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PABLO A. FERRARI

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Microscopic shocks in one dimensional driven systems

by

Pablo A. FERRARI

Universidade de São Paulo, Instituto de Matemática e Estatística,
Caixa Postal 20570, 01498 São Paulo SP, Brazil

ABSTRACT. — Systems of particles sitting on the integers and interacting only by simple exclusion are considered. An electric field is imposed on the motion. Each particle after a time that may be random or deterministic jumps to its right nearest neighbor site provided that it is empty. The time can be either continuous or discrete. Assume that at time zero we start from a configuration chosen according to an appropriate distribution that has density ρ to the left of the origin and λ to its right, $\rho<\lambda$. Then it is possible to define a position $X(t)$ that we call microscopic shock such that the distribution of the configuration at time t has roughly densities ρ and λ to the left and right of $X(t)$, respectively, uniformly in t. The connection between the systems and the Burgers equation is reviewed. The microscopic shock is related to the characteristics of the Burgers equation. Laws of large numbers and lower bounds for the diffusion coefficient of the shock are given.

Key words : driven systems, microscopic shock, asymmetric simple exclusion, Burgers equation.

RÉSUMÉ. — On considère des systèmes de particules sur les entiers, qui interagissent par simple exclusion et qui sont soumises à un champ électrique. Chaque particule saute sur son voisin de droite s'il est vide après un temps qui peut être aussi bien déterministe qu'aléatoire, discret ou

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continu. Supposons que la configuration initiale soit choisie aléatoirement selon une distribution qui a une densité ρ à gauche de l'origine et λ à droite, où $\rho<\lambda$. Il est alors possible de définir un site $X(t)$ que nous appellerons un choc microscopique tel que la configuration à l'instant t a approximativement des densités ρ à gauche et λ à droite de $X(t)$ et ceci uniformément en t. Nous décrivons le rapport entre le système et l'équation de Burgers, dont les caracteristiques sont reliées au choc microscopique. On présente des Lois des Grands Nombres et des bornes pour les coefficients de diffusion du choc.

1. INTRODUCTION

Here we study one dimensional lattice gas type systems. We concentrate on the simple exclusion process but the results hold for the analogous cellular automata models, in particular for the so called Boghosian Levermore cellular automaton [bl] and some cases of the sand piles introduced by Bak *et al.* [btw].

The main topic in this paper is to show the microscopic formation of shock waves on these systems that are conservative (particles do not die or are created). The simple exclusion process can be described in the following way. At most one particle is allowed at each site $x\in\mathbb{Z}$. Each particle has an internal clock that rings after a random time with exponential distribution of rate 1. As the clock rings for the particle sitting at x and if $x+1$ is empty, the particle jumps from x to $x+1$. Then the internal clock is reset for the next jump. All particles do the same independently.

The main feature of this process is the following: if one starts the system with a distribution that has densities ρ to the left, and λ to the right, $\rho<\lambda$, then there exist a random position $X(t)$ that also obeys local rules so that the system as seen from $X(t)$ has densities λ to the right and ρ to the left, asymptotically, uniformly in time. The random position $X(t)$ is what we call a "second class particle". Its motion is determined by the following rules: the second class particle has an internal exponential clock and jumps to empty sites as the other particles do, but when one of the other particles attempts to jump over the second class particle, the jump is realized so that the second class particle and the other particle interchange positions. The following heuristics justify the election of a second class particle for a microscopic shock. If we start with a measure with densities ρ and λ to the left and right of the second class particle respectively and this densities stay thru time, the velocity of this particle equals the rate of jumping to the right $(1-\lambda)$ minus the rate of jumping to the left (ρ). This gives $1-\rho-\lambda$ that is the right macroscopic velocity of the shock in the equation related to this model as we see below.

This model has been proven to be related to the Burgers equation for $u(r, t) \in \mathbb{R}$:

$$\frac{\partial u}{\partial t}+\frac{\partial[u(1-u)]}{\partial r}=0 \qquad (1.1\,a)$$

We study the case of non decreasing initial conditions that present only one shock: the initial condition $u(r, 0)$ is λ to the right of the origin and ρ to its left. The (weak) solution of this equation with this initial condition is $u(r, t)=u_0(r-vt)$, where $v=1-\rho-\lambda$ is the velocity of the shock.

The simple exclusion process has been introduced by Spitzer in [s]. The set of invariant measures was described by Liggett [l] and [L]. The hydrodynamical limit has been studied by Rost [r], Benassi and Fouque [bf] and Andjel and Vares [av]. Fouque [fo] reviews these approachs. The existence of a microscopic shock was studied in the case of vanishing left density by Ferrari [f1], Wick [w], De Masi, Kipnis, Presutti and Saada [dkps] and Gärtner and Presutti [gp]. In the case of non vanishing left density, the existence of a microscopic shock was simulated by Boldrighini, Cosimi, Frigio e Numes [bcfg] and proven by Ferrari, Kipnis and Saada [fks]. The present approach reviews Ferrari ([f2], [f3]). Bramson [b], Lebowitz, Presutti and Spohn [lps] and Spohn [S] reviewed some of the results. Other related results are due to Kipnis [k] who proved a central limit theorem and law of large numbers for the position of a tagged particle and to De Masi and Ferrari [df] who computed the variance of the limiting Gaussian distribution.

