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Instability of symmetric stationary states for some nonlinear Schrödinger equations with an external magnetic field

by

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ABSTRACT. – This work is concerned with instability properties of solutions $u(t, x)=e^{-i\omega}\Phi(x)$ of the equation $i\partial_t u=-L_A u+|u|^{p-1}u$, where iL_A is the Schrödinger operator in the presence of a uniform magnetic field, defined by the solenoidal vector potential $A: L_A u=-\Delta u-2iA.\nabla u+|A|^2 u$. Φ is a solution of the nonlinear elliptic equation $L_A\Phi+\omega\Phi-|\Phi|^{p-1}\Phi=0$, invariant by rotations around the z-axis, and solving a certain variational problem. Put $\Sigma=\{e^{-i\theta}\Phi(\hat{x}-\zeta e_z):\theta, \zeta\in\mathbf{R}\}$. We prove that Σ is unstable by the flow of the evolution equation, for some values of ω, p. Moreover, the trajectories used to exhibit instability are global and uniformly bounded.

RÉSUMÉ. – Dans cet article on étudie l'instabilité des solutions $u(t, x)=e^{-i\omega t}\Phi(x)$ de l'équation $i\partial_t u=-L_A u+|u|^{p-1}u$. iL_A est l'opérateur de Schrödinger avec un champ magnétique uniforme défini par le potentiel vecteur solenoïdal $A: L_A u=-\Delta u-2iA.\nabla u+|A|^2 u$. Φ est une solution de l'équation elliptique non linéaire $L_A\Phi+\omega\Phi-|\Phi|^{p-1}\Phi=0$; ce Φ est invariant par rotations autour de l'axe des zz et est solution d'un problème variationnel. Posons $\Sigma=\{e^{-i\theta}\Phi(\hat{x}-\zeta e_z):\theta, \zeta\in\mathbf{R}\}$. On prouve

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que Σ est instable pour l'équation d'évolution lorsque ω et p sont dans des intervalles convenables. De plus, les trajectoires qui exhibent l'instabilité sont globales et bornées uniformément.

CONTENTS

1. INTRODUCTION

1.1. In $\mathbf{R}^3$, we call the cylindric coordinates ρ, φ, z. A *uniform* magnetic field along the z-axis, $B = be_z$, $b \in \mathbf{R} - \{0\}$, can be derived from a simple solenoidal vector potential $A : B = \operatorname{rot} A$, $\operatorname{div} A = 0$, $A = b/2\,\rho\, e_\varphi$. In such a magnetic field, the evolution of a spinless quantum particle is described by the equation

$$\mathrm{i}d_t u = -L_A u = -(-\Delta u - 2iA.\nabla u + |A|^2 u). \tag{1.1}$$

We associate to (1.1) the nonlinear evolution equation

$$\mathrm{i}d_t u = -L_A u - \alpha |u|^{p-1} u, \tag{1.2}$$

and the nonlinear elliptic problem

$$L_A \Phi + \omega \Phi + \alpha |\Phi|^{p-1} \Phi = 0. \tag{1.3}$$

From the structure of the nonlinear term, looking for solutions to (1.2) of the type $u(t, x) = e^{-i\omega t}\Phi(x)$ is equivalent to solving (1.3). These *uu* are called **stationary state** solutions of (1.2).

We shall be concerned with *instability* properties of such stationary states, particularly of those with cylindrical symmetry, $\Phi(\rho, z)$.

The Cauchy problem relative to (1.2) was solved in [5] and the existence and variational characterization of solutions to (1.3), particularly the symmetric ones, was established in [7]. *Stability* of stationary states was proved in [5], for $p < 1 + 4/3$, by adapting the variational argument in [6].

In the absence of a magnetic field, instability of minimal action stationary states (ground states) was proved, for $p \geqq 1+4/3$, in two different approaches: in [1], based on a finite time blow-up argument; in [13], plotting the ground state action against the angular frequency and discussing the convexity of such function. Subsequently, this last approach flew into an abstract frame for studying the stability of Hamiltonian systems with symmetries, [9].

Our first attempt to prove instability followed the first approach: *see* [8] for an explanation of the unsuccess. This paper originates from the efforts to apply the Grillakis-Shatah-Strauss (GSS) formalism to the present problem. As it will be seen, the formalism is not directly applicable and some detours will be needed.

In the remainder of the Introduction, we shall detail some facts concerning equations (1.1)/(1.3) (subsections 1.2/1.10), state the main result (subsection 1.11) and describe the obstacles to utilization of GSS theory (subsection 1.12).

1.2. We now sketch an appropriate functional setting for the linear evolution equation (1.1).

Put $\nabla_A v = \nabla v + i A v$ and consider the two following *real* Hilbert spaces:

$$L^2 = \left\{ v : \mathbf{R}^3 \to \mathbf{C} : \int |v|^2 < \infty \right\}, \qquad (v, w) = \Re \int v \bar{w};$$

$$H^1_A = \left\{ v \in L^2 : \int |\nabla_A v|^2 < \infty \right\} \qquad (v, w)_{H^1_A} = \Re \int (v \bar{w} + \nabla_A v . \overline{\nabla_A w}).$$

[Integrations, except when otherwise specified, are over $\mathbf{R}^3$; (,) will stand for the scalar product in L^2.] Let H^{-1}_A be the dual space of H^1_A. Taking L^2 as a pivot, one has $H^1_A \overset{d}{\hookrightarrow} L^2 \overset{d}{\hookrightarrow} H^{-1}_A$. Now, define *on* L^2 the operator $\mathscr{L}$ by

$$D(\mathscr{L}) = \{ v \in H^1_A : L_A v \in L^2 \}, \qquad \mathscr{L} v = L_A v.$$

Put $H^2_A = D(\mathscr{L})$ and endow it with the graph norm; one has $H^2_A \overset{d}{\hookrightarrow} H^1_A$. Equality

$$(L_A v, w) = \Re \int \nabla_A v . \overline{\nabla_{A}, w}, \qquad \forall v \in H^2_A \quad w \in H^1_A \tag{1.4}$$

is easily proved and from here one deduces that $\mathscr{L}$ is a symmetric *m*-accretive operator and that $L_A + 1$ is an isometry from H^1_A onto H^{-1}_A. The last fact leads us to define, *on* H^{-1}_A *the operator* $\tilde{\mathscr{L}}$ by

$$D(\tilde{\mathscr{L}}) = H^1_A, \qquad \tilde{\mathscr{L}} v = L_A v.$$

(1.4) generalizes to

$$\langle L_A v, w \rangle = \Re \int \nabla_A v . \overline{\nabla_A w}, \qquad \forall v, w \in H^1_A,$$

[$\langle \, , \, \rangle$ will stand for the (H^{-1}_A, H^1_A) duality bracket] and one deduces that $\mathscr{L}$ still is a symmetric m-accretive operator. Last, apply the Hille-Yosida Theorem to operators $i\mathscr{L}, -i\mathscr{L}, i\tilde{\mathscr{L}}, -i\tilde{\mathscr{L}}$ and solve the Cauchy problem for equation (1.1):

Given $u_0 \in H^1_A$, there exists a unique $u \in C(\mathbf{R}, H^1_A) \cap C^1(\mathbf{R}, H^{-1}_A)$ such that $u(0) = u_0$ and $i d_t u = -L_A u$ in $C(\mathbf{R}, H^{-1}_A)$. Such u verifies the two conservation laws:

$$\int |\nabla_A u(t)|^2 = \int |\nabla_A u_0|^2, \qquad \forall t \in \mathbf{R}; \tag{1.5}$$

$$\int |u(t)|^2 = \int |u_0|^2, \qquad \forall t \in \mathbf{R}. \tag{1.6}$$

Last, if $u_0 \in H^2_A$, then $u \in C(\mathbf{R}, H^2_A) \cap C^1(\mathbf{R}, L^2)$ and the equation is verified in $C(\mathbf{R}, L^2)$.

1.3. Note that (1.1) is a *Hamiltonian system*. Indeed, define on H^1_A the **kinetic-magnetic energy** by $E(v) = 1/2 \int |\nabla_A v|^2$. This is a C^∞-functional, the derivative of which at v is $L_A v$. Now, write (1.1) as $d_t u = i E'(u)$, and the Hamiltonian character is evident, since multiplication by i is a skew-symmetric operator.

This illuminates the conservation law (1.5): *if* the trajectory were in $C^1(\mathbf{R}, H^1_A)$ – and this is *not* the case for general u_0 – we should have:

$$d_t E(u) = \langle E'(u), d_t u \rangle = \langle E'(u), i E'(u) \rangle = 0.$$

1.4. Conservation law (1.6) stems from a symmetry property (*see* [10] for an abstract discussion):

Let L be a skew-symmetric bounded operator in H^1_A, wich is also skew-symmetric in L^2, and let $(T(\theta))_{\theta \in \mathbf{R}}$ be the uniformly continuous group of isometries of H^1_A generated by L. Suppose E is invariant by this group,

$$E(T(\theta) v) = E(v), \qquad \forall v \in H^1_A, \quad \theta \in \mathbf{R}, \tag{1.7}$$

and define a functional on H^1_A, called **charge**, by $Q(v) = 1/2(-i L v, v)$. This is a C^∞-functional, the derivative of wich at v is $-i L v$. Differentiation of (1.7) at $\theta = 0$ yields $\langle E'(v), Lv \rangle = 0$ for all v, and charge is conserved along a $C^1(\mathbf{R}, H^1_A)$ – trajectory:

$$d_t Q(u) = \langle Q'(u), d_t u \rangle = \langle -i L u, i E'(u) \rangle = 0.$$

Identify L with multiplication by i and get the source of conservation law (1.6): *conservation of the* L^2*-norm comes from invariance by* $v \to e^{i\theta} v$.

1.5. We now associate to (1.1) a nonlinear evolution equation. We want conservation of energy and charge to hold; thus, we must keep the Hamiltonian character and invariance by $(T(\theta))_{\theta\in\mathbf{R}}$. Take a C^1-functional on H^1_A, the **potential energy** W, verifying

$$W(T(\theta)v)=W(v), \qquad \forall v\in H^1_A, \quad \theta\in\mathbf{R}, \tag{1.8}$$

and define on H^1_A the **(global) energy** or **Hamiltonian** by

$$H(v)=E(v)+W(v).$$

The nonlinear evolution equation will be $d_t u=iH'(u)$.

In this paper, we restrict to power-type nonlinearities,

$$W(v)=\alpha/(p+1)\int|v|^{p+1},$$

where α is a real parameter and p an admissible exponent; such W clearly verifies (1.8). To determine the admissible range for p, the departure point is inequality $|\nabla|\psi|\leqq|\nabla_A\psi|$ a.e., for any $\psi\in\mathscr{D}(\mathbf{R}^3,\mathbf{C})$. Consequence: if $v\in H^1_A$, then $|v|\in H^1$ and $\||v|\|_{H^1}\leqq\|v\|_{H^1_A}$. From this and Sobolev embeddings, $H^1_A\overset{d}{\hookrightarrow}L^q$ for $q\in[2,6]$, then $L^q\overset{d}{\hookrightarrow}H^{-1}_A$ for $q\in[6/5,2]$. Now, by applying the usual measure-theoretical techniques, one easily proves that, for $p\in[1,5]$, W is a C^1-functional (C^2, if $p>2$), the derivative of wich at v is $\alpha|v|^{p-1}v$. And the nonlinear equation reads

$$id_t u=-L_A u-\alpha|u|^{p-1}u. \tag{1.2}$$

1.6. For general nonlinearities, solving the Cauchy problem is not a triviality. Resolution is based upon very subtle estimates on the inhomogeneous linear equation obtained by interpolation techniques. We refer the reader to [5] and just state the result, adapted to power-type nonlinearities:

THEOREM 1.1. — *Suppose $p\in[1,5)$. Given $u_0\in H^1_A$, there exist a unique $T^*\in(0,\infty]$ and a unique $u\in C([0,T^*),H^1_A)\cap C^1([0,T^*),H^{-1}_A)$ such that $u(0)=u_0$, $id_t u=-L_A u-\alpha|u|^{p-1}u$ in $C([0,T^*),H^{-1}_A)$ and $\|u(t)\|_{H^1_A}\to\infty$ as $t\to T^*$, if $T^*<\infty$. For this u, conservation of energy and charge holds.* •

(The theorem goes on, discussing the case $u_0\in H^2_A$ and dependance of T^*, u on u_0. For our purposes, the present shortened version is sufficient.)

