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The temperature of an asteroid

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ABSTRACT. — The heat equation in a solid body with radiation boundary conditions and periodic heating at the boundary — modeling a rotating asteroid — is shown to have a unique time-periodic solution and it is globally attracting. We find approximations to this solution for various cases relevant to the possible existence of ice in the asteroid belt.

RÉSUMÉ. — Nous montrons que l'équation de la chaleur dans un corps solide avec conditions aux bords radiatives et une source de chaleur périodique sur la frontière possède une unique solution périodique qui est un attracteur. Cette équation sert à modéliser un astéroïde en rotation. Nous exhibons des approximations de cette solution que sont relevantes pour l'existence de glace dans la ceinture d'astéroïdes.

INTRODUCTION

We treat the temperature of a rotating solid body with no atmosphere, heated at the surface by absorption of Solar radiation and also losing heat by radiation. We have in mind an asteroid, typically a few kilometers (or few tens of kilometers) in diameter, with escape velocity at most some tens of meters per second, far too small to retain an atmosphere. We don't know what asteroids are made of, so our caculations will consider a wide range of materials. It turns out that only one dimensionless parameter (λ/δ) is important – depending on the material near the surface, distance from the Sun, and period of rotation – but we also don't know this value. (Strictly speaking, this is for a circular orbit; we study the effect of eccentricity later, and see it is negligible for almost any asteroid.) The calculations are fairly simple when λ/δ is large – and it would be large for most plausible Terrestrial materials – but are far more difficult when it is small, and many problems remain for efficient computation in this case. (The theory might be applied to Moon, but λ/δ is small – about 1/70 – largely due to slow rotation, and the results are not satisfactory. Nevertheless, it predicts temperature 68 K at the poles of Moon, independent of λ/δ, and I have some confidence in this number.)

We use coordinates fixed in the body; then the only effect of rotation is a periodic or almost-periodic variation of the heating rate at a given point of the surface. We show there is a unique long-term response, the limit from any initial temperature distribution as time goes to infinity, which is equally periodic or almost-periodic in time. With any plausible values of the physical parameters of an asteroid, the thermal relaxation time is far smaller than a billion years; we may be confident the limit is attained. (Even for Moon, the surface temperature should be determined by this theory, though radioactivity and residual "ancient" heat are important in the interior.) The limiting solution varies far more rapidly with depth than it does along the surface, so a one-dimensional model – depth below a given point of the surface – should be a good approximation. The solution of this one-dimensional model rapidly approaches a limit, independent of time, as depth increases. (A few meters below the surface should be enough.) This is used as a boundary value to determine the temperature, independent of time, in the interior of the asteroid.

We calculate in detail only the case of a spherical asteroid, with constant physical properties, though most asteroids are probably non-spherical and inhomogeneous. The calculation of the surface temperature – actually, a few meters below the surface – may be used directly for any convex body with any distribution of conductivity. (Non-convex bodies may produce troublesome shadows, and call for more complex computations, though not different in principle.) The subsequent calculation of the interior temperature will depend on the particular shape and conductivity. The general theorems apply to bodies of any (smooth) shape with any (smooth) inhomogeneities.

Mann and Wolf [5] treated a one-dimensional heat equation in $(0, \infty)$ with a radiation boundary condition, constant conductivity, constant initial temperature, and asymptotically constant heating rate at the boundary. They showed the solution approaches a constant as time tends to infinity. Levinson [4] started from a different physical problem, which led to the same mathematical problem. He showed, for a constant initial value and periodic heating at the boundary, the solution approaches the unique periodic solution with the same period. These authors worked exclusively with the integral equation. We will also use an integral equation on the boundary for questions of existence and smoothness, but we study the asymptotic behavior using the maximum principle and dynamical systems theory. We generalize the work of [4], [5] in several directions: the spatial dimension may be greater than one, the region may have any smooth shape and may be bounded or not, the physical properties (conductivity, etc.) may depend on position, the initial temperature is quite arbitrary (non-negative, bounded), and the time-dependence of the heating rate is also general, though the periodic and almost-periodic cases recieve special attention.

The interest of the mathematical problem is sufficient justification, but I believe the problem is also important. At least, the temperature is a crucial datum for an important question:

Is there ice in the asteroids?

It may not seem the question is important; have patience.

Comets are active only a short time – some 200 revolutions a few thousand years.

It is a respectable conjecture that "dead" comets, which have ceased visible emission, may very well end up in the asteroid belt. They cannot remain very long in the inner Solar System before being swept away by gravitational perturbations. Where's "away"? The asteroid belt may be the dust bin of the inner Solar System. (I am warned there are important unsolved dynamical problems here.)

Dead comets are sometimes called "de-volatilized", for no good reason. Photos of the nucleus of Halley's comet show emission only from patches, about 10% of the surface, the rest being covered by dust. A comet dies when its surface is completely covered.

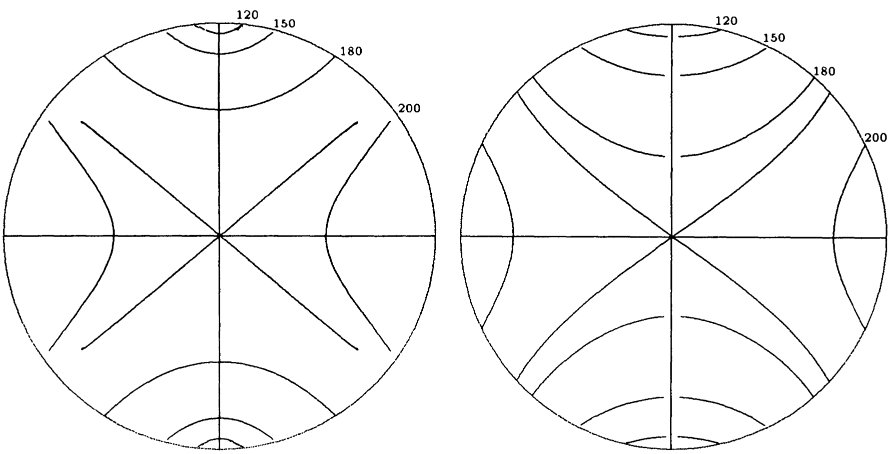
If the comet nucleus is not too small – at least a few kilometers across – it will still have a lot of volatiles (ices of H_2O, HCN, NH_3 and CO_2, though CH_4 and CO will probably be lost at temperatures above 100 K) when it arrives in the asteroid belt. If the ice can survive for billions of years under conditions encountered in the asteroid belt, an enormous stock of ice may have accumulated out there. This would determine the course of development of the Solar System for centuries to come. For example, it's easy to make air from water, cyanide or ammonia, and sunlight. (Now do you believe it's important?)

The average temperature at the inner edge of the asteroid belt (2 A.U. from the Sun) is nearly 200 K, dropping to 150 K at the outer edge (3.5 A.U.), and this is not very promising for long term survival of ice exposed to a vacuum. However: (1) The average temperature is not the only temperature; and: (2) The ice is not exposed.

(1) Even at 2 A.U., with average temperature 200 K, there will be substantial regions near the poles of an asteroid with temperatures below 150 K, and some below 120 K. (We may use the same estimates for dead comets; *see* "The effect of eccentricity" below.)

(2) We deal with dirty ice, which will shortly be covered by dust left behind as the ice vaporizes, and the dust slows the evaporation of underlying ice. (This point is treated in the Appendix.)

The prospects for survival of water ice, covered with dust, and ammonia and cyanide ices embedded in water ice, are good.



axis vertical
average temperature 196 K
polar temperature 0
equatorial temperature 210 K

axis inclination 30°
average temperature 188 K
polar temperature 101 K
equatorial temperature 207 K

Maximum surface temperature 280 K, $\lambda/\delta = \infty$.

Approximate isotherms below the surface for a spherical asteroid at 2 *A.U.*

The amount of material involved is not trivial. A (reasonable) rate of one new periodic comet every 30 years, of (modest) diameter 2 km and density (at least) that of water, continued for 4 billion years, results in total mass – almost 10^{21} kg – comparable to that of the entire asteroid belt. But 2 km diameter may be excessively modest, and comets may have been far more frequent in the early days. We show later that dead comets, like asteroids, can easily retain their volatiles for billions of years, so it's strange we don't see more of them around. But perhaps we do see them, in the asteroid belt.

Wishing doesn't make it so, and expert opinion seems clearly opposed to the notion of ice in the asteroids. In fact, I could find no local expert on physical properties of asteroids. My caricature of expert opinion is culled mainly from the last two decades of *Icarus* and the books: *Asteroids*, T. GEHRELS Ed., Univ. Arizona, 1979 and *The Evolution of the Small Bodies of the Solar System*, Proc. Intl. School of Physics E. FERMI, 1985, M. FULCHIGNONI and L. KRESÁK Ed., North-Holland, 1987.

The opposing argument is indirect, consisting of three claims:

(i) Unless everything we know, or think we know, about the asteroids is wrong, they are virtually unchanged in nature and position since their formation more than 3 billion years ago.

(ii) The conditions of their formation, under any usual hypothesis, preclude volatiles like water.

(iii) The contribution of dead comets is negligible.

Conclusion : **NO ICE!**

Did you notice "Unless.."?

The important point is (i). If the claim of (i) is wrong, (ii) hardly matters; the conditions of formation of Earth also seem to preclude water. And the only reason to believe in (iii) is to be consistent with (i); there is no evidence, for or against. The main evidence for (i) – the correlation of "type" (= color) of asteroid with distance from the Sun – I consider inconclusive.

It is assumed that "type", a surface property, reflects the internal constitution of the asteroid. The fact that it's been roasting in a particular orbit for a long time is irrelevant, since type supposedly does not change with time. It is also claimed that these orbits have hardly changed in several billions of years, which should interest those who study chaotic orbits. These points strike me as dubious, and certainly unproved.

If the question of ice in the asteroids were a theoretical point of minor interest, we might be content with an application of Occam's Razor to a state of general ignorance. But it is immensely important.

The experts prefer to call them minor planets, but asteroids may be radically different from all planets – not primordial rocks, but rather young dust balls, with buried treasure. It's worth finding out.

THE INITIAL BOUNDARY-VALUE PROBLEM

The body occupies an open set $\Omega \subset \mathbf{R}^n$, lying on one side of the boundary $\partial\Omega$, and $\partial\Omega$ is a compact smooth surface: class C^{1+r}, $(0<r<1)$ – or C^2 or C^{2+r}, in some later results. In $\bar{\Omega}$, are defined positive real functions of position ρ (density), C (specific heat), and the positive-definite symmetric conductivity matrix $K=[K_{ij}]$. On $\partial\Omega$, we have emissivity ε and absorptivity α. It is very likely that ε is close to 1; α, or an average value for α, is measurable as $1-(albedo)$. At a point x of $\partial\Omega$, the intercepted flux of Solar energy, per unit area of $\partial\Omega$ and per unit time, is $g(x, t)$; we suppose αg is absorbed and $(1-\alpha)g$ is reflected. Then the absolute temperature T satisfies

$$\left.\begin{aligned} &\rho C\,\partial T/\partial t=\operatorname{Div}(K\nabla T)=\sum_{i,j=1}\partial/\partial x_i(K_{ij}\,\partial T/\partial x_j)\\ &\text{for } x\in\Omega.\\ &(KN(x)).\nabla T+\sigma\varepsilon(x)T^4=\alpha(x)g(x,t)\qquad \text{on}\quad \partial\Omega.\end{aligned}\right\}\quad \text{(PDE)}$$

where $N(x)$ is the outward unit normal to $\partial\Omega$ at x and σ is the Stefan-Boltzmann constant, approximately $\sigma=5.67\times10^{-8}\,W/m^2K^4$.