2. INVARIANT MEASURES

The state space of the process is $\mathbf{X}:=\{0, 1\}^{\mathbb{Z}}$. Elements of $\mathbf{X}$ are functions that at each integer associate a number 0 or 1. We call these elements configurations, denote them by greek letters η, ξ, σ, etc. and say, for a configuration η, that a site x is occupied by a particle if $\eta(x)=1$ otherwise we say that x is empty. We identify a configuration η with $\{x : \eta(x)=1\}$, the subset of $\mathbb{Z}$ of occupied sites. We denote η_t the simple exclusion process whose behavior is described in the introduction. We denote $\mathrm{E}_\eta f(\eta_t)$ or $\mathrm{E} f(\eta_t^\eta)$ the expected value of $f(\eta_t)$ with respect to the process when the initial configuration is η. If ν is a measure on $\mathbf{X}$, we denote $E_\nu f(\eta_t) := \int d\nu(\eta) E_\eta f(\eta_t)$ and $S(t) f(\eta) := E_\eta f(\eta_t)$; finally $\nu S(t)$ is the measure defined by $\int d(\nu S(t))(\eta) f(\eta) := \int \nu(\eta) S(t) f(\eta)$.

A measure ν is (time) invariant for the process if $\nu S(t) = \nu$ for all t. This means that if one starts the process with the initial measure ν and looks at the distribution at later times, one finds that the process is still distributed according to ν.

The product measures ν_α are the measures defined by

$$\nu_\alpha \{ \eta : \eta(x) = 1, x \in A \} = \alpha^{|A|}$$

This means that a configuration η picked from the distribution ν_α can be constructed in the following way: at each site of $\mathbb{Z}$ put a particle with probability α and do this independently for each site.

There are also other invariant measures, called "blocking" measures. These are also product but not translation invariant. This means that the density depends on the site. We call them $\nu^{(n)}$. They give mass one to a single configuration:

$$\nu^{(n)}(\eta^{(n)}) = 1$$

where

$$\eta^{(n)}(x) := \begin{cases} 1, & \text{if } x \geqq n \\ 0, & \text{if } x < n \end{cases}$$

The $\nu^{(n)}$ are translation of each other.

Liggett [l] proved that all the invariant measures for the process are convex combinations of ν_α, $0 \leqq \alpha \leqq 1$ and $\nu^{(n)}$, $n \in \mathbb{Z}$.

3. THE BURGERS EQUATION

The inviscid Burgers equation is the hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial [u(1-u)]}{\partial r} = 0 \qquad (2.1\,a)$$

We consider the initial value problem $u(r, 0) = u_0(r)$, where

$$u_0(r) = \rho\, 1\{ r \leqq 0 \} + \lambda\, 1\{ r > 0 \} \qquad (2.1\,b)$$

(shock initial conditions). The way to find solutions in this case is called the method of characterics [lax]. If one calls $a(u) = \frac{\partial}{\partial u}(u(1-u)) = (1-2u)$, then the equation can be written

$$\frac{\partial u}{\partial t}+a(u)\frac{\partial u}{\partial r}=0$$

so that u is constant along trajectories $w(r, t)$ with $w(r, 0)=r$, that propagate with speed $a(u)$. These trajectories are called characteristics. They are straight lines allow to construct a solution of the equation for t small. If different characteristics meet, giving two different values to the same point, then the solution develops a discontinuity. Ours is the simplest case, when the discontinuity is present in the initial condition. Indeed, for $r>0$, the characteristics starting at r and $-r$ have speed $(1-2\lambda)$ and $(1-2\rho)$ respectively and meet at time $t(r)=r/(\lambda-\rho)$. Using the conservation law of the equation it is not difficult to show that the discontinuity propagates at velocity $v:=1-\lambda-\rho$. The solution $u(r, t)$ is λ for $r>vt$ and ρ for $r<vt$ *i.e.* $u(r, t)=u_0(r-vt)$. This means that for all continuously differentiable test functions $\Phi(r, t)$,

$$\iint\left(\frac{\partial\Phi}{\partial t}u+\frac{\partial\Phi}{\partial r}u(1-u)\right)dr\,dt=0$$

This solution is called entropic. It arises as a limit for $a\to 0$ of the (unique) solution of the (viscous) Burgers equation

$$\frac{\partial u}{\partial t}+\frac{\partial[u(1-u)]}{\partial r}=a\frac{\partial^2 u}{\partial r^2} \tag{2.2}$$

This is the solution one gets by deriving the equation as the hydrodynamical limit of the simple exclusion process.

4. THE HYDRODYNAMICAL LIMIT

We describe here the heuristic derivation of equations (2.1) from the process η_t by a hydrodynamical limit. We use the notations of the previous section. Define also $\eta_t^\varepsilon(r):=\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)$, where $\varepsilon^{-1}r$ is an abuse of notation for integer part of $\varepsilon^{-1}r$. We get

$$\frac{d}{dt}\mathrm{E}_\nu(\eta_t^\varepsilon(r))=\varepsilon^{-1}\mathrm{E}_\nu[-\eta_t^\varepsilon(r)(1-\eta_t^\varepsilon(r+\varepsilon))+\eta_t^\varepsilon(r-\varepsilon)(1-\eta_t^\varepsilon(r))].$$

