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Quasi-discrete quantum Markov processes

by

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ABSTRACT. — Following the ideas of R. Feynman, we formulate a theory of quantum probability in terms of amplitude functions. Within this framework, we then define the concept of a quasi-discrete quantum Markov process. Various examples of such processes are presented. In particular, we consider quantum coins, discrete quantum mechanics and a quantum Poisson process. Finally, a general theory of quasi-discrete quantum Markov processes is developed.

RÉSUMÉ. — Suivant des idées de R. Feynman, nous formulons une théorie de probabilités quantique basée sur les fonctions d'amplitude. Dans ce formalisme nous définissons le concept de processus de Markov quantique quasi discret. Nous présentons plusieurs exemples, en particulier : dés quantiques, mécanique quantique discrète et processus de Poisson quantique. Nous développons aussi une théorie générale des processus de Markov quantiques quasi discrets.

1. INTRODUCTION

Following the ideas of R. Feynman ([1], [2]), we have previously formulated a theory of quantum probability in terms of amplitude functions

([3]-[6]). Without repeating the details, we can summarize our motivation as follows. An outcome of a quantum mechanical measurement is the result of various interfering alternatives each having an amplitude for occurring. The probability of this outcome is the absolute value squared of the sum of these amplitudes. This forms the basic axiom of our quantum probability framework. Within this framework we can now define quantum stochastic processes and in particular quantum Markov processes.

We shall find it convenient to classify quantum stochastic processes according to two types. The first type, which we call quasi-discrete, is mainly applicable to discrete situations, while the second type, which we call regular, applies to the continuum case. The present paper is devoted to quasi-discrete processes and a sequel will study regular processes. Section 2 presents the notation and definitions needed for the subsequent material. Section 3 considers various quasi-discrete examples. In particular, we consider quantum coins, discrete quantum mechanics, and a quantum Poisson process. The general theory of quasi-discrete quantum Markov processes is developed in Section 4.

In this paper we formulate a more general approach than that used in [7]. The present framework contains that of [7] as a special case and is closer in spirit to Feynman's quantum probability.

2. NOTATION AND DEFINITIONS

A *point measure space* is a measure space in which singleton sets are measurable. Let Ω be a nonempty set which we call a *sample space* and whose elements we call *sample points*. A map $X : \Omega \to \mathscr{X}$ with range $R(X) \subseteq \mathscr{X}$ is a *measurement* if

(2.1) $R(X)$ is the base space of a point measure space $(R(X), \Sigma_X, \mu_X)$

(2.2) for every $x \in R(X)$, $X^{-1}(x)$ is the base space of a measure space $(X^{-1}(x), \Sigma_X^x, \mu_X^x)$. We call the elements of $R(X)$, *X-outcomes*, the sets in Σ_X, *X-events* and $X^{-1}(x)$ the *fiber* (or *sample*) *over x*. Notice that

$$\mathscr{E}(X) = \{X^{-1}(B) : B \in \Sigma_X\}$$

is a σ-algebra of subsets of Ω. We call the elements of $\mathscr{E}(X)$, *X-sample events*. A function $f : \Omega \to \mathbb{C}$ is an *amplitude density* for the measurement X if

(2.3) $f \mid X^{-1}(x) \in L^1(X^{-1}(x), \Sigma_X^x, \mu_X^x)$ for every $x \in R(X)$.

(2.4) $f_X(x) \equiv \int_{X^{-1}(x)} f \, d\mu_X^x \in L^2(R(X), \Sigma_X, \mu_X) \equiv H_X$.

(2.5) $\| f_X \|^2 = \int | f_X |^2 \, d\mu_X = 1$.

We call H_X the X-*Hilbert space* and f_X the (X, f)-*wave function*. Traditional quantum mechanics usually begins with H_X and f_X and hence loses information about the sample space Ω. We denote by $\mathscr{A}(\Omega, f)$ the set of all measurements for which f is an amplitude density and call $\mathscr{A}(\Omega, f)$ a *quantum probability space*. We call a subset $\mathscr{A}_0 \subseteq \mathscr{A}(\Omega, f)$ an (Ω, f)-*catalog*. For motivation and further details the reader is refered to ([3], [4], [6]).

For a measurement $X \in \mathscr{A}(\Omega, f)$, a set $A \subseteq \Omega$ is an (X, f)-*sample event* if

(2.6) $A \cap X^{-1}(x) \in \Sigma_X^x$ for all $x \in R(X)$.

(2.7) $f_X(A)(x) \equiv \int_{A \cap X^{-1}(x)} f \, d\mu_X^x \in H_X$.

We interpret $f_X(A)$ as the amplitude density of A as determined by X. Notice that $f_X(\Omega) = f_X$. Denote the set of (X, f)-sample events by $\mathscr{E}(X, f)$. It is clear that $\mathscr{E}(X) \subseteq \mathscr{E}(X, f)$. If $A \in \mathscr{E}(X, f)$, the (X, f)-*pseudo-probability* of A is

$$P_{X, f}(A) = \int |f_X(A)|^2 \, d\mu_X = \| f_X(A) \|^2.$$

We denote the indicator function of a set B by $\mathbf{1}_B$.

LEMMA 2.1. – *If* $A \in \mathscr{E}(X, f)$, $B \in \Sigma_X$ *then* $X^{-1}(B) \cap A \in \mathscr{E}(X, f)$ and $f_X[X^{-1}(B) \cap A] = 1_B f_X(A)$.

Proof. – It is clear that $X^{-1}(B) \cap A$ satisfies (2.6) and moreover,

$$X^{-1}(B) \cap A \cap X^{-1}(x) = \begin{cases} A \cap X^{-1}(x) & \text{if } x \in B. \\ 0 & \text{if } x \notin B \end{cases}$$

Hence,

$$f_X(X^{-1}(B) \cap A) = \int_{X^{-1}(B) \cap A \cap X^{-1}(x)} f \, d\mu_X^x = 1_B f_X(A).$$

Since $f_X(A) \in H_X$ we have $\mathbf{1}_B f_X(A) \in H_X$ so (2.7) holds. □

We write $P_{X, f}(B) = P_{X, f}[X^{-1}(B)]$ for $B \in \Sigma_X$. Applying Lemma 2.1, we have for $B \in \Sigma_X$

$$f_X[X^{-1}(B)] = f_X[X^{-1}(B) \cap \Omega] = \mathbf{1}_B f_X(\Omega) = \mathbf{1}_B f_X.$$

Hence,

$$P_{X, f}(B) = \int_B |f_X|^2 \, d\mu_X \tag{2.8}$$

We conclude from (2.5) and (2.8) that $P_{X, f}$ is a probability measure on Σ_X which we call the *distribution* of X. If $A \in \mathscr{E}(X. f)$, $B \in \Sigma_X$, then by