The case of physical interest is when Ω is a bounded set in $\mathbf{R}^3$, but the argument is the same for any dimension $n\geqq1$, and it is important for the calculations to also treat $\Omega=(0,\infty)$ in the case of constant K, ρ, C, etc. For most of our calculations, Ω could be bounded or not as long as $\partial\Omega$ is compact; but convergence results as $t\to\infty$ are delicate in unbounded domains and the case $n=1$, $\Omega=(0,\infty)$, sufficiently illustrates the point.

If we model the Sun as a point source, in the direction $\mathbf{s}(x,t)$ from $x\in\partial\Omega$ at time t ($|\mathbf{s}|=1$), then

$$g(x,t)=\begin{cases} f\mathbf{N}(x).\mathbf{s}(x,t), & \text{when the Sun is visible from } (x,t),\\ 0, & \text{when it is not visible.}\end{cases}$$

Here f is the flux of Solar energy, or Solar constant: about $1{,}353\pm21$ W/m², at the orbit of Earth, outside the atmosphere, decreasing as the square of the distance from the Sun (338 ± 5 W/m² at 2 A.U.). When the Sun is visible from x at time t, then $\mathbf{s}.\mathbf{N}\geqq0$, so $0\leqq g\leqq f$ in any case. If Ω is not convex, g will have discontinuous jumps as we cross a shadow boundary. We usually want g to be at least continuous, and this is physically reasonable since the Sun is *not* a point source, and g will be roughly proportional to the area of the Solar disc visible at the moment. It follows that g will be Lipschiptzian and probably better: $C^{5/4}$ for generic smooth surfaces $\partial\Omega$. (If the horizon from a given point, in the direction of sunrise or sunset, has curvature different from that of the Sun's disc, g will be locally $C^{3/2}$; if equal, g will be ordinarily $C^{5/4}$, and equality occurs only at isolated points, for most smooth surfaces $\partial\Omega$.)

We will state our results, make some comments, and outline proofs later.

THEOREM 1. — *Let* $0<r<1$, $n\geqq1$, *and suppose* Ω *is an open set in* $\mathbf{R}^n$, *lying on one side of its boundary* $\partial\Omega$, *which is a compact* C^{1+r} *surface.*

Suppose ρ, C, α, ε *are positive and bounded from zero, or their reciprocals are bounded, and they are uniformly of class* C^r; $K(x)=[K_{ij}(x)]_{i,j=1}^n$ *is symmetric, bounded, uniformly positive definite, and* K, $\partial K/\partial x$ *are both uniformly of class* C^r.

Finally suppose B is a positive constant and $\psi:\bar{\Omega}\to\mathbf{R}$ and $g:\partial\Omega\times[t_0,\infty)\to\mathbf{R}$ are continuous with

$$0\leqq\psi(x)\leqq B,\qquad 0\leqq g(x,t)\leqq\sigma B^4\varepsilon(x)/\alpha(x)$$

Then the problem (PDE) has a unique solution T on $\Omega\times(t_0,\infty)$, of class $C^{2+r,1+r/2}$, continuous in the closure with $T(x,t_0)=\psi(x)$ in $\bar{\Omega}$, satisfying the differential equation in the classical sense in the interior, and the boundary condition for $t>t_0$, $x\in\partial\Omega$, when we use

$$KN(x).\nabla T(x,t)=\lim\{KN(x').\nabla T(x',t)\mid x'\in\Omega, x'\to x \text{ non-tangentially}\}.$$

This solution has $0\leqq T(x,t)\leqq B$ always, and is an increasing (=non-decreasing) function of (ψ, g), that is: if $0\leqq\psi_1\leqq\psi_2\leqq B$ in Ω and $0\leqq g_1\leqq g_2\leqq\sigma\varepsilon B^4/\alpha$ on $\partial\Omega\times(t_0,\infty)$, the corresponding solutions T_j satisfy $0\leqq T_1\leqq T_2\leqq B$ in $\Omega\times(t_0,\infty)$.

For any compact $K\subset\bar{\Omega}$, $t_1>t_0$, and $0<\theta<1$, $T\mid K\times[t,t+1]\in C^{\theta,\theta/2}$ is bounded uniformly for $t\geqq t_1$.

If $g_\nu\to g$ and $\psi_\nu\to\psi$ uniformly, then $T_\nu\to T$ uniformly on compact sets in $\bar{\Omega}\times[t_0,\infty)$.

Remark. – The local problem is like that in Friedman [3], and we use a similar argument, with a Volterra integral equation on the boundary. To extend the solution for all time, we use the maximum principle to show $0\leqq T\leqq B$, as long as the solution exist.

We assume $\partial\Omega$ is C^{1+r} to apply the integral method. Aside from mathematical convenience, we can only plead engineering experience in radiation problems to justify this hypothesis. In fact, for many asteroids, there is evidence from polarimetry that the surface is dusty. But it is difficult to avoid assuming at least a Lipschitz surface, even in very weak formulations.

THEOREM 2. – *Assume* Ω, K, ρ, C, ε, α *satisfy the hypotheses of the first paragraph of Theorem* 1, *and for convenience, suppose* Ω *is bounded. Assume* ψ *and* g *are bounded and measurable with*

$$0\leqq\psi\leqq B,\qquad 0\leqq g(x,t)\leqq\sigma B^4\varepsilon(x)/\alpha(x)$$

almost everywhere on Ω *or* $\partial\Omega\times(t_0,\infty)$, *respectively. For any continuous* ψ_j, g_j *such that* $0\leqq\psi_1\leqq\psi\leqq\psi_2\leqq B$, *and* $0\leqq g_1\leqq g\leqq g_2\leqq\sigma B^4\varepsilon/\alpha$, *the corresponding solutions* T_j *have* $0\leqq T_1\leqq T_2\leqq B$, *and for* $t\geqq t_0$,

$$\|T_2(.,t)-T_1(.,t)\|_{L_2(\Omega)}\leqq B(t)\{\|\psi_2-\psi_1\|_{L_2(\Omega)}+\|g_2-g_1\|_{L_2\,\partial\Omega)\times(t_0,\infty)}\}$$

for a continuous function $B(t)$, *independent of the choice of the* ψ_j, g_j.

There is a unique $T:\Omega\times(t_0,\infty)\to\mathbf{R}$ with $T_1\leqq T\leqq T_2$ a.e., for every such choice of the ψ_j, g_j, and it is of class $C^{2+r,\,1+r/2}$ in the interior $\Omega\times(t_0,\infty)$, $T(x,t)\to\psi(x)$ a.e. as $t\to t_0+$. For any $0<\theta<1$, $t_1>t_0$, and compact $K\subset\bar{\Omega}$, $T\,|\,K\times[t,t+1]$ is uniformly bounded in $C^{\theta,\,\theta/2}$ for $t\geqq t_0$. (The boundary condition holds in a weak sense we won't specify.)

Remark. — This result serves mainly to justify restricting attention to continuous g, but the integral estimate, or a similar estimate, will be useful in another context.

THEOREM 3. — *In addition to the hypotheses of the first paragraph of Theorem* 1, *suppose* $\partial\Omega$ *is of class* C^{2+r} *and:* $g(x,t)=KN(x).\nabla_x G(x,t)$ *where* G *is locally* $C^{2+r,\,1+r/2}$ *or equivalently:*

$t\mapsto g(x,t)$ *is locally* $C^{(1+r)/2}$, $x\mapsto g(x,t)$ *is differentiable, and* $(x,t)\mapsto\partial_x g(x,t)$ *is locally* $C^{r,\,r/2}$ (*where "*∂_x*" indicates first order operators acting tangentially on* $\partial\Omega$).

A sufficient condition is that g be C^{1+r} in both variables.

We also suppose that $0\leqq\psi(x)\leqq B$ and $0\leqq g(x,t)\leqq\sigma B^4\varepsilon/\alpha$.

Then if ψ is continuous, the solution T is $C^{2+r,\,1+r/2}$ in compact subsets of $\bar{\Omega}\times(t_0,\infty)$, though merely continuous as $t\to t_0+$.

If $\psi\in C^{2+r}(\bar{\Omega})$ and is compatable with g on $\partial\Omega\times\{t=t_0\}$, that is

$$KN(x).\nabla\psi(x)+\sigma\varepsilon(x)\psi^4(x)=\alpha(x)g(x,t_0)$$

on $\partial\Omega$, then the solution is $C^{2+r,\,1+r/2}$ on compact subsets of $\bar{\Omega}\times[t_0,\infty)$.

Remark. — Theorem 3 follows from the Schauder estimates, once we have a sufficiently smooth solution $[C^{3,\,3/2}]$ for the case of smooth data; and this is proved by examination of the integral equation used for Theorem 1.

Outline of proof of Theorem 1

The argument is similar to that of Friedman [3], Section 2 of Chapter 5, so many details will be omitted.

We first extend ρ, C, K to the whole space, preserving smoothness and positivity properties, and construct the fundamental solution $\Gamma(x,t;y,s)$,

$$\rho C\,\partial_t\Gamma=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial}{\partial x_j}\Gamma\right)$$

for $t>s$, and for any continuous bounded φ and any $x\in\mathbf{R}^n$, as $t\to s+$,

$$\int_{\mathbf{R}^n}\Gamma(x,t;y,s)\,\varphi(y)\,dy\to\varphi(x).$$

In case ρ, C, K are constants and $a = \mathrm{K}/\rho\,\mathrm{C}$,

$$\Gamma(x, t; y, s) = \frac{(4\pi)^{-n/2}}{\sqrt{\det a}} \exp\left(-\frac{(x-y).a^{-1}(x-y)}{4(t-s)}\right),$$

and in general, this is the leading (most singular) part of Γ when ρ, C, K are evaluated at the point y. Since the coefficients don't depend on time,

$$\Gamma(x, t; y, s) = \Gamma(x, t-s; y, 0).$$

The solution is represented as

$$\mathrm{T}(x, t) = \tilde{\psi}(x, t) + \int_{t_0}^{t} ds \int_{\partial\Omega} d\mathrm{A}_y\, \Gamma(x, t; y, s)\, p(y, s),$$

where

$$\tilde{\psi}(x, t) = \int_{\mathbf{R}^n} \Gamma(x, t; y, t_0)\, \tilde{\psi}(y)\, dy,$$

for $x \in \Omega$, $t > t_0$, and $\tilde{\psi}$ is a continuous extension of ψ to all $\mathbf{R}^n$ with $0 \leqq \tilde{\psi}(x) \leqq \mathrm{B}$ and p is a continuous function on $\partial\Omega \times (t_0, \infty)$ which is not too singular as $t \to t_0$:

$$|p(x, t)| = o((t-t_0)^{-1/2})$$

uniformly for $x \in \partial\Omega$ as $t \to t_0+$.