Now, if there exists a limit $u(r, t)=\lim_{\varepsilon\to 0}\mathrm{E}_\nu(\eta_t^\varepsilon(r))$ and the measure at time $\varepsilon^{-1}t$ is approximately a product measure, such that $\mathrm{E}_\nu(\eta_t^\varepsilon(r)(1-\eta_t^\varepsilon(r+\varepsilon))$ converges, as $\varepsilon\to 0$, to $u(r, t)(1-u(r, t))$, then this limit must satisfy the Burgers equation (2.1). In fact it is proven that if $u(r, t)$ is the solution of the Burgers equation (2.1) with $u(r, t)=u_0(r)$

and ν^{ε} is a family of product measures such that $\nu^{\varepsilon}(\eta(\varepsilon^{-1}r))=u_0(r)$, then in the continuity points of $u(r, t)$,

$$\lim_{\varepsilon \to 0} \nu S(\varepsilon^{-1} t)\tau_{\varepsilon^{-1}r} f = \nu_{u(r,t)} f \tag{4.1}$$

where $\tau_x \eta$ is the configuration defined by $\tau_x \eta(z)=\eta(z+x)$ and $\tau_x f(\eta)=f(\tau_x \eta)$. Equation (4.1) has been proven first for initial profiles $u_0(r)$ that have only one discontinuity by Rost [r] in the non increasing case and extended by Benassi and Fouque [bf] and Andjel and Vares [av] to the non decreasing case. Recently Benassi, Fouque, Vares and Saada [bfvs] extended the result to any monotone profile and Landim [la2] to initial piecewise constant profiles presenting two or three discontinuities.

Notice that (4.1) gives the weak convergence of the process on the points of continuity of $u(r, t)$. Nothing is said about the points of discontinuity. The expected result is the following

$$\nu_{\rho,\lambda} S(\varepsilon^{-1} t)\tau_{\varepsilon^{-1}vt} = \frac{1}{2}\nu_{\rho} + \frac{1}{2}\nu_{\lambda} \tag{4.2}$$

where $\nu_{\rho,\lambda}$ is the product measure with densities ρ and λ to the left and right of the origin respectively. This is called "dynamical phase transition" and has been proven by Wick [w] in the case $\rho=0$ for a zero range model that is isomorphic to this one and by De Masi, Kipnis, Presutti and Saada [dkps]. We sketch Wick's proof in Section 5. Afterwards, using symmetry arguments, Andjel, Bramson and Liggett [abl] showed the result for the case $\lambda+\rho=1$ *i.e.* $v=0$.

5. THE SEMIINFINITE CASE

In this section we consider the measure $\nu_{0,\lambda}$, the product measure with densities 0 and λ to the left and right of the origin, respectively. For convenience we assume also that there is a particle at the origin, and define $\nu' := \nu(\,.\,|\,\eta(0)=1)$. Hence our initial measure is $\nu'_{0,\lambda}$. Call $X(t)$ the position at time t of the particle that is initially at the origin. We keep track of its position by considering the process $(\eta_t, X(t))$ in $\{(\eta, z) : \eta \in \mathbf{X},\ \eta(z)=1\}$.

We are interested in the process as seen from the tagged particle. Hence we consider the process $\eta'_t := \tau_{X(t)}\eta_t$. For this process there is a translation each time that the tagged particle moves, in such a way that the tagged particle is always at the origin. The position of the tagged particle can be recovered from this process by defining $X(t) :=$ number of translations of the system in the time interval $[0, t]$.

The following remarkable result is the key tool in this case. It comes from queuing theory. The connection between the simple exclusion process and a system of queues has been established by Kesten (*see* [s], [L] and also [k], [f1], [df] and [dkps]). In words one can think that each particle is a server in a series of infinitely many queuing systems. The holes to the right of particle i and to the left of particle $i+1$ are thought of as customers in the system of particle i. A system consists of a customer being served (if any) plus a queue. Each time that particle i jumps, a hole passes from its system to the system of particle $i-1$. In queuing theory language one says that the customer is served and enters in the next system. The Burke's theorem says that a single stationary queue system with Poisson arriving times at rate a and exponential service times at any rate $b>a$ has Poisson exiting times at the same rate a. For the series of infinitely many queuing systems we have $a=1-\lambda$ and $b=1$. Since the arriving times of system i are the exiting times of system $i+1$, both arriving and exiting times of any system are Poisson at rate $1-\lambda$. In our context Burke's Theorem says that if the initial measure is $\nu'_{0,\lambda}$ then $\tau_{X(t)}\eta_t$ and $X(t)$ are independent. Indeed

$$E(f(\tau_{X(t)}\eta_t)\,|\,X(t))=\nu'_{0,\lambda}f \quad \text{for all } t\geqq 0$$

and $X(t)$ is a nearest neighbor totally asymmetric random walk with parameter $1-\lambda$.

As a consequence of the fact that $X(t)$ is a Poisson point process a Law of large numbers for $X(t)$ follows:

$$\lim_{t\to\infty}\frac{X(t)}{t}=1-\lambda, \qquad P_{\nu_{0,\lambda}}\,a.s.$$

altogether with a central limit theorem:

$$\lim_{\varepsilon\to 0}\frac{X(\varepsilon^{-1}t)-(1-\lambda)\varepsilon^{-1}t}{\sqrt{(1-\lambda)\varepsilon^{-1}t}}=\mathcal{N}(0,1)$$

in distribution, where $\mathcal{N}(0,1)$ is a centered Gaussian random variable with variance 1. Burke's theorem is also useful to prove the hydrodynamical limit in the case $\rho=0$ and $\lambda>0$. The idea of the proof is to use the following: (*a*) $\tau_{X(t)}\eta_t$ has distribution $\nu'_{0,\lambda}$ for all t independently of $X(t)$. (*b*) by the law of large numbers for $X(t)$, $\varepsilon^{-1}r-X(\varepsilon^{-1}t)$ converges almost surely, as $\varepsilon\to 0$, to $-\infty$ if $r<(1-\lambda)t$ and to ∞ if $r>(1-\lambda)t$. From (*a*) and (*b*), what is seen at the macroscopic position r is ν_ρ if $r<(1-\lambda)t$ and ν_λ for $r>(1-\lambda)t$.

