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Hamiltonians with zero-range interactions supported by a Brownian path

by

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ABSTRACT. — Hamiltonians with zero-range interactions supported by some closed set Ω of Lebesgue measure zero are considered. The approach we use to construct such operators is a limiting procedure with ultraviolet cut-off in Fourier-representation. We consider two different cases: without renormalization and with renormalization procedure. Conditions sufficient to the existence of nontrivial hamiltonians with zero-range interactions are given. The obtained results are applied to the case where Ω is a Brownian path in $\mathbb{R}^m$, $m=3$, 4 and 5.

RÉSUMÉ. — On considère les hamiltoniens avec l'interaction de radius zéro, concentrée sur l'ensemble fermée de la mesure zéro Ω. L'approche qu'on utilise pour construire ces opérateurs c'est une procédure limite avec le couplage ultraviolet dans la représentation de Fourier. On considère deux cas différent : sans renormalisation et avec une procédure de renormalisation. Les conditions suffisantes à l'existence des hamiltoniens non triviaux sont données. Les résultats obtenus sont appliqués au cas où Ω est une trajectoire de la particule brownienne dans $\mathbb{R}^m$, $m=3$, 4, et 5.

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INTRODUCTION

Study of hamiltonians describing zero-range interactions is the field where research had been doing for a long time both by physicists and by mathematicians. We shall not give here the detailled review of known results, referring to [1].

Let us only note that most of papers are devoted to the case where interactions are concentrated in a discrete (finite or countable) set of points. One can write formally such hamiltonians in the following form:

$$H = -\Delta_x + \sum_j C_j \delta(x - y_j), \qquad y_j \in \mathbb{R}^m.$$

At the same time little attention has been devoted to the continuous case:

$$H = -\Delta_x + \int_\Omega dm(y)\, \delta(x-y)\, F(y) \tag{1}$$

where Ω is some closed set in $\mathbb{R}^m$ of Lebesgue measure 0, dm- some measure on Ω. In particular, Ω can be a line or a surface in $\mathbb{R}^m$, *see* [2]-[6]. Except of [8] where a nonstandard analysis is applied, the method usually used to construct self-adjoint hamiltonians, corresponding to (1) is based on the theory of extensions of symmetric operators in Hilbert space. The symmetric operator to be extended is given by $-\Delta \mid C_0^\infty(\mathbb{R}^m \setminus \Omega)$. The known examples of applications of the theory of extensions to this case are valid only for smooth manifolds Ω. At the same time construction of self-adjoint operators in the case where Ω is not smooth is of interest from different points of view, both mathematical and physical. In particular, it is important in the theory of field (*see* [8]) to construct hamiltonians of type (1), where Ω is a Brownian path in $\mathbb{R}^m$.

We shall describe a quite general scheme which will allow us to construct self-adjoint hamiltonians with zero-range interaction supported by a closed set with Lebesgue measure 0 which can be parametrised. Namely, let $\Omega \equiv \omega(X)$, where $\omega : X \to \mathbb{R}^m$, X is a space with measure μ. We shall assume that the map ω is measurable, but not necessarily smooth or continuous. The approach we use is a limiting procedure with ultraviolet cut- off in Fourier-representation. This idea was first used by Berezin and Faddeev in [9] in the case where Ω is a single point. Later it was applied by Grossmann, Hoegh-Krohn and Mebkhout [7] to the N-point case. We may consider our method as a continuous version of [9], [7].

The obtained results can be applied to the case where Ω is a Brownian path in $\mathbb{R}^m$, where $m = 3, 4, 5$. We get a standard proof of corresponding results of [8]. However, we not only prove the existence of nontrivial self-adjoint operators (1) but also get an explicit formula for their resolvents. This formula can be considered as a concrete realisation of abstract Krein's formula [10]. It allows us to begin studying of spectral properties of these operators.

1. CONSTRUCTION OF ZERO-RANGE INTERACTIONS WITHOUT RENORMALIZATION

Let Ω be a closed subset of Lebesgue measure 0 in $\mathbb{R}^m$, ω- a Borel measurable map from X to $\mathbb{R}^m$ such that $\Omega=\omega(\mathrm{X})$, X being a space with finite measure μ. Let us try to give a rigourous meaning to the expression (1). We can write formally

$$\mathrm{H}=-\Delta_x+\int_{\mathrm{X}} d\mu(t)\,\delta(x-\omega(t))\,\mathrm{F}(t),$$

where F is some real function on X. [One can take in particular $\mathrm{F}(\omega(t))$, where $\mathrm{F}(x)$ is a real function on $\mathbb{R}^m$]. One can construct an approximation of δ-function by a sequence of separable interactions with ultraviolet cut-off:

$$\delta(x-y)\approx\lim_{n\to\infty}(\ ,k_n(y))\,k_n(y),$$

where in p-representation

$$k_n(y,p)=(2\pi)^{-m/2}\exp(-i(p,y))\,\chi(|p|\leqq n),\qquad y,p\in\mathbb{R}^m.$$

Let us consider a sequence of operators in $\mathrm{L}_2(\mathbb{R}^m)$, given in p-representation by

$$(\mathrm{H}_n f)(p)=p^2 f(p)+\int_{\mathrm{X}} d\mu(t)\,\mathrm{F}(t)\,e_n(t,p)\,(f,e_n(t))\equiv p^2 f(p)+(\mathrm{V}_n f)(p),\quad (2)$$

where $e_n(t,p)=(2\pi)^{-m/2}\exp(-i(p,\omega(t)))\,\chi(|p|\leqq n)$. We shall assume that F is a real measurable bounded function on X. One can easily see that operators V_n are bounded and self-adjoint in $\mathrm{L}_2(\mathbb{R}^m)$, hence H_n are self-adjoint operators with a domain

$$\mathrm{D}(\mathrm{H}_n)=\left\{f:\int_{\mathbb{R}^m} dp\,|f(p)|^2(p^2+1)^2<+\infty\right\}\quad (3)$$

Note also that

$$\lim_{n\to\infty}(\mathrm{V}_n f,g)=\int_{\mathrm{X}} d\mu(t)\,\mathrm{F}(t)\,f(\omega(t))\,\overline{g(\omega(t))}$$

for all $f,g\in\mathrm{S}(\mathbb{R}^m)$. We shall use the following well-known statement [11].

TROTTER-KATO THEOREM. – Let H_n be a sequence of self-adjoint operators in Hilbert space $\mathcal{H}$. Assume that there are two points z_+ and z_- such that $\operatorname{Im} z_+ > 0$, $\operatorname{Im} z_- < 0$ and the strong limits exist:

$$T_{+,-} = s\text{-}\lim_{n\to\infty} (H_n - z_{+,-})^{-1}.$$

Assume that $\overline{\operatorname{Ran} T_+} = \mathcal{H}$. Then there exists a self-adjoint operator H such that H_n converge to H in the strong resolvent sense.

