^4}{r_1(s)r_2(s)} ds\right) = O(t^{-\kappa^2/4 F + 0}),$$

and as for  $M(t)$ , we may apply (3.17) of Lemma 6 to show that

$$|M(\infty) - M(t)| = O\left(\left\{\int_t^\infty \frac{\rho(s)^4}{r_1(s)^2 r_2(s)^2} ds\right\}^{1/2-0}\right) = O(t^{-\kappa^2/4 F + 0}).$$

This completes the proof of Lemma 11.  $\square$

As a corollary of this lemma, we obtain the following analogue of Oseledec's theorem.

LEMMA 12. — *With probability one,*

$$P(\infty) = \lim_{t \rightarrow +\infty} P(t)$$

*exists, and the speed of convergence is estimated as*

$$(3.36) \quad \|P(\infty) - P(t)\| = O(t^{-\kappa^2/4 F + 0})$$

*And for almost all  $\omega$ , we have*

$$(3.37) \quad \lim_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| = -\kappa^2/8 F,$$

*for all  $v \in \text{Im}(P(\infty)) \setminus \{0\}$ , and*

$$(3.38) \quad \lim_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| = \kappa^2/8 F,$$

*for all  $v \notin \text{Im}(P(\infty))$ .*

*Proof.* — The existence of  $P(\infty)$  and the estimate (3.36) are immediate consequences of Lemma 11 and (3.31)-(3.32).

Set  $\lambda(t) = \|V(t)\|$ , the maximum eigenvalue of  $\{V(t)^* V(t)\}^{1/2}$ . Then there is an orthogonal matrix  $K(t)$  such that

$$(3.39) \quad V(t) = K(t) \{ \lambda(t)(1 - P(t)) + \lambda(t)^{-1} P(t) \}.$$

Suppose  $v \in \text{Im}(P(\infty)) \setminus \{0\}$  is of the form  $v = P(\infty)w$ , with  $\|w\| = 1$ . Then noting (3.39), (3.36) and (3.30), we see that

$$\begin{aligned} \|V(t)v\| &\leq \lambda(t) \|(1 - P(t))P(\infty)w\| + \lambda(t)^{-1} \|P(t)\| \\ &\leq \lambda(t) \|P(t) - P(\infty)\| + \lambda(t)^{-1} \\ &\leq O(t^{-\kappa^2/8 F + 0}), \quad t \rightarrow +\infty. \end{aligned}$$

Hence

$$\limsup_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| \leq -\kappa^2/8F.$$

On the other hand, since

$$\|V(t)v\| \geq \lambda(t)^{-1} \|P(t)P(\infty)w\|,$$

and since

$$\lim_{t \rightarrow +\infty} \|P(t)P(\infty)w\| = \|P(\infty)w\| > 0,$$

we have

$$\liminf_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| \geq -\kappa^2/8F,$$

proving (3.37).

If, on the contrary,  $v \notin \text{Im}(P(\infty))$  with  $\|v\|=1$ , then one obtains from (3.39),

$$\|V(t)v\| \leq \lambda(t) + \lambda(t)^{-1},$$

and

$$\|V(t)v\| \geq \lambda(t) \|(1 - P(t))v\|,$$

whence follow

$$\limsup_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| \leq \kappa^2/8F,$$

and

$$\liminf_{t \rightarrow +\infty} \frac{1}{\log t} \log \|V(t)v\| \geq \kappa^2/8F,$$

where we have used the fact

$$\lim_{t \rightarrow +\infty} \|(1 - P(t))v\| = \|(1 - P(\infty))v\| > 0. \quad \square$$

The assertion (ii) of Theorem 6 is now obvious: We only have to take

$$u_0(t) = \rho(t)y_0(t),$$

where

$$(3.40) \quad \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} r_0(t) \cos \theta_0(t) \\ r_0(t) \sin \theta_0(t) \end{pmatrix} = V(t)v, \quad v \in \text{Im}(P(\infty)) \setminus \{0\}. \quad \square$$

In order to prove assertions (iii), (iv) and (v), we have to observe this  $u_0(t)$  more carefully. For this purpose, we first note that if

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then we have  $P(\infty)e_1 \neq 0$  almost surely. For otherwise,  $e_2 \in \text{Im}(P(\infty))$  with positive probability, so that we would have  $r_2(t) \rightarrow 0$  with positive



probability by Lemma 12, which contradicts Lemma 10. Hence we may take  $v = (1 + b^2)P(\infty)e_1$  in (3.40), where  $b$  is defined by (3.33), so that

$$V(t)v = \begin{pmatrix} r_1(t) \cos \theta_1(t) & r_2(t) \cos \theta_2(t) \\ r_1(t) \sin \theta_1(t) & r_2(t) \sin \theta_2(t) \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix}$$

and

$$u_0(t) = \rho(t) \{ r_1(t) \sin \theta_1(t) + br_2(t) \sin \theta_2(t) \}.$$

Now let us estimate the integral

$$(3.41) \quad \int_x^{x+1} u_0(t)^2 dt$$

from below. Noting  $r_1 r_2 \sin(\theta_1 - \theta_2) = \det V(t) = 1$ , we see

$$\begin{aligned} (3.42) \quad u_0(t)^2 &= \rho(t)^2 \{ r_1(t) \sin \theta_1(t) + br_2(t) \sin \theta_2(t) \}^2 \\ &= \rho^2 [r_1 \{ \sin(\theta_1 - \theta_2) \cos \theta_2 + \cos(\theta_1 - \theta_2) \sin \theta_2 \} + br_2 \sin \theta_2]^2 \\ &= \rho^2 \left[ -\frac{1}{r_2} \cos \theta_2 + \{ r_1 \cos(\theta_1 - \theta_2) \sin \theta_2 + br_2 \sin \theta_2 \} \right]^2 \\ &\geq \frac{\rho^2}{r_2^2} \cos^2 \theta_2 - \frac{2\rho^2}{r_2} \cos \theta_2 \{ r_1 \cos(\theta_1 - \theta_2) \sin \theta_2 + br_2 \sin \theta_2 \} \\ &= \frac{1}{2} \frac{\rho^2}{r_2^2} + \frac{1}{2} \frac{\rho^2}{r_2^2} \cos 2\theta_2 - \rho^2 \sin 2\theta_2 \left\{ \frac{r_1}{r_2} \cos(\theta_1 - \theta_2) + b \right\}. \end{aligned}$$