Finally, also the dynamical phase transition (4.2) can be proven. The idea is the same as before. The difference is that in order to show that $|X(\varepsilon^{-1}t)-v\varepsilon^{-1}t+\varepsilon^{-1/2}r|$ diverges one needs to use the fact that $X(t)$ is roughly a Gaussian random variable with mean and variance $(1-\lambda)t$.

After this it suffices to notice that the probability that $X(\varepsilon^{-1}t)$ is at the left of the origin is 1/2 and that this is independent of the configuration as seen from $X(t)$.

6. MICROSCOPIC SHOCK

We saw that in the case of vanishing left density a microscopic shock is defined. This is a random position that in this case coincides with the position of the leftmost particle. At time t, the system as seen from the shock has a measure $\mu' S'(t) = \nu'_{0,\lambda}$ for all t.

When the left density does not vanish it is not obvious what may be defined as a microscopic shock. One can try to tag a particle and follow it, but immediatly it is realized that the tagged particle has the wrong velocity: if the other particles have distribution ν_ρ, the tagged particle has velocity $(1-\rho)$ and if the other particles have distribution ν_λ the tagged particle has velocity $(1-\lambda)$. Nevertheless, in some sense, the idea of considering a last particle used for vanishing left density also works in the general case. We use a technique called coupling. We couple two processes. The σ process with initial measure ν_ρ and the η process with initial measure $\nu_{\rho,\lambda}$. At time 0 we couple the initial configurations as follows. Let $\{U_x\}_{x \in \mathbb{Z}}$ be a sequence of independent identically distributed random variables with distribution uniform in [0, 1]. Given a realization of these variables we define

$$\sigma(x) = 1\{U_x \leqq \rho\}$$

and

$$\eta(x) = 1\{U_x \leqq \rho\} . 1\{x < 0\} + 1\{U_x \leqq \lambda\} . 1\{x \geqq 0\}.$$

Hence σ has distribution ν_ρ and η has distribution $\nu_{\rho,\lambda}$. Under this realization $\sigma(x) \leqq \eta(x)$ for all x. The idea now is to use "the same random numbers" to realize jointly the processes η_t and σ_t. In a finite box this would work as follows: a site is chosen at random from the box and the jumping rule is applied to η and σ at that site. With this rule, if $\sigma_0 \leqq \eta_0$ coordinatewise, then $\sigma_t \leqq \eta_t$ for all t. The process σ_t coincides with the set of sites occupied by the two marginals and we call γ_t the set of sites occupied only by the marginal η. The reader can check that when a site x is chosen, the following can happen:

1. The site x is occupied by a σ or γ particle and $x+1$ is empty. Then the particle jumps.
2. The site x is occupied by a σ or γ particle and $x+1$ is occupied by a σ particle. Then nothing happens.
3. The site x is occupied by a γ particle and $x+1$ is occupied by a γ particle. Then nothing happens.

4. The site x is occupied by a σ particle and $x+1$ is occupied by a γ particle. Then the particles interchange positions: after the jump $\sigma(x+1)=1$ and $\gamma(x)=1$.

Since γ particles must arrange themselves in the sites left free by the σ particles, we say that the σ particles are first class and the γ are second class particles. It is also said that the σ particles have priority over the γ particles. This system of priorities was first used by Andjel and Kipnis [ak] in a related model called the zero range process for which they performed the hydrodynamical limit.

We look at the system as seen from the leftmost γ particle, whose position is called $X(t)$. When $\rho=0$ the η process coincides with the γ process and we saw in Section 5 that this particle is a microscopic shock. Observe however that $X(t)$ can be defined directly from the η process in the following manner: consider a configuration η and the configuration η' that coincides with η out of the origin and differs from η at the origin. One can check that at later times η_t and η_t' will differ at only one site. Call this site $X(t)$. It can be proven that this definition coincides with the previous one if η is picked from $\nu_{\rho,\lambda}$. The motion of $X(t)$ is the following: when its site is chosen the same rule as for the other particles is a applied: it jump to its right nearest neighbor site if it is empty. The difference is that when its left neighbor is chosen, the jump is realized and $X(t)$ must jump to the left in order to keep the exclusion rule. Let $S'(t)f(\eta)=E_\eta(f(\tau_{X(t)}\eta_t))$. One proves then the following result that implies that $X(t)$ is a microscopic shock:

Assume that η_0 has distribution $\nu'_{\rho,\lambda}$ and $X(0)=0$. Then for all $\delta>0$ and for all cylindric f there exists $y^*=\mu(\delta,f)$ such that for all $t\geqq 0$, $y>y^*$, $|\nu'_{\rho,\lambda}S'(t)(\tau_y f)-\nu_\lambda f|<\delta$ and $|\nu'_{\rho,\lambda}S'(t)(\tau_{-y} f)-\nu_\rho f|<\delta$.