In this case, T is a solution of the differential equation of class $\mathrm{C}^{2+r,\,1+r/2}$ in $\Omega \times (t_0, \infty)$ which extends continuously to the closure $\bar{\Omega} \times [t_0, \infty)$ with $\mathrm{T}(x, t) \to \psi(x)$ as $t \to t_0$, uniformly for x in compact sets in $\bar{\Omega}$. We will choose p so the boundary condition holds, and we find that this implies the "not too singular" condition above, so this is indeed a solution of problem (PDE).

According to Friedman [3], Thm. 1 of Sec. 2, Chap. 5, the boundary condition holds at $(x, t) \in \partial\Omega \times (t_0, \infty)$ if and only if

$$\frac{1}{2} p(x, t) + \int_{t_0}^{t} ds \int_{\partial\Omega} d\mathrm{A}_y \frac{\mathrm{KN}(x)}{\rho\,\mathrm{C}(x)} . \nabla_x \Gamma(x, t; y, s)\, p(y, s)$$
$$+ \frac{\mathrm{KN}(x)}{\rho\,\mathrm{C}(x)} . \nabla_x \tilde{\psi}(x, t) + \frac{\sigma\varepsilon(x)}{\rho\,\mathrm{C}(x)} \left(\tilde{\psi}(x, t) + \int_{t_0}^{t} \int_{\partial\Omega} \Gamma(x, t, .)\, p\right)^4$$
$$= \frac{\alpha(x)}{\rho\,\mathrm{C}(x)} g(x, t).$$

This is a Volterra integral equation with a mildly singular (absolutely integrable) kernel, and a local solution $(t_0 \leqq t \leqq t_1)$ may be found by iteration.

It is easy to show

$$\int_{\mathbf{R}^n} \left| \frac{\partial}{\partial x} \Gamma(x, t; y, s) \right| dy = O(t-s)^{-1/2}$$

and

$$\int_{\mathbf{R}^n} \frac{\partial}{\partial x} \Gamma(x, t; ., s) = o(t-s)^{-1/2},$$

uniformly in compact sets of x, as $t \to s+$, so by continuity of ψ, $\tilde{\psi}$, also $\frac{\partial}{\partial x} \tilde{\psi}(x, t) = o(t-t_0)^{-1/2}$ when $t \to t_0+$, uniformly in compacts. It folows that $p(x, t) = o(t-t_0)^{-1/2}$, as desired.

It is also easily proved that, if $\tilde{\psi}_v \to \tilde{\psi}$ uniformly on $\mathbf{R}^n$ and $g_v \to g$, uniformly on $\partial\Omega \times [t_0, t_1]$, the corresponding $T_v \to T$, uniformly on compact sets in $\bar{\Omega} \times [t_0, t_1]$.

Suppose the integral equation (**) has a solution p on (t_0, t^*), $t_0 < t^* < \infty$, such that $T \mid \partial\Omega = \tilde{\psi}(x, t) + \int_{t_0}^{t} \int_{\partial\Omega} \Gamma(x, t, .) p$ is bounded on $\partial\Omega \times (t_0, t^*)$. It follows that the solution $p(x, t)$ converges as $t \to t^*-$, uniformly on $\partial\Omega$, and the solution may be extended beyond t^*. Thus an *a priori* bound for T will ensure global existence, and such a bound follows from the maximum principle.

First assume the solution exists on $\{t_0 \leqq t \leqq t_1\}$ and there is a constant δ such that $0 < \delta < B$, $\delta \leqq \tilde{\psi} \leqq B - \delta$ on $\mathbf{R}^n$, and $\delta \leqq \frac{g(x, t)\,\alpha(x)}{\sigma\varepsilon(x)} \leqq (B-\delta)^4$ on $\partial\Omega \times [t_0, t_1]$. Then we show $0 < T(x, t) < B$ on $\bar{\Omega} \times [t_0, t_1]$. In fact, $\delta \leqq \tilde{\psi} \leqq B - \delta$ always, and $T(x, t) - \tilde{\psi}(x, t) \to 0$ as $|x| \to \infty$, (so dist $(x, \partial\Omega) \to \infty$), uniformly in $t_0 \leqq t \leqq t_1$. Thus, if the result fails, there exists (finite) x in $\bar{\Omega}$ and t in $(t_0, t_1]$ such that $0 < T < B$ in $\bar{\Omega} \times [t_0, t)$, but $T = 0$ or $T = B$ at (x, t). Perhaps $x \in \Omega$ — but then we violate Thm. 1, Sec. 1, Chap. 2, of [3]. Therefore $x \in \partial\Omega$ and $KN.\nabla T \leqq 0$ [or $\geqq 0$] where $T = 0$ [or, $T = B$] at (x, t), so $g \leqq 0$ [or $g \geqq \sigma B^4 \varepsilon/\alpha$], both of which are false.

By continuous dependence, $0 \leqq \tilde{\psi} \leqq B$ and $0 \leqq g \leqq \sigma B^4 \varepsilon/\alpha$ imply $0 \leqq T \leqq B$ on the interval of existence, so the solution exists for all $t \geqq t_0$. The representation of the solution and Friedman's [3], Thm. 3, Sec. 4, Chap. 5, give uniform bounds in $C^{\theta, \theta/2}$.

Outline of proof of Theorem 2

We only prove the integral estimate for $v = T_2 - T_1$. We have $\rho C v_t = \mathrm{div}(K \nabla v)$ in $\Omega \times (t_0, \infty)$, $0 \leqq T_j \leqq B$ always, and on $\partial\Omega \times (t_0, \infty)$,

$$KN.\nabla v + Q v = \alpha(g_2 - g_1)$$

with

$$Q = \sigma\varepsilon(T_1^3 + T_1^2 T_2 + T_1 T_2^2 + T_2^3),$$

so $0 \leqq Q \leqq 4\sigma\varepsilon B^4$. Let $I(t) = \int_\Omega \frac{1}{2} \rho C v(., t)^2$ and then

$$dI/dt = -\int_\Omega \nabla v . K \nabla v - \int_{\partial\Omega} Q v^2 + \int_{\partial\Omega} v \alpha (g_2 - g_1).$$

For any $\varepsilon > 0$ there exists C_ε so that

$$\int_{\partial\Omega} \varphi^2 \leqq \varepsilon \int_\Omega \nabla \alpha . K \nabla \varphi + C_\varepsilon \int_\Omega \frac{\rho C}{2} \varphi^2$$

for every $\varphi \in H^1(\Omega)$, hence with $\varepsilon = 1$, we find for all $t \geqq t_0$,

$$\frac{1}{2} \int_\Omega \rho C v^2 (., t) \leqq e^{C_1 (t - t_0)} \left\{ \frac{1}{2} \int_\Omega \rho C v^2 (., t_0) + \int_{t_0}^t \int_{\partial\Omega} \frac{\alpha^2}{4} (g_2 - g_1)^2 \right\}.$$

Outline of proof of Theorem 3

As noted earlier, we only need to show T is of class $C^{3, 3/2}$ when the data are smooth. Assume that $\partial\Omega$, K, ρC, $\varepsilon\alpha$ are C^∞ and also ψ and g are C^∞. We will not yet assume compatability at $\partial\Omega \times \{t = t_0\}$.

(1) Examination of the construction of the fundamental solution Γ in [3], Chap. 1, shows

$$\Gamma(x, t; y, s) = (t - s)^{-n/2} \tilde{\Gamma}(x, t; y, s; (x - y)/\sqrt{t - s})$$

where $\tilde{\Gamma}(x, t; y, s; z)$ is C^∞ for $t > s$ with

$$(t - s)^{(\alpha + \alpha')/2} (1 + |z|)^N \partial_t^\alpha \partial_s^{\alpha'} (\partial_t + \partial_s)^{\alpha''} \partial_x^\beta \partial_y^{\beta'} (\partial_x + \partial_y)^{\beta''} \partial_z^\gamma \tilde{\Gamma}(x, t; y, s; z)$$

uniformly bounded for $0 < t - s$ bounded and all choices of x, y, $z \in \mathbf{R}^n$ and all non-negative N, $\alpha\alpha'$, ..., γ. [In our case, $\tilde{\Gamma}(x, t; y, s) = \tilde{\Gamma}(x, t - s; y, 0)$.]

(2) Define

$$F_1(t) \varphi(x) = \int_{\partial\Omega} \Gamma(x, t; y, 0) \varphi(y) dA_y$$

$$F_2(t) \varphi(x) = \int_{\partial\Omega} \frac{KN(x)}{\rho C(x)} . \nabla_x \Gamma(x, t; y, 0) \varphi(y) dA_y$$

for $x \in \partial\Omega$, $t > 0$.

LEMMA. — *For any $t^* > 0$ and integer $m \geqq 0$, there is a number $C_m(t^*)$ such that $F_j(t)$ is a bounded operator on $C^m(\partial\Omega)$ to itself with norm $\leqq C_m t^{-1/2}$ for $0 < t \leqq t^*$ and $j = 1, 2$.*

Proof. — The kernel of the integral operator F_j may be expressed $t^{-n/2} \tilde{\Phi}(x, t; y, 0; (x - y)/\sqrt{t - s})$ where $\tilde{\Phi}$ satiafies estimates similar to those above for $\tilde{\Gamma}$, in particular, for any $N \geqq n$, there is a constant C_N so

$$|\tilde{\Phi}(x, t; y, O; z)| \leqq C_N (1+|z|)^{-N}.$$

If $r>0$, then

$$\int_{\partial\Omega \setminus B_r(x)} t^{-n/2} |\tilde{\Phi}| \, dA_y \leqq \int_{\partial\Omega} C_N t^{-n/2} (1+r/\sqrt{t})^{-N} \leqq O(t^{(N-n)/2})$$

is bounded on $O<t\leqq t^*$ for $x\in\partial\Omega$. If r is small, we may "straighten" the boundary near x so

$$\int_{\partial\Omega \cap B_r(x)} t^{-n/2} |\tilde{\Phi}| \leqq C'_N t^{-n/2} \int_{|\eta|<r,\, \eta\in \mathbf{R}^{n-1}} d\eta/(1+|\eta|/\sqrt{t})^{-N} \leqq O(t^{-1/2})$$

This gives the result for $m=1$.

Let σ be a smooth tangent vector field on $\partial\Omega$. If φ is continuously differentiable on $\partial\Omega$ and $t>0$,

$$\partial_\sigma (F_j(t)\varphi)(x) = \int_{\partial\Omega} t^{-n/2} \tilde{\Phi}_\sigma (x, t; y, 0; (x-y)/\sqrt{t})\, \varphi(y) + \int_{\partial\Omega} t^{-n/2} \tilde{\Phi}(x, t; y, 0; (x-y)/\sqrt{t})\, \partial_\sigma \varphi(y)$$

where $\tilde{\Phi}_\sigma$ is obtained by differentiation of σ and $\tilde{\Phi}$, and satisfies the same sort of estimates as $\tilde{\Phi}$ or $\tilde{\Gamma}$, so the same argument gives the result for $m=1$, and by induction, for all m.