By Lemma 10, we have

$$\frac{\rho^2(t)}{r_2(t)^2} \geq O(t^{-1/2 - \kappa^2/4F - 0}), \quad t \rightarrow +\infty,$$

so that

$$\frac{1}{2} \int_x^{x+1} \frac{\rho(t)^2}{r_2(t)^2} dt \geq O(x^{-1/2 - \kappa^2/4F - 0}), \quad x \rightarrow +\infty.$$

Hence in order to prove (iii) of Theorem 6, it suffices to prove that the contribution of the terms on the right hand side of (3.42) other than  $\rho^2 r_2^{-2}/2$  to the integral (3.41) is negligible. In fact, we can prove the following assertion.

LEMMA 13. — *With probability one, we have*

$$(3.43) \quad \int_x^{x+1} \frac{\rho(t)^2}{r_2(t)^2} \cos 2\theta_2(t) dt = O(x^{-1 - \kappa^2/4F + 0})$$

and

$$(3.44) \quad \int_x^{x+1} \rho(t)^2 \sin 2\theta_2(t) \left( \frac{r_1(t)}{r_2(t)} \cos(\theta_1(t) - \cos \theta_2(t)) + b \right) dt = O(x^{-1 - \kappa^2/4F + 0}).$$

*Proof.* — Instead of proving (3.43) itself, we shall estimate

$$(3.45) \quad I(x) = \int_x^{x+1} \frac{\rho(t)^2}{r_2(t)^2} e^{i\beta\theta_2(t)} dt$$

for  $\beta \neq 0$ . As before, we set  $\tilde{\theta}(t) \equiv \theta_2(t) - \alpha(t)$ , and perform the integration by parts:

$$(3.46) \quad \begin{aligned} I(x) &= \frac{i}{\beta} \int_x^{x+1} \frac{\rho^4(t)}{r_2(t)^2} e^{i\beta\tilde{\theta}(t)} \left( \frac{d}{dt} e^{i\beta\alpha(t)} \right) dt \\ &= \frac{i}{\beta} \left[ \frac{\rho(t)^4}{r_2(t)^2} e^{i\beta\theta_2(t)} \right]_x^{x+1} \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \frac{d\rho(t)^4}{dt} \frac{1}{r_2(t)^2} e^{i\beta\theta_2(t)} dt \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\theta_2(t)} d(r_2(t)^{-2}) \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\alpha(t)} r_2(t)^{-2} d(e^{i\beta\tilde{\theta}(t)}) \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\alpha(t)} d(e^{i\beta\tilde{\theta}(t)}) d(r_2(t)^{-2}) \\ &\equiv I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x), \end{aligned}$$

where the last term  $I_5(x)$  is due to Itô's formula.

Now since we have

$$r_2(t)^{-2} = O(t^{-\kappa^2/4 F + 0})$$

by Lemma 10, we see

$$(3.47) \quad I_1(x) = O(x^{-1 - \kappa^2/4 F + 0})$$

by (3.5). Using (3.8) with  $k=4$ , we get

$$(3.48) \quad I_2(x) = O(x^{-2 - \kappa^2/4 F + 0}).$$

Since

$$\begin{aligned} d(r_2(t)^{-2}) &= \kappa r_2(t)^{-2} \rho(t)^2 \sin 2\theta_2(t) dB(t) \\ &\quad + \kappa^2 \rho(t)^4 r_2(t)^{-2} (3 \sin^2 \theta_2(t) \cos^2 \theta_2(t) - \sin^4 \theta_2(t)) dt, \end{aligned}$$

we may write

$$I_3(x) = \int_x^{x+1} a(t) dB(t) + \int_x^{x+1} b(t) dt,$$

where

$$a(t) = O(t^{-3/2 - \kappa^2/4 F + 0})$$

and

$$b(t) = O(t^{-2 - \kappa^2/4 F + 0}).$$

Hence applying (3.18) to the real and imaginary parts of the martingale

$$M(t) \equiv \int_0^t a(s) dB(s),$$

we obtain

$$(3.49) \quad I_3(x) = O(x^{-3/2 - \kappa^2/4 F + 0}) + O(x^{-2 - \kappa^2/4 F + 0}) = O(x^{-3/2 - \kappa^2/4 F + 0}).$$

In the same way, it is easy to show

$$(3.50) \quad I_4(x) = O(x^{-3/2 - \kappa^2/4 F + 0})$$

and

$$(3.51) \quad I_5(x) = O(x^{-2 - \kappa^2/4 F + 0}).$$

Combining these estimates for  $I_j(t)$ ,  $1 \leq j \leq 5$ , we arrive at

$$(3.52) \quad I(x) = O(x^{-1 - \kappa^2/4 F + 0})$$

showing (3.43).