Also a law of large numbers holds:

$$\lim_{t\to\infty}\frac{X(t)}{t}=v,\qquad P_{\nu'_{\rho,\lambda}}\ a.s. \tag{6.1}$$

As a consequence of the law of large numbers we can prove the hydrodynamic limit (out of the shock) as we did in the semi infinite case.

7. SHOCK FLUCTUATIONS

In this system a perturbation on the initial condition translates as time goes to infinity into a shift of the shock position. For any configuration η, and for any site $y\in\mathbb{Z}$, let $\eta^{y|i}$ be defined by ($i\in\{0,1\}$).

$$\eta^{y|i}(x)=\begin{cases}\eta(x) & \text{for } x\neq y\\ i & \text{for } x=y.\end{cases}$$

Let $X(\eta, t)$ be the random position of the shock when the initial configuration is η.

For all $\varepsilon>0$, $y\in\mathbb{Z}$, it holds

$$\lim_{t\to\infty} |E_{\nu_{\rho,\lambda}}(X(\eta^{y|0}, t)-X(\eta^{y|1}, t))-(\lambda-\rho)^{-1}|=0 \tag{7.1}$$

This is the key tool in order to obtain a lowerbound for the diffusion coefficient for the shock. This is defined by (if the limit exists)

$$D=\lim_{t\to\infty} t^{-1} E_{\nu'_{\rho,\lambda}}(X(t)-vt)^2$$

We also define

$$\bar{D} := \frac{\rho(1-\rho)+\lambda(1-\lambda)}{\lambda-\rho}.$$

It has been conjectured by Spohn [S] that $D=\bar{D}$, but it is only proven that $\bar{D}$ is a lowerbound for D ([f2], [f3]). Indeed $D=\bar{D}+I$, where $I=\lim_{t\to\infty} I(t)$ and

$$I(t) := \int d\nu'_{\rho,\lambda}(\eta)\, E\left(X(\eta, t)-\frac{n_0(\eta, (\lambda-\rho)t)-n_1(\eta, -(\lambda-\rho)t)}{\lambda-\rho}\right)^2,$$

where $n_0(\eta, x) := \sum_{y=0}^{x} (1-\eta(y))$ is the number of empty sites of η between 0 and x and $n_1(\eta, x) := \sum_{y=x}^{0} \eta(y)$ is the number of η particles between the origin and $x<0$.

From these results we conclude that the diffusion coefficient of the shock is the same as the conjectured diffusion coefficient if and only if the position of the shock at time t is given (in the scale $\sqrt{t}$) by $(\lambda-\rho)^{-1}$ times the number of holes between 0 and $(\lambda-\rho)t$, minus the number of particles between 0 and $-(\lambda-\rho)t$. In any case, $I(t)$ is non negative, hence $\bar{D}$ is always a lowerbound. When $\rho=0$, $X(t)$ has the distribution of a plain tagged particle in the simple exclusion process with density λ. In this case it is known that $D := \lim_{t\to\infty} t^{-1} E(X(t)-EX(t))^2=\bar{D}=(1-\lambda)$. This implies that $\lim_{t\to\infty} I(t)/t=0$; hence, in the scale $\sqrt{t}$, the position of $R(t)$ is determined by the initial configuration in the sense discussed above. This has been proved before by Gärtner and Presutti [gp].

We finish this section by mentioning a couple of open problems.

Prove that there exists a microscopic shock in more dimensions or for a jump function that allows to go further than the nearest neighbors. The argument of this approach does not work even for the case of two parallel lines with a symmetric dynamics for jumps between the two lines and asymmetric jumps inside each line. Landim [la1] proved the existence of the hydrodynamical limit in two dimensions for some initial conditions.

General initial conditions. Benassi, Fouque, Saada and Vares [bfsv] have proved that the hydrodynamical limit can be taken when the initial profile is monotone. Prove the hydrodynamical limit for any initial profile.

Fluctuations. It is expected that the fluctuations around the deterministic hydrodynamic limit depend on the initial configuration as the fluctuations of the shock do. In the case $\rho=0$ this has been studied by [bf2].

8. THE NEAREST NEIGHBOR ASYMMETRIC SIMPLE EXCLUSION PROCESS. GENERAL CASE

Almost all the results described above have been also proven for the process whose particles can also jump backwards. In this case one assumes that the particles jump at rate p to the right nearest neighbor and with rate q the left one. We assume $p+q=1$ and $p>q$, but this is only for convenience.

The set of extremal invariant measures is $\{\nu_\alpha : 0 \leqq \alpha \leqq 1\} \cup \{\nu^{(n)} : n \in \mathbb{Z}\}$. The measure $\nu^{(n)}$ concentrates on the set

$$\mathbf{A}_n := \{\eta : \sum_{x \geqq 0} (1-\eta(x)) - \sum_{x<0} \eta(x) = n\}.$$

They are defined by $\nu^{(n)} := \nu^{[k]}(\,.\,|\,\mathbf{A}_n)$ for all k. The measure $\nu^{[k]}$ is also a product measure with marginals

$$\nu^{[k]}(\eta(x)) = \frac{(p/q)^{x-k}}{1-(p/q)^{x-k}} \tag{8.1}$$

and is even reversible for the process. They approach exponentially fast the densities 0 and 1 to the left and right of the origin respectively, so that, under $\nu^{(n)}$, the origin is a shock for $\rho=0$ and $\lambda=1$. If we put a second class particle at n with probability proportional (in average) to

$$m(i) := ((1+(p/q)^{i-1/2})(1+q/p)^{i+1/2})^{-1}, \tag{8.2}$$

in a way that depends on the configuration η chosen according to $\nu^{[0]}$, then we can get a reversible measure for the process $(\eta_t, \mathrm{R}(t))$, where $\mathrm{R}(t)$ stands for the position of the second class particle at time t. It is clear from (8.2) that the second class particle will remain tight; hence, in this case it is a shock.