Lemma 2.1, $X^{-1}(B) \cap A \in \mathscr{E}(X, f)$. If $P_{X, f}(A) \neq 0$, the *conditional probability of* B *given* A *is*

$$P_{X, f}(B \mid A) = \frac{P_{X, f}[X^{-1}(B) \cap A]}{P_{X, f}(A)}.$$

It follows from Lemma 2.1 that

$$P_{X, f}(B \mid A) = \frac{1}{P_{X, f}(A)} \int_B |f_X(A)|^2 \, d\mu_X$$

and hence $P_{X, f}(. \mid A)$ is a probability measure on Σ_X. Let X, $Y \in \mathscr{A}(\Omega, f)$. We say that X *does not interfere with* Y if $\mathscr{E}(Y) \subseteq \mathscr{E}(X, f)$ and for every $B \in \Sigma_Y$

$$P_{X, f}(Y \in B) \equiv P_{X, f}[Y^{-1}(B)] = P_{Y, f}(B).$$

In this case, the distribution of Y is determined by executing the measurement X. Simple examples show that this is not a symmetric relation in general [6]. We say that X is *independent* of Y if $\mathscr{E}(Y) \subseteq \mathscr{E}(X, f)$ and

$$P_{X, f}(X \in B,\ Y \in A) = P_{X, f}(B) P_{X, f}(Y \in A)$$

for all $B \in \Sigma_X$, $A \in \Sigma_Y$. (This is slightly different than the definition given in [6].) If X is independent of Y, then $P_{X, f}[B \mid Y^{-1}(A)] = P_{X, f}(B)$ for all $B \in \Sigma_Y$ and $A \in \Sigma_Y$ with $P_{X, f}(Y \in A) \neq 0$.

Let T be a nonempty subset of $\mathbb{R}$ and suppose for every $t \in T$ there exists a measurement $X_t \in \mathscr{A}(\Omega, f)$. When convenient, we use the notation $X(t) = X_t$, $f_t = f_{X(t)}$, $P_t = P_{X(t), f}$, $\mu_t = \mu_{X(t)}$, $\Sigma_t = \Sigma_{X(t)}$, etc. We say that $(X_t)_{t \in T}$ is a *quantum stochastic process* (QSP) if for every $t, s_1, \ldots, s_n \in T$ with $s_j \leqq t$, $j = 1, \ldots, n$, and every $B_j \in \Sigma_{s(j)}$ $(s(j) = s_j)$ we have

$$\bigcap X_{s(j)}^{-1}(B_j) \in \mathscr{E}(X_t, f). \tag{2.9}$$

Equation (2.9) states that a present measurement can be used to obtain information about the past.

This is weaker than the condition

$$\bigcap X_{s(j)}^{-1}(B_j) \in \mathscr{E}(X_t)$$

which states past information is contained in the present.

Let $(X_t)_{t \in T}$ be a QSP in $\mathscr{A}(\Omega, f)$. For $t_1, \ldots t_n \in T$ with $t_1 < \ldots < t_n$ and $x_j \in R_j$, $j = 1, \ldots, n$, we define

$$f_{t(n)}[x_n \mid X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1] = \frac{f_{t(n)}[X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1](x_n)}{f_{t(n-1)}[X(t_{n-2}) = x_{n-2}, \ldots, X(t_1) = x_1](x_{n-1})}$$

whenever the denominator does not vanish and otherwise we define the left hand side to be zero.

We say that $(X_t)_{t \in T}$ is a *quasi-discrete quantum Markov process* (QMP) if

(2.11) $f_{t(n)}[x_n \mid X(t_{n-1})=x_{n-1}, \ldots, X(t_1)=x_1]=f_{t(n)}[x_n \mid X(t_{n-1})=x_{n-1}]$.

(2.12) $f_t(X_s=x) \in L^1(R_t, \Sigma_t, \mu_t)$ for all $s, t \in T$ with $s \leqq t$.

(2.13) $x \mapsto f_t[A \cap X_s^{-1}(x)](y) \in L^{1,2}(R_s, \Sigma_s, \mu_s)$ for all $s \leqq t$, $y \in R_t$, and $A \in \mathscr{E}(X_u)$ with $u \leqq t$ and $\int f_t[A \cap X_s^{-1}(x)](y)\, d\mu_s(x)=f_t(A)(y)$ where $L^{1,2}=L^1 \cap L^2$.

We shall study quasi-discrete QMP's in the next two sections.

3. QUASI-DISCRETE EXAMPLES

This section presents examples of quasi-discrete QSP's. The first two examples refer to coin tossing, the third refers to discrete quantum mechanics and the fourth to a quantum Poisson process.

Example 1 (Quantum Coin). – Let $a, b \in \mathbb{C}$ satisfy $a+b=1$ and $|a|^2+|b|^2=1$. A necessary and sufficient condition for these properties is that $b=1-a$, $0 \leq \operatorname{Re} a \leq 1$ and $\operatorname{Im} a = \pm(\operatorname{Re} a)^{1/2}(1-\operatorname{Re} a)^{1/2}$. Thus, the single real number $0 \leq \operatorname{Re} a \leq 1$ determines a and b up to a complex conjugation. For instance, we could have $\operatorname{Re} a=1/2$ and then $a=(1 \pm i)/2$, $b=(1 \mp i)/2$. Let $n \in \mathbb{N}$ and let $\Omega=\{0, 1\}^n$. For $\omega=(x_1, \ldots, x_n) \in \Omega$ define $X_j(\omega)=x_j$, $j=1, \ldots, n$. Placing counting measures on the range and fibers of X_j, we see that X_j is a measurement, $j=1, \ldots, n$. Then Ω represents n flips of a coin and X_j measures the result of the j-th flip. Define the amplitude density $f: \Omega \to \mathbb{C}$ by $f(\omega)=a^k b^{n-k}$ where k is the number of 0's in the sequence ω. The wave functions f_j, $j=1, \ldots, n$, have the values

$$f_j(0)=a \sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-k-1}=a(a+b)^{n-1}=a$$
$$f_j(1)=b(a+b)^{n-1}=b.$$

Since

$$|f_j(0)|^2+|f_j(1)|^2=|a|^2+|b|^2=1$$

f is an amplitude density for X_j and $X_j \in \mathscr{A}(\Omega, f)$, $j=1, \ldots, n$. It is clear that $(X_j)_{j=1}^n$ is a QSP.

The distribution of X_j is given by

$$P_j(0)=|f_j(0)|^2=|a|^2$$
$$P_j(1)=|f_j(1)|^2=|b|^2.$$

Thus, the X_j's are identically distributed.