It is easy to see from (3.32) and Lemma 11 that

$$c(t) \equiv \frac{r_1(t)}{r_2(t)} \cos(\theta_1(t) - \theta_2(t)) + b = O(t^{-\kappa^2/4 F + 0}).$$

For the proof of (3.44), we shall estimate

$$J(x) \equiv \int_x^{x+1} \rho(t)^2 e^{i\beta\theta_2(t)} c(t) dt$$

for  $\beta \neq 0$ . The same integration by parts as in (3-46) yields

$$\begin{aligned} J(x) &= \frac{i}{\beta} [\rho(t)^4 e^{i\beta\theta_2(t)} c(t)]_x^{x+1} \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \frac{d\rho(t)^4}{dt} e^{i\beta\theta_2(t)} c(t) dt \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\alpha(t)} c(t) d(e^{i\beta\tilde{\theta}(t)}) \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\theta_2(t)} dc(t) \\ &\quad - \frac{i}{\beta} \int_x^{x+1} \rho(t)^4 e^{i\beta\alpha(t)} d(e^{i\beta\tilde{\theta}(t)}) dc(t) \\ &\equiv J_1(x) + J_2(x) + J_3(x) + J_4(x) + J_5(x). \end{aligned}$$

Corresponding to (3.47), (3.48) and (3.49), we obtain

$$\begin{aligned} J_1(x) &= O(x^{-1 - \kappa^2/4 F + 0}), \\ J_2(x) &= O(x^{-2 - \kappa^2/4 F + 0}), \end{aligned}$$

and

$$J_3(x) = O(x^{-3/2 - \kappa^2/4 F + 0}).$$

In order to estimate  $J_4(x)$  and  $J_5(x)$ , we have to know  $dc(t)$ . But this is equal to

$$\begin{aligned} & d\left(\frac{r_1(t)}{r_2(t)} \cos(\theta_1(t) - \theta_2(t))\right) \\ &= \operatorname{Re} \left[ d \left( \exp \left( \log \frac{r_2(t)}{r_1(t)} + i(\theta_2(t) - \theta_1(t)) \right) \right) \right] \\ &= \operatorname{Re} \left[ d \left( \exp \left( \log \frac{x_2(t) + iy_2(t)}{x_1(t) + iy_1(t)} \right) \right) \right] \\ &= \operatorname{Re} \left[ \frac{x_2 + iy_2}{x_1 + iy_1} \left[ d \left( \log \frac{x_2 + iy_2}{x_1 + iy_1} \right) + \frac{1}{2} \left( d \left( \log \frac{x_2 + iy_2}{x_1 + iy_1} \right) \right)^2 \right] \right]. \end{aligned}$$

From (3.35) and the boundedness of  $(x_2 + iy_2)/(x_1 + iy_2)$ , we may write

$$dc(t) = \gamma_1(t) dB(t) + \gamma_2(t) dt,$$

where

$$\gamma_1(t) = O(t^{-1/2 - \kappa^2/4 F + 0})$$

and

$$\gamma_2(t) = O(t^{-1 - \kappa^2/4 F + 0}).$$

Hence the same argument as before yields

$$J_4(x) = O(x^{-3/2 - \kappa^2/4 F + 0})$$

and

$$J_5(x) = O(x^{-2 - \kappa^2/4 F + 0}).$$

Combining these estimates for  $J_j(x)$ 's, we finally obtain

$$J(x) = O(x^{-1 - \kappa^2/4 F + 0}).$$

This completes the proof of Lemma 13, and of the assertion (iii) of Theorem 6.  $\square$

Since  $u_1(t) \equiv \rho(t) r_1(t) \sin \theta_1(t)$  is linearly independent of  $u_0(t)$  with probability one, it is sufficient, for the proof of (iv) of Theorem 6, to estimate  $u_1(t)$ . But we have

$$\begin{aligned} & \int_x^{x+1} \rho(t)^2 r_1(t)^2 \sin^2 \theta_1(t) dt \\ &= \frac{1}{2} \int_x^{x+1} \rho(t)^2 r_1(t)^2 dt - \frac{1}{2} \int_x^{x+1} \rho(t)^2 r_1(t)^2 \cos 2\theta_1(t) dt. \end{aligned}$$

and

$$\int_x^{x+1} \rho(t)^2 r_1(t)^2 dt \geq O(x^{-1/2 + \kappa^2/4 F - 0}),$$

as before. Moreover, it is not difficult to show

$$\int_x^{x+1} \rho(t)^2 r_1(t)^2 \cos 2\theta_1(t) dt = O(x^{-1+x^2/4F+0})$$

in the analogous way as Lemma 13. The assertion (iv) of Theorem 6 is thus proved.  $\square$

It remains to prove the assertion (v). Let  $\varphi(t)$  be the sum of the second and third terms of the right hand side of (3.42). Then from Lemma 13 and  $F = \kappa^2/2$ , we have

$$\int_x^{x+1} \varphi(t) dt = O(x^{-1-x^2/4F+0}) = O(x^{-3/2+0}),$$

so that the integral

$$\int_0^\infty \varphi(t) dt$$

converges almost surely. Therefore by (3.42), to prove that  $u_0(t) \notin L^2([0, \infty))$ , it suffices to prove that

$$(3.53) \quad \int_0^\infty \frac{\rho(t)^2}{r_2(t)^2} dt = \infty$$

almost surely. In the following, we shall write  $r(t)$  and  $\theta(t)$  instead of  $r_2(t)$  and  $\theta_2(t)$  for notational simplicity. By (3.27)

$$\begin{aligned} r(t)^{-2} &= \exp[-2 \log r(t)] \\ &= \exp \left[ \kappa \int_0^t \rho(s)^2 \sin 2\theta(s) dB(s) \right. \\ &\quad \left. - \kappa^2 \int_0^t \rho(s)^4 \sin^2 \theta(s) (\sin^2 \theta(s) - \cos^2 \theta(s)) ds \right] \end{aligned}$$