In the case $0 \leqq \rho < \lambda \leqq 1$ one constructs a process $(\sigma_t, \xi_t, \mathrm{X}(t))$, with σ_t first class particles and ξ_t second class. At time zero we set $\sigma(x) = 1\{\mathrm{U}(x) \leqq \rho\}$ and $\xi(x) = 1\{\rho < \mathrm{U}(x) \leqq \lambda\}$ where $\mathrm{U}(x)$ are independent identically distributed uniform random variables in [0, 1]. We set

$X(t)$ as a tagged ξ particle. Then call $x_i(t)$ the positions of the ξ particles, assuming that $x_0(t) \equiv X(t)$. Label the ξ particles independently in the following way: the n-th particle is labeled γ with probability $(p/q)^n/(1-(p/q)^n)$ otherwise it is labeled ζ. Finally choose the i-th ξ_t particle to be $R(t)$ with probability proportional (in average) to $m(i)$ [in a way that depends on (γ, ζ)]. The remarkable property of the resulting distribution is that the labeling remains invariant for later times. Hence the density of γ particles vanishes to the left and the density of ζ particles vanishes to the right exponentially fast. Furthermore $X(t)-R(t)$ remains tight. Since our original process can be recovered by writing $\eta_t = \sigma_t + \gamma_t$ as before, one can prove that either $X(t)$ or $R(t)$ are microscopic shocks. The advantage of $R(t)$ is that it can be defined directly as a second class particle with respect to η_t, while for $X(t)$ one needs to call the process (σ_t, ξ_t). With this observation and a little care one can prove all the other results. We refer to [fks] and [f2] for details.

9. THE WEAKLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

We have studied the hydrodynamical limit of a process by rescaling time and space in a convenient way. Another possibility is to consider a family of processes depending on the same parameter ε. Indeed this is what is done to derive the (viscous) Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial r}(u(1-u)) = \theta \frac{\partial^2 u}{\partial r^2} \tag{9.1}$$

This kind of limit is called kinetic. The family consists of asymmetric simple exclusion processes as defined in the previous section with $p = \frac{1+\varepsilon}{2}$ and $q = \frac{1-\varepsilon}{2}$. Since the asymmetry $p-q$ goes to 0 with the scaling parameter ε, the resulting (family of) process(es) is called weakly asymmetric. It was introduced by De Masi, Presutti and Scacciatelli [dps] and studied by Gärtner [g], Dittrich [d], Gärtner and Dittric [gd], Ravishankar [ra] and Ferrari, Kipnis and Saada [fks]. The results are quite complete and reinforce the conjectures on the simple exclusion process we gave in the previous sections.

Denoting by $S^\varepsilon(t)f$ the expected value of f with respect to the process, and ν^ε a family of product measures with marginals given by $\nu^\varepsilon(\eta(\varepsilon^{-1} r)) = u_0(r)$ the kinetic limit is given by

$$\lim_{\varepsilon \to 0} \nu^\varepsilon S^\varepsilon(\varepsilon^{-1} t) \tau_{\varepsilon^{-1/2} r} f = \nu_{u(r,t)} f$$

(local equilibrium), where $u(r, t)$ is the solution of (9.1) with initial condition $u(r, 0)=u_0(r)$. This result was proven by De Masi, Presutti and Scacciatelli [dps] and by Gärtner [g]. Notice that the scaling is different from that one we used to derive the unviscous Burgers equation. To obtain a Laplacian one needs to scale space as the square root of time. Since we are looking at time ε^{-1} and the asymmetry is of the order of ε, the effective drift is not rescaled and it appears in the macroscopic limit as a transport term. The same authors proved the law of large numbers for the density fields: Let Φ be a continuous function with compact support. Then,

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} \Phi(\varepsilon x)\, \eta_{\varepsilon^{-1}t}(x) = \int_{\mathbb{R}} \Phi(r)\, u(r, t)\, dr$$

$P^{\varepsilon}_{\nu^\varepsilon}$ almost surely. The stationary case was studied by Ferrari, Kipnis and Saada [fks]: For each ε there exists a position $X(t)$ such that the process as seen from $X(t)$ has a measure μ^ε with the property that

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(\eta(\varepsilon^{(-1/2)} r)) = u(r)$$

where

$$u(r) := \rho + \frac{\lambda - \rho}{1 + e^{-2\theta r(\lambda - \rho)}}$$

is the stationary travelling wave solution of the Burgers equation (9.1) with asymptotic densities ρ and λ. Also the density fields converge and the hydrodynamical limit is achieved for this family of initial measures.

A stronger result has been proved by Dittrich [d], who exibits a function ξ_t of the initial configuration η_0, such that for any test function ψ,

$$\lim_{\varepsilon \to 0} \varepsilon \sum_x (\eta_{\varepsilon^{-1}t}(x) - \xi_{\varepsilon^{-1}t}(x))\, \psi(\varepsilon x) = 0$$

In other words: in that scale, the motion is determined by the initial configuration. The approach also allows to study the shock wave case. In this case the shock fluctuates as a Brownian motion with diffusion coefficient given by $\bar{D}$ (defined in Section 7), and these fluctuations depend only on the initial configuration [d].