It is easy to check that the X_j's do not interfere with each other. For example

$$\begin{aligned} P_j(X_k=0) &= |f_j(X_k=0)(0)|^2 + |f_j(X_k=0)(1)|^2 \\ &= |a^2|^2 + |ab|^2 = |a|^2 = P_k(0) \end{aligned}$$

and

$$\begin{aligned} P_j(X_k=1) &= |f_j(X_k=1)(0)|^2 + |f_j(X_k=1)(1)|^2 \\ &= |ab|^2 + |b^2|^2 = |b|^2 = P_k(1). \end{aligned}$$

Also, the X_j's are independent since for example

$$\begin{aligned} P_j(X_j=0,\ X_k=1) &= \left| ab \sum_{i=0}^{n-2} \binom{n-2}{i} a^i b^{n-i-2} \right|^2 = |a|^2 |b|^2 \\ &= P_j(0)\, P_j(X_k=1). \end{aligned}$$

Moreover, $(X_j)_{j=1}^n$ is Markov. Indeed (2.12) and (2.13) clearly hold. To verify (2.11) we have

$$f_{t(m)}[X(t_{m-1})=x_{m-1},\ \ldots,\ X(t_1)=x_1](x_m)=a^k b^{m-k}$$

where k is the number of 0's in the sequence $(x_1, \ldots, x_m)$. Hence,

$$f_{t(m)}[x_m \,|\, X(t_{m-1})=x_{m-1},\ \ldots,\ x(t_1)=x_1] = \begin{cases} a & \text{if } x_m=0 \\ b & \text{if } x_m=1. \end{cases}$$

It is clear that $f_{t(m)}[x_m \,|\, X(t_{m-1})=x_{m-1}]$ has the same value.

We conclude that the QMP $(X_j)_{j=1}^n$ has essentially the same behavior as a classical coin tossing process in which the probability of a head (that is, the value 0) is $|a|^2$. However, $(X_j)_{j=1}^n$ does not possess all the well-behaved structure of a classical process. For example, consider the ordinary sum $Y=X_1+X_2$ and, for simplicity, suppose $n=2$. As is natural, we make Y into a measurement by again placing counting measures on its range and fibers. We then have $f_Y(0)=a^2$, $f_Y(1)=2ab$, $f_Y(2)=b^2$. Hence,

$$\int |f_Y|^2 d\mu_Y = |a|^4 + 4|a|^2|b|^2 + |b|^4 = 1 + 2|a|^2|b|^2.$$

This is not unity unless a or b equals zero. Hence in general, f is not an amplitude density for Y and $Y \notin \mathscr{A}(\Omega, f)$. For more discussion on this problem we refer the reader to [6].

We have considered the case in which a coin is tossed a fixed finite number of times. Now suppose the coin is tossed an arbitrary number of times. Such situations frequently occur; for example, a coin might to tossed until a head appears. In classical probability theory, one constructs the sample space $\Omega=\{0, 1\}^{\mathbb{N}}$. This presents no problem since we can form the σ-algebra on Ω generated by the cylinder sets $\mathscr{C}$ and construct the natural probability measure from its values on $\mathscr{C}$. However, in quantum probability theory we must define an amplitude density $f: \Omega \to \mathbb{C}$ in order to compute probabilities. There appears to be no way of defining such an f that extends our definition from the case of n tosses. We can circumvent this difficulty by altering our definition of Ω and X_j, $j=1, 2, \ldots$ But now X_j "destroys" the coin when this measurement is executed and measurements interfere with later measurements. The framework then has a resemblence to a simplified version of quantum field theory.

Define the sample space

$$\Omega = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$$

and define the amplitude density $f: \Omega \to \mathbb{C}$ as before. We next define $X_j: \Omega \to \{0, 1\}$, $j=1, 2, \ldots$, as follows

$$X_j(x_1, \ldots, x_n) = \begin{cases} x_j & \text{if } n \geqq j \\ 0 & \text{if } n < j. \end{cases}$$

Again, place the counting measure on $R(X_j) = \{0, 1\}$. On the fiber $X_j^{-1}(x)$, let $\Sigma_j^x = 2^{X_j^{-1}(x)}$ and for $\omega \in X_j^{-1}(x)$ define

$$\mu_j^x(\omega) = \begin{cases} 1 & \text{if } \omega \in \{0, 1\}^j \\ 0 & \text{otherwise} \end{cases}.$$

Then $(X_j^{-1}(x), \Sigma_j^x, \mu_j^x)$ is a measure space and with this structure, X_j becomes a measurement, $j=1, 2, \ldots$ As before, $f_j(0)=a$, $f_j(1)=b$ and hence $X_j \in \mathcal{A}(\Omega, f)$, $j=1, 2, \ldots$, and it is easy to show that $(X_j)_{j \in \mathbb{N}}$ is a QSP. Moreover, the distributions $P_j(0)=|a|^2$, $P_j(1)=|b|^2$ are the same as before so the X_j are identically distributed.

Let $j, k \in \mathbb{N}$ with $j<k$. Then

$$\begin{aligned} f_k(X_j=0)(0) &= a^2, & f_k(X_j=0)(1) &= ab \\ f_k(X_j=1)(0) &= ab, & f_k(X_j=1)(1) &= b^2 \\ f_j(X_k=0)(0) &= a, & f_j(X_k=0)(1) &= b \\ f_j(X_k=1)(0) &= 0, & f_j(X_k=1)(1) &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} P_k(X_j=0) &= |a|^2 = P_j(0) \\ P_k(X_j=1) &= |b|^2 = P_j(1). \end{aligned}$$

so X_k does not interfere with X_j. However,

$$\begin{aligned} P_j(X_k=0) &= |a|^2 + |b|^2 = 1 \\ P_j(X_k=1) &= 0. \end{aligned}$$

If $a \neq 1$, this does not agree with the distribution of X_k, so X_j interferes with X_k. In fact, X_j "destroys" the coin since if X_j is executed, then all subsequent measurements at later tosses will give the value zero (heads). Just as before, X_k is independent of X_j. Moreover, X_j is independent of

X_k since

$$\begin{aligned}
P_j(X_j=0,\ X_k=1)&=|f_j(X_k=1)(0)|^2=0=P_j(0)P_j(X_k=1)\\
P_j(X_j=0,\ X_k=0)&=|f_j(X_k=0)(0)|^2=|a|^2=P_j(0)P_j(X_k=0)\\
P_j(X_j=1,\ X_k=1)&=|f_j(X_k=1)(1)|^2=0=P_j(1)P_j(X_k=1)\\
P_j(X_j=1,\ X_k=0)&=|f_j(X_k=0)(1)|^2=|b|^2=P_j(1)P_j(X_k=0).
\end{aligned}$$

Finally, as before $(X_j)_{j\in\mathbb{N}}$ is Markov.