Using Lemma 9, it is easily seen that

$$\int_0^t \rho(s)^4 \sin^2 \theta(s) (\sin^2 \theta(s) - \cos^2 \theta(s)) ds = \frac{1}{4} \int_0^t \rho(s)^4 ds + O(1),$$

and by (3.6), we have

$$\int_0^t \rho(s)^4 ds = \frac{1}{F} \log t + O(1), \quad \text{as } t \rightarrow +\infty,$$

so that when  $F = \kappa^2/2$ ,

$$-\kappa^2 \int_0^t \rho(s)^4 \sin^2 \theta(s) (\sin^2 \theta(s) - \cos^2 \theta(s)) ds = -\frac{1}{2} \log t + O(1).$$

Hence if we set

$$M(t) = \kappa \int_0^t \rho(s)^2 \sin 2\theta(s) dB(s),$$

we have with probability one,

$$\begin{aligned} (3.54) \quad \int_0^T \rho(t)^2 r(t)^{-2} dt &= \int_0^T \rho(t)^2 \exp \left[ M(t) - \frac{1}{2} \log t + O(1) \right] dt \\ &\geq \text{const.} \int_0^T t^{-1/2} \rho(t)^2 e^{M(t)} dt \\ &\geq \text{const.} \int_0^T \rho(t)^4 e^{M(t)} dt. \end{aligned}$$

Now by martingale representation theorem (see Ikeda and Watanabe [11]), there is a Brownian motion  $\{\tilde{B}(t)\}_{t \geq 0}$  such that

$$M(t) = \tilde{B}(\langle M \rangle(t)),$$

where

$$\langle M \rangle(t) = \kappa^2 \int_0^t \rho(s)^4 \sin^2 2\theta(s) ds = \frac{\kappa^2}{2} \int_0^t \rho(s)^4 ds + o(1) \rightarrow +\infty,$$

almost surely as  $t \rightarrow +\infty$ .

If we let  $A(t)$  be the inverse function of  $t \rightarrow \langle M \rangle(t)$ , then

$$\frac{dA(t)}{dt} = [\kappa^2 \rho(A(t))^4 \sin^2 2\theta(A(t))]^{-1}.$$

Hence

$$\begin{aligned} \int_0^\infty \rho(t)^4 e^{M(t)} dt &= \int_0^\infty \rho(t)^4 \exp[\tilde{B}(\langle M \rangle(t))] dt \\ &= \int_0^\infty \rho(A(t))^4 \exp[\tilde{B}(t)] [\kappa^2 \rho(A(t))^4 \sin^2 2\theta(A(t))] dt \\ &\geq \kappa^{-2} \int_0^\infty e^{\tilde{B}(t)} dt, \end{aligned}$$

and (3.53) follows from (3.54) and the next lemma, completing the proof of Theorem 6.

LEMMA 14. — *If  $\{B(t)\}_{t \geq 0}$  is the standard Brownian motion, then*

$$\int_0^\infty e^{B(t)} dt = +\infty,$$

*with probability one.*

*Proof.* — It is easy to see that the event

$$A = \left\{ \int_0^\infty e^{B(t)} dt = \infty \right\}$$

belongs to the tail  $\sigma$ -field  $\mathcal{F}^\infty$ , namely

$$A \in \mathcal{F}^\infty = \bigcap_{t \geq 0} \sigma[B(s) - B(t); s \geq t],$$

so that  $A$  has probability zero or one. Hence it suffices to show  $P(A) > 0$ . But since

$$\int_0^\infty e^{B(t)} dt + \int_0^\infty e^{-B(t)} dt \geq \int_0^\infty 2 dt = \infty,$$

either  $A$  or

$$A' \equiv \left\{ \int_0^\infty e^{-B(t)} dt = \infty \right\}$$

must have positive probability. Since  $P(A) = P(A')$ , we conclude  $P(A) > 0$ .  $\square$

### 3.4. Proof of Lemma 4.

Let  $u_\lambda(t)$  and  $u_\mu(t)$  be the solutions of equations  $\mathcal{H}_0^F u = \lambda u$  and  $\mathcal{H}_0^F u = \mu u$  respectively which satisfy the same boundary condition (1.4)<sub>0</sub>, and let  $\theta_\lambda(t)$  and  $\theta_\mu(t)$  be the functions obtained from these by the transformations (3.9) and (3.13). By (3.10), the solution  $u_\lambda(t)$  has a zero if and only if  $\theta_\lambda(t) = 0 \pmod{\pi}$ , and by (3.15),  $\theta_\lambda(t)$  can only decrease when  $\theta_\lambda(t) = 0 \pmod{\pi}$ , because then  $\theta'_\lambda(t) = -\rho_\lambda(t)^2 < 0$ . Here we have set  $\rho_\lambda(t)^2 \equiv \varphi_{\lambda,+}(t)^2 + \varphi_{\lambda,-}(t)^2$ ,  $\varphi_{\lambda,\pm}(t)$  being the solutions of  $H_0^F u = \lambda u$  satisfying the conditions of Lemma 5. Hence we have

$$N(T, \lambda) = \left[ \frac{-\theta(T)}{\pi} \right] + 1,$$

so that in order to prove that (2.5) holds almost surely, it is sufficient to prove that

$$(3.55) \quad \lim_{t \rightarrow +\infty} \{ \theta_\lambda(t) - \theta_\mu(t) \} = +\infty,$$

holds almost surely. On the other hand, we have from (3.15),

$$\begin{aligned} \theta_\lambda(t) - \theta_\mu(t) &= \kappa \int_0^t \{ \rho_\lambda(s)^2 \sin^2 \theta_\lambda(s) - \rho_\mu(s)^2 \sin^2 \theta_\mu(s) \} dB(s) \\ &\quad + \kappa^2 \int_0^t \{ \rho_\lambda(s)^4 \sin^3 \theta_\lambda(s) \cos \theta_\lambda(s) - \rho_\mu(s)^4 \sin^3 \theta_\mu(s) \cos \theta_\mu(s) \} ds \\ &\quad + \int_0^t \{ \rho_\mu(s)^{-2} - \rho_\lambda(s)^{-2} \} ds = L_1(t) + L_2(t) + L_3(t). \end{aligned}$$