To use this theorem we have to study resolvents of H_n. Let $\operatorname{Im} z \neq 0$, $g \in L_2(\mathbb{R}^m)$, $f_n = (H_n - z)^{-1} g \equiv R_n(z) g$. We get from (2) that

$$f_n(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \int_X d\mu(t)\, e_n(t, p)\, F(t)\, (f_n, e_n(t)) \tag{4}$$

Deduce a formula for a function $\rho_n(t) \equiv (f_n, e_n(t))$. By taking the inner product with $e_n(s)$ in $L_2(\mathbb{R}^m)$ we get

$$\rho_n(s) = a_n(s, z) - \int_X d\mu(t)\, L_n(s, t, z)\, F(t)\, \rho_n(t), \tag{5}$$

where

$$a_n(s, z) = \int_{\mathbb{R}^m} dp\, \frac{g(p)\, \overline{e_n(s, p)}}{p^2 - z},$$

$$L_n(s, t, z) = \int_{\mathbb{R}^m} dp\, \frac{\exp(i(p, \omega(s) - \omega(t)))}{p^2 - z} (2\pi)^{-m} \chi(|p| \leqq n)$$

Let us investigate the behaviour of the kernels L_n as $n \to \infty$. Writing the integral in polar coordinates we get

$$L_n(s, t, z) = \alpha_m \int_0^n dR\, \frac{R^{m-1}}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta},$$

where $\Delta = \Delta(s, t) = |\omega(s) - \omega(t)|$, α_m is some positive constant, $m \geqq 2$ and for $\Delta = 0$ we define $\sin(R\Delta)/(R\Delta) = 1$. Let $\Delta > 0$, $z \in \mathbb{C} \setminus [0, +\infty)$. If $m = 2$ then the limit exixts

$$L(s, t, z) \equiv \lim_{n\to\infty} L_n(s, t, z) = \alpha_2 \int_0^\infty dR\, \frac{R}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta} = G_0^{(2)}(\omega(s) - \omega(t), z),$$

where $G_0^{(m)}$ is a Green function of $-\Delta_x$ in $L_2(\mathbb{R}^m)$. From $|\sin(x)/x| \leqq C_\delta |x|^{-\delta}$, $0 < \delta < 1$, we get the following estimate:

$$|L_n(s, t, z)| \leqq C(\delta, z)\, \Delta^{-\delta} \tag{6}$$

uniformly on s, t, n. In the case $m=3$

$$L_n(s, t, z) = \frac{\alpha_3}{\Delta} \int_0^{n\Delta} du \frac{\sin u}{u} + \alpha_3 z \int_0^n dR \frac{1}{R^2 - z} \frac{\sin(R\Delta)}{R\Delta},$$

so

$$|L_n(s, t, z)| \leqq C_1 \Delta^{-1} + C_2(z), \tag{7}$$

where C_1, C_2 do not depend on s, t, n and

$$L(s, t, z) \equiv \lim_{n \to \infty} L_n(s, t, z) = G_0^{(3)}(\omega(s) - \omega(t), z) = \frac{\exp(iz^{1/2}\Delta)}{4\pi\Delta}.$$

If $m \geqq 4$ then the kernels $L_n(s, t, z)$ do not have pointwise limits as $n \to \infty$. Therefore we shall assume in the part 1 that $m=2$ or $m=3$. We shall assume also that the following condition is satisfied. If $m=2$ then

(A1) For some $\delta: 0 < \delta < 1/2$

$$\int_{X^2} d\mu(t)\, d\mu(s)\, |\omega(s) - \omega(t)|^{-2\delta} < +\infty.$$

If $m=3$ then

(A2) The function

$$N(t, u) = \int_X d\mu(s)(1 + |\omega(s) - \omega(u)|^{-1})(1 + |\omega(s) - \omega(t)|^{-1})$$

belongs to $L_2(X^2)$.

LEMMA 1. – *Let* F *be a bounded real function. Consider a sequence of integral operators* $B_n(z)$ *in* $L_2(X)$ *with the kernels*

$$B_n(s, t, z) = L_n(s, t, z) F(t), \qquad z \in \mathbb{C} \setminus [0, +\infty)$$

The following statements hold:

1. *For all* $n \in N$, $B_n(z)$ *are Hilbert-Schmidt operators.*
2. $B_n(z)$ *converge as* $n \to \infty$ *in the operator norm to the operators* $B(z)$ *with the kernels*

$$B(s, t, z) = L(s, t, z) F(t) = G_0^{(m)}(\omega(s) - \omega(t), z) F(t)$$

3. $B(z)$ *are compact in* $L_2(X)$.
4. $\|B(z)\| \to 0$ *as* $\operatorname{Im}(z^{1/2}) \to +\infty$.
5. $B(z)$ *is an analytic operator-valued function on* $\mathbb{C} \setminus [0, +\infty)$.

Proof. – The statement 1 follows from boundedness of $B_n(s, t, z)$ and from finiteness of measure μ. Define $\Delta L_n(s, t, z) = L_n(s, t, z) - L(s, t, z)$

and estimate the norm in $L_2(X)$:

$$\| B_n(z) h - B(z) h \|^2 \leqq \int_X d\mu(s) \int_X d\mu(t) | h(t) | | F(t) | | \Delta L_n(s, t, z) |) \times \int_X d\mu(u) | h(u) | | F(u) | | \Delta L_n(s, u, z) | \leqq C \int_{X^2} d\mu(t) d\mu(u) | h(t) | | h(u) | K_n(t, u, z), \quad (8)$$

where

$$K_n(t, u, z) = \int_X d\mu(s) | \Delta L_n(s, u, z) | | \Delta L_n(s, t, z) |.$$

From (8) we get

$$\| B_n(z) - B(z) \|^2 \leqq C \| K_n(z) \|_{\mathscr{Y}} \quad (9)$$

where $\mathscr{Y} \equiv \mathscr{B}(L_2(X^2))$.

Let us show that $\| K_n(z) \|_{\mathscr{Y}} \to 0$ as $n \to \infty$.

1. Let $m=2$. Then

$$K_n^2(t, u, z) \leqq \int_X d\mu(s) | \Delta L_n(s, t, z) |^2 \int_X d\mu(s) | \Delta L_n(s, u, z) |^2$$

Hence

$$\| K_n(z) \|_{\mathscr{Y}} \leqq C \int_{X^2} d\mu(s) d\mu(t) | \Delta L_n(s, t, z) |^2 \quad (10)$$

It follows from condition A1 that $\Delta = | \omega(s) - \omega(t) | > 0$ for a.e. (s, t). Hence,

$$\lim_{n \to \infty} | \Delta L_n(s, t, z) |^2 = 0$$

for a.e. $(s, t) \in X^2$. From (6) by dominated convergence theorem we get

$$\lim_{n \to \infty} \| K_n(z) \|_{\mathscr{Y}} = 0.$$

2. Let $m=3$. From A2 we get $N(t, u) < +\infty$ for a.e. (t, u), and

$$| \omega(s) - \omega(u) | . | \omega(s) - \omega(t) | > 0$$

for a.e. $s \in X$, so

$$\lim_{n \to \infty} | \Delta L_n(s, t, z) | . | \Delta L_n(s, u, z) | = 0$$

for a.e. $s \in X$. From (7) and A2 by dominated convergence theorem

$$\lim_{n \to \infty} K_n(t, u, z) = 0 \quad (11)$$

for a. e. (t, u) and

$$|K_n(t, u, z)| \leqq C(z) N(t, u) \tag{12}$$

where $C(z)$ does not depend on t, u, n. Applying again the dominated convergence theorem, we obtain from (11), (12) and (A2) that

$$\lim_{n \to \infty} \|K_n(z)\| = 0.$$

The statement 2 is proved.

The statement 3 follows directly from 1 and 2.

Let us prove 4. If $m=3$ then

$$|B(s, t, z)| \leqq C \frac{\exp(-\operatorname{Im}(z^{1/2})\Delta)}{\Delta} \tag{13}$$

From (13) and A2 we get 4. If $m=2$ then

$$|B(s, t, z)| \leqq \frac{C(\delta)}{\Delta^\delta} \int_0^\infty dR \frac{R^{1-\delta}}{|R^2 - z|} \tag{14}$$

One can show that the integral in (14) tends to 0 as $\operatorname{Im} z^{1/2} \to \infty$, hence, 4 is true.

To prove 5 it is sufficient to show that the following limit exists for all $z \in \mathbb{C} \setminus [0, +\infty)$ in operator norm:

$$B'(z) = \lim_{z' \to z} \frac{B(z) - B(z')}{z - z'}$$

This statement can be proved quite asily, so we shall omit it. The proof of lemma is completed.