Example 2 (Three-Sided Quantum Coin). – Let $a, b, c\in\mathbb{C}$ satisfy $a+b+c=1$, $|a|^2+|b|^2+|c|^2=1$. Let $n\in\mathbb{N}$ and let $\Omega=\{0, 1, 2\}^n$. For $\omega=(x_1, \ldots, x_n)\in\Omega$, define $X_j(\omega)=x_j$, $j=1, \ldots, n$. Placing counting measures on the range and fibers of X_j, we see that X_j is a measurement, $j=1, \ldots, n$. Define the amplitude density $f:\Omega\to\mathbb{C}$ by $f(\omega)=a^{j_0}b^{j_1}c^{j_2}$ where j_k is the number of k's in the sequence ω. The wave functions f_j, $j=1, \ldots, n$, have the values

$$\begin{aligned}
f_j(0)&=a\sum\left\{\binom{n-1}{j_1 j_2 j_3}a^{j_1}b^{j_2}c^{j_3}:j_1+j_2+j_3=n-1\right\}\\
&=a(a+b+c)^{n-1}=a\\
f_j(1)&=b(a+b+c)^{n-1}=b\\
f_j(2)&=c(a+b+c)^{n-1}=c.
\end{aligned}$$

Since

$$|f_j(0)|^2+|f_j(1)|^2+|f_j(2)|^2=|a|^2+|b|^2+|c|^2=1$$

$X_j\in\mathscr{A}(\Omega, f)$, $j=1, \ldots, n$, and it is clear that $(X_j)_{j=1}^n$ is a QSP. The distribution of X_j is given by

$$\begin{aligned}
P_j(0)&=|f_j(0)|^2=|a|^2\\
P_j(1)&=|f_j(1)|^2=|b|^2\\
P_j(2)&=|f_j(2)|^2=|c|^2.
\end{aligned}$$

Thus, the X_j's are identically distributed and as in Example 1 they are mutually independent and form a QMP.

However, unlike Example 1, the X_j's interfere with each other. To show this, let $j, k\in\{1, \ldots, n\}$ with $j\neq k$. Then

$$\begin{aligned}
f_j(X_k\in\{0, 1\})(0)&=a(a+b)\\
f_j(X_k\in\{0, 1\})(1)&=b(a+b)\\
f_j(X_k\in\{0, 1\})(2)&=c(a+b).
\end{aligned}$$

Hence,

$$P_j(X_k\in\{0, 1\})=|a+b|^2.$$

In general, this does not equal

$$P_k(\{0, 1\})=|a|^2+|b|^2.$$

For example, if we let

$$a=\frac{1}{3}+\frac{i}{\sqrt{3}}, \qquad b=\frac{1}{3}-\frac{i}{\sqrt{3}}, \qquad c=\frac{1}{3}$$

then

$$|a+b|^2=\frac{4}{9}\neq\frac{8}{9}=|a|^2+|b|^2.$$

As in Example 1, we can form the infinite QMP $(X_j)_{j\in\mathbb{N}}$.

Example 3 (Discrete Quantum Mechanics). – This example is a slight generalization of a model for discrete quantum mechanics considered in [5]. Since this model has been previously studied, we refer the reader to the literature for motivation and further details.

Let S be a nonempty set which we interpret as a set of "states" that a quantum particle can occupy. A function $K_1 : S\times S\to\mathbb{C}$ is a *stochastic one-step transition amplitude* if for every $s_1, s_2\in S$ we have

$$\sum_s K_1(s_1, s)\bar{K}_1(s_2, s)=\sum_s K_1(s_1, s)\bar{K}_1(s_2, s)=\delta_{s_1 s_2}$$

and

$$\sum_s K_1(s_1, s)=1$$

where the summations converge absolutely. We denote the set of all such functions K_1 by T (S).

For $n\in\mathbb{N}$, an *n-path* is an $(n+1)$-tuple $(s_0, s_1, \ldots, s_n)\in S^{n+1}$. Let $\mathscr{P}_n(S)$ denote the set of n-paths in S and form the sample space $\Omega=\mathscr{P}_n(S)$. Let $f_0\in l^2(S)$ be a unit vector representing the initial distribution for a quantum particle. For $K_1\in T(S)$ and $\omega=(s_0, s_1, \ldots, s_n)\in\Omega$, define the amplitude density

$$f(\omega)=f_0(s_0)K_1(s_0, s_1)\ldots K(s_{n-1}, s_n).$$

For $j=0, 1, \ldots, n$ define $X_j : \Omega\to S$ by

$$X_j(s_0, s_1, \ldots, s_n)=s_j.$$

Let μ_j, μ_j^s be the counting measures on S and $X_j^{-1}(s)$, respectively. Equipped with this structure, X_j is a measurement, $j=0, 1, \ldots, n$.

For $j\in\mathbb{N}$ with $1\leqq j\leqq n$, define $K_j : S\times S\to\mathbb{C}$ by

$$K_j(s_0, s)=\sum\{K_1(s_0, s_1)K_1(s_1, s_2)\ldots K_1(s_{j-1}, s) : (s_0, \ldots, s)\in\mathscr{P}_j(S)\}$$

and define $K_0(s_0, s)=\delta_{s_0 s}$. We interpret $K_j(s_0, s)$ as the conditional amplitude that a particle is at s at time j given that it was at s_0 at time 0. It can be shown that $K_j(s_0, s)$ satisfies the Chapman-Kolmogorov equation

$$K_j(s_0, s)=\sum_{s'}K_m(s_0, s')K_{j-m}(s', s), \qquad m\leqq j. \tag{3.1}$$

The wave function for X_j becomes

$$f_j(s)=\sum\{f(\omega) : X_j(\omega)=s\}=\sum_{s_0} f_0(s_0) K_j(s_0, s).$$

If we define the linear operator $U : l^2(S) \to l^2(S)$ by

$$U g(s)=\sum_{s_0} K_1(s_0, s) g(s_0)$$

then it can be shown that U is unitary. Now from (3.1) we have

$$K_j(s_0, s)=\sum_{s'} K_{j-1}(s_0, s') K_1(s', s)=UK_{j-1}(s_0, s)$$
$$= \ldots = U^j K_0(s_0, s)=(U^j \delta_{s_0})(s).$$

Hence,

$$f_j(s)=\sum_{s_0} f_0(s_0)(U^j \delta_{s_0})(s)=(U^j \sum_{s_0} f_0(s_0) \delta_{s_0})(s)=(U^j f_0)(s).$$

Since U is unitary and $\|f_0\|=1$ we conclude that $\|f_j\|=1$. Hence, $X_j \in \mathscr{A}(\Omega, f), j=0, 1, \ldots, n$. If $j<k$, then we shall show in the next section that X_k does not interfere with X_j; however, simple examples show that X_j can interfere with X_k. Moreover, X_j and X_k need not be independent and $(X_j)_{j=0}^n$ is a quasi-discrete QMP [5].

This process can be extended to an infinite process $(X_j)_{j=0}^\infty$ as in Example 1. Define the sample space

$$\Omega = \bigcup_{n \in \mathbb{N}} \mathscr{P}_n(S)$$

and define the amplitude density $f : \Omega \to \mathbb{C}$ as before. Next define $X_j : \Omega \to S, j=0, 1, \ldots,$ as follows:

$$X_j(s_0, s_1, \ldots, s_n)=\begin{cases} s_j & \text{if } n \geqq j \\ s_0 & \text{if } n<j. \end{cases}$$

Place the counting measure on $R(X_j)$. On the fiber $X_j^{-1}(s)$ let $\Sigma_j^s = 2^{X_j^{-1}(s)}$ and for $\omega \in X_j^{-1}(s)$ define

$$\mu_j^s(\omega)=\begin{cases} 1 & \text{if } \omega \in \mathscr{P}_j(S) \\ 0 & \text{otherwise.} \end{cases}$$

Then $(X_j^\infty)_{j=0}$ becomes a QMP that retains many of the properties of the process $(X_j)_{j=0}^n$.