1.7. Since H is invariant by $(T(\theta))_{\theta\in\mathbf{R}}$, looking for solutions to the evolution equation of the type $u(t,x)=T(-\omega t)\Phi(x)$, where $\omega\in\mathbf{R}$ and $\Phi\in H^1_A$, leads us to a stationary equation:

$$L_A\Phi+\omega\Phi+\alpha|\Phi|^{p-1}\Phi=0, \quad \text{in } H^{-1}_A. \tag{1.9}$$

Such solutions are called **stationary states.**

To solve equation (1.9), one naturally considers two variational problems in H^1_A:

First, one tries to find a minimum point of the quadratic functional $E(v)+\omega Q(v)$ subjected to the condition $\int |v|^{p+1}=\mu$, $\mu\in(0,\infty)$. If such a minimum point exists, say w, it will verify $L_A w+\omega w=\lambda(p+1)|w|^{p-1}w$ in H_A^{-1}, for some real λ. Then, making use of the different homogenities of $E(v)+\omega Q(v)$ and $\int |v|^{p+1}$, one expects to determine λ as a function of μ and, by varying μ, to give λ the appropriate value. Details may be found in [7], particularly how to determine the values of ω for wich a minimizing sequence is bounded and how to circumvent the lack of compactness in the embedding $H_A^1 \overset{d}{\hookrightarrow} L^{p+1}$ by applying the concentration-compactness method. Here we give the result:

THEOREM 1.2. — *Suppose* $p\in(1,5)$, $\omega\in(-|b|,\infty)$, $\alpha\in(-\infty,0)$. *Then, for any* $\mu\in(0,\infty)$, *there exists the minimum of* $E(v)+\omega Q(v)$ *on the surface* $\int |v|^{p+1}=\mu$. *Moreover, there is a* μ, *say* μ^*, *such that every corresponding minimum point* w *satisfies* $L_A w+\omega w+\alpha|w|^{p-1}w=0$ *in* H_A^{-1}. ●

Secondly, one tries to find a minimum point of $H(v)$ subjcted to the condition $\int |v|^2=\mu$, $\mu\in(0,\infty)$. If such a minimum point exists, say w, it will verify $L_A w+\alpha|w|^{p-1}w=\lambda 2w$, in H_A^{-1}, for some real λ. In this case, one cannot argue by homogenity to adjust λ, and the Lagrange multiplier will remain undetermined. Again we refer to [7] and state the result:

THEOREM 1.3. — *Suppose* $p\in(1,1+4/3)$, $\alpha\in(-\infty,0)$. *Then, for any* $\mu\in(0,\infty)$, *there exists the minimum of* $H(v)$ *on the surface* $\int |v|^2=\mu$. *Every minimum point* w *satisfies* $L_A w+\omega w+\alpha|w|^{p-1}w=0$ *in* H_A^{-1}, *for some real* ω. ●

1.8. We now consider *stationary states with particular symmetry properties*.

For an integer k, define a closed subspace of H_A^1 by

$$H_{A,k}^1=\{v\in H_A^1 : v(\rho,\varphi,z)e^{ik\varphi} \text{ depends solely of } (\rho,z)\}.$$

We remark that $H_{A,k}^1$ is invariant by $(T(\theta))_{\theta\in\mathbf{R}}$ and that it is also invariant by the flow of the evolution equation. Consequence: looking for solutions to the evolution equation of the type $u(t,x)=T(-\omega t)\Phi(x)$, where $\omega\in\mathbf{R}$ and $\Phi\in H_{A,k}^1$, leads us to

$$L_A\Phi+\omega\Phi+\alpha|\Phi|^{p-1}\Phi=0, \quad in\ H_{A,k}^{-1}, \tag{1.10}$$

where $H_{A,k}^1$ is the dual space of $H_{A,k}^1$. To solve equation (1.10) one argues as in 1.7, everywhere replacing H_A^1 by $H_{A,k}^1$. Again we refer to [7] and state the results:

THEOREM 1.4. – *Fix an integer k, define* $\omega_k = \mathrm{Min}\{-|b|, kb\}$ *and suppose* $p \in (1,5)$, $\omega \in (\omega_k, \infty)$, $\alpha \in (-\infty, 0)$. *Then, for any* $\mu \in (0, \infty)$, *there exists the minimum of* $E(v) + \omega Q(v)$ *on the surface (in* $H^1_{A,k}$*)* $\int |v|^{p+1} = \mu$*; a minimum point can be chosen so that, multiplied by* $e^{ik\varphi}$, *becomes a* nonnegative *function. Moreover, there is a* μ, *say* μ_k^*, *such that every correponding minimum point w satisfies* $L_A w + \omega w + \alpha |w|^{p-1} w = 0$ *in* $H^{-1}_{A,k}$. ●

THEOREM 1.5. – *Fix an integer k and suppose* $p \in (1, 1+4/3)$, $\alpha \in (-\infty, 0)$. *Then, for any* $\mu \in (0, \infty)$, *there exists the minimum of* $H(v)$ *on the surface (in* $H^1_{A,k}$*)* $\int |v|^2 = \mu$*; a minimum point can be chosen so that, multiplied by* $e^{ik\varphi}$, *becomes a* nonnegative *function. Every minimum point w satisfies* $L_A w + \omega w + \alpha |w|^{p-1} w = 0$, *in* $H^{-1}_{A,k}$, *for some real* ω. ●

Nonnegativity and the (eventually) wider range of ω in Theorem 1.4 are consequences of the following facts:

– $v \in H^1_{A,0}$ iff v depends solely of (ρ, z), $v \in H^1(\mathbf{R}^3, \mathbf{C})$, $\rho v \in L^2(\mathbf{R}^3, \mathbf{C})$;

– for $k \neq 0$, $v \in H^1_{A,k}$ iff $v(\rho, \varphi, z) = v^*(\rho, z) e^{-ik\varphi}$, $v^* \in H^1(\mathbf{R}^3, \mathbf{C})$, $\rho v^* \in L^2(\mathbf{R}^3, \mathbf{C})$, $\rho^{-1} v^* \in L^2(\mathbf{R}^3, \mathbf{C})$;

– if $v \in H^1_{A,k}$, then $v \in H^1(\mathbf{R}^3, \mathbf{C})$ and $\rho v \in L^2(\mathbf{R}^3, \mathbf{C})$;

– *on* $H^1_{A,k}$, E decomposes into **kinetic energy**, $K(v) = 1/2 \int |\nabla v|^2$, and **magnetic energy**, $M(v) = b^2/8 \int \rho^2 |v|^2 - (kb)/2 \int |v|^2$.

Note that such decomposition (E = K + something depending on b) will possibly make no sense for general functions in H^1_A: inclusion $H^1_A \subset H^1$ remains an open problem.

1.9. For $v \in H^1_A$, $\delta \in (0, \infty)$, we call $B(v, \delta)$ the ball of H^1_A centered at v with radius δ; for any nonempty set Y in H^1_A and any $\delta \in (0, \infty)$, $\mathscr{V}(Y, \delta)$ is the δ-neighbourhood of Y in H^1_A: $\mathscr{V}(Y, \delta) = \bigcup_{v \in Y} B(v, \delta)$. Y is **stable by the flow** of the evolution equation iff, for any $\mathscr{V}(Y, \delta)$, there exists a $\mathscr{V}(Y, \varepsilon) \subset \mathscr{V}(Y, \delta)$ such that all the trajectories with initial value in $\mathscr{V}(Y, \varepsilon)$ are global and remain in $\mathscr{V}(Y, \delta)$ for all positive *tt*. Let u be a periodic solution of the evolution equation and let $\mathcal{O}$ be the corresponding closed orbit: $\mathcal{O} = \{u(t) : t \in [0, \infty)\}$. **Orbital stability** of u usually means stability of $\mathcal{O}$ by the flow.

Now, fix a minimum point given by Theorem 1.2 (with $\mu = \mu^*$) or by Theorem 1.3, say Φ, and associate to the periodic solution $T(-\omega t)\Phi(x)$

the closed orbit $\mathcal{O}=\{T(-\omega t)\Phi : t\in[0, \infty)\}$. One expects orbital stability of Φ to be discussed in terms of stability of $\mathcal{O}$ by the flow. But some *adaptation* of the usual concept of orbital stability to the present situation is needed, the reason being *an additional symmetry property of* H.

For $y\in\mathbf{R}^3$, consider the mapping $U(y): L^2\ni v\to e^{-iA(y)\cdot\hat{x}}v(\hat{x}-y)$, composed from the y-translation and the corresponding linear gauge transformation. $(U(y))_{y\in\mathbf{R}^3}$ is a continuous unitary representation of the Lie group $(\mathbf{R}^3, +)$ into H^1_A and L^2, *wich leaves* H *invariant*. Define

$$\Omega=\{T(\theta)U(y)\Phi : \theta\in\mathbf{R},\ y\in\mathbf{R}^3\},$$

$$\tilde{\Omega}=\{v\in H^1_A : v \text{ is a solution to the minimization problem solved by } \Phi\}.$$

From invariance, one has $\Omega\subset\tilde{\Omega}$. (We do not know whether equality holds.) And one naturally *presumes* that orbital stability of Φ must be thought in terms of stability *of* $\tilde{\Omega}$ by the flow. Indeed, it can be proved (*see* [5] and references therein, particularly [6]) that, for a minimum point Φ given by Theorem 1.3, $\tilde{\Omega}$ is stable by the flow. Of course, such theorem *does not* imply $\mathcal{O}$ is unstable by the flow; but, in similar cases (absence of magnetic field), examples can be given to show that $\tilde{\Omega}$-stability cannot be strengthened into $\mathcal{O}$-stability (*see* [6]).

1.10. What we have just said about stability of *general* stationary states can also be said, *mutatis mutandis*, about stability of *symmetric* ones. In order to remain in $H^1_{A,k}$, translations must be restricted to translations along the z-axis; this gives no change of gauge, since A is invariant by such translations. So, we define a one parameter strongly continuous group $(V(\zeta))_{\zeta\in\mathbf{R}}$ of isometries of $H^1_{A,k}$ and L^2 by putting

$$V(\zeta): L^2\ni v\to v(\hat{x}-\zeta e_z).$$

For a fixed minimum point Φ given by Theorem 1.4 (with $\mu=\mu_k^*$) or by Theorem 1.5, define

$$\Sigma=\{T(\theta)V(\zeta)\Phi : \theta,\ \zeta\in\mathbf{R}\},$$

$$\tilde{\Sigma}=\{v\in H^1_{A,k} : v \text{ is a solution to the minimization problem solved by } \Phi\}.$$

(Again we ignore wether equality holds in $\Sigma\subset\tilde{\Sigma}$.) Then, it can be proved (*see* [5]) that, for a minimum point Φ given by Theorem 1.5, $\tilde{\Sigma}$ is stable by the flow (restricted to $H^1_{A,k}$). We stress out that this stability refers to perturbations in $H^1_{A,k}$: $\mathcal{V}(\tilde{\Sigma}, \delta)$, $\mathcal{V}(\tilde{\Sigma}, \varepsilon)$ are to be understood as neighbourhoods of $\tilde{\Sigma}$ *in* $H^1_{A,k}$. It is an open problem whether such k-symmetric stationary states are stable (in some decent sense) for general perturbations.

1.11. We now come to the main result of this work. For a nonempty set Y in H^1_A and a $u_0\in Y$, we define the **exit time** by

$$T_*=\operatorname{Sup}\{t\in[0, T^*) : u(\tau)\in Y, \text{ for } \tau\in[0, t]\},$$

with T^*, u given by Theorem 1.1; of course, the same concept applies with an $H^1_{A,k}$ in the place of H^1_A.

THEOREM 1.6. – *Refer to Theorem* 1.4, *and suppose* $k=0$, $\omega\in(0,\infty)$, $p\in[p_{uns}, 5)$, *where* $p_{uns}=1+4/3+(4\sqrt{10}-8)/9$. *With* $\mu=\mu_0^*$, *take a nonnegative minimum point* Φ. *Then,* Σ *is unstable by the flow and, to exhibit instability, one can choose global and uniformly bounded trajectories. Precisely: there exist a* $\mathscr{V}(\Sigma,\delta)$ *in* $H^1_{A,0}$ *and a sequence* $(u_{0,j})_j$ *in* $\mathscr{V}(\Sigma,\delta)$ *such that:*

- $u_{0,j}\to\Phi$ *in* $H^1_{A,0}$, *as* $j\to\infty$;
- $T_*(u_{0,j}, \mathscr{V}(\Sigma,\delta))<\infty, \forall j$;
- $T^*(u_{0,j})=\infty, \forall j$;
- $$\lim_j \operatorname{Sup}_{t\geq 0} \|u_j(t)\|_{H^1_{A,0}}=\|\Phi\|_{H^1_{A,0}}, \qquad (1.11)$$

where u_j *is the trajectory with initial value* $u_{0,j}$. ●

Two remarks about Theorem 1.6:

First, the theorem states instability of Σ to perturbations in $H^1_{A,0}$, hence to general perturbations.