LEMMA 2. — *Let* F *be a real bounded function. Then for all* z: $\operatorname{Im} z \neq 0$ *the bounded operators* $(I + B(z))^{-1}$ *exist.*

Proof. — Operators $B(z)$ being compact, it is sufficient to show that there are no nontrivial solutions to the equation

$$f(s) = -(B(z) f)(s), \qquad f \in L_2(X). \tag{15}$$

Multiplying (15) by $\overline{f(s) F(s)}$ and integrating over X, we get

$$-(f, Ff) = (B(z) f, F f)$$

The operators $\mathbf{B}_n(z)$ converge to $B(z)$, hence

$$-(f, Ff) = \lim_{n\to\infty} (\mathbf{B}_n(z) f, Ff)$$

$$= \lim_{n\to\infty} \int_{X^2} d\mu(t)\, d\mu(s)\, \overline{F(s) f(s)}\, F(t) f(t) (2\pi)^{-m} \times \int_{\mathbb{R}^m} dp \frac{\chi(|p|\leqq n)}{p^2 - z} \exp(i, (p, \omega(s) - \omega(t)))$$

$$\equiv \lim_{n\to\infty} \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2 - z} \chi(|p|\leqq n) \equiv \lim_{n\to\infty} I_n \quad (16)$$

where

$$l(p) = (2\pi)^{-m/2} \int_X d\mu(t) F(t) f(t) \exp(-i(p, \omega(t))), \qquad l \in L_\infty(\mathbb{R}^m),$$

and the limit (16) exists.

Therefore,

$$\lim_{n\to\infty} \operatorname{Im} I_n = \operatorname{Im} z \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2} \quad (17)$$

Since F is real, it follows from (16) and (17) that

$$\int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2} = 0$$

and $l(p) = 0$ for a. e. $p \in \mathbb{R}^m$. Hence,

$$B(z) f = L_2(X) - \lim_{n\to\infty} B_n(z) f$$

$$= L_2(X) - \lim_{n\to\infty} \int_{\mathbb{R}^m} dp \frac{\exp(i(p, \omega(s)))\, l(p)}{p^2 - z} \chi(|p|\leqq n) \equiv 0 \quad (18)$$

It follows from (15) and (18) that $f = 0$. The proof is completed.

Remark. — It follows from the proof that for $F(t) \geqq 0$ (a repulsive case) the operators $(I + B(z))^{-1}$ exist for all $z = -a$, $a > 0$.

Let us return to the expression for resolvents $R_n(z)$. Fix $z : \operatorname{Im} z \neq 0$. By Lemma 1 and Lemma 2 for $n > N(z)$ operators $(I + B_n(z))^{-1}$ exist and

$$\lim_{n\to\infty} \| (I + B_n(z))^{-1} - (I + B(z))^{-1} \| = 0 \quad (19)$$

From (4) and (5) we get for all $g \in L_2(\mathbb{R}^m)$

$$(R_n(z) g)(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \times \int_X d\mu(t)\, e_n(t, p)\, F(t) ((I + B_n(z))^{-1} a_n(z))(t), \quad (20)$$

where

$$a_n(s, z)=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dp \frac{\exp(i(p, \omega(s)))g(p)}{p^2-z} \chi(|p|\leqq n).$$

Obviously,

$$\lim_{n\to\infty}\int_{\mathbb{R}^m} dp \frac{|e_n(t, p)-(2\pi)^{-m/2}\exp(-i(p, \omega(t)))|^2}{|p^2-z|^2}$$
$$=\lim_{n\to\infty}\int_{\mathbb{R}^m} dp \frac{1}{|p^2-z|^2}\chi(|p|\geqq n)(2\pi)^{-m}=0 \quad (21)$$

uniformly on X. In a similar manner,

$$\lim_{n\to\infty}\sup_s |a_n(s, z)-a(s, z)|=0, \quad (22)$$

where

$$a(s, z)=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dp \frac{g(p)\exp(i(p, \omega(s)))}{p^2-z}$$

It follows from (20)-(22) that $R_n(z)g$ converge in $L_2(\mathbb{R}^m)$ as $n\to\infty$, and

$$T(z)g\equiv L_2(\mathbb{R}^m)-\lim_{n\to\infty} R_n(z)g=\frac{g(p)}{p^2-z}-\frac{(2\pi)^{-m/2}}{p^2-z}$$
$$\times\int_X d\mu(t)\exp(-i(p, \omega(t)))F(t)((I+B(z))^{-1}a(z))(t) \quad (23)$$

To satisfy to the conditions of Trotter-Kato theorem we should demonstrate that $\operatorname{Ran} T(z)$ is dense in $L_2(\mathbb{R}^m)$. Suppose that it is not true and for some $\varphi\neq 0$, $(\varphi, T(z)g)=0$ for all $g\in L_2(\mathbb{R}^m)$. Let $f\in C_0^\infty(\mathbb{R}^m\setminus\Omega)$ in x-represantation, $g(p)=(p^2-z)f(p)$. From (23) we get

$$0=(T(z)g, \varphi)=(f, \varphi) \quad (24)$$

since

$$a(s)=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dp \frac{g(p)\exp(i(p, \omega(s)))}{p^2-z}$$
$$=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dp\, f(p)\exp(i(p, \omega(s)))=f(\omega(s))\equiv 0.$$

Since $C_0^\infty(\mathbb{R}^m\setminus\Omega)$ is dense $L_2(\mathbb{R}^m)$, we get from (24) that $\varphi=0$.

We have proved the main result of part 1.

THEOREM 1. — *Let* ω *be a measurable map from* X *to* $\mathbb{R}^m$. Assume conditions A1 ($m=2$) *or* A2 ($m=3$). *Let* F *be a real bounded measurable function on* X. *Then there exists a self-adjoint operator* H *in* $L_2(\mathbb{R}^m)$ *such* *that* H_n *converge to* H *in the strong resolvent sence. For all* $g \in L_2(\mathbb{R}^m)$ *and* $z : \operatorname{Im} z \neq 0$ *the resolvent of* H *is given by* (23).

Having constructed the self-adjoint operator H, we can study now some of its properties.

Define $H_0 = -\Delta$, $D(H_0) = \{ f : f \in H^2(\mathbb{R}^m), f|_\Omega = 0 \}$. It is clear that H_0 is a closed symmetric operator.

LEMMA 3. — H *is a self-adjoint extension of* H_0.

Proof. — Let $f \in D(H_0)$, $g(p) = (p^2 - z) f(p)$, $\operatorname{Im} z \neq 0$. From (23) we have $R(z) g = f$, since $a(s) = f(\omega(s)) = 0$. Hence, $f \in D(H)$ and $(H - z) f = g = (H_0 - z) f$. The proof is completed.

It is important that H be nontrivial extension of H_0, that is, $H \neq -\Delta | H^2(\mathbb{R}^m)$. We shall denote by $K_R(x_0)$ a ball of radius R centered at x_0. Define

$$\beta_R(x_0) = \{ t : t \in X, \ \omega(t) \in K_R(x_0) \}$$

LEMMA 4. — *Assume that two conditions are satisfied:*

1. $\mu(\beta_R(x_0)) > 0$ *for some* $R > 0$, $x_0 \in \mathbb{R}^m$.

2. *For some* $M > R$ *the function* $F(t)$ *is strictly positive (or negative) for a.e.* $t \in \beta_M(x_0)$.

Then H *is a nontrivial extension of* H_0.

Proof. — Suppose that $H = -\Delta$. Let

$$\varphi \in H^2(\mathbb{R}^m), \quad \operatorname{Im} z \neq 0, \quad g(p) = (p^2 - z)\varphi(p), \quad f(p) = (p^2 - \bar{z})\varphi(p).$$

By (23) and $(R(z) g, f) = (R_0(z) g, f)$ we get

$$\int_X d\mu(t) \overline{\varphi(\omega(t))} F(t) (((I + B(z))^{-1} \varphi(\omega(.)))(t) = 0 \tag{25}$$

Define $h(s) \equiv ((I + B(z))^{-1} \varphi(\omega(.)))(s)$. It follows from (25) that

$$\int_X d\mu(t) F(t) h(t) \overline{((I + B(z)) h)(t)} = 0,$$

and

$$\int_X d\mu(t) F(t) |h(t)|^2 = -(F h, B(z) h) \tag{26}$$

One obtain from (26) and the proof of lemma 2 that $B(z) h = 0$, hence $\varphi(\omega(s)) = ((I + B(z)) h)(s) = h(s)$ and by (26)

$$\int_X d\mu(t) F(t) |\varphi(\omega(t))|^2 = 0 \tag{27}$$

Let us choose $\varphi(x)$ such that $\varphi\in C_0^\infty(\mathbb{R}^m)$, $\varphi(x)\equiv 1$ for $x\in K_R(x_0)$ and $\varphi(x)\equiv 0$ for $x\in\mathbb{R}^m\setminus K_M(x_0)$. From (27) and condition 2 of lemma we get

$$\varphi(\omega(t))=0$$

for a.e. $t\in\beta_M(x_0)$. On the other hand, $\varphi(\omega(t))=1$ for $t\in\beta_R(x_0)$, and by condigion 1 $\mu(\beta_R(x_0))>0$. This contradiction completes the proof.