(3) Choosing $t_0=0$, the integral equation for p may be written

$$p(t)/2 + \int_0^t F_2(t-s)\, p(s)\, ds + \frac{\sigma\varepsilon}{\rho C}\left(\tilde{\psi}(., t) + \int_0^t F_1(t-s)\, p(s)\, ds\right)^4 = r(t)$$

where $r(t) = \dfrac{1}{\rho C}(\alpha g(., t) - KN.\tilde{\psi}(., t))$. For any $m\geqq 0$, $t\mapsto r(t)$, $\tilde{\psi}(., t)\in C^m(\partial\Omega)$ are C^∞; we may solve the integral equation for a continuous $p:[0, \infty)\to C^m(\partial\Omega)$ such that

$$\frac{\rho C}{2} p(0) = \alpha g(., 0) - KN.\nabla\tilde{\psi} - \sigma\varepsilon\tilde{\psi}^4 \qquad \text{on} \quad \partial\Omega$$

so the compatability condition says precisely that $p(0)=0$.

Suppose $p(0)=0$ and for small $\delta>0$ define $q_\delta(t)=\frac{1}{\delta}(p(t+\delta)-p(t))$. Then

$$q_\delta(t)/2=\frac{1}{\delta}\int_0^\delta F_2(t+\delta-s)p(s)\,ds+\int_0^t F_2(t-s)q_\delta(s)\,ds$$
$$+\frac{1}{\delta}\frac{\sigma\varepsilon}{\rho C}\left\{(\tilde{\psi}(t+\delta)+\int_0^t F_1(t-s)p(s+\delta)\right.$$
$$\left.+\int_0^\delta F_1(t+\delta+s)p(s))^4-(\tilde{\psi}(t)+\int_0^t F_1(t-s)p(s))^4\right\}$$
$$=\frac{1}{\delta}(r(t+\delta)-r(t)).$$

Since $p(0)=0$, $q_\delta(t)\to\dot{p}(t)$ as $\delta\to 0$, where $\dot{p}$ is the solution of

$$\dot{p}(t)/2+\int_0^t F_2(t-s)\dot{p}(s)+\frac{4\sigma\varepsilon}{\rho C}(\tilde{\psi}(t)+F_1*p(t))^3(\dot{\tilde{\psi}}(t)+F_1*\dot{p}(t))=\dot{r}(t).$$

Thus $p:[0,\infty)\to C^m(\partial\Omega)$ is continuously differentiable. Since we only assumed zero-order compatability $[p(0)=0]$, in general $\dot{p}(0)\neq 0$, but a similar argument shows $\dot{p}(t)$ is C^1 for $t>0$ with $\|\ddot{p}(t)\|_{C^m(\Omega)}=O(t^{-1/2})$ as $t\to 0+$, so $p\in C^{3/2}_{\text{loc}}([0,\infty), C^m(\partial\Omega))$.

(4) In the interior Ω, the solution is C^∞, so it is enough to prove smoothness near the boundary. Since p is quite smooth, it is easy to show tangential derivatives extend continuously to the boundary. We find the same holds for the time-derivatives using

$$\begin{pmatrix}\partial_t\\ \partial_t^2\end{pmatrix}(T(x,t)-\tilde{\psi}(x,t))$$
$$=\int_0^t\int_{\partial\Omega}\Gamma(x,t-s,.)\begin{pmatrix}\dot{p}\\ \ddot{p}\end{pmatrix}(s)+\begin{pmatrix}0\\ \int_{\partial\Omega}\Gamma(x,t-s,.)\dot{p}(0)\end{pmatrix}$$

From the boundary condition, $KN.\nabla T$ is smooth *on* the boundary. Introducing appropriate coordinates near $\partial\Omega$ and changing variables in the differential equation, we see the normal derivatives are also smooth to the boundary, so T is locally $C^{3,3/2}$, as claimed.

Asymptotic behavior of solutions

Examination of the calculations for Theorem 2 shows we could find much better estimates if we had a positive lower bound for the temperature on $\partial\Omega$. We obtain such an estimate from the maximum principle on any compact set in $\bar{\Omega}$, such as $\partial\Omega$, provided $\partial\Omega$ is C^2. With mild "compactness" and "positivity" hypotheses on g, we obtain a positive lower bound, uniformly fot $t \geqq$ const., and thus obtain convergence as $t \to \infty$. If the heating rate is periodic or almost-periodic in time, the limiting "steady-state" solution is equally periodic or almost-periodic.

LEMMA. — *Assume* $\partial\Omega$ *is* C^2 (*or has an interior sphere at each point*) *and the hypotheses of Theorem* 1 *hold. If, for some* $t > t_0$, *and* (*finite*) $x \in \bar{\Omega}$, *we have* $T(x, t) = 0$, *then* $T \equiv 0$ *in* $\bar{\Omega} \times [t_0, t]$. *Equally, if* $\psi \not\equiv 0$ *on* $\bar{\Omega}$ *or* $g \not\equiv 0$ *on* $\partial\Omega \times [t_0, t]$, *then* $T(x, t) > 0$.

Proof. — If $x \in \Omega$, the result is immediate from Thm. 1, Sec. 1, Chap. 2 of [3]. If $T \not\equiv 0$ in $\bar{\Omega} \times [t_0, t]$ then $T > 0$ in $\bar{\Omega} \times (0, t)$ and in $\Omega \times \{t\}$, while $T(x, t) = 0$, so $KN.\nabla T < 0$ at (x, t) and $g(x, t) < 0$, which is false. (Friedman's, Thm. 14, Sec. 5, Chap. 2 of [3], does not apply, but a similar argument — using a parabolic "cup" instead of an ellipsoid — gives the result. For the one-dimensional heat equation, the argument is in Cannon [2].)

We will use the following hypotheses:

The compactness assumption: For any compact interval $J \subset \mathbf{R}$ and any sequence $t_n \to \infty$, there is a subsequence $t_{n'} \to \infty$ such that $\{g(.,\ . + t_{n'}) \mid \partial\Omega \times J\}$ is a Cauchy sequence in $C(\partial\Omega \times J, \mathbf{R})$.

The positivity assumption: For some positive constants τ_0, c_0, we have

$$\int_t^{t+\tau_0} \int_{\partial\Omega} g \geqq c_0 \quad \text{for all } t \geqq t_0.$$

THEOREM 4. — *Suppose* $\partial\Omega$ *is* C^2 *or has interior spheres and the hypotheses of Theorem* 1 *hold. We also suppose that either* Ω *is bounded or that* $n = 1$, $\Omega = (0, \infty)$, *and* ρ, C, K, ε, α *are constants. We assume given a solution* $\tilde{T}$ *of the problem* (PDE) *on* $\Omega \times (t_0, \infty)$.

(*a*) *If the compactness assumption holds, any sequence* $t_n \to \infty$ *has a subsequence* $t_{n'}$ *such that*

$$g^*(x, t) = \lim_{n' \to \infty} g(x, t + t_{n'})$$
$$T^*(x, t) = \lim_{n' \to \infty} T(x, t + t_{n'})$$

exist uniformly on compact sets of $\partial\Omega \times \mathbf{R}$, $\bar{\Omega} \times \mathbf{R}$, *respectively, and* T^* *is a solution of* (PDE) — *with* g^* *in place of* g — *on all* $\Omega \times \mathbf{R}$, *satisfying* $0 \leqq T^* \leqq B$ *everywhere.*

(*b*) *If the compactness and positivity assumptions hold, then for any compact* $K \subset \bar{\Omega}$, $\tilde{T}(x, t) \geqq \text{const.} > 0$ *on* $K \times [t_0 + \tau_0, \infty)$. *Given any sequence* $t_n \to \infty$, *such that* $\lim_{n \to \infty} g(.,. + t_n) = g^*$ *exists uniformly on every compact set in* $\partial\Omega \times \mathbf{R}$, *it follows that*

$$\lim_{n \to \infty} \tilde{T}(x, t + t_n) = T^*(x, t)$$

exists uniformly on compacts in $\bar{\Omega} \times \mathbf{R}$, *the limit solution* T^* *depends only on* g^* (*independent of the initial value and the sequence* t_n) *and satisfies* $T^* \geqq \text{const.} > 0$, *uniformly in time, on each compact subset of* $\bar{\Omega}$.

(*c*) *If* $t \mapsto g(., t)$ *is uniformly almost periodic* [*or periodic with period* $p > 0$, *or constant*], $0 \leqq g \leqq B$ *as usual, and* $g \not\equiv 0$, *then the compactness and positivity assumptions hold.*

There is a unique non-negative bounded solution T of (PDE) on all $\Omega \times \mathbf{R}$, and it is almost periodic in t, uniformly for $x \in \bar{\Omega}$, with frequency module contained in that of g [or is periodic with period p, or is constant in time, respectively], and it is bounded from zero on every set $K \times \mathbf{R}$, K compact in $\bar{\Omega}$. Any non-negative solution $\tilde{T}$ of (PDE) on $\Omega \times (t_0, \infty)$ satisfies

$$\tilde{T}(x, t) - T(x, t) \to 0 \qquad \text{as} \quad t \to \infty,$$

uniformly on compact sets of x, and the convergence is exponential when Ω is bounded.

Proof. – (*a*) Since $\{ T(., t) \,|\, K \}$ is bounded in $C^\theta(K)$, for any compact K and $0 < \theta < 1$, it is in a compact subset. We may choose, by the diagonal process, a subsequence $\{ t_{n'} \}$ of $\{ t_n \}$ such that $g(., . + t_{n'}) \to g^*$ and also

$$T_k^*(x) = \lim_{n' \to \infty} \tilde{T}(x, t_{n'} - k)$$

exists uniformly on compact sets in $\bar{\Omega}$ for each $k = 1, 2, 3, \ldots$ It follows easily that $T^* = \lim_{n'} \tilde{T}(., . + t_{n'})$ exists uniformly on compacts in $\bar{\Omega} \times \mathbf{R}$ and is the limit solution claimed in (*a*).

(*b*) Suppose $K \subset \bar{\Omega}$ is compact and $\liminf_{t \to \infty, x \in K} \tilde{T}(x, t) = 0$. There exist $x_n \in K$, $t_n \to \infty$ so $\tilde{T}(x_n, t_n) \to 0$. We may assume, by (*a*), that $\tilde{T}(., . + t_n) \to T^*$, uniformly on compacts, and $x_n \to x^*$, so $T^*(x^*, 0) = 0$. By the lemma, $T^* \equiv 0$ on $\Omega \times (-\infty, 0]$ so $g^* = 0$ on $\partial\Omega \times (-\infty, 0]$, contracting the positivity hypothesis. Thus $\tilde{T} \geqq \text{const.} > 0$ on $K \times [t_0 + \tau_0, \infty)$. If $g(., . + t_n) \to g^*$, by (*a*), we have convergence of translates of $\tilde{T}$ by *some subsequence* of $\{ t_n \}$. But we prove the limit solution T^* is unique, depending only on g^*, so in fact, $\tilde{T}(., . + t_n) \to T^*$ uniformly on compacts.