We now present a concrete model for discrete quantum mechanics in two dimensions. Let a and m be relatively prime positive integers with m even. Let α be an angle with radian measure $2\pi/m$ and let $k_0, k_1, \ldots, k_{m-1}$ be unit vectors in $\mathbb{R}^2$ such that each forms an angle α with its predecessor. Let $V \subseteq \mathbb{R}^2$ denote the set of points in $\mathbb{R}^2$ of the form $q=\Sigma e_j, e_j \in \{k_0, \ldots, k_{m-1}\}$. We think of V as a discrete configuration space and form a discrete "phase space"

$$S=\{(q, k_j) : q\in V, j=0, \dots, m-1\}.$$

We define the *discrete Feynman one-step transition amplitude* K_1 : $S\times S\to\mathbb{C}$

$$K_1((q, k_r), (q+k_r, k_t))=m^{-1/2}\exp\left[\frac{ia\pi(t-r)^2}{m}\right]$$

and K_1 is zero, otherwise. It can be shown that a constant multiple of K_1 of modulus 1 is contained in $T(S)$. Since such a multiple does not affect the probabilities, we can assume that $K_1\in T(S)$.

Let $n\in\mathbb{N}$ and form the sample space $\Omega=\mathscr{P}_n(S)$. Define the amplitude density $f:\Omega\to\mathbb{C}$ and the measurements $X_j:\Omega\to S$, $j=0, \dots, n$ as before. In this case, for

$$\omega=((q_0, k_{i(0)}), \dots, (q_n, k_{i(n)}))$$

we have $X_j(\omega)=(q_j, k_{i(j)})$, $j=0, \dots, n$. From our previous work we conclude that $(X_j)_{j=0}^n$ is a QMP in $\mathscr{A}(\Omega, f)$. We now define $Q_j:\Omega\to V$ and $P_j:\Omega\to\{k_0, \dots, k_{m-1}\}$ by $Q_j(\omega)=q_j$ and $P_j(\omega)=k_{i(j)}$, $j=0, \dots, n$. It can be shown that $(Q_j)_{j=0}^n$ and $(P_j)_{j=0}^n$ are QMP's in $\mathscr{A}(\Omega, f)$. In this case, Q_j represents a position measurement and P_j a momentum measurement at the discrete time j, $j=0, \dots, n$. It is shown in [5] that K_1 is a discrete approximation to the usual free-particle continuum Feynman amplitude. Moreover, a potential term can be introduced to describe a particle under the influence of a force field [5].

Example 4 (Quantum Poisson Process). – This example shows that there exist quasi-discrete QMP's with a continuum time. Let $\tau>0$ and let $T=[0, \tau]$. For each $n\in\mathbb{N}$ let

$$\Gamma_n=\{(x_1, \dots, x_n): 0<x_1<\dots<x_n\leqq\tau\}$$

and define $\Gamma_0=\{0\}$. Let $\Omega=\bigcup_{n\in\mathbb{N}_0}\Gamma_n$ where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and for $\omega\in\Omega$ and $t\in T$ define

$$X(t)(\omega)=X_t(\omega)=|\{x_i\in\omega : x_i\leqq t\}|$$

where $|A|$ denotes the cardinality of the set A. Let $a\in\mathbb{C}$ and define the amplitude density $f:\Omega\to\mathbb{C}$ by $f(\omega)=e^{aX(\tau)(\omega)}$. For $n\in\mathbb{N}$, let λ_n denote the restriction of n-dimensional Lebesgue measure to Γ_n. Then $\nu_n=n!\,\lambda_n/\tau^n$ is a probability measure on $\mathscr{B}(\Gamma_n)$. Define $\mathscr{B}(\Gamma_0)=2^{\Gamma_0}$ and ν_0 the unique probability measure on 2^{Γ_0}. Then

$$\Sigma=\left\{\bigcup_{n=0}^{\infty}B_n : B_n\in\mathscr{B}(\Gamma_n), n\in\mathbb{N}_0\right\}$$

is clearly a σ-algebra on Ω. For each element $\bigcup B_n \in \Sigma$ define

$$\nu(\bigcup B_n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \nu_n(B_n) \tag{3.2}$$

where $\lambda > 0$ is a constant. Clearly, ν is a probability measure on (Ω, Σ). For $t \in T$, $x \in \mathbb{N}_0$, let

$$\Sigma_t^x = \{ A \cap X_t^{-1}(x) : A \in \Sigma \}.$$

Clearly, $X_t^{-1}(x) \in \Sigma$ and Σ_t^x is a σ-algebra on $X_t^{-1}(x)$. Let $\mu_t^x = \beta(t, x) \nu \,|\, \Sigma_t^x$ where

$$\beta(t, x) = (x!)^{1/2} \left| \exp\left\{ \lambda \left[\frac{(\tau - t)}{\tau} e^a - 1 \right] \right\} \right|^{-1} \exp\left[-\frac{1}{2} \left(\frac{t \lambda |e^a|}{\tau} \right)^2 \right].$$

For $t \in T$, let μ_t denote the counting measure on $\Sigma_t = 2^{\mathbb{N}_0}$. Then X_t is a measurement on Ω with fiber spaces $(X_t^{-1}(x), \Sigma_t^x, \mu_t^x)$ and range space $(\mathbb{N}_0, \Sigma_t, \mu_t)$.

LEMMA 3.1. — *Let* $0 < t_1 < \ldots < t_n \leqq \tau$ *and let* $x_1, \ldots, x_n$, $N \in \mathbb{N}_0$ *with* $0 \leqq x_1 \leqq \ldots \leqq x_n \leqq N$. *Then*

$$\begin{aligned} \nu[X(t_1) = x_1, \ldots, X(t_n) = x_n, X(\tau) = N] \\ = \frac{\lambda^N e^{-\lambda}}{N!\,\tau^N} \binom{N}{x_1, x_2 - x_1, \ldots, x_n - x_{n-1}, N - x_n} \\ \times t_1^{x_1} (t_2 - t_1)^{x_2 - x_1} \ldots (t_n - t_{n-1})^{x_n - x_{n-1}} (\tau - t_n)^{N - x_n}. \end{aligned}$$

Proof. — Let

$$A = \{ (y_1, \ldots, y_N) \in T^N : | \{ j : y_j \leqq t_i \} | = x_i,\ i = 1, \ldots, n \}.$$

It is easy to see that

$$\begin{aligned} \lambda_N(A) = \binom{N}{x_1, x_2 - x_1, \ldots, x_n - x_{n-1}, N - x_n} t_1^{x_1} (t_2 - t_1)^{x_2 - x_1} \ldots \\ \times (t_n - t_{n-1})^{x_n - x_{n-1}} (\tau - t_n)^{N - x_n}. \end{aligned}$$