Last, (1.11) – wich is more than uniform boundedness – implies a somehow curious behaviour of the sequence $(u_j)_j$: for j large enough, u_j cannot leave $\mathscr{V}(\Sigma,\delta)$ through an arbitrary point of its frontier – *points wich "turn away from the origin" are forbidden, only points "closer to the origin" are allowed.*

1.12. In [9], stability of stationary states is studied in an abstract setting. To prove Theorem 1.6 within such setting, one needs a C^1 mapping from a nonempty open interval I of $\mathbf{R}$ into $H^1_{A,0}, I\ni\omega\to\Phi_\omega\in H^1_{A,0}$ such that Φ_ω is a nontrivial critical point of the action (with angular frequency ω, S_ω) and $S''_\omega(\Phi_\omega)$ satisfies certain spectral conditions, namely its kernel is spanned by $i\Phi_\omega$. Then, $\mathcal{O}$-*stability* may be discussed in terms of convexity of the real valued function $I\ni\omega\to S_\omega(\Phi_\omega)\in\mathbf{R}$.

In the absence of a magnetic field, the scaling and dilation technique yields Φ_ω and explicitly gives $S_\omega(\Phi_\omega)$. To have $\operatorname{Ker} S''_\omega(\Phi_\omega)$ spanned by $i\Phi_\omega$, one must exclude translations [since $\partial_1\Phi_\omega$, $\partial_2\Phi_\omega$, $\partial_3\Phi_\omega\in\operatorname{Ker} S''_\omega(\Phi_\omega)$ when working in $H^1(\mathbf{R}^3, \mathbf{C})$] – and this leads to *a priori* restriction to radial functions.

When a magnetic field is present, owing to the different homogenities of $-\Delta$ and ρ^2 to scaling, ω *and* b change simultaneously, and scaling and dilation does not yield Φ_ω. On the other hand, *a priori* restriction to radial functions is out of question and working in $H^1_{A,0}$ gives $i\Phi_\omega$, $\partial_z\Phi_\omega\in\operatorname{Ker} S''_\omega(\Phi_\omega)$.

Thus, to prove Theorem 1.6, we give up discussion of stability in terms of a real valued function of ω and we shall look, *directly*, for a Ψ tangent to the constant charge surface at Φ and such that $\langle S''(\Phi)\Psi, \Psi\rangle<0$.

Combined with minimization of S on a surface at Φ, this will appear as an advantageous way of handling spectralities.

1.13. This paper is organized as follows:

– In section 2, instability is proved under a Geometric Condition for Instability (GCI) and assuming the existence of an Auxiliary Dynamical System (ADS).

– In section 3, ADS is constructed under a Geometric Condition for existence of ADS (GCADS) and a Regularity Condition (RC).

– In section 4, we establish GCI, GCADS (under a regularity restriction).

– In section 5, regularity questions are solved.

– Last, in section 6, the proof comes to an end.

The work concludes with some remarks (section 7) and the references (section 8).

2. INSTABILITY

2.1. We refer to Theorem 1.4, with the only restriction $p>2$, fix $\mu=\mu_k^*$, and take a minimum point Φ. We call $\mathscr{Q}$ the constant charge surface at Φ, $\mathscr{Q}=\{v\in H_{A,k}^1 : Q(v)=Q(\Phi)\}$, and call $\mathscr{W}$ the constant potential energy surface at Φ, $\mathscr{W}=\left\{v\in H_{A,k}^1 : \int |v|^{p+1}=\mu_k^*\right\}$. **Action** is defined (in the whole of H_A^1) by $S(v)=H(v)+\omega Q(v)$. According to the restriction on p, this is a C^2-functional; its derivative at v is $S'(v)=L_A v+\alpha|v|^{p-1}v+\omega v$. Consequently, Φ *is a critical point of* $S: S'(\Phi)=0$. Moreover, Φ *minimizes* S *on* $\mathscr{W}: S(\Phi)=\inf_{v\in\mathscr{W}} S(v)$.

We now *suppose* the two following conditions to hold:

THE GEOMETRIC CONDITION FOR INSTABILITY (GCI) – There is a $\Psi\in H_{A,k}^1$ tangent to $\mathscr{Q}$ at Φ such that the Hessian of the action at Φ is strictly negative along $\Psi: \langle S''(\Phi)\Psi, \Psi\rangle<0$. •

[The $(H_{A,k}^1, H_{A,k}^1)$ duality bracket will be $\langle\ ,\ \rangle$ whenever the context allows to distinguish from the (H_A^{-1}, H_A^1) case].

THE AUXILIARY DYNAMICAL SYSTEM (ADS) – There exist a $\mathscr{V}(\Sigma, \varepsilon)$ and a functional $\mathscr{H}: \mathscr{V}(\Sigma, \varepsilon)\to\mathbf{R}$ such that:

– $\mathscr{H}(T(\theta)V(\zeta)v)=\mathscr{H}(v)$, $\forall v\in\mathscr{V}(\Sigma, \varepsilon)$, $\forall\theta, \zeta\in\mathbf{R}$; (2.1)

– $\mathscr{H}$ is differentiable everywhere and $\mathscr{H}'(v)\in H_{A,k}^1$, $\forall v\in\mathscr{V}(\Sigma, \varepsilon)$;

– $\mathscr{H}': \mathscr{V}(\Sigma, \varepsilon)\to H_{A,k}^1$ is a C^1-function, with bounded derivative; (2.2)

– $i\mathscr{H}'(\Phi)=\Psi$. •

Now, consider the vector field $i\mathcal{H}':\mathcal{V}(\Sigma, \varepsilon)\to \mathrm{H}^1_{\mathrm{A}, k}$. From (2.2), the corresponding Cauchy problem is solvable, thus giving rise to a dynamical system: ADS. More precisely: given $u_0\in\mathcal{V}(\Sigma, \varepsilon)$, there exist a unique $\sigma\in(0, \infty]$ and a unique $\varphi(u_0, \hat{s})\in \mathrm{C}^1((-\sigma, \sigma), \mathcal{V}(\Sigma, \varepsilon))$ such that $\varphi(u_0, 0)=u_0$, $d_s\varphi(u_0, s)=i\mathcal{H}'(\varphi(u_0, s))$ in $\mathrm{C}((-\sigma, \sigma), \mathcal{V}(\Sigma, \varepsilon))$ and the distance from $\varphi(u_0, s)$ to Σ tends to ε as $|s|$ tends to σ, if $\sigma<\infty$.

We remark that $i\mathcal{H}'$ is a *uniformly* Lipschitz field: there is a $\mathrm{C}\in(0, \infty)$ such that $\| i\mathcal{H}'(v_2)-i\mathcal{H}'(v_1)\|_{\mathrm{H}^1_{\mathrm{A}, k}}\leqq \mathrm{C}\| v_2-v_1\|_{\mathrm{H}^1_{\mathrm{A}, k}}$ whenever v_1, v_2 are the extreme points of a segment in $\mathcal{V}(\Sigma, \varepsilon)$. Consequence : if $\varepsilon_1\in(0, \varepsilon)$, then $\sigma(u_0)\geqq\sigma_1$ for all $u_0\in\mathcal{V}(\Sigma, \varepsilon_1)$ and some $\sigma_1\in(0, \infty)$. We fix, once for all, such ε_1, σ_1.

Consider the function $\varphi:\mathcal{V}(\Sigma, \varepsilon_1)\times(-\sigma_1, \sigma_1)\to\mathcal{V}(\Sigma, \varepsilon)$ just described. From (2.2), by standard methods, one can prove some additional regularity of φ: φ is C^1 as a function of the two arguments and φ is C^2 in time. These properties will be essential in the sequel.

Two final remarks about ADS:

First, ADS is a *Hamiltonian system*, with Hamiltonian $\mathcal{H}$; of course, $\mathcal{H}$ is conserved along the C^1 (indeed, C^2) $\mathrm{H}^1_{\mathrm{A}, k}$-valued trajectories of ADS.

Last, from (2.1), ADS posseses *the same relevant symmetries* of the original dynamical system (1.2). In particular, *charge is conserved along the (regular) trajectories of* ADS.

We now state

THEOREM 2.1. – *We refer to Theorem* 1.4, *with* $p>2$, $\mu=\mu_k^*$ *and fix a minimum point* Φ. If GCI, ADS hold, *then there exist a* $\mathcal{V}(\Sigma, \delta)$ *in* $\mathrm{H}^1_{\mathrm{A}, k}$ *and a sequence* $(u_{0, j})_j$ *in* $\mathcal{V}(\Sigma, \delta)$ *such that:*

- $u_{0, j}\to\Phi$ *in* $\mathrm{H}^1_{\mathrm{A}, k}$, *as* $j\to\infty$;
- $\mathrm{T}_*(u_{0, j}, \mathcal{V}(\Sigma, \delta))<\infty$, $\forall j$;
- $\mathrm{T}^*(u_{0, j})=\infty$, $\forall j$;
- $\lim_j \operatorname{Sup}_{t\geqq 0}\| u_j(t)\|_{\mathrm{H}^1_{\mathrm{A}, k}}=\|\Phi\|_{\mathrm{H}^1_{\mathrm{A}, k}}$,

where u_j *is the trajectory coming from* $u_{0, j}$. ●

We break into several steps the proof of Theorem 2.1.

2.2. *Proof of Theorem* 2.1. 1*st step.* – We turn to the variation of the action along the trajectories of ADS.

Given $u_0\in\mathcal{V}(\Sigma, \varepsilon_1)$, the mapping $(-\sigma_1, \sigma_1)\ni s\to \mathrm{S}(\varphi(u_0, s))$ is C^2. A calculation gives $d_s\mathrm{S}(\varphi(u_0, s))=\mathrm{P}(\varphi(u_0, s))$, $d_s^2\mathrm{S}(\varphi(u_0, s))=\mathrm{R}(\varphi(u_0, s))$, where P, R are functionals defined on $\mathcal{V}(\Sigma, \varepsilon)$ by

$$\mathrm{P}(v)=\langle \mathrm{S}'(v), i\mathcal{H}'(v)\rangle,$$
$$\mathrm{R}(v)=\langle \mathrm{S}''(v)\, i\mathcal{H}'(v), i\mathcal{H}'(v)\rangle+\langle \mathrm{S}'(v), i\mathcal{H}''(v)(i\mathcal{H}'(v))\rangle.$$

Apply Taylor's Theorem: for $u_0\in\mathcal{V}(\Sigma, \varepsilon_1)$ and $s\in(-\sigma_1, \sigma_1)$, there exists a $\xi\in[0, 1]$ such that

$$\mathrm{S}(\varphi(u_0, s))=\mathrm{S}(u_0)+\mathrm{P}(u_0)s+1/2\,\mathrm{R}(\varphi(u_0, \xi s))s^2.$$

R is a continuous functional and $R(\Phi)<0$. Consequence: there exist $\varepsilon_2 \in (0, \varepsilon_1)$, $\sigma_2 \in (0, \sigma_1)$ such that

$$\forall u_0 \in B(\Phi, \varepsilon_2),\ \forall s \in (-\sigma_2, \sigma_2): \quad S(\varphi(u_0, s)) \leqq S(u_0) + P(u_0) s. \quad (2.3)$$

Note that we used the fact φ is continuous as a function of the two arguments. ●

2.3. *Proof of Theorem* 2.1. *2nd step.* – We intersect the trajectories of ADS with $\mathscr{W}$ to get some uniformity in the LHS of inequality (2.3).

Consider the mapping

$$\mathscr{V}(\Sigma, \varepsilon_1) \times (-\sigma_1, \sigma_1) \ni (u_0, s) \to \int |\varphi(u_0, s)|^{p+1}.$$

This is a C^1-mapping (we use the fact φ is C^1 as a function of the two arguments) and its value at $(\Phi, 0)$ is μ_k^*. On the other hand, the s-partial derivative at $(\Phi, 0)$ is $(p+1)\Re \int |\Phi|^{p-1} \Phi \bar{\Psi}$; if such a quantity were zero, then Ψ would be tangent to $\mathscr{W}$ at Φ; in this case, one would have $\langle S''(\Phi)\Psi, \Psi \rangle \geqq 0$ (otherwise, Φ would not minimize S on $\mathscr{W}$); this is a contradiction with GCI; conclusion – the s-partial derivative at $(\Phi, 0)$ is not zero. Now, we can apply IFT: there exist $\varepsilon_3 \in (0, \varepsilon_2)$, $\sigma_3 \in (0, \sigma_2)$ such that

$$\forall u_0 \in B(\Phi, \varepsilon_3), \quad \exists^1 s \in (-\sigma_3, \sigma_3): \qquad \int |\varphi(u_0, s)|^{p+1} = \mu_k^*.$$

$$i.\,e.\ \varphi(u_0, s) \in \mathscr{W}. \qquad (2.4)$$

Given $u_0 \in B(\Phi, \varepsilon_3)$, apply (2.3) to the pair $(u_0, s(u_0))$ given by (2.4) and take into account that Φ minimizes S on $\mathscr{W}$:

$$\forall u_0 \in B(\Phi, \varepsilon_3), \quad \exists s \in (-\sigma_3, \sigma_3): \qquad S(\Phi) \leqq S(u_0) + P(u_0) s.$$

And from this and the symmetry invariances:

$$\forall u_0 \in \mathscr{V}(\Sigma, \varepsilon_3), \quad \exists s \in (-\sigma_3, \sigma_3): \qquad S(\Phi) \leqq S(u_0) + P(u_0) s. \quad (2.5)$$ ●

2.4. *Proof of Theorem* 2.1. *3rd step.* – We use (2.5) to prove that, along some particular trajectories of (1.2), and as long as one remains in $\mathscr{V}(\Sigma, \varepsilon_3)$, P is bounded away from zero.