Now we can apply Lemma 6, to obtain

$$L_1(t) = O((\log t)^{1/2+0}),$$

and Lemma 9, to see that the finite limit

$$\lim_{t \rightarrow +\infty} L_2(t)$$

exists almost surely. Finally noting that

$$\rho_\lambda(t) = \rho_0(t + \lambda/f),$$

we see

$$\begin{aligned} L_3(t) &= \int_0^t ds \int_\lambda^\mu d\xi \frac{\partial}{\partial \xi} \{ \rho_0(s + \xi/F)^{-2} \} \\ &= \frac{1}{F} \int_0^t ds \int_\lambda^\mu d\xi \frac{\partial}{\partial s} \{ \rho_0(s + \xi/F)^{-2} \} \\ &= \frac{1}{F} \int_\lambda^\mu \{ \rho_0(t + \xi/F)^{-2} - \rho_0(\xi/F)^{-2} \} d\xi, \end{aligned}$$

so that

$$L_3(t) \geq 0 (t^{1/2}).$$

This proves (3.55) and hence Lemma 4.  $\square$

### 3.5. Discussion on the case $F < 0$

Consider  $\mathcal{H}_\omega^{F, \kappa, \theta}$  with  $F < 0$  and the operator

$$\mathcal{H}_\omega^F = -d^2/dt^2 + \kappa B'_\omega(t) - Ft, \quad -\infty < t < \infty$$

in  $L^2(\mathbb{R}^1)$ . Although the author was unable to prove it, he believes that the following assertion holds.

CONJECTURE. — *When  $F < 0$ ,  $\mathcal{H}_\omega^{F, \kappa, \theta}$  has almost surely only discrete spectrum for every  $\theta \in [0, \pi)$ . Moreover,  $\mathcal{H}_\omega^F$ , considered in  $L^2(\mathbb{R}^1)$ , has exactly the same spectral nature as expressed in Theorem 2, when  $F > 0$ .*

In this last subsection, we would like to discuss how this conjecture is plausible. We first note, corresponding to Lemma 5, that if  $F < 0$ ,  $H_0^F u = \xi u$  has two solutions  $\varphi_+(t)$  and  $\varphi_-(t)$  such that  $[\varphi_+, \varphi_-] = 1$  and that

$$\varphi_\pm(t) = \exp \left[ \pm \frac{2}{3} |F|^{1/2} \left( t + \frac{\xi}{F} \right) \right] \left( \pm \frac{|F|^{-1/4}}{\sqrt{2}} \left( t + \frac{\xi}{F} \right)^{-1/4} + 0(t^{-7/4}) \right)$$

and

$$\varphi'_\pm(t) = \exp \left[ \pm \frac{2}{3} |F|^{1/2} \left( t + \frac{\xi}{F} \right) \right] \left( \frac{|F|^{1/4}}{\sqrt{2}} \left( t + \frac{\xi}{F} \right)^{1/4} + 0(t^{-5/4}) \right),$$



as  $t \rightarrow +\infty$ . In this case, we have

$$\rho(t) \equiv \{ \varphi_+(t)^2 + \varphi_-(t)^2 \}^{1/2} \\ \sim \frac{|F|^{-1/4}}{\sqrt{2}} \left( t + \frac{\xi}{F} \right)^{-1/4} \exp \left[ \frac{2}{3} |F|^{1/2} \left( t + \frac{\xi}{F} \right)^{3/2} \right],$$

as  $t \rightarrow +\infty$ .

As before, let us introduce the variables  $r(t)$  and  $\theta(t)$  by performing the transformation of subsection 3.1 to the solution of  $\mathcal{H}_\omega^F u = \xi u$ . In particular,  $\theta(t)$  satisfies

$$d\theta(t) = \kappa \rho(t)^2 \sin^2 \theta(t) dB(t) + \kappa^2 \rho(t)^4 \sin^3 \theta(t) \cos \theta(t) dt - \rho(t)^{-2} dt.$$

Let

$$L(t) = \int_0^t \rho(s)^4 ds,$$

and let  $L^{-1}(t)$  be the inverse function of  $L(t)$ . Further, let  $\hat{\theta}(t) \equiv \theta(L^{-1}(t))$  and

$$\hat{B}(t) \equiv \int_0^t \rho(L^{-1}(s))^2 dB(L^{-1}(s)).$$

Then  $\{\hat{B}(t)\}$  is again a Brownian motion, and  $\hat{\theta}(t)$  is the unique solution of the stochastic differential equation

$$d\hat{\theta}(t) = \kappa \sin^2 \hat{\theta}(t) d\hat{B}(t) + \kappa^2 \sin^3 \hat{\theta}(t) \cos \hat{\theta}(t) dt - \rho(L^{-1}(t))^{-6} dt.$$

Now we have

$$\int_0^\infty \rho(L^{-1}(t))^{-6} dt < \infty.$$

On the other hand, the diffusion process  $z(t)$  defined by

$$dz(t) = \kappa \sin^2 z(t) d\hat{B}(t) + \kappa^2 \sin^3 z(t) \cos z(t) dt$$

is recurrent in each interval  $(n\pi, (n+1)\pi)$ ,  $n \in \mathbb{Z}^1$ . In fact,  $z(t)$  is explicitly given by

$$\cot z(t) = \cot z(0) - \kappa \hat{B}(t),$$

namely by a transformation of state space of the standard Brownian motion.