Since the elements of Γ_N are ordered, we have

$$\lambda_N(\{ \omega \in \Gamma_N : X(t_1)(\omega) = x_1, \ldots, X(t_n)(\omega) = x_n, X(\tau)(\omega) = N \}) = \frac{\lambda_N(A)}{N!}.$$

Applying (3.2) gives the result. □

THEOREM 3.2. — *Let* $0<t_1<\ldots<t_n\leqq\tau$ *and let* $x_1, \ldots, x_n\in\mathbb{N}_0$ *with* $0\leqq x_1\leqq\ldots\leqq x_n$. *Then*

$$f_{t_n}[X(t_{n-1})=x_{n-1}, \ldots, X(t_1)=x_1](x_n) = \frac{\beta(t_n, x_n)e^{-\lambda}(\lambda e^a)^{x_n}}{\tau^{x_n}x_1!(x_2-x_1)!\ldots(x_n-x_{n-1})!}t^{x_1}(t_2-t_1)^{x_2-x_1}\ldots \times(t_n-t_{n-1})^{x_n-x_{n-1}}\exp\left[\frac{(\tau-t_n)}{\tau}\lambda e^a\right].$$

Proof. — Applying Lemma 3.1 gives

$$\begin{aligned}f_{t_n}&[X(t_{n-1})=x_{n-1}, \ldots, X(t_1)=x_1](x_n)\\ &=\sum_{N\geqq x_n}\mu_{t_n}^{x_n}[X(t_1)=x_1, \ldots, X(t_n)=x_n, X(\tau)=N]e^{aN}\\ &=\beta(t_n, x_n)\sum_{N\geqq x_n}\nu[X(t_1)=x_1, \ldots, X(t_n)=x_n, X(\tau)=N]e^{aN}\\ &=\frac{\beta(t_n, x_n)e^{-\lambda}(\lambda e^a)^{x_n}}{\tau^{x_n}x_1!(x_2-x_1)!(x_n-x_{n-1})!}t_1^{x_1}(t_2-t_1)^{x_2-x_1}\ldots(t_n-t_{n-1})^{x_n-x_{n-1}}\\ &\quad\times\sum_{N\geqq x_n}\frac{1}{(N-x_n)!}\left(\frac{\tau-t_n}{\tau}\right)^{N-x_n}\lambda^{N-x_n}e^{a(N-x_n)}\end{aligned}$$

and the result follows since the summation equals

$$\exp\left[\frac{(\tau-t_n)}{\tau}\lambda e^a\right]. \qquad \square$$

As a particular case of Theorem 3.2 we have

$$f_t(x)=\frac{\beta(t, x)e^{-\lambda}(\lambda e^a)^x}{x!}\left(\frac{t}{\tau}\right)^x\exp\left[\frac{(\tau-t)}{\tau}\lambda e^a\right]. \tag{3.3}$$

Substituting the value of $\beta(t, x)$ into (3.3) gives

$$|f_t(x)|^2=\frac{1}{x!}\left(\frac{t}{\tau}\lambda|e^a|\right)^{2x}\exp\left[-\left(\frac{t}{\tau}\lambda|e^a|\right)^2\right].$$

Hence,

$$\int|f_t|^2d\mu=\exp\left[-\left(\frac{t}{\tau}\lambda|e^a|\right)^2\right]\sum_{x=0}^{\infty}\frac{1}{x!}\left(\frac{t}{\tau}\lambda|e^a|\right)^{2x}=1.$$

We conclude that $X_t\in\mathscr{A}(\Omega, f)$, $t\in T$.

THEOREM 3.3. — $(X_t)_{t\in T}$ *is a quasi-discrete QMP.*

Proof. — To verify (2.11), applying Theorem 3.2 gives

$$f_{t_n}[x_n \mid X(t_{n-1})=x_{n-1}, \ldots, X(t_1)=x_1]$$
$$=\frac{\beta(t_n, x_n)}{\beta(t_{n-1}, x_{n-1})}\frac{(\lambda e^a)^{x_n-x_{n-1}}}{(x_n-x_{n-1})!}\left(\frac{t_n-t_{n-1}}{\tau}\right)^{x_n-x_{x-1}}\exp\left(\frac{t_{n-1}-t_n}{\tau}\lambda e^a\right).$$

Since

$$f_{t_n}[x_n \mid X(t_{n-1})=x_{n-1}]=\frac{f_{t_n}[X(t_{n-1})=x_{n-1}](x_n)}{f_{t_{n-1}}(x_{n-1})}$$

applying Theorem 3.2 and Equation (3.3) shows that these two expressions coincide. To show that the equation in (2.13) holds, let $0<s<t\leqq\tau$ and let $x\in\mathbb{N}$. Then for every $B\in\Sigma$ we have

$$f_t[B\cap X_s^{-1}(x)](y)=\beta(t, x)\sum_{N\geqq y}\nu(B\cap\{X_s=x, X_t=y, X_\tau=N\})e^{aN}.$$

Hence,

$$\begin{aligned}\int f_t[B\cap X_s^{-1}(x)](y)\,d\mu_s(x)&=\sum_{x\leqq y}f_t[B\cap X_s^{-1}(x)](y)\\&=\beta(t, y)\sum_{N\geqq y}\nu(B\cap\{X_t=y, X_\tau=N\})e^{aN}\\&=f_t(B)(y).\end{aligned}$$

It is straightforward to verify the other conditions for a quasi-discrete QMP. □

One can show, in general, that $(X_t)_{t\in T}$ is not stationary and that X_s and X_t mutually interfere.

4. QUASI-DISCRETE QMP'S

In this section, X_t, $t\in T\subseteq\mathbb{R}$, will be a quasi-discrete QMP on $\mathscr{A}(\Omega, f)$. For s, $t\in T$ with $s\leqq t$, define $F_{s,t}: R_s\times R_t\to\mathbb{C}$ by

$$F_{s,t}(x, y)=f_t(y\mid X_s=x).$$

If $f_s(x)\neq 0$, then

$$F_{s,t}(x, y)=\frac{f_t(X_s=x)(y)}{f_s(x)}$$

and $F_{s,t}(x, y)=0$, otherwise. We can apply (2.13) to compute f_t in terms of f_s for $s\leqq t$

$$f_t(y)=\int f_t(X_s=x)(y)\,d\mu_s(x)=\int f_s(x)\,F_{s,t}(x, y)\,d\mu_s(x) \qquad (4.1)$$

The *transition amplitude kernel* $K_{s,t}: R_s \times \Sigma_t \to \mathbb{C}$ for $s, t \in T$, $s \leqq t$, is given by

$$K_{s,t}(x, B) = \int_B F_{s,t}(x, y)\, d\mu_t(y).$$

It follows from (2.12) that $K_{s,t}$ exists and is finite and from (2.13) that $F_{s,t}$ is measurable in both variables. In analogy with a Markov kernel, we see that $K_{s,t}(x, .)$ is a bounded complex measure on Σ_t and $K_{s,t}(., B)$ is measurable on R_s. Moreover, $K_{s,t}(x, .) \ll \mu_t$ and

$$\frac{dK_{s,t}(x, .)}{d\mu_t}(y) = F_{s,t}(x, y).$$

We now prove a Chapman-Kolmogorov theorem in this context.