Define

$$\mathscr{P}^+ = \{ v \in \mathscr{V}(\Sigma, \varepsilon_3): S(v) < S(\Phi), P(v) > 0 \},$$
$$\mathscr{P}^- = \{ v \in \mathscr{V}(\Sigma, \varepsilon_3): S(v) < S(\Phi), P(v) < 0 \},$$
$$\mathscr{P} = \mathscr{P}^+ \cup \mathscr{P}^-.$$

Suppose $u_0 \in \mathscr{P}$. (In a latter step, it will be proved $\mathscr{P}$ is nonempty.) Take an arbitrary $t \in [0, T_*(\mu_0, \mathscr{V}(\Sigma, \varepsilon_3))$ and apply (2.5) to the value at t of the trajectory u of (1.2) coming from u_0: there exists an $s \in (-\sigma_3, \sigma_3)$

such that $S(\Phi) \leqq S(u(t)) + P(u(t))s$. Since action is conserved along the trajectories of (1.2), this yields $S(\Phi) \leqq S(u_0) + P(u(t))s$, then

$$|P(u(t))| \geqq (S(\Phi) - S(u_0))/\sigma_3.$$

Conclusion, from continuity:

$$\begin{gathered}\forall u_0 \in \mathscr{P}, \quad \exists \eta \in (0, \infty), \quad \forall t \in [0, T_*(u_0, \mathscr{V}(\Sigma, \varepsilon_3)) \\ \text{either } P(u(t)) \geqq \eta, \quad \text{or } P(u(t)) \leqq -\eta. \quad \bullet \end{gathered} \tag{2.6}$$

2.5. *Proof of Theorem* 2.1. *4th step.* – We evaluate the variation of $\mathscr{H}$ along the trajectories of (1.2).

Suppose – for heuristical purposes – the trajectories of (1.2) to be $H^1_{A,k}$-valued C^1-functions. Then, as long as such trajectories remain in $\mathscr{V}(\Sigma, \varepsilon)$, one has:

$$\begin{aligned} d_t \mathscr{H}(u(t)) &= \langle \mathscr{H}'(u(t)), d_t u(t) \rangle = \langle \mathscr{H}'(u(t)), i H'(u(t)) \rangle \\ &= -\langle H'(u(t)), i\mathscr{H}'(u(t)) \rangle = -\langle S'(u(t)), i\mathscr{H}'(u(t)) \rangle, \end{aligned}$$

since charge is constant along the (regular) trajectories of ADS. That is:

$$d_t \mathscr{H}(u(t)) = -P(u(t)). \tag{2.7}$$

To sum up: P *measures the variation of* S, H *along the trajectories of* ADS; $-P$ *measures the variation of* $\mathscr{H}$ *along the trajectories of* (1.2).

We now need a rigorous derivation of (2.7). Take $t \in (0, \infty)$, $w \in C^1([0, t], \mathscr{V}(\Sigma, \varepsilon))$. One has:

$$d_\tau \mathscr{H}(w(\tau)) = \langle \mathscr{H}'(w(\tau)), d_\tau w(\tau) \rangle = \langle d_\tau w(\tau), \mathscr{H}'(w(\tau)) \rangle,$$

$$\mathscr{H}(w(t)) - \mathscr{H}(w(0)) = \int_0^t \langle d_\tau w(\tau), \mathscr{H}'(w(\tau)) \rangle \, d\tau. \tag{2.8}$$

By density, (2.8) remains valid for

$$w \in C([0, t], \mathscr{V}(\Sigma, \varepsilon)) \cap C^1([0, t], H^{-1}_{A,k}).$$

In particular, (2.8) applies to the trajectory of (1.2) coming from $u_0 \in \mathscr{V}(\Sigma, \varepsilon)$, as long as it remains in $\mathscr{V}(\Sigma, \varepsilon)$:

$$\begin{aligned} \mathscr{H}(u(t)) - \mathscr{H}(u_0) &= \int_0^t \langle d_\tau u(\tau), \mathscr{H}'(u(\tau)) \rangle \, d\tau \\ &= \int_0^t \langle i H'(u(\tau)), \mathscr{H}'(u(\tau)) \rangle \, d\tau = -\int_0^t P(u(\tau)) \, d\tau. \end{aligned}$$

And, by applying the Fundamental Theorem of Calculus, one concludes that $\mathscr{H}(u(t))$ is C^1 and that (2.7) holds. $\bullet$

2.6. *Proof of Theorem* 2.1. *5th step.* – We prove that the trajectories of (1.2) coming from points in $\mathscr{P}$ do leave $\mathscr{V}(\Sigma, \varepsilon_3)$ in a finite time.

Take a $u_0 \in \mathscr{P}$ and suppose that $T_*(u_0, \mathscr{V}(\Sigma, \varepsilon_3)) = \infty$. From the previous steps, there exists an $\eta \in (0, \infty)$ such that either $d_t \mathscr{H}(u(t)) \leqq -\eta$, or $d_t \mathscr{H}(u(t)) \geqq \eta$, for $t \in [0, \infty)$. In both cases, $|\mathscr{H}(u(t))| \to \infty$ as $t \to \infty$. But $\mathscr{H}$ is bounded on $\mathscr{V}(\Sigma, \varepsilon)$. Consequence: $T_*(u_0, \mathscr{V}(\Sigma, \varepsilon_3)) < \infty$. •

2.7. *Proof of Theorem* 2.1. *6th step.* – We now prove that there are points in $\mathscr{P}$ arbitrarily close to Φ.

Follow the action along the trajectory of ADS which passes through Φ: $(-\sigma_1, \sigma_1) \ni s \to S(\varphi(\Phi, s))$. It is known that $d_s S(\varphi(\Phi, s))_{|s=0} = P(\Phi) = 0$, $d_s^2 S(\varphi(\Phi, s))_{|s=0} = R(\Phi) < 0$. One can then take $\sigma_4 \in (0, \sigma_2)$ such that:

$$- S(\varphi(\Phi, s)) < S(\Phi), \ \forall s \in (-\sigma_4, 0) \cup (0, \sigma_4); \tag{2.9}$$

$$- \varphi(\Phi, s) \in B(\Phi, \varepsilon_3), \ \forall s \in (-\sigma_4, \sigma_4). \tag{2.10}$$

From (2.10), one can apply (2.3) with $\varphi(\Phi, s)$, where $s \in (-\sigma_4, \sigma_4)$, in the place of u_0:

$$S(\varphi(\varphi(\Phi, s_1), s_2)) \leqq S(\varphi(\Phi, s_1)) + P(\varphi(\Phi, s_1)) s_2,$$
$$\forall s_1 \in (-\sigma_4, \sigma_4),\ s_2 \in (-\sigma_2, \sigma_2).$$

Take $s_2 = -s_1$:

$$S(\Phi) \leqq S(\varphi(\Phi, s_1)) - P(\varphi(\Phi, s_1)) s_1, \qquad \forall s_1 \in (-\sigma_4, \sigma_4). \tag{2.11}$$

From (2.9), (2.11), one concludes that $P(\varphi(\Phi, s)) \neq 0$, for $s \in (-\sigma_4, 0) \cup (0, \sigma_4)$. Combined with (2.9), (2.10), this gives $\varphi(\Phi, s) \in \mathscr{P}$, for $s \in (-\sigma_4, 0) \cup (0, \sigma_4)$. •

2.8. *Proof of Theorem* 2.1. *7th step.* – We prove that the trajectory of (1.2) coming from $\varphi(\Phi, s)$, s sufficiently close to zero, s with the appropriate sign, is global and bounded.

We recall that the mapping $(-\sigma_1, \sigma_1) \ni s \to \int |\varphi(\Phi, s)|^{p+1}$ is C^1 (indeed, C^2), its value at $s=0$ is μ_k^* and its derivative at $s=0$ is not zero. Then, there is a $\sigma_5 \in (0, \sigma_4)$ such that $\int |\varphi(\Phi, s)|^{p+1} < \mu_k^*$ if $\beta s \in (0, \sigma_5)$, where $\beta = -1$ if the derivative is positive and $\beta = 1$ otherwise. Fix an s such that $\beta s \in (0, \sigma_5)$, put $u_0 = \varphi(\Phi, s)$, and consider the trajectory u of (1.2) coming from u_0.

Suppose that one had $\int |u(t)|^{p+1} = \mu_k^*$, for some $t \in (0, T^*(u_0))$; then, $u(t) \in \mathscr{W}$ and, from conservation of the action and $u_0 \in \mathscr{P}$, it would come $S(u(t)) < S(\Phi)$; this would be a contradiction with minimization of S on $\mathscr{W}$ by Φ. Thus, by continuity, $\int |u(t)|^{p+1} < \mu_k^*$, for any $t \in [0, T^*(u_0))$. Combined with conservation of energy and charge, this gives

$$1/2 \| u(t) \|_{H^1_{A,k}}^2 < 1/2 \| \Phi \|_{H^1_{A,k}}^2 + (H(u_0) + Q(u_0)) - (H(\Phi) + Q(\Phi)),$$
$$\forall t \in [0, T^*(u_0)). \tag{2.12}$$

This shows that u is a global and bounded trajectory. •

2.9 *Proof of Theorem* 2.1. *Conclusion.* – Take $\mathscr{V}(\Sigma, \delta) = \mathscr{V}(\Sigma, \varepsilon_3)$. Given a positive integer j, choose an s_j such that $\beta s_j \in (0, \sigma_5)$ and

$$-u_{0,j} \in \mathrm{B}(\Phi, 1/j), \tag{2.13}$$

$$-(\mathrm{H}(u_{0,j}) + \mathrm{Q}(u_{0,j})) - (\mathrm{H}(\Phi) + \mathrm{Q}(\Phi)) \leqq 1/(2j^2) + 1/j \,\|\Phi\|_{\mathrm{H}^1_{\mathrm{A},k}}, \tag{2.14}$$

where $u_{o,j} = \varphi(\Phi, s_j)$. $(u_{0,j})_j$ converges to Φ. For every j, the trajectory u_j coming from $u_{0,j}$ is global and bounded; $u_{0,j} \in \mathscr{P}$, u_j does leave $\mathscr{V}(\Sigma, \delta)$ in a finite time. Last, from (2.12)/(2.14):

$$\|\Phi\|_{\mathrm{H}^1_{\mathrm{A},k}} - 1/j \leqq \sup_{t \geqq 0} \|u_j(t)\|_{\mathrm{H}^1_{\mathrm{A},k}} < \|\Phi\|_{\mathrm{H}^1_{\mathrm{A},k}} + 1/j. \quad \bullet\bullet$$

2.10. A theorem analogous to Theorem 2.1 holds for general stationary states given by Theorem 1.2. One simply must replace everywhere $\mathrm{H}^1_{\mathrm{A},k}$, μ^*_k, Σ by $\mathrm{H}^1_{\mathrm{A}}$, μ^*, Ω, respectively.

2.11. To prove Σ is unstable by the flow, the previously described method applies to other situations. For instance, when Φ is a critical point of S wich minimizes S *locally* on a surface $\mathscr{X}$. GCI, ADS stay unchanged; minor modifications are needed in the proof, so that intersection of the trajectories of ADS with $\mathscr{X}$ takes place in an appropriate neighbourhood of Φ in $\mathscr{X}$; and the last part of the proof (globality, boundedness) is no longer valid – here it is essential Φ to minimize S *globally on* $\mathscr{W}$. Unhappily, the material is not available (to our knowledge) for such generalization: we do not know any theorem to assert the existence of a critical point of S wich minimizes S locally (nonglobally) on a surface $\mathscr{X}$.

2.12. One also may interchange everywhere <and>: the method still works. This means the critical point Φ would maximize S on $\mathscr{X}$ and GCI would change from concavity to convexity. Again we ignore the relevant existence theorem.

2.13. Section 2 is inspired by [13], [9]. But, such as they can be found there, the arguments are not applicable to our case: we could not rely on some C^1-trajectory, $\mathrm{I} \ni \omega \to \Phi_\omega \in \mathrm{H}^1_{\mathrm{A},k}$, where $\mathrm{S}''_\omega(\Phi_\omega)$ fulfills specific spectral requirements (*see* 1.12); nor did we intend to prove $\mathscr{O}$-instability, but rather instability of a larger set (Σ).