Indeed, $T^*(x, t) \geqq c > 0$ on $\partial\Omega \times \mathbf{R}$ (recall $\partial\Omega$ is compact) so for any bounded, non-negative solution $\hat{T}$ of (PDE) on all $\Omega \times \mathbf{R}$ – with g^* in place of g – the difference $v = T^* - \hat{T}$ satisfies

$$\begin{aligned} \rho C \partial v &= \operatorname{div}(K \nabla v) && \text{on} \quad \Omega \times \mathbf{R} \\ KN . \nabla v + Q v &= 0 && \text{on} \quad \partial\Omega \times \mathbf{R} \end{aligned}$$

and $|v| \leqq \text{const.} < \infty$ on $\Omega \times \mathbf{R}$, $Q \geqq \sigma \varepsilon c^4$ so $0 < c_1 \leqq Q$ for some constant c_1. We show $v \equiv 0$.

If Ω is bounded, there is a $C_3>0$ so that

$$\int_\Omega \nabla\varphi.K\nabla\varphi+C_1\int_{\partial\Omega}\varphi^2\geqq C_3\int_\Omega \frac{\rho C}{2}\varphi^2,\quad \text{for all } \varphi\in H^1(\Omega).$$

Thus,

$$\frac{d}{dt}\int_\Omega \rho C v^2/2=-\int_\Omega \nabla v.K\nabla v-\int_{\partial\Omega} Q v^2\leqq -C_3\int_\Omega \rho C v^2/2$$

so

$$\int_\Omega \rho C v(.,t)^2\leqq\int_\Omega \rho C v(.,s)^2\exp(-C_3(t-s))$$

when $t\geqq s$. Let $s\to-\infty$ to see $v\equiv 0$.

If Ω is not bounded, $\Omega=(0,\infty)$ and ρ, C, K, α, ε are constant. From the maximum principle, $|v(x,t)|\leqq M w(x,t-s)$ for $t\geqq s$, where $M=\max|v(.,s)|$ and w is the solution of

$$\rho C w_t=w_{xx}\qquad (t>0,\ x>0)$$

and

$$K w_x=C_2 w\qquad (x=0,\ t>0)$$

with $w=1$ for $t=0$, $x>0$. It suffices to show $w(x,t)\to 0$ as $t\to\infty$, uniformly for x in compact sets.

We rescale t and x to get $w_t=w_{xx}$, $w_x=w$ at $x=0$, with $w=1$ at $t=0$. By Laplace transformation in t, we find

$$w(x,t)=\frac{2}{\pi}\int_0^\infty ds\frac{e^{-s^2t}}{1+s^2}\left(\cos xs+\frac{\sin xs}{s}\right)$$

which is $O(t^{-1/2})$ as $t\to\infty$ with x bounded. In fact,

$$0<w(x,t)<(1+x)/\sqrt{\pi t}$$

and, from another representation, $0<1-w(x,t)<\exp-x^2/y$, for $x>0$, $t>0$. It is tather weak "convergence to 0" since $w(x,t)\to 1$ as $x\to\infty$ for each $t\geqq 0$.

(*c*) By almost-periodicity, we may choose $t_n\to\infty$ so that

$$g(x,t+t_n)-g(x,t)\to 0$$

uniformly; for any solution $\tilde{T}$, $\lim\tilde{T}(.,.+t_n)=T$ exists uniformly on compacts, and T is the unique positive bounded solution of (PDE) on $\Omega\times\mathbf{R}$. The argument above shows $T(x,t+t_n)-T(x,t)\to 0$ uniformly on compact sets of $\bar{\Omega}\times\mathbf{R}$, but we need uniformity on $K\times\mathbf{R}$ for every compact $K\subset\bar{\Omega}$.

Let $v_n(x, t) = T(x, t+t_n) - T(x, t)$; then

$$\rho C \partial v_n/\partial t = \operatorname{div}(K \nabla v_n) \quad \text{on} \quad \Omega \times \mathbf{R}$$
$$KN . \nabla v_n + Q_n v_n = \alpha (g_n - g) \quad \text{on} \quad \partial\Omega \times \mathbf{R}$$

where $g_n = g(.,. + f_n)$ and $Q_n \geqq C_2 > 0$ for all n, t, x.

Define $w(x, t)$ as the solution of

$$\rho C \partial w/\partial t = \operatorname{div}(K \nabla w) \quad \text{in} \quad \Omega \times \mathbf{R}_+$$
$$KN . \nabla w + C_2 w = 0 \quad \text{on} \quad \partial\Omega \times \mathbf{R}_+$$

with $w \equiv 1$ when $t = 0$. Then for any real s,

$$|v_n(x, t)| \leqq M w(x, t) + \varepsilon_n/C_2 \quad \text{when} \quad t \geqq s,$$

where $M = \sup T(\Omega \times \mathbf{R})$ and $\varepsilon_n = \sup_{\partial\Omega \times \mathbf{R}} |g_n - g|$. Allowing $s \to -\infty$, we find $|v_n| \leqq \varepsilon_n/C_2$ on $\Omega \times \mathbf{R}$. When $n \to \infty$, $\varepsilon_n \to 0$ and $v_n = T(.,. + t_n) - T \to 0$ uniformly, proving almost-periodicity: *see* Ameiro and Prouse [1].

If g is periodic with period p, and if T is a "limit" solution, bounded and defined for all time, then $T(x, t+p)$ is another limit solution; there is only one limit solution, so it is periodic in time with period p. If g is constant in time, it is p-periodic for every $p > 0$, so the same is true of the limit solution, which is constant in time.

The convergence results are proved as above, by comparison with w.

Remark. — Consider a homogeneous, isotropic ball of radius R in $\mathbf{R}^3$; the solutions of

$$\rho C \partial T/\partial t = K \Delta T \quad \text{in} \quad \{|x| < R\}, \qquad T = T_S \quad \text{on} \quad \{|x| = R\}$$

tend to the surface temperature T_S like $\exp(-t/\tau_R)$, where

$$\tau_R = R^2/(\pi^2 K/\rho C)$$

is the thermal relaxation time. For water-ice and various types of rocks, the diffusivity $K/\rho C$ is about 0.005 m^2/s, so for $R = 1$ km [10 km, 100 km], the relaxation time is about 2,300 years [230,000 yr, 23×10^6 yr]. Thus we can expect thermal equilibrium in any asteroid.

APPROXIMATE SOLUTIONS

An asteroid's "day" — with few exceptions, from 3 to 50 hours — is many hundreds of times smaller than its "year" — 3 to 6 of our years. We will first neglect the yearly variation, consider periodic (daily) heating, and find the temperature a short distance below the surface (perhaps a meter). This is independent of the daily rotation but still varies during the year. A second (yearly) average gives a temperature independent of time, still near the surface, which we use as a boundary value for an equilibrium problem in the interior.

We rescale time so the daily variation has period 2π. We also rescale depth in units δ (the skin depth, specified later-probably less than 10 cm). The daily variation is confined to a depth of order δ, far smaller than the diameter of the asteroid, and the variation of temperature with depth is far more rapid than the variation along the surface. Thus the heat equation becomes approximately

$$\omega\rho c\frac{\partial T}{\partial t}=\frac{N.KN}{\delta^2}\frac{\partial^2 T}{\partial s^2}$$

at depth δs below a given point of the surface, $\omega=2\pi/$(period of rotation). We choose $\delta=\sqrt{\frac{N.KN}{\rho c}\cdot\frac{1}{\omega}}$ ("skin depth") and also define "conduction depth" $\lambda=T_0 N.KN/\alpha f$ and "maximum surface temperature" $T_0=(\alpha f/\varepsilon\sigma)^{1/4}$. (Applied to Moon and Mercury, we find $T_0=390$ and 700 K respectively, which are in fact close to the observed maximum surface temperatures. A temperature gradient T_0/λ produces heat flux $N.KNT_0/\lambda$ equal to the maximum absorbed heat flux αf.)

Normalizing temperature $T/T_0=u$ and the heating rate $g/\alpha f=\gamma$, as functions of normalized time t and normalized depth s, we obtain

$$\left.\begin{aligned}&\partial u/\partial t=\partial^2 u/\partial s^2 \qquad (s>0)\\ &\lambda/\delta\frac{\partial u}{\partial s}=u^4-\gamma \qquad (s=0)\end{aligned}\right\} \qquad (*)$$

where γ is 2π-periodic in $t:\gamma=\cos$ (angle from zenith to Sun) when the Sun is visible, $\gamma=0$ when it is not visible, $0\leqq\gamma\leqq 1$. This equation has a unique non-negative bounded solution u on $\{-\infty<t<\infty,\ s\geqq 0\}$, which is in fact strictly positive $(0<u<1)$ and 2π-periodic in time. We write $u(s,t)=\sum_{-\infty}^{\infty} u_k e^{ikt-p_k s}$, where $p_k=\sqrt{|k|}e^{\pm i\pi/4}$ ($\pm=\operatorname{sgn} k$) so $p_k^2=i$, $\operatorname{Re} p_k\geqq 0$, and the boundary condition becomes

$$\sum_{-\infty}^{\infty} p_k u_k e^{ikt}+(u_0+\Sigma' u_k e^{ikt})^4=\gamma(t)=\sum_{-\infty}^{\infty}\gamma_k e^{ikt}. \qquad (**)$$

The important quantity is $u_0=\lim_{s\to+\infty} u(st)$; the limit is essentially attained when $s=5$ ($|e^{-5p_1}|\approx 0.03$) so the temperature $T\approx u_0.T_0$ at a depth $5.\delta$, independent of daily variation.

Before studying (**), we compute λ and δ for various materials for the case $T_0=280$ K, $f=300$ kcal/hr.m^2, $\alpha=\theta=1$, which we might expect at 2 A.U., and for period 6 hours. (Recall δ is proportional to $\sqrt{\text{period}}$, so

δ should be doubled if the period of rotation is 1 day.) The following values come mostly from Kreith, *Principles of Heat Transfer*. (I've seen various values cited for Moon dust and don't have much confidence in these.)

Material	K [kcal/h.m.K]	K/ρc [m^2/h]	λ [m]	δ [m]	λ/δ
Window glass	.66	.0012	.62	.034	18
Water ice	1.9	.0045	1.8	.066	27
Granite	3.3	.006	3.1	.08	40
Basalt	1.9	.004	1.8	.06	30
Iron	62.5	.073	58	.26	220
Rockwool insulation	.031	.0009	.029	.029	1
Moon dust	.01	.0003	.008	.017	0.5

It is not claimed that asteroids are made of window glass or ice at 0°C; we only wish to display possible values of λ/δ. For Moon dust, under conditions on Moon, $-T_0=390$ K, period$=29\,1/2$ days$-\lambda/\delta$ is about 1/70, largely because of slow-rotation.