From these facts, it is natural to conjecture that the time inhomogeneous diffusion process  $\{\hat{\theta}(t)\}$  hits the lattice points  $n\pi$ ,  $n \in \mathbb{Z}^1$  only finitely often. (This is the point the author could not justify.) This would mean, in view of (3.10) and (3.13), that the solution  $u(t)$  of  $\mathcal{H}_\omega^F u = \xi u$  should have only finitely many zeros on  $[0, \infty)$  with probability one, for any  $\xi \in \mathbb{R}^1$ . Hence it would follow, by the classical result of Hartman [9], that  $\mathcal{H}_\omega^{F, \theta}$  is self-adjoint and has discrete spectrum only, and that for any  $\xi \in \mathbb{R}^1$ ,  $\mathcal{H}_\omega^F u = \xi u$  has a solution in  $L^2([0, \infty))$ .

By this consideration, when  $F < 0$ ,  $\mathcal{H}_\omega^F u = \xi u$  would have a solution in  $L^2((-\infty, 0])$  for any  $\xi \in \mathbb{R}^1$ . On the other hand, as to the behavior of the solutions for  $t \geq 0$ , we have already Theorem 6. Hence in view of (i) and (v) of Theorem 4,  $\mathcal{H}_\omega^F$ , considered in  $L^2(\mathbb{R}^1)$ , would have the same spectral property as expressed in Theorem 2.

### Appendix A.

#### Proof of Lemma 1

We will actually prove that for  $I = [0, a]$ , with  $a > 0$  arbitrary, one has  $A_1^+, A_2^+ \in \mathcal{G}(I) \times \mathcal{B}(\mathbb{R}^1)$ , where  $\mathcal{G}(I) = \sigma[q_\omega(t); t \notin I]$ . Throughout this appendix, we use the same notation as that of Gilbert and Pearson [6]. In particular the spectral measure is denoted by  $\rho$  rather than  $\sigma$ , and the parameter  $z = x + iy$  plays the same role as the spectral parameter  $\lambda = \xi + i\eta$ .

By Theorem 1 of [6], we have first

$$A_2^+ = \{(\omega, x) \mid \lim_{y \downarrow 0} m(x + iy; \omega) \text{ exists in } \mathbb{R} \cup \{\infty\}\},$$

where  $m(z; \omega)$  is the  $m(z, 0)$  defined in (2,1)-(2,2) of [6] with the potential  $V = q_\omega(\cdot)$ . Let  $m^a(z; \omega)$  be defined in the same way as  $m(z; \omega)$  with  $u_j(r, z, 0)$ ,  $j = 1, 2$  replaced by the solutions  $u_j^a(r, z)$ ,  $j = 1, 2$  of  $L_\omega u = zu$  which satisfy  $u_1^a(a, z) = (d/dr)u_2^a(a, z) = 0$ ,  $u_2^a(a, z) = (d/dr)u_1^a(a, z) = 1$ .

Then it is easy to see that

$$(A.1) \quad m^a(z; \omega) = \frac{u_2'(a, z, 0)m(z; \omega) + u_1'(a, z, 0)}{u_2(a, z, 0)m(z; \omega) + u_1(a, z, 0)}.$$

Hence  $\lim_{y \downarrow 0} m(x + iy; \omega)$  exists in  $\mathbb{R} \cup \{\infty\}$  if and only if  $\lim_{y \downarrow 0} m^a(x + iy; \omega)$  exists in  $\mathbb{R} \cup \{\infty\}$ . Moreover, the function  $(y, \omega, x) \mapsto m^a(x + iy; \omega)$  is continuous in  $y > 0$ , and  $\mathcal{G}(I) \times \mathcal{B}(\mathbb{R}^1)$ -measurable in  $(\omega, x)$ , so that

$$A_1^+ = \{(\omega, x) \mid \lim_{y \downarrow 0, y: \text{rational}} m^a(x + iy; \omega) \text{ exists in } \mathbb{R} \cup \{\infty\}\} \in \mathcal{G}(I) \times \mathcal{B}(\mathbb{R}^1),$$

proving a part of Lemma 1.

Now  $L_\omega u = zu$  has a solution in  $L^2([0, \infty))$  if and only if  $L_\omega u(r) = zu(r)$ ,  $r \geq a$ , has a solution in  $L^2([a, \infty))$ . This, in turn, is equivalent to the condition

$$(A.2) \quad \lim_{y \downarrow 0} y \operatorname{Im} m^a(x + iy; \omega) > 0 \quad \text{or} \quad \lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im} m^a(x + iy; \omega) < \infty$$

as we will prove below. Therefore the set  $A_1^+$ , being equal to the totality of  $(\omega, x)$  for which (A. 2) is valid, belongs to  $\mathcal{G}(I) \times \mathcal{B}(\mathbb{R}^1)$ .