3. THE AUXILIARY DYNAMICAL SYSTEM

3.1. Call $\mathrm{T}'(0)$ the infinitesimal generator of $(\mathrm{T}(\theta))_{\theta \in \mathbf{R}}$ and $\mathrm{V}'(0)$ the infinitesimal generator of $(\mathrm{V}(\zeta))_{\zeta \in \mathbf{R}}$, both groups taken as strongly continuous groups of isometries of $H^1_{A,k}$:

$$D(T'(0)) = H^1_{A,k}, \qquad T'(0)v = iv;$$
$$D(V'(0)) = \{ v \in H^1_{A,k} : \partial_z v \in H^1_{A,k} \}, \qquad V'(0)v = -\partial_z v.$$

Since $(T(\theta))_{\theta\in\mathbf{R}}$, $(V(\zeta))_{\zeta\in\mathbf{R}}$ are also strongly continuous groups of isometries of L^2, the infinitesimal generators, in the L^2 setting, are skew-symetric; particularly:

$$(T'(0)v_1, v_2) = -(v_1, T'(0)v_2), \qquad \forall v_1, v_2 \in D(T'(0)); \tag{3.1}$$
$$(V'(0)v_1, v_2) = -(v_1, V'(0)v_2), \qquad \forall v_1, v_2 \in D(V'(0)); \tag{3.2}$$

(3.1). (3.2) and the fact that the two groups commute on L^2 will be used throughout this section.

Take a $\Phi \in H^1_{A,k}$, $\Phi \neq 0$, and a $\Psi \in H^1_{A,k}$ and *suppose* the two following conditions to hold:

REGULARITY CONDITION (RC). – $\Phi \in D((V'(0))^2)$, $\Psi \in D(V'(0))$. •

GEOMETRIC CONDITION FOR EXISTENCE OF ADS (GCADS). – In L^2, $i\Psi$ is orthogonal to $T'(0)\Phi$ and to $V'(0)\Phi$. Besides, $T'(0)\Phi$, $V'(0)\Phi$ are linearly independent. •

The main result of section 3 is:

THEOREM 3.1. – *Take a* $\Phi \in H^1_{A,k}$, $\Phi \neq 0$, *and a* $\Psi \in H^1_{A,k}$. If RC, GCADS *hold, then there is an* ADS *in a neighbourhood* $\mathscr{V}(\Sigma, \varepsilon)$. •

Note that variational characterization of Φ is irrelevant here. To prove Theorem 3.1, one needs an (almost) evident proposition about the topology of Σ:

3.2. PROPOSITION 3.2. – *Put* $\mathbf{T}^1 = \mathbf{R}/(2\pi\mathbf{Z})$ (*with the quotient topology*), *define* χ *by* $\chi : \mathbf{T}^1 \times \mathbf{R} \ni ([\theta], \zeta) \to T(\theta)V(\zeta)\Phi \in \Sigma$, *and endow* Σ *with either the topology of* L^2 *or with that of* $H^1_{A,k}$. *Then,* χ *is a homeomorphism from the cylinder* $\mathbf{T}^1 \times \mathbf{R}$ *onto* Σ.

Proof of Proposition 3.2. – Suppose $T(\theta_1)V(\zeta_1)\Phi = T(\theta_2)V(\zeta_2)\Phi$. From $\Phi \neq 0$, one concludes first that $\zeta_1 = \zeta_2$, then that $[\theta_1] = [\theta_2]$. Thus, χ is one-to-one. Of course, χ is continuous. To prove χ^{-1} is continuous, it is sufficient to prove continuity at Φ. Given $(\theta_j)_j$, $(\zeta_j)_j$ such that $T(\theta_j)V(\zeta_j)\Phi \to \Phi$ in L^2 as $j \to \infty$, one must prove that $([\theta_j], \zeta_j) \to ([0], 0)$ in $\mathbf{T}^1 \times \mathbf{R}$ as $j \to \infty$. Put $\mathbf{S}^1 = \mathbf{R} \cup \{\infty\}$ (with the usual topology); since $\mathbf{T}^1 \times \mathbf{S}^1$ is compact, one can additionally assume $([\theta_j], \zeta_j) \to ([\theta^*], \zeta^*)$ in $\mathbf{T}^1 \times \mathbf{S}^1$ as $j \to \infty$, for some $\theta^* \in \mathbf{R}$, $\zeta^* \in \mathbf{S}^1$. If $\zeta^* = \infty$, then $T(\theta_j)V(\zeta_j)\Phi \to 0$ in $\mathscr{D}'$ as $j \to \infty$, wich is a contradiction with $\Phi \neq 0$. Thus, $\zeta^* \in \mathbf{R}$ and $T(\theta_j)V(\zeta_j)\Phi \to T(\theta^*)V(\zeta^*)\Phi$ in L^2 as $j \to \infty$. Combined with injectivity of χ, this gives $[\theta^*] = [0]$, $\zeta^* = 0$. •

We remark one only needs $\Phi \neq 0$ to derive Proposition 3.2.

3.3. *Proof of Theorem* 3.1. 1*st step*. – For v sufficiently close to Φ and (θ, ζ) sufficiently close to $(0, 0)$, we minimize the L^2-distance from $T(\theta)V(\zeta)\Phi$ to Φ.

Consider the mapping

$$F : H^1_{A,k} \times \mathbf{R} \times \mathbf{R} \ni (v, \theta, \zeta) \to 1/2 \| T(\theta) V(\zeta) v - \Phi \|^2_{L^2} \in \mathbf{R}.$$

From $\Phi \in D((V'(0))^2)$ ($\partial_z \Phi$, $\partial_z^2 \Phi \in L^2$ would be sufficient here), one concludes that F is C^2 and

$$\partial_2 F = (T(-\theta) V(-\zeta) T'(0) \Phi, v), \qquad (3.3)$$
$$\partial_3 F = (T(-\theta) V(-\zeta) V'(0) \Phi, v), \qquad (3.4)$$
$$\partial^2_{2,2} F = (-T(-\theta) V(-\zeta) ((T'(0))^2 \Phi, v),$$
$$\partial^2_{2,3} F = (-T(-\theta) V(-\zeta) T'(0) V'(0) \Phi, v),$$
$$\partial^2_{3,3} F = (-T(-\theta) V(-\zeta) ((V'(0))^2 \Phi, v).$$

In particular:

$$\partial_2 F(\Phi, 0, 0) = 0,$$
$$\partial_3 F(\Phi, 0, 0) = 0,$$
$$\partial^2_{2,2} F(\Phi, 0, 0) = \| T'(0) \Phi \|^2_{L^2},$$
$$\partial^2_{2,3} F(\Phi, 0, 0) = (T'(0) \Phi, V'(0) \Phi),$$
$$\partial^2_{3,3} F(\Phi, 0, 0) = \| V'(0) \Phi \|^2_{L^2}.$$

Since $T'(0)\Phi$, $V'(0)\Phi$ are linearly independent, one has

$$(\partial^2_{2,2} F \, \partial^2_{3,3} F - \partial^2_{2,3} F \, \partial^2_{3,2} F)(\Phi, 0, 0) > 0.$$

One then applies IFT to the function $H^1_{A,k} \times \mathbf{R} \times \mathbf{R} \ni (v, \theta, \zeta) \to (\partial_2 F, \partial_3 F)$ $(v, \theta, \zeta) \in \mathbf{R} \times \mathbf{R}$ to conclude that there exist ε_1, ζ_1, $\eta \in (0, \infty)$, $\theta_1 \in (0, \pi)$ such that:

– $\partial^2_{2,2} F(v, \theta, \zeta) > \eta$, $(\partial^2_{2,2} F \, \partial^2_{3,3} F - \partial^2_{2,3} F \, \partial^2_{3,2} F)(v, \theta, \zeta) > \eta$,
$\forall (v, \theta, \zeta) \in B(\Phi, \varepsilon_1) \times (-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1)$; (3.5)

– $\forall v \in B(\Phi, \varepsilon_1)$, $\exists^1 (\theta, \zeta) \in (-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1)$:
$(\partial_2 F, \partial_3 F)(v, \theta, \zeta) = 0$. (3.6)

Call G the function described in (3.6):

$$G : B(\Phi, \varepsilon_1) \ni v \to (\theta(v), \zeta(v)) \in (-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1).$$

According to IFT, this is a C^1-function and one has explicit (in terms of $\partial^2 F$) expressions for G'_1, G'_2. Moreover, according to (3.5), (3.6), and arguing by convexity:

– Given $v \in B(\Phi, \varepsilon_1)$,
$G(v)$ minimizes $F(v, \hat{\theta}, \hat{\zeta})$ on $(-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1)$. • (3.7)

3.4. *Proof of Theorem* 3.1. 2*nd step*. – We turn to the variation of G with $(T(\theta))_{\theta \in \mathbf{R}}$, $(V(\zeta))_{\zeta \in \mathbf{R}}$.

From continuity of χ^{-1} at Φ:

$$\forall \theta_1 \in (0, \pi), \zeta_1 \in (0, \infty), \quad \exists \varepsilon \in (0, \infty), \quad \forall \theta, \zeta \in \mathbf{R}:$$

$$\| T(\theta) V(\zeta) \Phi - \Phi \|_{L^2} < \varepsilon \Rightarrow (\exists m \in \mathbf{Z} : |\theta - m 2\pi| < \theta_1) \text{ and } |\zeta| < \zeta_1. \quad (3.8)$$

We apply (3.8) to the θ_1, ζ_1 determined in the 1st step of the proof and obtain an $\varepsilon(\theta_1, \zeta_1)$. Then, we choose an $\varepsilon_2 \in (0, \varepsilon_1) \cap (0, \varepsilon(\theta_1, \zeta_1)/4)$, where ε_1 was determined in the 1st step of the proof. We say that, if $v \in B(\Phi, \varepsilon_2)$, $\theta \in \mathbf{R}$, $\zeta \in \mathbf{R}$, $T(\theta) V(\zeta) v \in B(\Phi, \varepsilon_2)$, then:

– $\exists m \in \mathbf{Z} : G_1(T(\theta) V(\zeta) v) = G_1(v) - \theta - m 2\pi$; (3.9)

– $G_2(T(\theta) V(\zeta) v) = G_2(v) - \zeta$. (3.10)

To prove such assertion, we start from inequality

$$\| T(G_1(v) - \theta) V(G_2(v) - \zeta) \Phi - \Phi \|_{L^2} \leqq 2 \| v - \Phi \|_{L^2} + \| T(\theta) V(\zeta) v - \Phi \|_{L^2} + \| T(G_1(v)) V(G_2(v)) v - \Phi \|_{L^2}.$$

By assumption, the sum of the two first terms in the RHS is less than $3\varepsilon_2$. On the other hand, from (3.7), the last term in the RHS is less than ε_2. Thus, the LHS is less than $\varepsilon(\theta_1, \zeta_1)$. We now apply (3.8) and conclude that, for some $m \in \mathbf{Z}$:

– $G_1(v) - \theta - m 2\pi \in (-\theta_1, \theta_1)$; (3.11)

– $G_2(v) - \zeta \in (-\zeta_1, \zeta_1)$. (3.12)

Last, consider the function

$$(-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1) \ni (\theta', \zeta') \to (\partial_2 F, \partial_3 F)(T(\theta) V(\zeta) v, \theta', \zeta') \in \mathbf{R} \times \mathbf{R}.$$

From the 1st step, we know that $(G_1(T(\theta) V(\zeta) v), G_2(T(\theta) V(\zeta) v))$ is *the only zero* of such function *in* $(-\theta_1, \theta_1) \times (-\zeta_1, \zeta_1)$. Thus, taking (3.11), (3.12) into account, to justify (3.9), (3.10) things are reduced to prove that

$$\partial_2 F(T(\theta) V(\zeta) v, G_1(v) - \theta - m 2\pi, G_2(v) - \zeta) = 0;$$
$$\partial_3 F(T(\theta) V(\zeta) v, G_1(v) - \theta - m 2\pi, G_2(v) - \zeta) = 0.$$

This is an immediate consequence of the expressions for $\partial_2 F$, $\partial_3 F$ and the definition of G. ●

3.5. *Proof of Theorem* 3.1. *3rd step.* – Definition of $\mathscr{H}$.