When λ/δ is large – excluding rockwool and Moon dust from the list above – equation (∗∗) is fairly simple. We have

$$u_0=\gamma^{1/4}+O(\delta/\lambda)^2, \qquad u_k=O(\delta/\lambda) \quad \text{for } k\neq 0$$

or in the second approximation

$$u_k\approx\gamma_k\Big/\left(4u_0^3+\frac{\lambda}{\delta}p_k\right) \qquad \text{for} \quad k\neq 0$$

$$u_0^4+6u_0^2\sum_{k\neq 0}\left|\gamma_k\Big/\left(4u_0^3+\frac{\lambda}{\delta}p_k\right)\right|^2\approx\gamma_0.$$

Consider, for example, a sphere with vertical axis of rotation and $\alpha=\varepsilon=1$. At colatitude θ (latitude $\pi/2-\theta$), $\gamma=(\cos t)_+ .\sin\theta$, where the normalized time has period 2π and midday is $t=0\,(\mathrm{mod}\,2\pi)$. Then

$$\gamma=(\cos t)_+\sin\theta=\sin\theta.\left(\frac{1}{\pi}+\frac{1}{2}\cos t+\frac{2}{3\pi}\cos 2t\right.$$
$$\left.-\frac{2}{15\pi}cos\,4+\frac{2}{35\pi}cos\,4t\ldots\right)$$

In the limit $\lambda/\delta\to\infty$ we have $u_0=\gamma_0^{1/4}=\left(\frac{1}{\pi}\sin\theta\right)^{1/4}$ at colatitude θ. Even for $\lambda/\delta=4$, at the equator $(\theta=\pi/2)$ the second approximation above gives $u_0=0.7413$, only 1% below $\pi^{-1/4}=0.7511$ for $\lambda/\delta=\infty$. (The same applies to nonspherical convex bodies, as noted later.)

Another simple limit is when $\lambda/\delta \to 0$: the boundary condition becomes $u^4 = \gamma$ or

$$u = (\sin\theta)^{1/4} (\cos t)_+^{1/4}$$
$$\approx (\sin\theta)^{1/4} . (0.4297 + 0.2964 \cos t + 0.0477 \cos 2t + \dots)$$

so $u_0 \approx 0.4297 (\sin\theta)^{1/4}$. This is substantially below $0.7511 (\sin\theta)^{1/4}$ predicted for the other limit, $\lambda/\delta \to \infty$. It is quite possible that Moon dust is a better thermal model for an asteroid surface than ice or solid rock. There is some evidence that asteroid do have dusty surfaces, but it may be dust sticky with hydrocarbon tars, which would have higher conductivity than Moon dust. Still it is important to know what happens when λ/δ is small. (For Moon itself, $\lambda/\delta \approx 0.014$; this is a difficult case, but it's the only case where we have believable measurements.)

To study the variation with λ/δ, we use the Galerkin method: for some k, set $u_j = 0$ for $|j| > k$ and choose $u_0, u_{\pm 1}, \dots, u_{\pm k}$ so that $u_{-j} = \bar{u}_j$ (so u is real valued) and (**) holds at least for frequencies $0, \pm 1, \dots, \pm k$. This leads to the systems of algebraic equations

$$(k=0) \qquad u_0^4 = \gamma_0$$

$$(k=1) \qquad \begin{cases} u_0^4 + 12 u_0^2 |u_1|^2 + 6 |u_1|^4 = \gamma_0 \\ u_1 \left(\dfrac{\lambda}{\delta} e^{i\pi/4} + 4 u_0^3 + 12 u_0 |u_1|^2 \right) = \gamma_1 \end{cases}$$

$$(k=2) \qquad \begin{cases} u_0^4 + 12 u_0^2 (|u_1|^2 + |u_2|^2) + 24 u_0 \operatorname{Re}(u_1^2 \bar{u}_2) \\ \qquad + 6 (|u_1|^2 + |u_2|^2)^2 + 12 |u_1|^2 |u_2|^2 = \gamma_0 \\ u_1 \left(\dfrac{\lambda}{\delta} e^{i\pi/4} + 4 u_0^3 + 12 u_0 (|u_1|^2 + 2 |u_2|^2) \right) \\ \qquad + 12 \bar{u}_1 u_2 (u_0^2 + |u_1|^2 + |u_2|^2) + 4 u_1^3 \bar{u}_2 = \gamma_1 \\ u_2 \left(\dfrac{\lambda}{\delta} \sqrt{2} e^{i\pi/4} + 4 u_0^3 + 12 u_0 (2 |u_1|^2 + |u_2|^2) \right) \\ \qquad + 6 u_0^2 u_1^2 + 6 \bar{u}_1^2 u_2^2 + 4 u_1^2 (|u_1|^2 + 3 |u_2|^2) = \gamma_2 \end{cases}$$

We take $\gamma = \sin\theta . (\cos t)_+$ as before, and use the system for $k=1$ with $\theta = \pi/2$:

$\lambda/\delta =$	$u_0 =$	$\|u_1\| =$
0	.54528	.24218
1/8	.61108	.19490
1/4	.64223	..17007
1/2	.67544	.14035
1	.70584	.10766
2	.72881	.07513
4	.74228	.04716
∞	.75113	0.

$\Big[$For colatitude θ, use $u_0(\sin\theta)^{1/4}$, where u_0 is found above with $\frac{\lambda}{\delta}(\sin\theta)^{-3/4}$ in place of $\frac{\lambda}{\delta}.\Big]$ This gives $u_0=.74228$ for $\lambda/\delta=4$, compared to .7413 computed earlier. The limit $\lambda/\delta \to 0$ should have $u_0=.4297$ rather than .5453, but this will (one hopes) improve for larger k. Note the rapid variations as $\lambda/\delta \to 0$: a 12% increase in u_0 when λ/δ increases from 0 to 1/8, but only 5% from 1/8 to 1/4 (or 1/4 to 1/2, or 1/2 to 1, or 1 to 4).

The axis of Moon is not vertical (to the ecliptic) but its inclination is only 1.53°, so this should apply if we avoid the poles. At latitude 20° (the landing site of *Apollo*-17) we would expect

$$\text{temperature}\begin{cases}209\,\text{K}\\235\,\text{K}\\247\,\text{K}\\260\,\text{K}\\288\,\text{K}\end{cases}\quad\text{for}\quad\lambda/\delta=\begin{cases}0\\1/8\\1/4\\1/2\\\infty.\end{cases}$$

In fact, $\lambda/\delta\approx 1/70$ and the average subsoil temperature is about 250 K. This is far from satisfactory, but at least shows variation of λ/δ suffices to explain the difference from 288 K which would be expected if $\lambda/\delta=\infty$. (The value of λ/δ is uncertain, and there may be other changes when we consider Galerkin systems with $k>1$.)

Take $\lambda/\delta=1/8$ and consider the variation with latitude. Note the "effective λ/δ" $=\frac{1}{8}(\cos\text{lat}))^{-3/4}$ becomes large near the poles. We compare the solution u_0 (for $k=1$) with that expected $u_0^{\infty}=\left(\frac{1}{\pi}\cos(\text{lat})\right)^{1/4}$ for $\lambda/\delta=\infty$.

Latitude	u_0/u_0^{∞}
0°	.8340
30°	.8401
60°	.8661
85°	.9447
89°	.9867

("Effective λ/δ" $=0.78$ or 2.6 at latitude 85° or 89°.) This gives approximately $u_0\approx 0.626(\cos(\text{lat}))^{1/5}$. The temperature of Moon, measured optically from Earth, varies more nearly as the sixth root of cos (latitude), rather than the fourth root expected for $\lambda/\delta=\infty$. This may be, in part, due to λ/δ-though it's not entirely clear which "temperature" is being measured.

Calculation of γ for a convex body

The geometry is simplest for a sphere, but calculation of γ – the cosine of the angle from zenith to Sun, while the Sun is visible – is the same for any convex body. (There would be complications for non-convex bodies, due to shadows.)

At a point of the surface $\partial\Omega$, define the "local co-latitude" as the angle from the zenith N (unit outward normal) to the positive axis of rotation. Taking $(0,0,1)$ as the axis of rotation and $(\sin\alpha, 0, \cos\alpha)$ as the direction of the Sun [α = angle from positive axis of rotation to the Sun, $0<\alpha<\pi$] for a point of $\partial\Omega$ with local co-latitude θ, hence coordinates $(\sin\theta\cos t, \sin\theta\sin t, \cos\theta)$ at local time or hour angle t, we find

$$\gamma(t) = (\cos\theta\cos\alpha + \sin\theta\sin\alpha\cos t)_+.$$

The time-average $\gamma_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\gamma$ is, for $0<\alpha<\pi/2$,

$$\gamma_0 = \begin{cases} \cos\theta\cos\alpha \\ \text{if } 0\leqq\theta\leqq\pi/2-\alpha \text{ (perpetual day)} \\ \dfrac{1}{\pi}\sin\theta\sin\alpha[\sin\psi_0-\psi_0\cos\psi_0] \\ \text{if } 0\leqq\psi_0\leqq\pi,\ \cos\psi_0=-\cot\theta\cot\alpha,\ \pi/2-\alpha\leqq\theta\leqq\pi/2+\alpha \\ 0 \\ \text{if } \pi/2+\alpha\leqq\theta\leqq\pi \text{ (perpetual night)} \end{cases}$$

To treat $\alpha>\pi/2$, exchange poles: γ is unchanged when we use $(\pi-\theta,\ \pi-\alpha)$ in place of $(\theta,\ \alpha)$. In the extreme case $\alpha=\pi/2$, we have $\gamma_0=\dfrac{1}{\pi}\sin\theta$, as expected for vertical axis.

Note

$$\gamma_0 = O\left(\theta-\frac{\pi}{2}-\alpha\right)^{3/2}$$

as $\theta\to\dfrac{\pi}{2}+\alpha$, and

$$\gamma_0-\cos\theta\cos\alpha = O\left(\theta-\frac{\pi}{2}+\alpha\right)^{3/2}$$

as $\theta\to\dfrac{\pi}{2}-\alpha$, so $\theta\mapsto\gamma_0(\theta,\ \alpha)$ is $C^{3/2}$.

Other Fourier coefficients of γ may be computed similarly, but γ_0 suffices when λ/δ is large.

Of course, α will vary during the year. Suppose the axis of rotation makes an angle β with the normal to the plane of revolution, and let φ

be the season angle [$\varphi=0\,(\mathrm{mod}\,2\pi)$ at Northern hemisphere mid-summer]. The axis of rotation has a fixed celestial direction $(-\sin\beta,\ 0,\ \cos\beta)$ and the direction to the Sun is $-(\cos\varphi,\ \sin\varphi,\ 0)$ so

$$\cos\alpha=\sin\beta\cos\varphi, \qquad \pi/2-\beta\leqq\alpha\leqq\pi/2+\beta.$$

For $\lambda/\delta=\infty$, the daily average γ_0 was computed above and $u_0=\gamma_0^{1/4}$; then the yearly average of u_0 (for a circular orbit) is

$$\bar{u}_0=\frac{1}{\pi}\int_0^{\pi/2}(\gamma_0^{1/4}(\theta,\alpha)+\gamma_0^{1/4}(\pi-\theta,\alpha))\,d\varphi,$$
$$\cos\alpha=\sin\beta\cos\varphi,$$

where we used the average of the "summer averages" at θ and $\pi-\theta$ to keep $\alpha\leqq\pi/2$. (We study non-circular orbits similarly – *see* "The effect of eccentricity" below.)