Let us now prove the equivalence of (A. 2) and  $(\omega, x) \in A_1^+$ . Suppose  $u(r)$  is a solution of  $L_\omega u(r) = xu(r)$ ,  $r \geq a$ , which is square integrable. Then  $x$  is an eigenvalue corresponding to some boundary condition at  $r = a$ :

$$(A. 3) \quad \cos \theta u(a) + \sin \theta u'(a) = 0.$$

If  $\theta = 0$ , then the measure  $\rho_0(dx)$  which represents  $m^a(z; \omega)$  through the formula (2. 5) of [6], has a point mass at  $x$ , and vice versa. Namely

$$0 < \rho_0(\{x\}) = \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{y^2}{(t-x)^2 + y^2} \sigma_0(dt) = \lim_{y \downarrow 0} y \operatorname{Im} m^a(x + iy; \omega).$$

On the other hand, if  $\theta \neq 0$ , then the measure  $\rho_\theta$  which represents the function

$$m^a(z; \omega, \theta) \equiv \frac{\cot \theta m^a(z) - 1}{\cot \theta + m^a(z)}$$

through the formula (2. 5) of [6], has a point mass at  $x$ . Hence as above,

$$\lim_{y \downarrow 0} y \operatorname{Im} m^a(x + iy; \omega; \theta) = \rho_\theta(\{\xi\}) > 0.$$

Since

$$y \operatorname{Im} m^a(x + iy; \omega; \theta) = \frac{y \operatorname{Im} m^a}{|\cot \theta + m^a|^2} \leq \frac{y}{\operatorname{Im} m^a},$$

this implies

$$\overline{\lim}_{y \downarrow 0} \frac{1}{y} \operatorname{Im} m^a(\xi + iy; \omega) < \infty.$$

But this  $\overline{\lim}$  can be replaced by  $\lim$  in view of

$$(A. 4) \quad \frac{1}{y} \operatorname{Im} m^a(x + iy; \omega) = \int \frac{\rho_0(dt)}{(t-x)^2 + y^2} \uparrow \int \frac{\rho_0(t)}{(t-x)^2} \quad (y \downarrow 0).$$

Thus far we have shown that the existence of the  $L^2$ -solution vanishing at  $r = a$  is equivalent to  $\lim_{y \downarrow 0} y \operatorname{Im} m^a(x + iy; \omega) > 0$ , and that the existence

of  $L^2$ -solution not vanishing at  $r = a$  implies  $\lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im} m^a(x + iy; \omega) < \infty$ . To

complete the proof, let us show that this last condition implies the existence of  $L^2$ -solution. So suppose

$$\lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im} m^a(x + iy; \omega) = \int_{-\infty}^{\infty} \frac{\rho_0(dt)}{(t-x)^2} < \infty.$$

Then using (2. 5) of [6], it can be shown that  $m^a(x + i0; \omega)$  exists and is real. In this case,  $u_m^a(r; x + i0)$  is the desired  $L^2$ -solution, where we have

set

$$u_m^a(r; z) = u_2^a(r; z) + m^a(z; \omega) u_1^a(r; z).$$

Indeed by Fatou's lemma, we see

$$\begin{aligned} \int_a^\infty |u_m^a(r; x + i0)|^2 dr &\leq \liminf_{y \downarrow 0} \int_a^\infty |u_m^a(r; x + iy)|^2 dx \\ &= \lim_{y \downarrow 0} \frac{1}{y} \operatorname{Im} m^a(x + iy; \omega) < \infty, \end{aligned}$$

completing the proof. The last equality is just (2.3) of [6].

### Appendix B.

#### Kotani's trick

In this appendix, we sketch how to prove Lemmas 2 and 3 of paragraph 2. We begin with Lemma 2.

Let  $0 \leq f(t) \leq \delta$  be a  $C^1$ -function with its support contained in  $[0, 1]$ . By change of drift formula, if we denote by  $P^\tau$  the distribution of  $\{X_\omega(t) + \tau f(t); -\infty < t < \infty\}$  in the path space, then  $P^\tau$  and  $P$  are mutually absolutely continuous.

In the same way as in Kotani-Simon [14], we can prove that for any  $\omega \in C$ ,

$$(B.1) \quad \int_0^1 d\tau \sigma_{\omega(\cdot) + \tau f(\cdot)}(d\xi) \ll \mu(d\xi),$$

where  $\mu$  is the Lebesgue measure.

Now if  $(P \times \mu)(B) = 0$ , then for  $P$ -a.a.  $\omega \in C$ , we have  $\mu(B(\omega)) = 0$ , and hence by (B.1),

$$(B.2) \quad \int_0^1 d\tau \sigma_{\omega(\cdot) + \tau f(\cdot)}(B(\omega)) = 0.$$

This means that for some  $\tau$  (in fact for  $\mu$ -a.a.  $\tau \in [0, 1]$ ), we have  $\sigma_{\omega + \tau f}(B(\omega)) = 0$  for  $P$ -a.a.  $\omega \in C$ . But if  $B \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^1)$ , we have the equivalence

$$(B.3) \quad (\omega(\cdot), \xi) \in B \leftrightarrow (\omega(\cdot) + \tau f(\cdot), \xi) \in B,$$

so that

$$(B.4) \quad \sigma_{\omega(\cdot) + \tau f(\cdot)}(B(\omega + \tau f)) = 0, \quad \text{for } P\text{-a.a. } \omega \in C,$$

or equivalently

$$(B.5) \quad \sigma_\omega(B(\omega)) = 0, \quad \text{for } P^\tau\text{-a.a. } \omega \in C.$$

Since  $P^\tau$  and  $P$  are mutually absolutely continuous, we have

$$(B.6) \quad \sigma_\omega(B(\omega))=0, \quad \text{for } P\text{-a.a. } \omega \in C,$$

completing the proof of Lemma 2.