Example. – If $F>0$ (or $F<0$) on X, then the statement of lemma holds.

LEMMA 5. – *The operator* H *is bounded from below.*

Proof. – By lemma 1 we have $\|B(z)\|\to 0$ as $\mathrm{Im}(z^{1/2})\to+\infty$. Hence, for some $A>0$ the inverse operators $(I+B(z))^{-1}$ exist and

$$\|(I+B(z))^{-1}\|\leqq 2 \tag{28}$$

for all $z\in V_A=\{z\colon \mathrm{Re}\,z\leqq -A,\ 0<|\mathrm{Im}\,z|\leqq 1\}$. One can easily show from (23) and (28) that $\|R(z)g\|\leqq C$ uniformly on V_A and

$$R(-a+i0)=R(-a-i0)$$

for all $a\geqq A$. This implies $(-\infty, -A)\in\rho(H)$. The proof is completed.

LEMMA 6. – *The spectrum of* H *on* $(-\infty, 0)$ *is discrete with only possible accumulation point* $z=0$.

The proof of this lemma is identical to that in lemma 12 of part 2, so we shall omit it.

LEMMA 7. – *Assume that* $F(t)\geqq 0$ *for a.e.* $t\in X$. *Then* $(-\infty, 0)\in\rho(H)$, *that is,* $H\geqq 0$ *and there are no bound states with negative energy.*

The result follows from the remark to lemma 2.

An important property of H is locality [1].

LEMMA 8. – *Let* $f\in D(H)$ *and for some ball* $K_R f(x)=0$ *for a.e.* $x\in K_R$. *Then* $(Hf)(x)=0$ *a.e. on* K_R.

The result follows from lemma C2 of appendix C in [1].

Applications

1. Ω is a part of line in $\mathbb{R}^2$. For instance, $X=[0, 1]$, $\omega(t)=(t, 0)$. The condition A1 is equivalent to

$$\int_0^1\int_0^1 dt\,ds\,|t-s|^{-2\delta}<+\infty,$$

that is true. One can apply our results also in the case where Ω is a part of a smooth curve.

2. Ω is a part of plane in $\mathbb{R}^3$. The fact to verify is whether a function

$$N(s, t)=\int_{\mathbb{R}^2} du \frac{\chi(|u|\leqq C)}{|s-u|.|u-t|}, \qquad s, t\in\mathbb{R}^2,$$

belongs to $L_{2,\,loc}(\mathbb{R}^4)$. One can easily show that N has a logarithmic singularity at $s=t$, hence the condition A2 is satisfied.

3. Ω is a sphere in $\mathbb{R}^3$. Our results can be applied to construct hamiltonians with zero-range interactions which are not spherically symmetric.

4. Ω is a Brownian path in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $X=[0, 1]$, $\omega(t)$ be a Brownian path. For $m=2$ we have to show that

$$\Phi(\omega(.))=\int_0^1\int_0^1 dt\, ds\, |\omega(t)-\omega(s)|^{-2\delta}<+\infty \tag{29}$$

Let μ_{Wien} be a Wiener measure on $C([0, 1]; \mathbb{R}^m)$. One can easily show (*see* [8]) that

$$\int d\mu_{\text{Wien}}\, \Phi(\omega(.))<+\infty$$

for $\delta<1$. Hence, (29) is satisfied for a. e. $\omega(.)\in C([0, 1]; \mathbb{R}^m)$.

Let $m=3$. To verify condition A2, it is sufficient to show that

$$I=\int d\mu_{\text{Wien}}\, \psi(\omega(.))<+\infty \tag{30}$$

where

$$\psi(\omega(.))=\int_0^1\int_0^1 dt\, ds\left(\int_0^1 du\, |\omega(s)-\omega(u)|^{-1}|\omega(u)-\omega(t)|^{-1}\right)^2$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1 dt\, ds\, du\, dv\, \Delta(s, u)^{-1}\Delta(s, v)^{-1}\Delta(t, u)^{-1}\Delta(t, v)^{-1},$$

and $\Delta(s, t)\equiv|\omega(s)-\omega(t)|$. We shall demonstrate that (30) holds for all $m\geqq 3$. One should consider all possible orderings of points s, t, u, v. We may assume that $s<t$ and $u<v$ since points s, t and u, v are equivalent. There are 6 cases:

1. $s<t<u<v$, 2. $u<v<s<t$, 3. $s<u<t<v$, 4. $u<s<v<t$, 5. $s<u<v<t$, 6. $u<s<t<v$.

The cases 1 and 2, 3 and 4, 5 and 6 are similar (we have to do a substitution $s\to u$, $t\to v$). Finally we have 3 essentially different cases: 1, 3 and 5. Denote by I_1, I_3, I_5 corresponding contributions in I.

Case 1:

$$I_1=\int_P ds\, dt\, du\, dv\, \chi(s<t<u<v)\, J_1,$$

where $P=[0, 1]^4$,

$$J_1(s, t, u, v)=C_m \int_{(\mathbb{R}^m)^4} dx_1\, dx_2\, dx_3\, dx_4\, |x_1-x_3|^{-1} \times |x_2-x_3|^{-1} |x_1-x_4|^{-1} |x_2-x_4|^{-1} \times \frac{\exp(-(x_1^2/s+(x_2-x_1)^2/(t-s)+(x_3-x_2)^2/(u-t)+(x_4-x_3)^2/(v-u)))}{s^{m/2}(t-s)^{m/2}(u-t)^{m/2}(v-u)^{m/2}},$$

C_m is some constant. Obviously,

$$\int_0^1 ds \frac{\exp(-x^2/s)}{s^{m/2}} = C|x|^{2-m} \times \int_0^{1/x^2} d\tau\, \tau^{-m/2} \exp(-1/\tau) \leqq C_m |x|^{2-m} \exp(-x^2/2), \qquad m \geqq 3.$$

Consequently,

$$I_1 \leqq C_m \int_{(\mathbb{R}^m)^4} dx_1\, dx_2\, dx_3\, dx_4 \times \exp(-1/2(x_1{}^2+(x_2-x_1)^2+(x_3-x_2)^2+(x_4-x_3)^2)) |x_1|^{2-m} |x_2-x_1|^{2-m} |x_3-x_2|^{2-m} |x_4-x_3|^{2-m} |x_1-x_3|^{-1} |x_2-x_3|^{-1} |x_1-x_4|^{-1} |x_2-x_4|^{-1} \quad (31)$$

Let us change the variables in (31): $u=x_4-x_3$, $v=x_3-x_2$, $w=x_2-x_1$, $x=x_1$. Then

$$I_1 \leqq \int_{\mathbb{R}^m} dv\, |v|^{1-m} \exp(-v^2/2)\, L(v), \quad (32)$$

where

$$L(v)=\int_{\mathbb{R}^m} dv\, |u|^{2-m} |u+v|^{-1} \exp(-u^2/2) \times \int_{\mathbb{R}^m} dw\, |w|^{2-m} |w+v|^{-1} |u+v+w|^{-1} \exp(-w^2/2) \equiv \int_{\mathbb{R}^m} du\, |u|^{2-m} |u+v|^{-1} \exp(-u^2/2)\, M(u, v) \quad (33)$$

Obviously,

$$M(u, v) \leqq \int_{\mathbb{R}^m} dw\, |w|^{2-m} \exp(-w^2/2)(|w+v|^{-2}+|u+v+w|^{-2}) \quad (34)$$