Vertical axis of rotation, $\lambda/\delta=\infty$

This is the simplest case. For rapid rotation or λ/δ large we take $u=\gamma_0^{1/4}$ on the surface. Assuming constant conductivity inside, and normalizing Ω to a unit ball, we must solve

$$\Delta u=0 \quad \text{in} \quad \{r<1\}$$
$$u=\left(\frac{1}{\pi}\sin\theta\right)^{1/4} \quad \text{on} \quad r=1 \quad (\text{colatitude } \theta).$$

The solution is $u(r,\theta=\sum_{n=0}^{\infty}C_{2n}r^{2n}P_{2n}(\cos\theta)$ where

$$C_n=(n+1/2)\int_{-1}^{1}P_n(x)\,(1-x^2)^{1/8}\,dx/\pi^{1/4}$$

$C_n=0$ for odd n,

$$C_{n+2}/C_n=(n+1)(n+5/2)(n-1/4)/[(n+2)(n+1/2)(n+13/4)]$$

for even n, and $C_0=\frac{1}{2}\pi^{1/4}(1/8)!/(5/8)!\cong 0.69920328$ is the average value.

We have used

$$(1-x^2)P_n'(x)+nx\,P_n(x)=n\,P_{n-1}(x)$$

and

$$(n+1)P_{n+1}(x)=(2n+1)x\,P_n(x)-n\,P_{n-1}(x)$$

for all $n\geqq 1$. The last [with $P_0=1$, $P_1(x)=x$] is used to compute the $P_n(\cos\theta)$; it is a stable iteration, trustworthy for hundreds of steps – which are needed near the poles. We have $C_n\approx-0.3860\,n^{-5/4}$ for large even n

and $|P_n(\cos\theta)| = O(n^{-1/2})$ away from $\theta=0$ or π, but convergence near the poles is very slow, so we record values near the pole ($\theta=0$, $r=1$).

Colatitude	$r=1$	0.95	0.9	0.8
$\theta=0$	0	.42758	.49635	.56930
5°	.40812	.46550	.51150	
10°	.48488	.51344	.54055	
15°	.53575	.55270	.56975	.60159
20°	.57442	.58507	.59612	
30°	.63162	.63560	.64000	.64966

The temperature is $T = T_0 . u$, where T_0 is the maximum surface temperature. (The effect of finite λ/δ is probably to decrease u except near the poles.)

The temperature near the poles is very sensitive to small changes in the inclination of the axis, as we see below, but the rest of the structure seems quite robust.

Inclination of axis 30°

Near the surface (for rapid rotation or large λ/δ) u = yearly average $\gamma_0^{1/4}$.

For $\beta = 30°$ we compute summer average $\gamma_0^{1/4}$ for steps of 15°:

$\theta=$		$\theta=$	
0	.72667	180°	0
15°	.73612	165°	.135321
30°	.74809	150°	.38609
45°	.76088	135°	.56873
60°	.76500	120°	.65143
75°	.75788	105°	.70422
90°	.73828	90°	.73828

Note the (Northern hemisphere) summer average has maximum about colatitude 60°, the Tropic of Cancer. From this we obtain an approximate solution u of $\Delta u = 0$ in $\{r<1\}$, $u = \text{avg}\,\gamma_0^{1/4}$ on $\{r=1\}$,

$$u(r, \theta) = 0.670 - .183\, r^2 P_2(\text{cost}\ \theta) - .0773\, r^4 P_4(\cos\theta) - .0286\, r^6 P_6(\cos\theta)$$

by approximating avg $\gamma_0^{1/4}$ at $\theta = 0, 15°, 30°, \ldots, 90°$ with a polynomial in $\cos^2\theta$ (so it will be even with respect to the equator), then writing it using $P_{2n}(\cos\theta)$. On $r=1$, for $\theta = 0, 15°, 30°, \ldots, 90°$ we have

$u=$	.3851	.4438	.5646	.6583	.7048	.7314	.7430
avg $\gamma_0^{1/4}=$	.3633	.4347	.5671	.6648	.7082	.7310	.7383

The error is probably less than .01 [or 3 K], except near the poles where it may reach .022. This is the basis for the isotherm sketch given earlier.

The effect of inclination of the axis

For the poles, $\gamma^{1/4} = (\sin\beta\cos\varphi)_+^{1/4}$ has no "daily" variation and avg $\gamma_0^{1/4} \cong 0.4297 \times (\sin\beta)^{1/4}$. (Applied to Moon, with $\beta = 1.53°$ from the ecliptic and $T_0 = 390$ K, we find 68 K as the polar temperature.) Away from the poles avg $\gamma_0^{1/4}$ is found by numerical integration. We take $T_0 = 280$ K and suppose λ/δ is large.

When the axis is vertical [or is 30°, 60°, 90° from the vertical] we can expect surface temperature $T < 150$ K within 15° of the poles [or 26°, 38°, 42°] and $T < 120$ K within 6° of the poles [or 14°, 6°, 0°]. For inclination 90°, with the axis of rotaton lying in the plane of revolution, the minimum temperature (at the poles) is 120 K.

Inclination	T (equator)	T (poles)
0°	210 K	0 K
30°	207 K	101 K
60°	196 K	116 K
90°	181 K	120 K

The theoretical "zero" polar temperature for $\beta = 0$ disappears under small perturbations: it would be 44 K for $\beta = 1°$. When β is small, the angle of incidence is always small so the polar temperature could be unusually sensitive to roughness of the surface, and our implicit hypothesis that absorptivity is independent of the angle of incidence could also fail significantly. The effect of finite λ/δ is probably to decrease the temperature.

Finally we note that, if the direction of the axis of rotation has a uniform distribution, we can expect a fraction $\cos\beta$ of asteroids to have inclination angle $\geqq \beta$, and half will have inclination $\geqq 60°$.

The effect of eccentricity

We assumed a circular orbit; for simplicity, but we may consider a general Keplerian ellipse with a bit more work. Most asteroids have small eccentricity ($e \leqq 0.2$), when – according to calculations below – the effect is negligible. Nearly all asteroids have $e \leqq 0.4$, when the effect is slight (4% in the extreme case $\beta = 90°$, $\varphi_0 = 0$). But we would also like to treat "dead" comet nuclei, covered with dust, when the eccentricity is not small, and the is also significant variation with the perihelion angle φ_0. In every case computed, however, the equatorial temperature for given semi-axis a — while varying eccentricity e, angle of perihelion φ_0 and angle of inclination β — is a maximum for a circular orbit with vertical axis of rotation ($\beta=0$). Among present comet orbits, the highest temperature (smallest a, smallest period) occurs for comet Encke: period 3.3 years, $a=2.2$ AU. Thus *all* (present-day) comet orbits give temperatures below that expected at the inner edge of the asteroid belt, at 2 A.U. A dead comet nucleus, covered with dust, which is not very small and does not break up into small pieces, and which does not pass very close to the Sun (so it doesn't lose its dust blanket), could retain its volatiles during billions of years in cometary orbits.

Consider an orbit with major semi-axis a. Let T_0 be maximum surface temperature at disance a from the Sun, λ/δ the corresponding parameter value, leading to temperature $T_0 u_0(\varphi, \lambda/\delta)$ [daily average] slightly below a given point of the surface, when the season angle is φ. Note u_0 depends implicitly on the point of the surface considered, as well as β, the inclination of the axis of rotation. At distance r from the Sum, the subsurface temperature is $T_0\sqrt{\frac{a}{r}}\, u_0(\varphi, \lambda/\delta\,(r/a)^{3/2})$, and the time-average on an orbit with perihelion angle φ_0 and eccentricity e — *i. e.*,

$$r=a(1-e^2)/(1+e\cos(\varphi-\varphi_0)) - \text{is } T_0\bar{u}_0$$

where

$$\bar{u}_0=\frac{1-e^2}{2\pi}\int_0^{2\pi} d\varphi\, u_0\left(\varphi, \frac{\lambda}{\delta}\left(\frac{1-e^2}{1+e\cos(\varphi-\varphi_0)}\right)^{3/2}(1+e\cos(\varphi-\varphi_0))^{-3/2}.\right.$$

The case "$e=0$" was considered above; we write $\bar{u}_0(e=0)$ for the average of $\varphi\mapsto u_0(\varphi; \lambda/\delta)$. For small e, it is reasonable to expect $\bar{u}_0\approx Q(e)\,\bar{u}_0(e=0)$, where

$$Q(e)=\frac{1-e^2}{2\pi}\int_0^{2\pi}(1-e\cos\varphi)^{-3/2}\,d\varphi.$$

We study Q below and see it is decreasing with limit

$$Q(1)=2\sqrt{2}/\pi=0.9003...$$

as $e\to 1-$. For large e, the body spends most of its time near aphelion, with season angle near $\varphi_0+\pi$. (For example, the comet Halley with $e=0.967$ spends 50% of its time within 32° of aphelion, and the effect is stronger for larger e). Thus in the limit $e\to 1-$,

$$\bar{u}_0(e=1)=Q(1)\,u_0(\varphi_0+\pi;\ 2^{3/2}\lambda/\delta).$$

We compute $T_0\bar{u}_0$ at the equator (colatitude 90°) for a convex body with $\lambda/\delta=\infty$ for various choices of eccentricity e, inclination β, and perihelion angle $\varphi=0°/\varphi_0=90°$. (Note $\varphi_0=0°$ or 180° give the same results, as do

$\varphi = \pm 90°$.) When $e=0$ or $\beta=0$, there is no variation with φ_0, so we give only one temperature.

Subsoil equatorial temperature for a $=2$ *A.U.*, $T_0=280\,K$, $\lambda/\delta=\infty$, *for perhelion angle* $\varphi_0=0°/90°$.

	$\beta=0$	30°	60°	75°	90°
$e=0$:	210 K	207	196	188	181
0.2	210	206/206	195/196	187/188	180/181
0.5	207	203/204	191/195	182/189	172/183
0.7	203	198/200	185/194	174/189	162/185
0.9	196	191/195	173/191	158/189	140/187
1.0	189	183/189	159/189	135/189	0/189

The "zero" for $e=1$, $\beta=90°$, $\varphi_0=0$, should not be taken too seriously; it would be 69 K for $\varphi_0=\pm 1°$, 103 K for $\varphi_0=\pm 5°$.

We note the usualy monotone dependence on e. For $\beta=73°$, $\varphi_0=90°$, it is not monotone. For $\beta=75°$, the temperature (apparently) decreases with e when $\varphi_0 \leqq 60°$, increases when $\varphi_0 \geqq 75°$ and for $\beta=90°$, it decreases when $\varphi_0 \leqq 30°$, increases when $\varphi_0 \geqq 60°$.

The values for $e=1$ are computed from

$$T_0\,\bar{u}_0\,(e=1)=280\times 2\sqrt{2}/\pi^{5/4}\times(1-\sin^2\beta\cos^2\varphi_0)^{1/4},$$

which has a maximum (and is independent of β) for $\varphi_0=\pi/2$. In general, $\bar{u}_0$ is unchanged when we replace φ_0 by $-\varphi_0$ or $\pi-\varphi_0$, so it is reasonable to expect extreme values at $\varphi_0=0$ and $\varphi_0=\pi/2$. This certainly holds when $e=1$ and is computationally verified below in other cases.