Turning to Lemma 3, let  $f \in C^1(\mathbb{R}^1)$  be such that  $f'(t) \geq 0$ ,  $f \neq \text{constant}$ ,  $f(t) = 0$  for  $t \leq 1/2$ , and  $\text{Supp}(f'(\cdot)) \subset [1/2, 1]$ . Then the distributions in the path space of  $\{B_\omega(\cdot)\}$  and  $\{B_\omega(\cdot) + \tau f(\cdot)\}$  are mutually absolutely continuous. In this case, we can prove

$$(B.7) \quad \int_0^1 d\tau \sigma_{\omega(\cdot) + \tau f(\cdot)}^0(d\xi) \ll \mu(d\xi)$$

for all  $\omega \in \Omega$  for which  $\mathcal{H}_\omega^0$  is self-adjoint, and Lemma 3 follows in the same manner as before.

### Appendix C.

#### Asymptotic behavior of Airy's function.

In this appendix, we sketch how to obtain Lemma 5 from classical results on the asymptotic behavior of Airy's function. See Copson [4] for details.

Let  $A(z)$  be the Airy's function, namely

$$(C.1) \quad A(z) = \frac{1}{2\pi i} \int_\Gamma \exp\left[zs - \frac{1}{3}s^3\right] ds,$$

where  $\Gamma$  is a path starting from  $\infty \cdot e^{-i\alpha}$ , passing through the origin, and ending at  $\infty \cdot e^{i\alpha}$ , with  $\pi/2 < \alpha < 5/6\pi$ . Then  $A(z)$  is entire and satisfies  $-A''(z) + zA(z) = 0$ . When  $z > 0$ , we may replace  $\Gamma$  by the straight line from  $-z^{1/2} - \infty \cdot i$  to  $-z^{1/2} + \infty \cdot i$ . In this case, we have

$$(C.2) \quad A(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{2}{3}z^{3/2} - z^{1/2}t^2 + \frac{i}{3}t^3\right] dt \\ = \frac{1}{2\pi} \exp\left[-\frac{2}{3}z^{3/2}\right] \int_0^{\infty} \exp[-z^{1/2}u] \cos\left(\frac{1}{3}u^{3/2}\right) \frac{du}{\sqrt{u}}.$$

But the integral on the right hand side converges uniformly on every compact set in  $\{|\arg z| < \pi\}$ , and accordingly defines a holomorphic function there. Hence the representation (C.2) is valid also for  $\{|\arg z| < \pi\}$ . Substituting

$$(C.3) \quad \frac{1}{\sqrt{u}} \cos\left(\frac{1}{3}u^{3/2}\right) = \frac{1}{\sqrt{u}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! 3^{2n}} u^{3n}$$

into (C.2), we obtain by Watson's lemma (see Copson [4], Ch. 6), the following asymptotic expansion

$$(C.4) \quad A(z) \sim \frac{1}{2\pi} \exp\left[-\frac{2}{3}z^{3/2}\right] \frac{1}{z^{1/4}} \sum_{n=0}^{\infty} \frac{\Gamma(3n+1/2)}{3^{2n}(2n)!} \frac{(-1)^n}{z^{3n/2}},$$

which holds as  $|z| \rightarrow \infty$  uniformly in any closed sector contained in  $\{|\arg z| < \pi\}$ . In the same manner as in page 11 of [4], we may differentiate (C.4) term by term, to obtain

$$(C.5) \quad A'(z) \sim -\frac{1}{2\pi} \exp\left[-\frac{2}{3}z^{3/2}\right] \times \sum_{n=0}^{\infty} \frac{\Gamma(3n+1/2)}{3^{2n}(2n)!} (-1)^n \left( z^{-3n/2+1/4} + \left(\frac{3n}{2} + \frac{1}{4}\right) z^{-3n/2+5/4} \right).$$

Explicitly calculating up to the second terms, we get

$$(C.6) \quad A(z) = \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left[-\frac{2}{3}z^{3/2}\right] + O\left(\left|z^{-7/4} \exp\left[-\frac{2}{3}z^{3/2}\right]\right|\right),$$

and

$$(C.7) \quad A'(z) = -\frac{1}{2\sqrt{\pi}} z^{1/4} \exp\left[-\frac{2}{3}z^{3/2}\right] + O\left(\left|z^{-5/4} \exp\left[-\frac{2}{3}z^{3/2}\right]\right|\right).$$

Now if we set  $\omega = e^{2\pi i/3}$  and

$$(C.8) \quad \begin{cases} \psi_+(t) = \pi^{1/2} (e^{-\pi i/12} A(-\omega t) + e^{\pi i/12} \overline{A(-\omega t)}) \\ \psi_-(t) = i\pi^{1/2} (-e^{-\pi i/12} A(-\psi t) + e^{\pi i/12} \overline{A(-\omega t)}), \end{cases}$$

then  $\psi_{\pm}(t)$  are solutions of  $-u''(t) - tu(t) = 0$ , and satisfy

$$(C.9) \quad \psi_{\pm}(t) = t^{-1/4} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \left(\frac{2}{3}t^{3/2}\right) + O(t^{-7/4}),$$

and

$$(C.10) \quad \psi'_{\pm}(t) = t^{1/4} \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(\frac{2}{3}t^{3/2}\right) + O(t^{-5/4}).$$

From these estimate, we see

$$(C.11) \quad [\psi_+, \psi_-](t) = \psi_+(t)\psi'_-(t) - \psi'_+(t)\psi_-(t) = 1 + o(1),$$

as  $t \rightarrow +\infty$ . But the Wronskian is constant in  $t$ , so that this actually means that  $[\psi_+, \psi_-](t) \equiv 1$ . The desired solution  $\varphi_+(t)$  and  $\varphi_-(t)$  of  $H_0^F u = \xi u$  are now obtained by setting

$$(C.12) \quad \varphi_{\pm}(t) = F^{-1/6} \psi_{\pm}(F^{1/3}(t + \xi/F)),$$

as can be easily verified.

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