The integral

$$D(y)=\int_{\mathbb{R}^m} dw\, |w|^{2-m} |w+y|^{-2} \exp(-w^2/2)$$

can be easily estimated:

$$\mathrm{D}(y) \leqq \mathrm{C}(1+\chi(|y| \leqq 1)|\ln(|y|)|) \tag{35}$$

It follows from (33)-(35) that

$$\mathrm{L}(v) \leqq \mathrm{C} \int_{\mathbb{R}^m} du\,|u|^{2-m}|u+v|^{-1} \exp(-u^2/2)$$
$$(1+\chi(|v| \leqq 1)|\ln(|v|)|+\chi(|u+v| \leqq 1)$$
$$|\ln(|u+v|)|) \leqq \mathrm{C}(1+\chi(|v| \leqq 1)|\ln(|v|)|) \tag{36}$$

From (32) and (36) we get

$$\mathrm{I}_1 \leqq \mathrm{C} \int_{\mathbb{R}^m} dv\,|v|^{1-m} \exp(-v^2/2)(1+\chi(|v| \leqq 1)|\ln(|v|)|)<+\infty$$

Case 3. — In an analogous manner we have to estimate the integral

$$\mathrm{I}_3 \leqq \mathrm{C} \int_{(\mathbb{R}^m)^3} du\,dv\,dw \exp(-1/2(u^2+v^2+w^2))$$
$$\times |u|^{2-m}|v|^{2-m}|w|^{2-m}|u|^{-1}|v|^{-1}|w|^{-1}|u+v+w|^{-1}$$
$$\leqq \mathrm{C} \int_{(\mathbb{R}^m)^2} dv\,dw \exp(-1/2(v^2+w^2))|v|^{1-m}|w|^{1-m}$$
$$\times \int_{\mathbb{R}^m} du\,|u|^{1-m}|u+v+w|^{-1} \exp(-u^2/2)$$
$$\leqq \mathrm{C} \int_{(\mathbb{R}^m)^2} dv\,dw \exp(-1/2(v^2+w^2))$$
$$|v|^{1-m}|w|^{1-m}(1+\chi(|v+w| \leqq 1)|\ln(|v+w|)|)<+\infty$$

Case 5:

$$\mathrm{I}_5 \leqq \mathrm{C} \int_{\mathbb{R}^m} du\,|u|^{1-m} \exp(-u^2/2) \int_{\mathbb{R}^m} dv\,|v|^{2-m}|u+v|^{-1} \exp(-v^2/2)$$
$$\times \int_{\mathbb{R}^m} dw\,|w|^{1-m}|u+v+w|^{-1} \exp(-w^2/2) \leqq \mathrm{C} \int_{\mathbb{R}^m} du\,|u|^{1-m} \exp(-u^2/2)$$
$$\times \int_{\mathbb{R}^m} dv\,|v|^{2-m}|u+v|^{-1} \exp(-v^2/2)(1+\chi(|u+v| \leqq 1)|\ln(|u+v|)|)$$
$$\leqq \mathrm{C} \int_{\mathbb{R}^m} du\,|u|^{1-m} \exp(-u^2/2)<+\infty.$$

We have just proved that for $m \geqq 3$

$$\psi(\omega(.))<+\infty$$

for a. e. $\omega(.) \in \mathrm{C}([0,1]; \mathbb{R}^m)$. Therefore we have constructed corresponding semibounded self-adjoint hamiltonians in $\mathrm{L}_2(\mathbb{R}^2)$ and $\mathrm{L}_2(\mathbb{R}^3)$. This can be considered as a standard proof of corresponding statements in [8] for $m \leqq 3$, obtained there by means of nonstandard analysis.

2. HAMILTONIANS OBTAINED BY A RENORMALIZATION PROCEDURE

The considerations of the part 1 show that in certain cases the construction procedure without renormalisation fails. If $m \geqq 4$ then the kernels $B_n(s, t, z)$ have not pointwise limit as $n \to \infty$. If $m=2$ and Ω is discrete or $m=3$ and Ω is discrete or a smooth curve, operators $B_n(z)$ have not a bounded limit as $n \to \infty$. Hence, we need another procedure. It is well known in the case where Ω consists of one or several separated points in $\mathbb{R}^m$, $m=2, 3$. The corresponding method was elaborated in [9], [7]. What we shall give below for $m=3, 4, 5$ is a continuous version of it.

The main idea is again to construct a sequence of operators H_n given by

$$(H_n f, g) = \int_{\mathbb{R}^m} dp p^2 f(p) \overline{g(p)} + \int_X d\mu(t) (F_n \varphi_n)(t) \overline{\psi_n(t)}, \tag{37}$$

where $\varphi_n(t) = (f, e_n(t))$, $\psi_n(t) = (g, e_n(t))$. The difference from part 1 is that F_n should be now some operators in $L_2(X)$ depending on n, $\omega(.)$.

Assume that F_n are bounded self-adjoint operators in $L_2(X)$. One can see that H_n are self-adjoint in $L_2(\mathbb{R}^m)$ with the same domain (3). By the same consideration as in the part 1 we get

$$f_n(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z} \int_X d\mu(t) e_n(t, p) (F_n \rho_n)(t), \tag{38}$$

and

$$\rho_n(s) = a_n(s, z) - \int_X d\mu(t) L_n(s, t, z) (F_n \rho_n)(t), \tag{39}$$

where $f_n = (H_n - z)^{-1} g$, $\operatorname{Im} z \neq 0$; ρ_n, a_n and L_n are given by the same formulae. We shall assume for a while that $g \in S(\mathbb{R}^m)$.

Assume that operators F_n have bounded inverses. Then

$$(F_n^{-1} h_n)(s) = a_n(s, z) - \int_X d\mu(t) L_n(s, t, z) h_n(t), \tag{40}$$

where $h_n = F_n \rho_n$. As we just have mentioned, the operators $L_n(z)$ may have not bounded limits as $n \to \infty$. Therefore we should separate the most singular part in $L_n(z)$ as $n \to \infty$:

$$L_n(z) = L_n^{\text{sing}} + L_n^{\text{reg}}(z)$$

Then we shall choose F_n^{-1} such that L_n^{sing} be compensated. Let

$$F_n^{-1} = A - L_n^{sing} \tag{41}$$

where A is some bounded self-adjoint operator in $L_2(X)$ which has a bounded inverse. From (40)-(41) we get formally

$$h_n = (A + L_n^{reg}(z))^{-1} a_n(z)$$

If $L_n^{reg}(z)$ converge in the operator norm to some operators $L^{reg}(z)$ as $n \to \infty$ and $(A + L^{reg}(z))^{-1}$ exist, then h_n and f_n have limits as $n \to \infty$.

Let us elaborate this strategy in detail. We begin by writing $L_n(s, t, z)$ in the form

$$L_n(s, t, z) = \alpha_m \int_0^n dR R^{m-3} \frac{\sin(R\Delta)}{R\Delta} + \alpha_m z \int_0^n dR R^{m-3} \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \equiv L_n^{sing}(s, t) + L_n^{reg}(s, t, z) \tag{42}$$

where $z \in \mathbb{C} \setminus [0, +\infty)$; $m = 3, 4, 5$; $\Delta \equiv \Delta(s, t) = |\omega(s) - \omega(t)|$. We need that operators in the right-hand side of (41) be invertible. Fix the operator A and consider a family of operators

$$D_n(\lambda) = \lambda A - L_n^{sing} = A(\lambda - A^{-1} L_n^{sing}), \qquad \lambda \in \mathbb{R}.$$

The operators L_n^{sing} are compact self-adjoint in $L_2(X)$ for all n, hence, $A^{-1} L_n^{sing}$ are also compact. Let E_n be the spectrum of $A^{-1} L_n^{sing}$, this is a countable set. Hence $E = \bigcup_n E_n$ is also countable. For all $\lambda \in \mathbb{R} \setminus E$ the inverse operators $D_n^{-1}(\lambda)$ exist for all n. We shall choose $\lambda \in \mathbb{R}\pi \setminus E$ and calculate the strong limit of $R_n(z)$. Then one can take another limit as $\lambda \to 1$, $\lambda \notin E$.