An interesting problem is to decide what happens with the second class particle in this limit. The rescaling of (8.2) implies that, calling the position of the second class particle at time t by $R(t)$, with $R(0)=0$,

$$\lim_{\varepsilon \to 0} P_{\mu^\varepsilon}(R^\varepsilon(t) - v\varepsilon^{-1} t \leqq \varepsilon^{-(1/2)} r) = \int_0^r \frac{\lambda - \rho}{(1 + e^{-2s(\lambda - \rho)})(1 + e^{2s(\lambda - \rho)})} ds$$

We conjecture that the motion of the second class particle is an Ornstein Uhlenbeck process with an appropriate drift and with the stationary measure given above.

De Masi, Presutti and Scacciatelli [dps] have studied the fluctuation fields and proved that these converge to a generalized Ornstein Uhlenbeck process.

10. THE BOGHOSIAN LEVERMORE CELLULAR AUTOMATA

In this section we review recent results by Ferrari and Ravishankar [fr] on a deterministic version of a probabilistic cellular automata first studied by Boghosian and Levermore [bl]. This is a dynamical system with random initial condition. The simplicity of the model allows to prove not only all the results but also the conjectures given for the asymmetric simple exclusion process.

A configuration of the one dimensional Boghosian Levermore Cellular Automata (BCLA) is an arrangement of particles with velocities $+1$ and -1 on $\mathbb{Z}$, satisfying the exclusion condition that there is at most one particle with a given velocity ($+1$ or -1) at each site. We denote a configuration by $\eta \in \{0, 1\}^{\mathbb{Z} \times \{-1, +1\}} := \mathbf{X}$, the state space. If $\eta(x, s) = 1$ we say that there is a particle with velocity s at site x, where $x \in \mathbb{Z}$ and $s \in \{-1, +1\}$.

Dynamics. – The time is discrete and the dynamics is given in two steps:

1. Collision: for a given η let $C\eta$ be the configuration

$$C\eta(x, +1) = 1\{\eta(x, 1) + \eta(x, -1) \geqq 1\} = \max\{\eta(x, 1), \eta(x, -1)\}$$
$$C\eta(x, -1) = 1\{\eta(x, 1) + \eta(x, -1) = 2\} = \min\{\eta(x, 1), \eta(x, -1)\}$$

In other words, if there is no particle or two particles at x, then nothing happens. If there is only one particle at x, then this particle receives velocity 1.

2. Advection. This part of the dynamics moves each particle along its velocity to a neighboring site in unit time. The operator A is defined by

$$A\eta(x, s) = \eta(x - s, s).$$

Defining $T := AC$, the dynamics is given by

$$\eta_{t+1} = T\eta_t$$

We say that a measure μ on $\mathbf{X}$ is stationary for the process if $\mu T = \mu$. Cheng, Lebowitz and Speer [cls] have noticed that this dynamics acts independently in the space-time sublattices $\{(x, t) : x + t \text{ is even}\}$ and $\{(x, t) : x + t \text{ is odd}\}$. Any translation invariant measure concentrating on one of the two sets $\{\eta : \eta(x, 1) = 1\}$, $\{\eta : \eta(x, -1) = 0\}$ is stationary for the process. Also, there are non translation invariant measures: the measures ν^n giving mass 1/2 to η^n and $T\eta^n$ each, where $\eta^n(x, \pm 1) = 1\{x \geqq n\}$. Observe that the configuration η^n is two steps invariant (*i.e.* $\eta^n = T^2\eta^n$).

The problem of determining if these measures are sufficient to describe the set of all invariant measures for the process is open. We remark that in contrast with the simple exclusion, there are stationary translation invariant ergodic measures that are not product measures

The hydrodynamical limit has been done for initial product measures with densities 0 and ρ for particles with velocity -1 and $+1$ respectively at negative sites, and with densities λ and 1 for particles with velocities -1 and $+1$, respectively to the right of the origin. Hence the particle density per site to the left of the origin is ρ and to its right is $1+\lambda$. The limiting density per site satisfies the equation

$$\frac{\partial u}{\partial t} = \mathrm{F}(u)$$

where

$$\mathrm{F}(u) = u\,1\{0 \leqq u \leqq 1\} + (2-u)\,1\{1 \leqq u \leqq 2\}$$

This equation has only two characteristics, 1 and -1 according to the density being smaller or larger than 1. The definition of the microscopic shock is the same as for the simple exclusion process. Just take two configurations that differ at only one velocity at only one site. At latter times they will also differ at only one site that we call second class particle, as it gives priority to other particles that attempt to occupy its place.

When the initial configuration is taken from the product measure described above, it turns out that the position of the second class particle can be expressed as a sum of independent and identically distributed random variables whose number is random and such that each one is a difference of two geometric random variables. Furthermore the number of summands is independent of the summands. Hence, as a corollary of this representation, laws of large numbers and central limit theorems for the position of the shock are proven. Another result is that the position of the shock at any given time is independent of the configuration at that time as seen from the shock. This gives a way to prove the hydrodynamical limit described above.

The results can be extended to initial measures with density having more than one step, to decreasing profiles and also to the probabilistic cellular automaton, for which the C rule is applied with probability p and is not applied with probability $1-p$ [frv]. The weakly asymmetric case was studied by [lop].