Define $\mathscr{H}$ *on* $B(\Phi, \varepsilon_2)$ by

$$\mathscr{H}(v) = (-i\Psi, T(G_1(v)) V(G_2(v)) v).$$

Let $v \in B(\Phi, \varepsilon_2)$, $\theta \in \mathbf{R}$, $\zeta \in \mathbf{R}$ such that $T(\theta) V(\zeta) v \in B(\Phi, \varepsilon_2)$. From (3.9), (3.10), one has $\mathscr{H}(T(\theta) V(\zeta) v) = \mathscr{H}(v)$, *i. e.* $\mathscr{H}$ *is invariant by* $(T(\theta))_{\theta \in \mathbf{R}}$, $(V(\zeta))_{\zeta \in \mathbf{R}}$ *on* $B(\Phi, \varepsilon_2)$. Consequence: *one can coherently extend* $\mathscr{H}$ *to* $\mathscr{V}(\Sigma, \varepsilon_2) = \bigcup_{\theta, \zeta \in \mathbf{R}} T(\theta) V(\zeta) B(\Phi, \varepsilon_2)$ *by means of* $(T(\theta))_{\theta \in \mathbf{R}}$, $(V(\zeta))_{\zeta \in \mathbf{R}}$. ●

3.6. *Proof of Theorem* 3.1. *Conclusion.* – Take $\varepsilon = \varepsilon_2$. On $\mathscr{V}(\Sigma, \varepsilon)$, $\mathscr{H}$ is, by construction, invariant by $(T(\theta))_{\theta \in \mathbf{R}}$, $(V(\zeta))_{\zeta \in \mathbf{R}}$. From the definition of $\mathscr{H}$, the expressions (given by IFT) for G'_1, G'_2 and Φ, $\Psi \in D(V'(0))$

($\partial_z \Phi, \partial_z \Psi \in L^2$ would be sufficient here), one deduces that

$$\begin{aligned}\mathcal{H}'(v) = & Y_1(v) T(-G_1(v)) V(-G_2(v)) T'(0) \Phi \\ & + Y_2(v) T(-G_1(v)) V(-G_2(v)) V'(0) \Phi \\ & + T(-G_1(v)) V(-G_2(v)) (-i\Psi), \\ & \forall v \in B(\Phi, \varepsilon),\end{aligned}$$

where Y_1, Y_2 are functionals defined on $B(\Phi, \varepsilon)$ by

$$\begin{aligned}Y_j(v) = & D_{j,1}(v, G_1(v), G_2(v))(iT'(0)\Psi, T(G_1(v)) V(G_2(v)) v) \\ & + D_{j,2}(v, G_1(v), G_2(v))(iV'(0)\Psi, T(G_1(v)) V(G_2(v)) v); \\ D_{1,1} = & -(\partial^2_{2,2} F \partial^2_{3,3} F - \partial^2_{2,3} F \partial^2_{3,2})^{-1} \partial^2_{3,3} F, \\ D_{1,2} = D_{2,1} = & (\partial^2_{2,2} F \partial^2_{3,3} F - \partial^2_{2,3} F \partial^2_{3,2} F)^{-1} \partial^2_{2,3} F, \\ D_{2,2} = & -(\partial^2_{2,2} F \partial^2_{3,3} F - \partial^2_{2,3} F \partial^2_{3,2} F)^{-1} \partial^2_{2,2} F.\end{aligned}$$

Since $\Phi \in D(V'(0))$, one concludes that $\mathcal{H}'(v) \in H^1_{A,k}$ for $v \in B(\Phi, \varepsilon)$; of course, the same conclusion holds for $v \in \mathcal{V}(\Sigma, \varepsilon)$. From orthogonality in GCADS, $Y_1(\Phi) = Y_2(\Phi) = 0$, hence $i\mathcal{H}'(\Phi) = \Psi$. Last, one must check up whether $\mathcal{H}'$ is an $H^1_{A,k}$-valued C^1-function with bounded derivative. This is a consequence of $\Phi \in D((V'(0))^2)$, $\Psi \in D(V'(0))$ (from what, in particular, $\partial^3_z \Phi, \partial^2_z \Psi \in L^2$) and, concerning boundedeness, of (3.5). Verification is easy but rather cumbersome, and is left to the reader. •

3.7. Theorem 3.1 is inspired by [9], [13], where ADS is constructed in the case of a one paremeter group symmetry.

4. THE TWO GEOMETRIC CONDITIONS

4.1. To find a Ψ tangent to $\mathcal{Q}$ at Φ, we proceed by scaling and dilation: for appropriate real γ_1, γ_2, charge is constant along the trajectory $(0, \infty) \ni \tau \to \tau^{\gamma_1} \Phi(\tau^{\gamma_2} \hat{x}) \in H^1_{A,0}$; *if* this trajectory is regular (for instance, $\Phi \in \mathcal{S}^2$), then derivation at $\tau = 1$ yields the desired Ψ:

$$\Psi = C(3/2 \Phi + x . \nabla \Phi), \qquad C \neq 0.$$

Now, we *assume* $x . \nabla \Phi \in H^1_{A,0}$, *define* Ψ by

$$\Psi = 3/2 \Phi + x . \nabla \Phi, \tag{4.1}$$

and investigate under what conditions GCI, GCADS hold.

The main result of section 4 is:

THEOREM 4.1. – *Suppose* $p \in [p_{uns}, 5)$, $\omega \in (0, \infty)$ *and let* Φ *be a nontrivial* nonnegative *critical point of the action in* $H^1_{A,0}$. Assume

$$\Phi \in D(V'(0)), \; x . \nabla \Phi \in H^1_{A,0}$$

and define Ψ *by* (4.1). *Then, GCI and GCADS hold.* •

Note that variational characterization of the critical point is immaterial.

To derive Theorem 4.1, one needs a Virial Theorem, *i. e.* a relation on the kinetic, magnetic and potential energies of *any* critical point. On $H^1_{A,0}$, define a functional N by

$$N(v)=1/2\int|\nabla v|^2-b^2/8\int\rho^2|v|^2+3\alpha/4\,(p-1)/(p+1)\int|v|^{p+1}.$$

One then has:

THEOREM 4.2. – $N(\Phi)=0$, *if* Φ *is a critical point of the action in* $H^1_{A,0}$. ●

Note that, by applying $S'(\Phi)=0$ to Φ, one has:

$$1/2\int|\nabla\Phi|^2+b^2/8\int\rho^2|\Phi|^2+\omega/2\int|\Phi|^2+\alpha/2\int|\Phi|^{p+1}=0. \quad (4.2)$$

So, there is a trivial relation on the energies and the charge. The nontrivial point in the Virial Theorem is absence of charge.

We break into several steps the proofs of Theorems 4.1, 4.2.

4.2. *Proof of Theorem* 4.2. 1*st step.* – Regularity.

To prove the Virial Theorem, we must anticipate just a little bit of regularity: we say that $\Phi\in H^2_{loc}(\mathbf{R}^3,\mathbf{C})$.

Since Φ is a critical point of the action in $H^1_{A,0}$, one has

$$-\ \Re\int\Phi\Delta\bar\psi+b^2/4\,\Re\int\rho^2\Phi\bar\psi+\omega\,\Re\int\Phi\bar\psi+\alpha\,\Re\int|\Phi|^{p-1}\Phi\bar\psi=0, \quad (4.3)$$

for $\psi\in\mathscr{D}(\mathbf{R}^3,\mathbf{C})$ and ψ invariant by rotations around the z-axis. Given *any* $\psi_1\in\mathscr{D}(\mathbf{R}^3,\mathbf{C})$, apply (4.3) first to $\psi(\rho,z)=(2\pi)^{-1}\int_0^{2\pi}\psi_1(\rho,\varphi,z)\,d\varphi$, then to this function multiplied by i. Consequence:

$$-\ \Delta\Phi+b^2/4\,\rho^2\Phi+\omega\Phi+\alpha|\Phi|^{p-1}\Phi=0,\ in\ \mathscr{D}'(\mathbf{R}^3,\mathbf{C}).$$

Take a $\xi\in\mathscr{D}(\mathbf{R}^3,\mathbf{R})$. $\xi\Phi$ verifies the equation

$$-\Delta(\xi\Phi)+\xi\Phi=(-\Delta\xi-b^2/4\,\rho^2\xi-(\omega-1)\xi)\Phi$$
$$-2\nabla\xi.\nabla\Phi-\alpha|\Phi|^{p-1}\xi\Phi,\ \text{in}\ \mathscr{D}'(\mathbf{R}^3,\mathbf{C}). \quad (4.4)$$

In this work, we shall uniquely make use of the following regularity property:

– For $q\in[1,\infty)$ and a nonnegative integer m,
let a tempered distribution f verify $(-\Delta+1)f\in W^{m,q}(\mathbf{R}^3,\mathbf{C})$.
Then, $f\in W^{m+2,q}(\mathbf{R}^3,\mathbf{C})$. (4.5)

In the RHS of (4.4), the two first terms are in L^2. If $p\leqq 3$, the same holds for the third term. If $p>3$, a bootstrap argument based on (4.5) and Sobolev embeddings gives that the third term still is in L^2. [In this argument, ξ must change from step to step, owing to $|\Phi|^{p-1}$ instead of $|\xi\Phi|^{p-1}$ in (4.4).] Thus, $\xi\Phi\in H^2(\mathbf{R}^3, \mathbf{C})$. ●

4.3. *Proof of Theorem* 4.2. *Conclusion.* – Apply $S'(\Phi)=0$ to $\Delta\psi_1\,\psi_2+2\nabla\psi_1.\nabla\psi_2$, where $\psi_1\in\mathscr{D}(\mathbf{R}^3,\mathbf{R})$, $\psi_2\in\mathscr{D}(\mathbf{R}^3,\mathbf{C})$ and ψ_1, ψ_2 are invariant by rotations around the z-axis:

$$\Re\int\nabla\Phi.\nabla(\Delta\psi_1\,\bar{\psi}_2+2\nabla\psi_1.\nabla\bar{\psi}_2)\\+\Re\int(b^2/4\rho^2\,\Phi+\omega\Phi+\alpha|\Phi|^{p-1}\Phi)\\\times(\Delta\psi_1\,\bar{\psi}_2+2\nabla\psi_1.\nabla\bar{\psi}_2)=0. \quad (4.6)$$

We intend to let $\psi_2\to\Phi$ in $H^1_{A,0}$. The second term in (4.6) is good, since $\Phi\in L^\infty_{loc}$, but the first term needs some preparation:

$$\Re\int\nabla\Phi.\nabla(\Delta\psi_1\,\bar{\psi}_2+2\nabla\psi_1.\nabla\bar{\psi}_2)\\=\Re\int\nabla\Phi.(\bar{\psi}_2\nabla\Delta\psi_1+\Delta\psi_1\nabla\bar{\psi}_2+2D^2\psi_1(\nabla\bar{\psi}_2))\\+2\Re\int\nabla\Phi.D^2\bar{\psi}_2(\nabla\psi_1). \quad (4.7)$$

Now, is the last term in (4.7) that must be modified:

$$2\Re\int\nabla\Phi.D^2\bar{\psi}_2(\nabla\psi_1)\\=2\Re\int(\nabla(\nabla\bar{\psi}_2.\nabla\Phi)-D^2\Phi(\nabla\bar{\psi}_2)).\nabla\psi_1\\=-2\Re\int\nabla\bar{\psi}_2.\nabla\Phi\,\Delta\psi_1-2\Re\int D^2\Phi(\nabla\bar{\psi}_2).\nabla\psi_1, \quad (4.8)$$

since $\Phi\in H^2_{loc}$. Collecting together (4.6)/(4.8) and letting $\psi_2\to\Phi$ in $H^1_{A,0}$ yields

$$\Re\int\nabla\Phi.(\bar{\Phi}\nabla\Delta\psi_1-\Delta\psi_1\nabla\bar{\Phi}+2D^2\psi_1(\nabla\bar{\Phi}))+2\Re\int D^2\Phi(\nabla\bar{\Phi}).\nabla\psi_1\\+\Re\int(b^2/4\rho^2\,\Phi+\omega\Phi+\alpha|\Phi|^{p-1}\Phi)(\Delta\psi_1\,\bar{\Phi}+2\nabla\psi_1.\nabla\bar{\Phi})=0,$$

then

$$-1/2\int|\Phi|^2\Delta^2\psi_1-\int|\nabla\Phi|^2\Delta\psi_1+2\int D^2\psi_1(\nabla\Phi).\nabla\bar{\Phi}+\int|\nabla\Phi|^2\Delta\psi_1$$
$$+b^2/4\int\rho^2|\Phi|^2\Delta\psi_1+\omega\int|\Phi|^2\Delta\psi_1+\alpha\int|\Phi|^{p+1}\Delta\psi_1$$
$$-b^2/4\int|\Phi|^2\operatorname{div}(\rho^2\nabla\psi_1)-\omega\int|\Phi|^2\Delta\psi_1-2\alpha/(p+1)\int|\Phi|^{p+1}\Delta\psi_1=0,$$

hence

$$-1/2\int|\Phi|^2\Delta^2\psi_1+2\int D^2\psi_1(\nabla\Phi).\nabla\bar{\Phi}$$
$$-b^2/4\int|\Phi|^2\nabla\rho^2.\nabla\psi_1+\alpha(p-1)/(p+1)\int|\Phi|^{p+1}\Delta\psi_1=0. \quad (4.9)$$

Put $\psi_1(x)=\xi(|x|^2/j^2)|x|^2/2$, where $\xi\in C^\infty((0,\infty),\mathbf{R})$, ξ equals 1 on $(0,1)$ and ξ equals zero on $(2,\infty)$. Let $j\to\infty$; by applying the Dominated Convergence Theorem, (4.9) becomes $4N(\Phi)=0$. ●●

4.4. *Proof of Theorem* 4.1. 1*st step.* – The trivial part.