Thus, $F_n = (\lambda A - L_n^{sing})^{-1}$ exist for all n and are bounded self-adjoint operators in $L_2(X)$. Let us study the behaviour of the kernels $L_n^{reg}(s, t, z)$ as $n \to \infty$.

If $m = 3$, then for all $s, t \in X$, $z \in \mathbb{C} \setminus [0, +\infty)$

$$L^{reg}(s, t, z) \equiv \lim_{n \to \infty} L_n^{reg}(s, t, z) = \alpha_m z \int_0^\infty dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \tag{43}$$

The estimate holds

$$|L_n^{reg}(s, t, z)| \leq C|z| \int_0^\infty dR |R^2 - z|^{-1} \leq C(z), \tag{44}$$

where $C(z)$ does not depend on s, t, n.

If $m = 4$ then for $\Delta(s, t) > 0$

$$L^{reg}(s, t, z) \equiv \lim_{n \to \infty} L_n^{reg}(s, t, z) = \alpha_m \int_0^n dR R \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \tag{45}$$

The uniform estimate holds:

$$|L_n^{reg}(s, t, z)| \leq C(\delta, z)\Delta^{-\delta}\alpha_m \times \int_0^n dR \frac{R^{1-\delta}}{|R^2 - z|} \leq C(\delta, z)\Delta^{-\delta}, \qquad 0<\delta<1 \quad (46)$$

If $m=5$ then for $\Delta>0$

$$L_n^{reg}(s, t, z) = \alpha_m z \Delta^{-1} \int_0^{n\Delta} du \frac{\sin u}{u} + \alpha_m z^2 \int_0^n dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (47)$$

Hence,

$$L^{reg}(s, t, z) \equiv \lim_{n\to\infty} L_n^{reg}(s, t, z) = \alpha_m z \Delta^{-1}\pi/2 + \alpha_m z^2 \int_0^n dR \frac{\sin(R\Delta)}{R\Delta(R^2 - z)} \quad (48)$$

It follows from (47) that

$$|L_n^{reg}(s, t, z)| \leq C(z)(1+\Delta(s, t)^{-1}), \quad (49)$$

where $C(z)$ does not depend on s, t, n.

Assume that for $m=4$ the map ω satisfies to the condition A1 of part 1 and for $m=5$ ω satisfies to A2. As to the case $m=3$, we shall not assume anything about ω except of measurability.

Let $L_n^{reg}(z)$, $L^{reg}(z)$ be integral operators with the kernels $L_n^{reg}(s, t, z)$, $L^{reg}(s, t, z); z\in\mathbb{C}\setminus[0, +\infty)$.

LEMMA 9. — *The following statements hold:*

1. $L_n^{reg}(z)$ *are Hilbert-Schmidt operators in* $L_2(X)$ *for all* n.
2. $L_n^{reg}(z)$ *converge in the operator norm in* $L_2(X)$ *to* $L^{reg}(z)$ *as* $n\to\infty$.
3. $L^{reg}(z)$ *are compact in* $L_2(X)$.
4. $L^{reg}(z)$ *is an analytic operator-valued function on* $\mathbb{C}\setminus[0, +\infty)$.

The proof is identical to that in lemma 1.

LEMMA 10. — *For all* z: $\operatorname{Im} z\neq 0$ *the inverse operators* $(A+L^{reg}(z))^{-1}$ *exist.*

Proof. — The operator A is invertible, hence

$$A+L^{reg}(z) = A(I+A^{-1}L^{reg}(z)),$$

where by lemma 9 the operator $A^{-1}L^{reg}(z)$ is compact. Therefore it is sufficient to show that the equation

$$A^{-1}L^{reg}(z)f = -f \quad (50)$$

has no nonzero solutions in $L_2(X)$. From (50) we get

$$-(Af, f) = \lim_{n\to\infty}((L_n(z)f, f) - (L_n^{sing}f, f)) \quad (51)$$

The proof of lemma 2 implies that

$$(L_n^{\rm sing} f, f) = \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2} \chi(|p|<n) \geqq 0, \tag{52}$$

$$(L_n(z) f, f) = \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{p^2 - z} \chi(|p| \leqq n), \tag{53}$$

where $l(p) = (2\pi)^{-m/2} \int_X d\mu(t) f(t) \exp(-i(p, \omega(t)))$. From (51)-(53) we get

$$0 = -\operatorname{Im}(A f, f) = \operatorname{Im} z \lim_{n \to \infty} \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 z|^2} \chi(|p| \leqq n)$$

$$= \operatorname{Im} z \int_{\mathbb{R}^m} dp \frac{|l(p)|^2}{|p^2 - z|^2}, \tag{54}$$

where the integral over $\mathbb{R}^m$ converges. Since $\operatorname{Im} z \neq 0$, (54) implies $l(p) = 0$ a. e. By (52)-(53) we have $L^{\rm reg}(z) f = 0$, hence $f = 0$. The proof is completed.

Remark. — It follows from the proof that $L_n^{\rm sing} \geqq 0$ and $L_n^{\rm reg}(-a) \leqq 0$, $L^{\rm reg}(-a) \leqq 0$, $a > 0$.

Thus the inverse operators $(\lambda A + L^{\rm reg}(z))^{-1}$ exist for all z: $\operatorname{Im} z \neq 0$, $\lambda \in \mathbb{R}$. By the statement 2 of lemma 9, the operators $(\lambda A + L_n^{\rm reg}(z))^{-1}$ exist for $n > N(z)$ and the resolvents $R_n(z)$ are given by

$$(R_n(z) g)(p) = \frac{g(p)}{p^2 - z} - \frac{1}{p^2 - z}$$

$$\times \int_X d\mu(t) e_n(t, p) ((A + L_n^{\rm reg}(z))^{-1} a_n(z))(t) \tag{55}$$

Define

$$v_n(p) = \frac{1}{p^2 - z} \int_X d\mu(t) h_n(t) (e_n(t, p) - (2\pi)^{-m/2} \exp(-i(p, \omega(t))))$$

where $h_n = (\lambda A + L_n^{\rm reg}(z))^{-1} a_n(z)$, $g \in S(\mathbb{R}^m)$. It is obvious that

$$\lim_{n \to \infty} \sup_S |a_n(s, z) - a(s, z)| = 0,$$

where

$$a(s, z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp g(p) (p^2 - z)^{-1} \exp(i(p, \omega(s))),$$

$$a(z) \in L_\infty(X) \in L_2(X) \equiv \mathscr{H}.$$

The operators $(A + L_n^{\rm reg}(z))^{-1}$ converge to $(A + L^{\rm reg}(z))^{-1}$, hence

$$\lim_{n \to \infty} \| h_n - h \|_{L_2(X)} = 0, \tag{56}$$

where $h=(\lambda A+L^{reg}(z))^{-1} a(z)$. It follows from (56), in particular, that

$$\|h_n\|_{\mathscr{H}} \leqq C \tag{57}$$

uniformly on $n \in \mathbb{N}$. Let us show that

$$\lim_{n \to \infty} \|v_n\|_{\mathscr{M}}=0, \qquad \mathscr{M} \equiv L_2(\mathbb{R}^m).$$

Consider the following integrals, where $d \geqq n$:

$$J(d, n) \equiv \int_{\mathbb{R}^m} dp \,|v_n(p)|^2 \chi(|p| \leqq d)=\int_{X^2} d\mu(t)\, d\mu(s)\, h_n(t) \overline{h_n(s)} (2\pi)^{-m}$$

$$\int_{\mathbb{R}^m} dp \frac{\chi(n \leqq |p| \leqq d)}{|p^2-z|^2} \exp(i(p, \omega(s)-\omega(t)))=\alpha_m \int_{X^2} d\mu(t)\, d\mu(s)\, h_n(t) \overline{h_n(s)}$$

$$\int_n^d dR \frac{R^{m-1}}{|R^2-z|^2} \frac{\sin(R\Delta)}{R\Delta},$$

$$\Delta \equiv \Delta(s, t)=|\omega(s)-\omega(t)|. \tag{58}$$

Let us estimate the integrals

$$I(\Delta, d, n)=\int_n^d dR \frac{R^{m-1}}{|R^2-z|^2} \frac{\sin(R\Delta)}{R\Delta}, \qquad \Delta>0.$$