THEOREM 4.1. – *For* $s, u, t \in T$, $s \leqq u \leqq t$, *and* $x \in R_s$, $z \in R_t$, $B \in \Sigma_t$ *we have*

$$F_{s,t}(x, z) = \int F_{u,t}(y, z) F_{s,u}(x, y)\, d\mu_u(y) = \int F_{u,t}(y, z) K_{s,u}(x, dy)$$

$$K_{s,t}(x, B) = \int K_{u,t}(y, B) K_{s,u}(x, dy).$$

Proof. – If $f_s(x) = 0$, the equalities clearly hold so assume $f_s(x) \neq 0$. Applying (2.11) and (2.13) give

$$\begin{aligned}
\int F_{u,t}(y, z) K_{s,u}(x, dy) &= \int F_{u,t}(y, z) F_{s,u}(x, y)\, d\mu_u(y) \\
&= \int f_t(z \mid X_u = y) f_u(y \mid X_s = x)\, d\mu_u(y) \\
&= \int f_t(z \mid X_u = y, X_s = x) f_u(y \mid X_s = x)\, d\mu_u(y) \\
&= \int \frac{f_t(X_u = y, X_s = x)(z)}{f_u(X_s = x)(y)} \frac{f_u(X_s = x)(y)}{f_s(x)}\, d\mu_u(y) \\
&= \frac{1}{f_s(x)} \int f_t(X_u = y, X_s = x)(z)\, d\mu_u(y) \\
&= \frac{1}{f_s(x)} f_t(X_s = x)(z) = F_{s,t}(x, z).
\end{aligned}$$

The second equality follows upon integrating the first. □

We now assume that the range spaces all coincide

$$(R_t, \Sigma_t, \mu_t) = (R, \Sigma, \mu), \qquad t \in T$$

a condition that frequently holds, and let $H = L^2(R, \Sigma, \mu)$. We then call $(X_t)_{t \in T}$ *stationary* if $F_{s+u, t+u} = F_{s, t}$ whenever, $s, t, s+u, t+u \in T$, $s \leqq t$. Now assume that $(X_t)_{t \in T}$ is stationary and $T = [0, a]$, $0 < a < \infty$, or $T = [0, \infty)$. We then define $F_t : R \times R \to \mathbb{C}$ by $F_t = F_{0, t}$, $t \in T$. We then have $F_{s, t} = F_{t-s}$. Similarly, we define $K_t : R \times \Sigma \to \mathbb{C}$ by $K_t = K_{0, t}$ so $K_{s, t} = K_{t-s}$. Then

$$K_t(x, B) = \int_B F_t(x, y)\, d\mu(y).$$

We can now write (4.1) in the form

$$f_t(y) = \int f_0(x)\, F_1(x, y)\, d\mu(x). \tag{4.2}$$

Moreover, by letting $s = 0$ and replacing t by $s + t$ and u by s, the Chapman-Kolmogorov equations become

$$F_{s+t}(x, z) = \int F_s(x, y)\, F_t(y, z)\, d\mu(y) \tag{4.3}$$

$$K_{s+t}(x, B) = \int K_t(y, B)\, K_s(x, dy) \tag{4.4}$$

For $t \in T$, define the map $K_t : \Sigma \times R \to \mathbb{C}$ by

$$K_t(B, y) = \int_B F_t(x, y)\, d\mu(x).$$

It follows from (2.12) and (2.13) that $K_t(B, x)$ is measurable in the second variable and is a bounded complex measure in the first variable. Moreover, $K_t(\,.\,, y) \ll \mu$ and

$$\frac{dK_t(\,.\,, y)}{d\mu}(x) = F_t(x, y).$$

We say that $(X_t)_{t \in T}$ is *unitary* if for every $t \in T$, $y \in R$, $B \in \Sigma$, with $\mu(B) < \infty$, we have

$$\int \bar{K}_t(x, B)\, F_t(x, y)\, d\mu(x) = \int \bar{K}_t(B, x)\, F_t(y, x)\, d\mu(x) = 1_B(y) \quad \text{a.e.}\,[\mu].$$

Define the linear operator $U_t : H \to H$ by

$$U_t g(y) = \int g(x)\, F_1(x, y)\, d\mu(x) = \int g(x)\, K_t(dx, y).$$

Applying (4.2) we have $f_t = U_t f_0$, $t \in T$. We now show that in this case, $t \mapsto U_t$ is a one-parameter unitary semigroup. We also show that $(X_t)_{t \in T}$ is unitary if and only if U_t is unitary for all $t \in T$.

THEOREM 4.2. — *(a) If* $(X_t)_{t\in T}$ *is unitary, then* U_t *is a unitary operator,* $t\in T$, *and* $U_{s+t}=U_s U_t$ *for all* $s, t\in T$ *with* $s+t\in T$. *(b) If* U_t *is unitary,* $t\in T$, *then so is* $(X_t)_{t\in T}$.

Proof. — (a) Suppose $(X_t)_{t\in T}$ is unitary. We first show that U_t is bounded, $t\in T$. Let $g\in H$ be a simple function. Then there exist $B_i\in\Sigma$, $i=1, \ldots, n$, with $B_i\cap B_j=0$ for $i\neq j$, $\mu(B_i)<\infty$ and $c_i\in\mathbb{C}$, $i=1, \ldots, n$, such that $g=\Sigma c_i \mathbf{1}_{B(i)}$. Then

$$\begin{aligned}\|U_t g\|^2 &= \int |U_t g(y)|^2 d\mu(y)\\ &= \int\left[\int g(x) K_t(dx, y)\right]\left[\int \bar{g}(z)\bar{K}_t(dz, y)\right]d\mu(y)\\ &= \int\left[\int g(x) F_t(x, y)\, d\mu(x)\right][\textstyle\sum \bar{c}_i \bar{K}_t(B_i, y)]\, d\mu(y)\\ &= \sum \bar{c}_i \int g(x)\left[\int \bar{K}_t(B_i, y) F_t(x, y)\, d\mu(y)\right]d\mu(x)\\ &= \sum \bar{c}_i \int g(x) \mathbf{1}_{B(i)}(x)\, d\mu(x) = \sum |c_i|^2 \mu(B_i) = \|g\|^2.\end{aligned}$$

Hence, U_t restricted to the dense subspace S of simple functions has norm 1. Thus, this restriction has a unique bounded linear extension $\hat{U}_t$ to H of norm 1. Let $g\in H$ be arbitrary. Then there exists a sequence $g_i\in S$ such that $|g_i(x)|\leqq|g(x)|$ for all $x\in R$ and $g_i\to g$ in norm. It follows that there exists a subsequence which we also denote by g_i such that $g_i\to g$ a. e. $[\mu]$. Since g, $F_t(., y)\in H$ we have $g F_t(., y)\in L^1(R, \Sigma, \mu)$ and moreover,

$$|g_i(x) F_t(x, y)|\leqq|g(x) F_1(x, y)|$$

for all $x\in R$. Applying the dominated convergence theorem, we have

$$\begin{aligned}\hat{U}_t g(y) &= \lim \hat{U}_t g_i(y) = \lim \int g_i(x) F_t(x, y)\, d\mu(x)\\ &= \int g(x) F_t(x, y)\, d\mu(x) = U_t g(y).\end{aligned}$$

Hence, $U_t=\hat{U}_t$ and U_t is bounded.