A simple calculation gives that Ψ is tangent to $\mathcal{Q}$ at Φ, hence that $i\Psi$ is orthogonal to $T'(0)\Phi$. Orthogonality of $i\Psi$ to $V'(0)\Phi$ comes from Φ being real-valued. To prove that $T'(0)\Phi$, $V'(0)\Phi$ are linearly independent, one may prove they are orthogonal and again this comes from Φ being real-valued. ●

We now turn to the proof of $\langle S''(\Phi)\Psi,\Psi\rangle<0$.

4.5. *Proof of Theorem* 4.1. 2*nd step.* – We derive a formula for $\langle S''(\Phi)\Psi,\Psi\rangle$ in terms of Φ, $x.\nabla\Phi$.

The mapping $\mathbf{R}\ni\tau\to\Phi+\tau\Psi\in H^1_{A,0}$ is a regular trajectory with speed Ψ and zero acceleration. Consequence:

$$d_\tau^2 S(\Phi+\tau\Psi)_{|\tau=0}=\langle S''(\Phi)\Psi,\Psi\rangle.$$

We have:

$$S(\Phi+\tau\Psi)=1/2\int|\nabla_A\Phi|^2+\Re\int\nabla_A\Phi.\overline{\nabla_A\Psi}\tau$$
$$+1/2\int|\nabla_A\Psi|^2\tau^2+\omega/2\int\Phi^2+\omega\int\Phi\Psi\tau+\omega/2\int\Psi^2\tau^2$$
$$+\alpha/(p+1)\int|\Phi+\tau\Psi|^{p+1}.$$

Differentiating twice at $\tau=0$, and taking into account (in the differentiation of the last term) that Φ is nonnegative, one then has:

$$\langle S''(\Phi)\Psi, \Psi\rangle = \int |\nabla \Psi|^2 + b^2/4\int \rho^2 \Psi^2 + \omega \int \Psi^2 + \alpha p \int \Phi^{p-1}\Psi^2.$$

Last, from the definition of Ψ:

$$\begin{aligned}\langle S''(\Phi)\Psi, \Psi\rangle &= 9/4\left(\int |\nabla \Phi|^2 + b^2/4\int \rho^2 \Phi^2 + \omega\int \Phi^2 + \alpha p \int \Phi^{p+1}\right)\\ &+3\left(\int \nabla\Phi.\nabla(x.\nabla\Phi) + b^2/4\int \rho^2\Phi x.\nabla\Phi + \omega\int \Phi x.\nabla\Phi + \alpha p\int \Phi^p x.\nabla\Phi\right)\\ &+\left(\int |\nabla(x.\nabla\Phi)|^2 + b^2/4\int \rho^2 (x.\nabla\Phi)^2\right.\\ &\left.+\omega\int (x.\nabla\Phi)^2 + \alpha p\int \Phi^{p-1}(x.\nabla\Phi)^2\right). \quad \bullet \quad (4.10)\end{aligned}$$

We now turn to simplification of the linear and quadratic terms in $x.\nabla\Phi$.

4.6. *Proof of Theorem* 4.1 *3rd step.* – Simplification of the linear terms.

Taking into account that Φ is a critical point of the action, one has:

$$\begin{aligned}&\int \nabla\Phi.\nabla(x.\nabla\Phi) + b^2/4\int \rho^2\Phi x.\nabla\Phi\\ &\qquad+\omega\int \Phi x.\nabla\Phi + \alpha p\int \Phi^p x.\nabla\Phi\\ &= \alpha(p-1)\int \Phi^p x.\nabla\Phi = -3\alpha(p-1)/(p+1)\int \Phi^{p+1}. \quad \bullet \quad (4.11)\end{aligned}$$

4.7. *Proof of Theorem* 4.1. *4th step.* – Simplification of the quadratic terms.

Take $\psi_1 \in \mathscr{D}(\mathbf{R}^3, \mathbf{R})$, ψ_1 invariant by rotations around the z-axis. One has:

$$\begin{aligned}&\int \nabla(x.\nabla\Phi).\nabla\psi_1 + b^2/4\int \rho^2 x.\nabla\Phi\psi_1\\ &\qquad+\omega\int x.\nabla\Phi\psi_1 + \alpha p\int \Phi^{p-1}x.\nabla\Phi\psi_1\\ &= \int \nabla(x.\nabla\Phi).\nabla\psi_1 - 5b^2/4\int \rho^2\Phi\psi_1 - b^2/4\int \rho^2\Phi x.\nabla\psi_1\end{aligned}$$

$$-3\omega\int\Phi\psi_1-\omega\int\Phi x.\nabla\psi_1-3\alpha\int\Phi^p\psi_1-\alpha\int\Phi^p x.\nabla\psi_1$$
$$=\int\nabla(x.\nabla\Phi).\nabla\psi_1+\int\nabla\Phi.\nabla(x.\nabla\psi_1)$$
$$-5b^2/4\int\rho^2\Phi\psi_1-3\omega\int\Phi\psi_1-3\alpha\int\Phi^p\psi_1, \quad (4.12)$$

since Φ is a critical point of the action.

Let $\psi_2\in\mathscr{D}(\mathbf{R}^3, \mathbf{R})$. Integration by parts gives:

$$\int\nabla(x.\nabla\psi_2).\nabla\psi_1+\int\nabla\psi_2.\nabla(x.\nabla\psi_1)=-\int\nabla\psi_2.\nabla\psi_1. \quad (4.13)$$

In (4.13), let $\psi_2\to\Phi$ in $H^1_{A,0}$; apply the resulting expression to (4.12) and, once more, take into account that Φ is a critical point of the action. This gives:

$$\int\nabla(x.\nabla\Phi).\nabla\psi_1+b^2/4\int\rho^2 x.\nabla\Phi\psi_1$$
$$+\omega\int x.\nabla\Phi\psi_1+\alpha p\int\Phi^{p-1}x.\nabla\Phi\psi_1$$
$$=-b^2\int\rho^2\Phi\psi_1-2\omega\int\Phi\psi_1-2\alpha\int\Phi^p\psi_1. \quad (4.14)$$

In (4.14), let $\psi_1\to x.\nabla\Phi$ in $H^1_{A,0}$; in the RHS of the resulting expression, one solely has to deal with linear terms in $x.\nabla\Phi$; simplify these terms as in 4.6, and get

$$\int|\nabla(x.\nabla\Phi)|^2+b^2/4\int\rho^2(x.\nabla\Phi)^2$$
$$+\omega\int(x.\nabla\Phi)^2+\alpha p\int\Phi^{p-1}(x.\nabla\Phi)^2$$
$$=10b^2/4\int\rho^2\Phi^2+3\omega\int\Phi^2+6\alpha/(p+1)\int\Phi^{p+1}. \bullet \quad (4.15)$$

4.8. *Proof of Theorem* 4.1. *Conclusion.* – Collecting together (4.10), (4.11), (4.15) yields:

$$\langle S''(\Phi)\Psi, \Psi\rangle$$
$$=9/4\int|\nabla\Phi|^2+(10+9/4)b^2/4\int\rho^2\Phi^2+(3+9/4)\omega\int\Phi^2$$
$$+\alpha(9p/4-9(p-1)/(p+1)+6/(p+1))\int\Phi^{p+1}.$$

So, we have derived an expression for $\langle S''(\Phi)\Psi, \Psi\rangle$ in terms of the energies and the charge of Φ. We now use (4.2) to eliminate charge and then the Virial Theorem to eliminate the potential energy. The final result is:

$$\langle S''(\Phi)\Psi, \Psi\rangle = -4(1-a)/(1+2a) \times \left(\int |\nabla\Phi|^2 - (2+a)/(1-a)\; b^2/4 \int \rho^2\Phi^2\right), \quad (4.16)$$

where $a=(5-p)/(2p-2)$.

Last, we need a relation on the kinetic and magnetic energies of the type $K(\Phi)>C(a)M(\Phi)$. Turn back to the Virial Theorem and (4.2); elimination of the potential energy and $\omega>0$ give:

$$\int |\nabla\Phi|^2 > (1+3/a)\, b^2/4 \int \rho^2\Phi^2. \quad (4.17)$$

From (4.16), (4.17), one shall have $\langle S''(\Phi)\Psi, \Psi\rangle<0$ provided $1+3/a \geqq (2+a)/(1-a)$, *i.e.* $p \geqq p_{\text{uns}}$. ●●

4.9. The preceeding arguments will *possibly* generalize to $H^1_{A,k}$, $k\neq 0$. Extension to H^1_A seems problematic: the argument is based on the splitting of the kinetic-magnetic energy and, on H^1_A, this is a dubious point.

5. REGULARITY

5.1. The main result of section 5 is

THEOREM 5.1. – *Suppose* $p\in[2,5)$, $\omega\in(0,\infty)$ *and let* Φ *be a* nonnegative *critical point of the action in* $H^1_{A,0}$. *Then,* $\partial_z\Phi$, $\partial_z^2\Phi$, $x.\nabla\Phi$, $\partial_z(x.\nabla\Phi)\in H^1_{A,0}$. ●

To prove Theorem 5.1, we need a fact concerning a differential inequality:

PROPOSITION 5.2. – *For* $\beta\in(0,\infty)$, *let* $u\in C^2([0,\infty), \mathbf{R})$, *with* $u\geqq 0$, *satisfy* $u''(t)\geqq \beta u(t)$, $\forall t\in[0,\infty)$. *Then:*

– either $u(t)\leqq C e^{-\sqrt{\beta}t}$, $\forall t\in[0,\infty)$, *for some positive* C; (5.1)

– or $u(t)\geqq C e^{\sqrt{\beta}t}$, $\forall t\in[t_1,\infty)$, *for some positive* C, t_1. ● (5.2)

We first prove Proposition 5.2 and then, in several steps, Theorem 5.1.

5.2. *Proof of Proposition* 5.2. – Put $v(t)=e^{\sqrt{\beta}t}(u'(t)+\sqrt{\beta}u(t))$. v satifies $v'(t)\geqq 2\sqrt{\beta}v(t)$, $\forall t\in[0,\infty)$. Now, *either* $v(t)\leqq 0$, $\forall t\in[0,\infty)$; *or* $v(t_0)>0$, for some $t_0\in[0,\infty)$.

In the first case, $u'(t)+\sqrt{\beta}u(t)\leqq 0$, $\forall t\in[0,\infty)$, wich implies (5.1).

In the last case, $v(t)\geqq v(t_0)e^{2\sqrt{\beta}(t-t_0)}$, $\forall t\in[t_0,\infty)$. Therefore, $e^{\sqrt{\beta}t}u(t)+(v(t_0)e^{-2\sqrt{\beta}t_0}/(2\sqrt{\beta}))e^{2\sqrt{\beta}t}$ is nondecreasing on $[t_0,\infty)$, which implies (5.2) ●

5.3. *Proof of Theorem* 5.1. 1*st sept.* — local regularity: Φ is C.
Turn back to (4.4),

$$-\Delta(\xi\Phi)+\xi\Phi=(-\Delta\xi-b^2/4\,\rho^2\xi-(\omega-1)\xi)\Phi$$
$$-2\nabla\xi.\nabla\Phi-\alpha\xi\Phi^p,\ \text{in}\ \mathscr{D}'(\mathbf{R}^3,\mathbf{R}),\quad (5.3)$$

and remember that $\Phi\in H^2_{\text{loc}}$. Then, the RHS is in L^6. Apply (4.5): $\Phi\in W^{2,6}_{\text{loc}}$. Then, the RHS is in L^∞. Apply again (4.5): $\Phi\in W^{2,q}_{\text{loc}}$, q arbitrarily large; so, Φ is C^1. Now, the RHS is in $W^{1,q}$, q arbitrarily large. Once more (4.5) gives $\Phi\in W^{3,q}_{\text{loc}}$, q arbitrarily large; so, Φ is C^2.

Particularly:

$$-\Delta\Phi+b^2/4\,\rho^2\Phi+\omega\Phi+\alpha\Phi^p=0,\quad \textit{with the classical meaning.}\quad \bullet \quad (5.4)$$

5.4. *Proof of Theorem* 5.1. 2*nd step.* — Behaviour at infinity: $\Phi\in C_0(\mathbf{R}_3,\mathbf{R})$.