It is obvious that for $m=3, 4$

$$|I(\Delta, d, n)| \leqq C(\delta, z) \Delta^{-\delta} \gamma_n(z), \tag{59}$$

where $\delta=0$ for $m=3$, $0<\delta<1$ for $m=4$, $\gamma_n(z)$ does not depend on d, Δ and

$$\lim_{n \to \infty} \gamma_n(z)=0 \tag{60}$$

If $m=5$ then

$$I(\Delta, d, n)=\Delta^{-1} \int_n^d dR \frac{\sin(R\Delta)}{R}+\Delta^{-1} \int_n^d dR \frac{\sin(R\Delta)}{R}\left(\frac{R^4}{|R^2-z|^2}-1\right)$$

$$\equiv I_1(\Delta, d, n)+I_2(\Delta, d, n) \tag{61}$$

where

$$I_1(\Delta, d, n)=\Delta^{-1}(Si(d\Delta)-Si(n\Delta)); \qquad Si(x)=\int_0^x du \frac{\sin u}{u}$$

It is easy to show that

$$|I_2(\Delta, d, n)| \leqq C \Delta^{-1} \gamma_n(z), \tag{62}$$

where $\gamma_n(z)$ does not depend on Δ, d and

$$\lim_{n \to \infty} \gamma_n(z)=0 \tag{63}$$

Now we can estimate the norm of ν_n in $L_2(\mathbb{R}^m)$. From (58)-(59) we get for $m=3, 4$

$$\|\nu_n\|^2 = \sup_{d>n} J(d, n) \leqq C(\delta, z)\gamma_n(z) \times \int_{X^2} d\mu(t)\, d\mu(s)\, |h_n(t)|\, |h_n(s)|\, |\omega(s)-\omega(t)|^{-\delta} \quad (64)$$

If $m=3$ than $\delta=0$ and from (57) we get

$$\|\nu_n\|^2 \leqq C(\delta, z)\gamma_n(z) \to 0$$

as $n\to\infty$. If $m=4$, then by condition A1 the operator with the kernel $|\omega(t)-\omega(s)|^{-\delta}$ is bounded in $L_2(X)$, hence by (58), (59) and (64) we get $\|\nu_n\|\to 0$.

Let us consider the case $m=5$. By (58), (62) and condition A2

$$\lim_{n\to\infty} \sup_{d>n} |J_2(d, n)| = 0, \quad (65)$$

where J_2 is a corresponding contribution in J. As to J_1, it can be written as follows:

$$J_1(d, n) = \alpha_m \int_{X^2} d\mu(t)\, d\mu(s)\, h_n(t)\overline{h_n(s)}\, \Delta^{-1}(\mathrm{S}i(d\Delta) - \mathrm{S}i(n\Delta)),$$
$$\Delta = |\omega(s) - \omega(t)| \quad (66)$$

Let $U(d, n)$ be the operator with the kernel $\Delta^{-1}(\mathrm{S}i(d\Delta)-\mathrm{S}i(n\Delta))$, $U(n)$ be the operator with the kernel $\Delta^{-1}(\pi/2-\mathrm{S}i(n\Delta))$. By condition A2 and the dominated convergence theorem

$$\lim_{d\to\infty} \|U(d, n) - U(n)\| = 0, \quad (67)$$

$$\lim_{n\to\infty} \|U(n)\| = 0. \quad (68)$$

From (65)-(67) we get

$$\|\nu_n\|^2 = \overline{\lim_{d\to\infty}}\,(J_1(d, n) + J_2(d, n)) \leqq \sup_{d>n} |J_1(d, n)| + \lim_{d\to\infty} J_2(d, n)$$
$$= \sup_{d>n} |J_1(d, n)| + J_2(n), \quad (69)$$

where $J_2(n) = \alpha_m (U(n) h_n, h_n)$. It follows from (65), (57), (68) and (69) that $\|\nu_n\|\to 0$.

To calculate the strong limit of resolvents we should consider now the following sequence in $L_2(\mathbb{R}^m)$:

$$Q_n(p) = \frac{(2\pi)^{-m/2}}{p^2 - z} \int_X d\mu(t) \exp(-i(p, \omega(t)))\, h_n(t)$$

It follows from considerations we have made above that $Q_n \in L_2(\mathbb{R}^m)$ for all n. Define

$$Q(q) = \frac{(2\pi)^{-m/2}}{p^2 - z} \int_X d\mu(t) \exp(-i(p, \omega(t))) h(t)$$

As we made it for $\| v_n \|^2$, we write

$$V(d) \equiv \int_{\mathbb{R}^m} dp \, | Q_n(p) - Q(p) |^2 \chi(|p| \leq d)$$
$$= \alpha_m \int_{X^2} d\mu(t) \, d\mu(s) (h_n(t) - h(t)) \overline{(h_n(s) - h(s))} I(\Delta, d, 0)$$

and it can be shown that

$$\| Q_n - Q \|^2 = \lim_{d \to \infty} V(d) = (U(h_n - h), h_n - h), \tag{70}$$

where U is an operator with the kernel

$$U(s, t) = \lim_{d \to \infty} \alpha_m \int_0^d dR \frac{R^{m-1}}{|R^2 - z|^2} \frac{\sin(R\Delta)}{R\Delta}$$

One can show from (59)-(62) that U is a bounded operator in $L_2(X)$. Hence from (56) and (70) we have

$$\lim_{n \to \infty} \| Q_n - Q \| = 0,$$

in particular, $Q \in L_2(\mathbb{R}^m)$.

We have just shown that for all z: $\operatorname{Im} z \neq 0$, $g \in S(\mathbb{R}^m)$ the sequence $R_n(z) g$ converge in $L_2(\mathbb{R}^m)$ as $n \to \infty$ and

$$T(z) g \equiv \lim_{n \to \infty} R_n(z) g = \frac{g(p)}{p^2 - z} - \frac{(2\pi)^{-m/2}}{p^2 - z}$$
$$\times \int_X d\mu(t) \exp(-i(p, \omega(t))) ((\lambda A + L^{reg}(z))^{-1} a(z))(t), \tag{71}$$

where

$$a(s, z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dp \frac{g(p) \exp(i(p, \omega(s)))}{p^2 - z} = (R_0(z) g)(\omega(s))$$

in space-representation. The operators $R_n(z)$ are resolvents of self-adjoint operators in $L_2(\mathbb{R}^m)$, hence for all n

$$\| R_n(z) \| \leq | \operatorname{Im} z |^{-1}$$

Therefore $R_n(z) g$ converge in $L_2(\mathbb{R}^m)$ for all $g \in L_2(\mathbb{R}^m)$ and

$$\| T(z) \| \leq | \operatorname{Im} z |^{-1}$$

However the formula for $a(z)$ holds for any g only for $m = 3$.

Let us show that $\operatorname{Ran} T(z)$ is dense in $L_2(\mathbb{R}^m)$. Suppose that $(\varphi, T(z)g)=0$ for all $g \in S(\mathbb{R}^m)$. For $f \in C_0^\infty(\mathbb{R}^m \setminus \Omega)$ in space-representation define $g(p)=(p^2-z)f(p) \in S(\mathbb{R}^m)$. By (71) we get

$$T(z)g = R_0(z)g = f,$$

since

$$a(s)=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dpf(p)\exp(i(p, \omega(s))) = f(\omega(s))=0.$$

Hence

$$0=(\varphi, T(z)g)=(\varphi, f)$$

for all $f \in C_0^\infty(\mathbb{R}^m \setminus \Omega)$. This set is dense in $L_2(\mathbb{R}^m)$, so $\varphi=0$.

We have proved the following result.