To show that U_t is unitary, it is clear that its adjoint U_t^* is given by

$$U_t^* g(y) = \int g(x)\bar{F}_t(y, x)\, d\mu(x) = \int g(x)\bar{K}_t(y, dx).$$

Again, if $g=\sum c_i \mathbf{1}_{B(i)}$ is a simple function as before, then

$$\begin{aligned}U_t U_t^* g(y) &= \int U_t^* g(x) F_t(x, y)\, d\mu(x)\\ &= \int\left[\int g(z) \bar{K}_t(x, dz)\right] F_t(x, y)\, d\mu(x)\\ &= \sum c_i \int \bar{K}_t(x, B_i) F_t(x, y)\, d\mu(x) = \sum c_i \mathbf{1}_{B(i)}(y) = g(y).\end{aligned}$$

Similarly,

$$\begin{aligned}U_t^* U_t g(y) &= \int U_t g(x) \bar{F}_t(y, x)\, d\mu(x)\\ &= \int\left[\int g(z) K_t(dz, x)\right] \bar{F}_t(y, x)\, d\mu(x)\\ &= \sum c_i \int K_t(B_i, x) \bar{F}_t(y, x)\, d\mu(x) = \sum c_i \mathbf{1}_{B(i)}(y) = g(y).\end{aligned}$$

Hence, $U_t U_t^* = U_t^* U_t = I$ on S so U_t is unitary.

Finally, if s, t, $s+t \in T$, then by (4.3) we have

$$\begin{aligned}U_{s+t} g(y) &= \int g(x) F_{s+t}(x, y)\, d\mu(x)\\ &= \int g(x)\left[\int F_s(x, z) F_t(z, y)\, d\mu(z)\right] d\mu(x)\\ &= \int\left[\int g(x) F_s(x, z)\, d\mu(x)\right] F_t(z, y)\, d\mu(z)\\ &= U_t U_s g(y).\end{aligned}$$

(*b*) Suppose U_t, $t \in T$ is unitary. If $B \in \Sigma$ with $\mu(B) < \infty$, then $\mathbf{1}_B \in H$. Hence,

$$\begin{aligned}\mathbf{1}_B(x) = U_t U_t^* \mathbf{1}_B(x) &= \int\left[\int \mathbf{1}_B(z) \bar{K}_t(x, dz)\right] F_t(x, y)\, d\mu(x)\\ &= \int \bar{K}_t(x, B) F_t(x, y)\, d\mu(x)\end{aligned}$$

and

$$\begin{aligned}\mathbf{1}_B(x) = U_t^* U_t \mathbf{1}_B(x) &= \int\left[\int \mathbf{1}_B(z) K_t(dz, x)\right] \bar{F}_t(y, x)\, d\mu(x)\\ &= \int K_t(B, x) \bar{F}_t(y, x)\, d\mu(x). \quad \square\end{aligned}$$

Let $(X_t)_{t \in T}$ be unitary with $T=[0, \infty)$. If we define $U_{-t}=U_t^*$, then $t \mapsto U_t$ becomes a one-parameter unitary group on $\mathbb{R}$. Notice that $F_t(x, .)-F_0(x, .) \in H$ and $\|F_t(x, .)-F_0(x, .)\|$ is measurable. We say that $(X_t)_{t \in T}$ is *continuous* if for any $\varepsilon>0$ there exists a $\delta>0$ such that $|t|<\delta$ implies

$$\int \| F_t(x, .)-F_0(x, .)\|^2 d\mu(x)<\varepsilon.$$

THEOREM 4.3. – *If $(X_t)_{t \in T}$ is continuous, unitary and stationary, then $t \mapsto U_t$ is strongly continuous.*

Proof. – We show that $t \rightarrow U_t$ is weakly continuous at 0 from which the result follows. For g, $h \in H$ we have upon applying Schwarz's inequality

$$\begin{aligned}
|\langle (U_t-I)g, h\rangle| &= \left| \iint g(x)[F_t(x, y)-F_0(x, y)]\, d\mu(x)\, \bar{h}(y)\, d\mu(y)\right| \\
&\leqq \int |g(x)| \int |F_t(x, y)-F_0(x, y)|\, |h(y)|\, d\mu(y)\, d\mu(x) \\
&\leqq \|h\| \int |g(x)|\, \|F_t(x, .)-F_0(x, .)\|\, d\mu(x) \\
&\leqq \|h\|\, \|g\| \left[\int \|F_t(x, .)-F_0(x, .)\|^2 d\mu(x)\right]^{1/2}.
\end{aligned}$$

Hence, given $\varepsilon>0$ there exists a $\delta>0$ such that $|t|<\delta$ implies

$$|\langle (U_t-I)g, h\rangle| \leqq \|h\|\, \|g\|\, \varepsilon. \quad \square$$

Under the conditions of Theorems 4.3, $t \mapsto U_t$ is a continuous one-parameter unitary group so by Stone's theorem, $U_t=e^{itH}$ for a unique self-adjoint operator H. We call such a process a *Hamiltonian process.*

THEOREM 4.4. – *If $(X_t)_{t \in T}$ is stationary and unitary, then X_t does not interfere with X_s for $s \leqq t$.*

Proof. – *For* $B \in \Sigma$ *we have by* (2.13) *and the unitary of* U_{t-s} *that*

$$\begin{aligned} P_t(X_s \in B) &= \int |f_t(X_s \in B)(y)|^2 \, d\mu(y) \\ &= \int \left| \int f_t[X_s^{-1}(B) \cap X_s^{-1}(x)](y) \, d\mu(x) \right|^2 d\mu(y) \\ &= \int \left| \int \mathbf{1}_B(x) f_t(X_s = x)(y) \, d\mu(x) \right|^2 d\mu(y) \\ &= \int \left| \int \mathbf{1}_B(x) f_s(x) F_{t-s}(x, y) \, d\mu(x) \right|^2 d\mu(y) \\ &= \int |U_{t-s} \mathbf{1}_B f_s(y)|^2 \, d\mu(y) = \| U_{t-s} \mathbf{1}_B f_s \|^2 \\ &= \| \mathbf{1}_B f_s \|^2 = \int_B |f_s|^2 \, d\mu = P_s(B). \quad \square \end{aligned}$$

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