It appears that $Q(e)\,\bar{u}_0\,(e=0)$ is a reasonable approximation to $\bar{u}_0$ unless both e and β are large, so there is substantial variation with φ_0.

Variation of subsoil equatorial temperature with perihelion angle φ_0.

	$e=$	$\varphi_0=0$	15°	30°	60°	75°	90°
$\beta=75°$	0.7	174	*	178	186	*	189
	0.9	158	162	170	184	188	189
	1.0	135	146	163	183	188	189
$\beta=90°$	0.7	162	*	169	180	*	185
	0.9	140	147	161	181	186	187
	1.0	0	135	159	183	188	189

(*: value not computed.)

Calculation of Q:

$$Q(e)/(1-e^2)=\frac{1}{2\pi}\int_0^{2\pi}(1+e\cos\varphi)^{-3/2}\,d\varphi=\sum_{k=0}^{\infty}c_k\,e^{2k}$$

where

$$c_0 = 1, \qquad c_k = \frac{2}{\pi} \frac{(2k+(1/2)!(k-(1/2))!}{(2k)!k!}$$

and as $e \to 1-$, $Q(e) \to \lim_{k \to \infty} c_k = 2\sqrt{2}/\pi$. Also $Q(e) = 1 - \sum_1^\infty (c_{k-1} - c_k) e^{2k}$ is decreasing, since $c_k/c_{k-1} = 1 - 1/4k^2 < 1$, and the series converges in $0 \leqq e \leqq 1$, though slowly for large e.

$e=0$	0.2	0.4	0.6	0.7	0.8	0.85	0.9	0.95	1
$Q(e)=1$	.997	.990	.975	.965	.951	.943	.933	.920	.900

Q may be readily expressed as a hypergeometric function, in case we need more detailed information.

The comets of period less than ten years are most interesting for our purposes. There are 53 of these, and 35 (about 2/3) have eccentricity in $0.41 < e < 0.67$, with 9 (about 1/6) below and 9 above (maximum $e = 0.85$ for Encke). For the orbit of Encke ($a = 2.2$ A.U., $e = 0.85$), we find

Inclination $\beta=$	0	30°	60°	75°	90°
Equatorial temperature=	188	183/186	167/182	154/179	138/177

for perihelion angle $(\varphi_0 = 0)/(\varphi_0 = 90°)$. The calculations are for $\lambda/\delta = \infty$; finite λ/δ would probably give lower temperatures. The orbit of Encke has the shortest period, yielding the highest temperature, of any (present-day) comet orbit. It thus appears that, in any cometary orbit, the temperature is no higher than would be expected at the inner edge of the asteroid belt, which permits long-term survival of ice.

APPENDIX

Rate of evaporation of ice

A Maxellian distribution of velocities at temperature T results in a mass flux $F = \frac{p}{v}\sqrt{\frac{3}{2\pi}}\left[\frac{kg}{m^2 - s}\right]$ where p is the pressure $[N/m^2]$ and v the r. m. s. speed $v = \sqrt{3kT/m} \approx 37.2\sqrt{T}\left[\frac{m}{s}\right]$ at T kelvins, for water molecules. This is the equilibrium rate at which mass crosses a surface, and if p is the vapor pressure, the rate at which mass hits or leaves a solid surface of the material. The speeds are typically several hundreds of meters per second; when they strike solid, they probably bounce off except for a small fraction a which "stick". Then $a\mathrm{F}$ is the rate of addition of mass to the surface and inequilibrium, also the rate of loss of mass by evaporation. We assume the same rate of loss in a vacuum; the only difference is that no mass is added. If ρ is the density of the solid, $a\mathrm{F}/\rho\left[\frac{m}{s}\right]$ is the rate of evaporation of ice in a vaccum.

Unfortunately we don't know the value of a; fortunately this doesn't matter in our problem. The ice is covered by dust and conditions near the ice surface will be near equilibrium, so the important flux there is F, not $a\mathrm{F}$. For water-ice, with specific gravity 0.9,

$$\mathrm{F}/\rho=\begin{cases} 50\ \mathrm{m/yr} & \text{at } 210\ \mathrm{K} \\ 40\ \mathrm{cm/yr} & \text{at } 180\ \mathrm{K} \\ .04\ \mathrm{cm/yr} & \text{at } 150\ \mathrm{K} \\ 10^{-6}\ \mathrm{cm/yr} & \text{at } 120\ \mathrm{K} \end{cases}$$

(For the last two, vapor pressure was extrapolated in the form $a\exp(-b/\mathrm{T})$ with constants a, b). This is not the rate of evaporation; we have not considered the effect of the dust blanket.

Moon dust, by mass, has 30% of its particles of diameter less than 50 μm, and 30% between 50 and 100 μm (in a sample from *Apollo*-12). Dust expelled from comets is also predominantly small, micron-size. Smaller particles make more effective "blankets", and we will use the probably conservative figure of 100 μm for the mean free path in the interstices between the dust particles.

A simple-minded model of the process has a water molecule proceeding in discrete steps: it may stay where is, with probability $1-p$, or go one place to the right or left with equal probabilty $p/2$, $0<p\leqq 1$. At one end, the ice, the flux F is given; at the other end, there is no inward flux from the vaccum. (ICE) (VACUUM). In equilibrium, the value of p doesn't matter, and we have flux $\mathrm{F}/(\mathrm{N}+1)$ into the vacuum. At the ice surface, the flux F is nearly all reffected $\left(\mathrm{F}\ \text{out},\ \frac{\mathrm{N}}{\mathrm{N}+1}\ \mathrm{F}\ \text{in}\right)$, justifying the hypothesis of thermodynamic near-equilibrium at the iceface, for large N. If each "step" has length l and the dust blanket has thickness $\mathrm{L}=\mathrm{N}\,l$, we find $\frac{\mathrm{F}}{\rho}\cdot\frac{l}{\mathrm{L}+l}$ as the rate of evaporation. (A meter of micron-sized particles reduces the rate by roughly 10^{-6}.)

A more realistic model gives nearly the same result. A water molecule travels in a straight line, with given speed v, until it hits a dust particle; the path length has an exponential distribution with average value l. It may "stick" to the particle, with probability $1-p$, for an exponentially distributed waiting time, before being emitted with speed v and direction uniformly distributed on the unit sphere. Or, with probability p, it may be immediately scattered to a new direction, uniformly distributed, with the same speed v. The inward flux at the ice surface $(X=0)$ is given, following Lambert's law; there is no inward flux from the vaccum, at $X=L$. In equilibrium, the possibility of "sticking" doesn't matter and we obtain a standard transport equation

$$\mu \frac{\partial f}{\partial X}+\frac{1}{l} f(X, \mu)=\frac{1}{2 l} \int_{-1}^{1} f(X, .)$$
$$0<X<L, \qquad -1<\mu<1,$$

with $f(0, \mu)=2 F \mu$ on $0<\mu<1$, $f(L, \mu)=0$ on $-1<\mu<0$. Here μ is the cosine of the angle of flight with respect to the X-direction and $\int_{\mu_1}^{\mu_2} f(X, .)$ is the flux at X of those particles with angle θ in $\mu_1<\cos\theta<\mu_2$. [The total inward flux at $X=0$ is $\int_0^1 f(0, \mu)\, d\mu=F$; the rate of evaporation is $\frac{1}{\rho}\int_0^1 f(L, \mu)\, d\mu$.]

We use the "discrete ordinate method", replacing the integral by a Gaussian integration scheme,

$$\int_{-1}^{1} \varphi \approx \sum_{-m}^{m} c_k \varphi(\mu_k) \qquad \text{(excluding } k=0 \text{ in the sum)}$$

where the μ_k $(\mu_{-k}=-\mu_k)$ are the roots of the Legendre polynomial P_{2m} and $c_k=\int_{-1}^{1} P_{2m}(X)/\{(X-\mu_k) P'_{2m}(\mu_k)\}$, so the result is exact when φ is a polynomial of degree $\leqq 4m-1$. Then $f(X, \mu_k) \approx f_k(X)$ where

$$\mu_k f'_k(X)+\frac{1}{l} f_k(X)=\frac{1}{2 l} \sum_j c_j f_j(X)$$
$$(k=\pm 1, \pm 2, \ldots, \pm m, 0<X<L)$$
$$f_k(0)=2 F \mu_k \qquad \text{for} \quad 1 \leqq k \leqq m,$$
$$f_k(L)=0 \qquad \text{for} \quad -m \leqq k \leqq -1.$$

The flux to the vacuum is $\int_0^1 f(L, \mu)\, d\mu \approx \sum_1^m f_k(L)\, c_k$.

Discarding terms exponentially small, $O(e^{-L/l})$, since $L \gg l$, we find evaporation rate $\frac{1}{\rho}\int_0^1 f(L, \mu)\, d \approx F/\rho$ times

$$\left.\begin{matrix} 1.333\, l/(L+1.155\, l) \\ 1.603\, l/(L+1.388\, l) \\ 1.626\, l/(L+1.408\, l) \\ 1.633\, l/(L+1.414\, l) \end{matrix}\right\} \text{ using } \left\{\begin{matrix} P_2 \\ P_4 \\ P_6 \\ P_8 \end{matrix}\right\} \text{ Gauss,}$$

We will use $\frac{F}{\rho}\frac{1.6\, l}{L+1.4\, l}$ $\left[\text{and recall the factor } \frac{l}{L+l} \text{ of the simple scheme above}\right]$.

Suppose the ice has fraction f (by volume) of dust; then assuming constant temperature, the thickness L of the dust blanket satisfies

$$\frac{dL}{dt} = \frac{f}{1-f}\,\frac{F}{\rho}\,\frac{1.6\, l}{L+1.4\, l}, \qquad L(0)=0,$$

and for large times, $L \gg l$, we have

$$L \approx \sqrt{\frac{f}{1-f}\,\frac{F}{\rho}\, 3.2\, l t}.$$

We choose $l = 100\ \mu m = \frac{1}{100}$ cm, $f = 1/2$, and then L is approximately

	T=210 K	180 K	150 K	120 K
$t = 10^6$ yr:	100 m	10 m	0.3 m	1/4 cm
$t = 10^9$ yr:	3 km	300 m	10 m	8 cm

For $l = 1\ \mu m$ [or 1 cm], multiply by $\frac{1}{10}$ [or 10]. The factor $\sqrt{\frac{f}{1-f}}$ is between $\frac{1}{2}$ and 2 when $.2 \leq f \leq .8$.

If L is not very small compared to the size of the asteroid, there may be significant variation of temperature, contrary to hypothesis. But this will usually mean a decrease in temperature, hence in L. As an example, consider a one-dimensional (radial) model for a ball of radius 3 km with constant temperature 210 K at the surface, 196 K at the center, supposing the conductivity of ice is $\frac{1}{20}$ that of ice, and the vapor pressure is $a e^{-b/T}$ (a, b constants). Then, instead of 1 billion years, it takes nearly 7 billion years to melt through-or 4 billion years, considering only the temperature effect (supposing the dust has the same conductivity as the ice).

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