THEOREM 2. – *Let* $m=3$, 4 *or* 5 *and* ω *satisfies to condition* A1 *for* $m=4$ *and to* A2 *for* $m=5$. *Assume that* A *is a bounded invertible self-adjoint operator in* $L_2(X)$. *Then there exists a countable set* $E \in \mathbb{R}$ *such that for all* $\lambda \in \mathbb{R} \setminus E$, $n \in \mathbb{N}$ *the operators* $F_n=(\lambda A - L_n^{\text{sing}})^{-1}$ *exist. If* $\lambda \in \mathbb{R} \setminus E$ *then the self-adjoint operators* H_n *given by* (37), *converge in the strong resolvent sense to some self-adjoint operator* H. *Its resolvent is given by* (71) *for all z:* $\operatorname{Im} z \neq 0$, $g \in S(\mathbb{R}^m)$.

One can consider now the limit of (71) as $\lambda \to 1$. Applying again the Trotter-Kato theorem we get that the right-hand size of (71) is for $\lambda=1$ also a resolvent of some self-adjoint operator in $L_2(\mathbb{R}^m)$. Therefore we can put $\lambda=1$.

We may study now some properties of H. The fact that H is a self-adjoint extension of $H_0=-\Delta$ with the domain $C_0\infty(\mathbb{R}^m \setminus \Omega)$ can be proved by the same way as in lemma 3.

LEMMA 11. – *Assume that* $(A\varphi, \varphi)=0$ *iff* $\varphi=0$. *Then* H *is nontrivial.*

Proof. – Suppose that H is trivial. Then for all z: $\operatorname{Im} z \neq 0$, $g \in S(\mathbb{R}^m)$

$$0=((R(z)-R_0(z))(p^2-z)g(p), (p^2-\bar{z})g(p))$$

$$= -\int_X d\mu(t)\overline{\rho(t)}((A+L^{\text{reg}}(z))^{-1}\rho)(t) \quad (72)$$

where

$$\rho(s)=(2\pi)^{-m/2}\int_{\mathbb{R}^m} dpg(p)\exp(i(p, \omega(s))) = g(\omega(s)),$$

g taken in space-represantation. Let $\varphi=(A+L^{\text{reg}}(z))^{-1}\rho$. From (72) we get

$$(\varphi, A\varphi + L^{\text{reg}}(z)\varphi)=0 \quad (73)$$

for all z: Im $z \neq 0$. From (73) and (52)-(54) we have $L^{reg}(z)\varphi = 0$, hence, $(\varphi, A\varphi) = 0$, and by condition of lemma $\varphi = 0$. Therefore

$$g(\omega(s)) = \rho(s) = (A + L^{reg}(z))\varphi = 0$$

This is not true all $g \in S(\mathbb{R}^m)$. The contradiction proves the lemma.

Remark. – The condition of lemma is satisfied, in particular, if $A > 0$ or $A < 0$.

The locality of H (*see* lemma 8) follows again from [1].

The semiboundedness of H is a difficult problem. The reason is that operators $L^{reg}(-a)$ may have not a bounded limit as $a \to \infty$. Let us discuss this question in detail. By lemma 9, lemma 10 and analytical Fredholm theorem [11] the operator-valued function $(A + L^{reg}(z))^{-1}$ is meromorphic on $\mathbb{C} \setminus [0, +\infty)$ and is analytic on $\mathbb{C} \setminus [0, +\infty) \setminus S$, where S is a discrete set on $(-\infty, 0)$, having as possible accumulation points 0 and $-\infty$.

LEMMA 12. – *Let* $z \in (-\infty, 0) \setminus S$. *Then* z *belongs to resolvent set of* H.

This result follows directly from (71).

LEMMA 13. – *Assume that* $(A\varphi, \varphi) \leqq -\lambda(\varphi, \varphi)$ *for some* $\lambda > 0$ *for all* $\varphi \in L_2(X)$. *Then* $(-\infty, 0) \in \rho(H)$, *that is,* $H \geqq 0$ *and it has no negative eigenvalues.*

This result follows from lemma 12 and the remark to lemma 10.

Applications

1. Ω is a single point in $\mathbb{R}^3$. One can take $X = \{0\}$, $\mu(0) = 1$, $\omega(0) = y \in \mathbb{R}^3$. In such case the operator F_n is in fact a real number $(A - 4\pi n)^{-1}$. This renormalization scheme has been considered in [9].

2. Ω is a finite set in $\mathbb{R}^3$. In this case our consideration is similar to [7]. One can take $X = \{1, 2, \ldots, N\}$, $\mu(j) = 1$ for all $j \in X$, $\omega(j) = y_j \in \mathscr{R}^3$. We assume that $y_j \neq y_1$ for $j \neq 1$. The operator A is a matrix such that $A_{ik} = \bar{A}_{ki}$, the operator F_n^{-1} also being a matrix:

$$(F_n^{-1})_{ik} = A_{ik} - d_{ik}^{(n)},$$

where $d_{ii}^{(n)} = 4\pi n$; $d_{ik}^{(n)} = 4\pi \int_0^n dR \frac{\sin(R\Delta)}{R\Delta}$, $i \neq k$, $\Delta = |y_i - y_k|$. It is clear that for $i \neq k$ there exist finite limits

$$d_{ik} = \lim_{n \to \infty} d_{ik}^{(n)}$$

Therefore one can choose A such that operators F_n be asymptotically diagonal as $n \to \infty$. Let

$$A_{ii} = \alpha_i; \qquad A_{ik} = d_{ik}, \quad i \neq k.$$

Then

$$(F_n^{-1})_{ik} = \delta_{ik}\left(-\frac{1}{4\pi n} + \frac{\alpha_k}{(4\pi n)^2}\right) + o(n^{-2})$$

as $n \to \infty$. In this concrete case we could take L_n^{sing} diagonal for all n, $(L_n(z))_{ik}$ having finite limits for $i \neq k$. The operators F_n^{-1} and F_n would be diagonal for all n. However, if Ω is not a discrete set, we cannot do it and we should consider the asymptotic diagonality of F_n.

Let us explain the importance of diagonality of F_n. As it has been mentioned in [1], it can be interpreted as independence of zero-range interactions in different points. One can see it from the definition of H_n. It seems quite natural that the condition of independence should be satisfied, at least asymptotically as $n \to \infty$. In [1] it makes it possible to choose the N-parameter family of self-adjoint extensions, the whole family of extensions being N^2-parameter.

In the part 1 this condition is obviously satisfied since F is a function. Certainly, one could apply the renormalization procedure to the case considered in the part 1. However, the operators L_n^{sing} having a bounded limit L^{sing} as $n \to \infty$, the renormalization is equivalent to the consideration of hamiltonians (2), where F is a bounded self-adjoint operator in $L_2(X)$, $F = (A - L^{\text{sing}})^{-1}$. Only for $A = F^{-1} + L^{\text{sing}}$, where F is diagonal, we have self-adjoint extensions H interesting from physical point of view. Other extensions are less attractive.

The situation where renormalization is really necessary occurs when operators $(I + L_n(z)F)^{-1}$ have not nonzero bounded limits as $n \to \infty$. In this case L_n^{sing} have not bounded limit, and the main problem is to choose operators A such that $F_n = (A - L_n^{\text{sing}})^{-1}$ be asymptotically diagonal.

3. Ω is a Brownian path in $\mathbb{R}^4$ or $\mathbb{R}^5$.

The conditions A1 (for $m=4$) or A2 (for $m=5$) are satisfied, as we have proved in the part 1. Therefore for a. e. ω we have self-adjoint operators with expected properties, whose resolvents on a dense set are given by (71). If $A \leqq -\lambda I$, where $\lambda > 0$, then these operators are nontrivial by lemma 11 and nonnegative (hence, semibounded) by lemma 13. Since $L_n^{\text{sing}} \geqq 0$ and most probably have not a bounded limit, we can expect that in some sense F_n are "negative infinitesimal" operators. This is in agreement with corresponding results of [8] for $m=4$, 5, obtained there by nonstandard methods.

The question we cannot answer at present is for what A operators F_n are asymptotically diagonal.

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