11. OTHER CELLULAR AUTOMATA

The BLCA is isomorfic to two simple exclusion automata in $\{0, 1\}^{\mathbb{Z}}$ and a sand-pile automaton.

The asymmetric simple exclusion cellular automaton. – For a given configuration $\xi \in \{0, 1\}^{\mathbb{Z}}$, define $B_1 \xi$ as the configuration

$$B_1 \xi(2z+1) = 1\{\xi(2z)+\xi(2z+1) \geqq 1\} = \max\{\xi(2z), \xi(2z+1)\}$$

$$B_1 \xi(2z) = \xi(2z)+\xi(2z+1) - 1\{\xi(2z)+\xi(2z+1) \geqq 1\} = \min\{\xi(2z, \xi(2z+1)\}$$

and $B_2 \xi$ as the configuration

$$B_2 \xi(2z) = 1\{\xi(2z-1)+\xi(2z) \geqq 1\}$$

$$B_2 \xi(2z-1) = \xi(2z-1)+\xi(2z) - 1\{\xi(2z-1)+\xi(2z) \geqq 1\}$$

Now define the asymmetric simple exclusion cellular automaton (ASECA) by:

$$\xi_{2t} = B_1 \xi_{2t-1} \quad \text{and} \quad \xi_{2t+1} = B_2 \xi_{2t} \tag{11.1}$$

In words, at even times, all particles occupying even sites jump to the right if the successive odd site is empty. At odd times, the particles occupying the odd sites do the same.

We prove that this is isomorfic to a subsystem of the BLCA. As observed before, the BLCA consists of two independent subsystems: $\{\eta(x, s, t) : x+t \text{ odd}\}$ and $\{\eta(x, s, t) : x+t \text{ even}\}$. Consider the odd subsystem and define the configuration ξ_t by

$$\xi_t(2x+1) = \begin{cases} \eta(2x+1, -1, t) & \text{for } t \text{ even} \\ \eta(2x, 1, t) & \text{for } t \text{ odd} \end{cases}$$

and

$$\xi_t(2x) = \begin{cases} \eta(2x-1, 1, t) & \text{for } t \text{ even} \\ \eta(2x, -1, t) & \text{for } t \text{ odd} \end{cases}$$

The transformation ξ defines an isomorfism between the subsystem $\{\eta(x, s, t) : x+t \text{ odd}\}$ and ξ_t, $t \in \mathbb{Z}$, such that ξ_t is the asymmetric simple exclusion cellular automaton, with distribution described by (11.1).

The automaton 184 ([wo], [ks]). – Let $B\gamma$ be the configuration defined by

$$B\gamma(z) = \begin{cases} 1 & \text{if } \gamma(z-1)=1 \text{ and } \gamma(z)=0 \\ 0 & \text{if } \gamma(z)=1 \text{ and } \gamma(z+1)=0 \\ \gamma(z) & \text{otherwise} \end{cases}$$

In words, $B\gamma$ is the configuration obtained when all particles of γ allowed to jump one unit to the right do it. Define the automaton by $\gamma_t = B\gamma_{t-1}$. Assume now that at time 0, all even sites are empty. In this case this is isomorfic to ξ_t with the same initial configuration. On the other hand, if all even sites are occupied, it is also isomorfic to ξ_t. For other configurations this system is not isomorfic and presents a richer structure. For those type of initial conditions the results proved for the BLCA hold.

Sand piles. – We present in this section an infinite version of an automaton introduced by Bak, Tang and Wisenfeld [btw] and studied by Goles [g]. Consider a process ζ_t on the state space $\{\zeta \in \mathbb{Z}^{\mathbb{Z}} : \zeta(x) \geqq \zeta(x+1)\}$. Define $\mathrm{C} : \zeta \mapsto \mathrm{C}\zeta$ as follows

$$(\mathrm{C}\zeta)(x) = \zeta(x) + 1\{\zeta(x-1) - \zeta(x) \geqq 2\} - 1\{\zeta(x) - \zeta(x+1) \geqq 2\}.$$

In words, we can think that at each integer there is a pile of grains of sand. At each time each pile is ready to give one of its grains to its right neighboring pile. But this only happens if grain "rolls down", *i.e.* if the receiving pile is not higher than the one from which the grain comes. These operations are all done in parallel. The automaton is defined by

$$\zeta_t = \mathrm{C}(\zeta_{t-1})$$

It turns out that for some initial configurations, this automaton is isomorfic to the automaton 184 described above. This has been established in [fgv]. Let γ be a configuration of $\{0, 1\}^{\mathbb{Z}}$ and define $\zeta = \zeta(\gamma)$ as the configuration

$$\zeta(x) = -x + \gamma(x) \tag{11.2}$$

Then it is easy to see that

$$(\mathrm{C}\zeta)(x) = -x + (\mathrm{B}\gamma)(x)$$

so that we get that $\zeta_t(x) = -x + \gamma_t(x)$ for all t, all x. This implies that all the results for the hydrodynamics and shocks hold in this model for this kind of initial conditions. Other initial conditions are under investigation by [fgv].

It is not hard to see that a particle system can be constructed using the same law. Assume that at each site we have a Poisson point process of rate one, independent of the rest. When the clock rings at site x, the pile attempts to give up a grain to site $x+1$, but it does so only if $\zeta(x) - \zeta(x+1) \geqq 2$. One can show that the transformation (11.2) makes this system isomorphic to the simple exclusion process.

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