It is known (*see* [5]) that $H^2_A \subsetneq L^\infty(\mathbf{R}^3,\mathbf{C})$. Now, suppose that $\Phi\in H^2_A$, and take a sequence in $\mathscr{D}$ converging to Φ in H^2_A; the sequence also converges to Φ in L^∞; hence, Φ will be in the closure of $\mathscr{D}$ in L^∞-the space of continuous functions wich tend to zero at infinity, $C_0(\mathbf{R}^3,\mathbf{R})$. Thus, it is sufficient to prove $\Phi\in H^2_A$. From definition, this means $-\Delta\Phi+b^2/4\,\rho^2\Phi\in L^2$. Equivalently, $\Phi\in L^{2p}$. To prove this, we first consider the complement of a cylindric neighbourhood of the z-axis and take advantage of the symmetry of Φ.

Define a *planar* open set by $\Omega_1=\{(\rho,z):\rho>1,\ z\in\mathbf{R}\}$. One has:

- $\displaystyle\int_{\Omega_1}\Phi^2(\rho,z)\,d\rho\,dz\leqq(2\pi)^{-1}\int\Phi^2;$
- $\displaystyle\int_{\Omega_1}((\partial_\rho\Phi)^2+(\partial_z\Phi)^2)(\rho,z)\,d\rho\,dz\leqq(2\pi)^{-1}\int|\nabla\Phi|^2.$

So, taking Φ as a function of *two* variables, one has $\Phi\in H^1(\Omega_1,\mathbf{R})$. Consequence:

$$\int_{\Omega_1}\Phi^q(\rho,z)\,d\rho\,dz<\infty,\qquad \forall q\in[2,\infty).\qquad (5.5)$$

From Hölder's Inequality:

$$\int_{\Omega_1}\Phi^{2p}(\rho,z)\,\rho\,d\rho\,dz$$
$$\leqq\left(\int_{\Omega_1}\Phi^{3p-1}(\rho,z)\,d\rho\,dz\right)^{2/3}\times\left(\int_{\Omega_1}\Phi^2(\rho,z)\,\rho^3\,d\rho\,dz\right)^{1/3}.\qquad (5.6)$$

And, from (5.5), (5.6) and $\rho\Phi\in L^2$, we finally get

$$\int_{\Omega_1}\Phi^{2p}(\rho,z)\,\rho\,d\rho\,dz<\infty.$$

We now turn to the cylindric neighbourhood of the z-axis.

Take a $\xi \in C^\infty((0, \infty), \mathbf{R})$, $\xi \geqq 0$, such that $\xi(\rho)=1$ if $0<\rho<1$, and $\xi(\rho)=0$ if $\rho>2$. Things are reduced to prove that $\xi\Phi \in L^{2p}$. $\xi\Phi$ satisfies the equation

$$-\Delta(\xi\Phi)+\xi\Phi=(-\Delta\xi\Phi-2\nabla\xi.\nabla\Phi-b^2/4\rho^2\xi\Phi-(\omega-1)\xi\Phi \\ -\alpha(\xi-\xi^p)\Phi^p)-\alpha(\xi\Phi)^p, \text{ in } \mathscr{D}'(\mathbf{R}^3, \mathbf{R}). \quad (5.7)$$

Note that, since $\rho^2\xi$, $\xi-\xi^p$ are bounded and $\xi(\rho)-\xi^p(\rho)=0$, for $0<\rho<1$, the first term in the RHS is in L^2. Split (5.7) into a system of two equations, *i. e.* consider f_1, f_2 uniquely determined by

- $f_1, f_2 \in H^1(\mathbf{R}^3, \mathbf{R})$;
- $(-\Delta+1)f_1=-\Delta\xi\Phi-2\nabla\xi.\nabla\Phi-b^2/4\rho^2\xi\Phi$
  $-(\omega-1)\xi\Phi-\alpha(\xi-\xi^p)\Phi^p$, in $\mathscr{D}'(\mathbf{R}^3. \mathbf{R})$;
- $(-\Delta+1)f_2=-\alpha(\xi\Phi)^p$, in $\mathscr{D}'(\mathbf{R}^3, \mathbf{R})$.

Obviously, $\xi\Phi=f_1+f_2$ and $f_1 \in H^2$, hence $f_1 \in L^q$, $\forall q \in [2, \infty]$. Now, a bootstrap argument based on (4.5) and Sobolev embeddings gives that $f_2 \in L^q$, $\forall q \in [2, \infty]$. Particularly, $\xi\Phi \in L^{2p}$. ●

5.5. *Proof of Theorem* 5.1. 3*rd step*. – Behaviour at infinity: exponential decay of the spherical average.

For $r \in (0, \infty)$, $f \in C(\mathbf{R}^3-\{0\}, \mathbf{R})$, call $S(r)$ the sphere of radius r and call $\langle f \rangle_r$ the mean value of f on $S(r)$: $\langle f \rangle_r=(4\pi r^2)^{-1}\int_{S(r)} f$.

Integration of (5.4) on $S(r)$ yields a second order differential equation on $\langle \Phi \rangle_r$:

$$d_r^2\langle \Phi \rangle_r+2/r\, d_r\langle \Phi \rangle_r=b^2/4\, r^2\langle \sin^2\theta\, \Phi \rangle_r+\omega\langle \Phi \rangle_r+\alpha\langle \Phi^p \rangle_r, \\ \forall r \in (0, \infty),$$

where θ is colatitude. The RHS is $\geqq(\omega-|\alpha|\,\|\Phi\|_{L^\infty(S(r), \mathbf{R})}^{p-1})\langle \Phi \rangle_r$. Since $\Phi \in C_0(\mathbf{R}^3, \mathbf{R})$, given $\omega_1 \in (0, \omega)$, there exists an $r_1 \in (0, \infty)$ such that $\omega-|\alpha|\,\|\Phi\|_{L^\infty(S(r), \mathbf{R})}^{p-1} \geqq \omega_1$, for $r \geqq r_1$. Thus:

$$d_r^2\langle \Phi \rangle_r+2/r\, d_r\langle \Phi \rangle_r \geqq \omega_1\langle \Phi \rangle_r, \qquad \forall r \in [r_1, \infty). \qquad (5.8)$$

Put $u(r)=(r\langle \Phi \rangle_r)^2$, for $r>0$. Then:

$$u''(r)=2(d_r(r\langle \Phi \rangle_r))^2+2r\langle \Phi \rangle_r d_r^2(r\langle \Phi \rangle_r), \\ u''(r) \geqq 2r\langle \Phi \rangle_r r(d_r^2\langle \Phi \rangle_r+2/r\, d_r\langle \Phi \rangle_r),$$

and taking (5.8) into account:

$$u''(r) \geqq 2\omega_1 u(r), \qquad \forall r \in [r_1, \infty).$$

We now apply Proposition 5.2: *either* $u(.+r_1)$ decays exponentially to zero, *or* it grows exponentially to infinity. The second term in the alternative cannot occur, since $\int_0^\infty u(r)\,dr \leqq (4\pi)^{-1}\int \Phi^2 < \infty$. Thus

$$u(r) \leqq C e^{-\sqrt{2\omega_1}}(r-r_1), \qquad \forall r \in [r_1, \infty), \text{ for some positive C;}$$
$$\langle \Phi \rangle_r \leqq C e^{\sqrt{-2\omega_1}/2r}, \qquad \forall r \in [0, \infty), \text{ for some positive C.} \quad \bullet \quad (5.9)$$

5.6. *Proof of Theorem* 5.1. 4*th step*. – For $\gamma \in [0, \infty)$, $r^\gamma x.\nabla\Phi \in L^2$. The departure point is equation

$$-\Delta\Phi^2 = 2\Phi(-\Delta\Phi) - 2|\nabla\Phi|^2$$
$$= -b^2/2\,\rho^2\Phi^2 - 2\omega\Phi^2 - 2\alpha\Phi^{p+1} - 2|\nabla\Phi|^2,$$
$$|\nabla\Phi|^2 = -b^2/4\,\rho^2\Phi^2 - \omega\,\Phi^2 - \alpha\Phi^{p+1} + 1/2\,\Delta\Phi^2.$$

Thus:

$$|\nabla\Phi|^2 \leqq |\alpha|.\|\Phi\|_{L^\infty}^p\Phi + 1/2\,\Delta\Phi^2,$$

and integration on $S(r)$ yields:

$$\langle |\nabla\Phi|^2 \rangle_r \leqq |\alpha|.\|\Phi\|_{L^\infty}^p \langle \Phi \rangle_r + 1/2\, d_r^2 \langle \Phi^2 \rangle_r + 1/r\, d_r \langle \Phi^2 \rangle_r.$$

Multiplicate by $4\pi r^{4+2\gamma}$ and integrate on r from r_1 to r_2 $(0<r_1<r_2<\infty)$; integrating by parts the two last terms gives:

$$\begin{aligned}\int_{r_1<r<r_2} r^{2+2\gamma}|\nabla\Phi|^2 \leqq{}& 4\pi|\alpha|.\|\Phi\|_{L^\infty}^p \int_{r_1}^{r_2} r^{4+2\gamma}\langle\Phi\rangle_r\,dr \\ &+ 2\pi(2+2\gamma)(3+2\gamma)\int_{r_1}^{r_2} r^{2+2\gamma}\langle\Phi^2\rangle_r\,dr \\ &+ 2\pi(4+2\gamma)r_1^{3+2\gamma}\langle\Phi^2\rangle_{r_1} - r_1^{2+2\gamma}\int_{S(r_1)}\Phi\,\partial_r\Phi \\ &- 2\pi(4+2\gamma)r_2^{3+2\gamma}\langle\Phi^2\rangle_{r_2} + r_2^{2+2\gamma}\int_{S(r_2)}\Phi\,\partial_r\Phi.\end{aligned}$$

Thus:

$$\begin{aligned}\int_{r_1<r<r_2}(r^\gamma|x.\nabla\Phi|)^2 \leqq{}& C\int_{r_1}^{r_2}(r^{4+2\gamma}+r^{2+2\gamma})\langle\Phi\rangle_r\,dr \\ &+ 2\pi(4+2\gamma)r_1^{3+2\gamma}\langle\Phi^2\rangle_{r_1} - r_1^{2+2\gamma}\int_{S(r_1)}\Phi\,\partial_r\Phi \\ &+ 2\pi^{1/2}\|\Phi\|_{L^\infty}^{1/2} r_2^{3+2\gamma}(\langle\Phi\rangle_{r_2})^{1/2}\left(\int_{S(r_2)}|\nabla\Phi|^2\right)^{1/2},\end{aligned}$$

where $C = C(\alpha, p, \|\Phi\|_{L^\infty}, \gamma)$. We let $r_1 \to 0$ and then apply (5.9). Conclusion: there are $C_1, C_2 \in (0, \infty)$ such that, for any $r_2 \in (0, \infty)$:

$$\int_{r<r_2}(r^\gamma|x.\nabla\Phi|)^2 \leqq C_1 + C_2 r_2^{3+2\gamma} e^{-\sqrt{2\omega_1}/4\,r_2}\left(\int_{S(r_2)}|\nabla\Phi|^2\right)^{1/2}.$$

Thus, it is sufficient to prove the existence of an increasing unbounded sequence $(r_{2,j})_j$ such that:

$$r_{2,j}^{3+2\gamma} e^{-\sqrt{2\omega_1}/4\, r_{2,j}} \left(\int_{S(r_{2,j})} |\nabla \Phi|^2 \right)^{1/2} \leqq 1, \qquad \forall j.$$

If such a sequence does not exist, we shall have, for r_2 in some neighbourhood of ∞:

$$\int_{S(r_2)} |\nabla \Phi|^2 > r_2^{-6-4\gamma} e^{-\sqrt{2\omega_1}/2\, r_2}.$$

This is contradictory with

$$\int_0^\infty dr_2 \int_{S(r_2)} |\nabla \Phi|^2 = \int |\nabla \Phi|^2 < \infty.$$

Consequence: such a sequence does exist and $r^\gamma x.\nabla \Phi \in L^2$. •

5.7. *Proof of Theorem* 5.1. *Conclusion.* – It is known (*see* [5]) that $v \in H_A^2$ iff:

$v \in L^2$;
$(\partial_j + i A_j) v \in L^2, \qquad \forall j \in \{1, 2, 3\}$;
$(\partial_k + i A_k)(\partial_j + i A_j) v \in L^2, \qquad \forall j, k \in \{1, 2, 3\}$.

Therefore, $\partial_z v \in H_A^1$ if $v \in H_A^2$. Thus, it is sufficient to prove that $\Phi, \partial_z \Phi, x.\nabla \Phi \in H_A^2$. We have already proved, 2nd step, that $\Phi \in H_A^2$, hence $\partial_z \Phi \in H_A^1$. So, it is sufficient to prove that:

– $L_A \partial_z \Phi \in L^2$; (5.10)
– $x.\nabla \Phi \in H_A^1$; (5.11)
– $L_A (x.\nabla \Phi) \in L^2$. (5.12)

Let us prove that (5.11) is a consequence of (5.12). From the previous step, one has $x.\nabla \Phi$, $i A (x.\nabla \Phi) \in L^2$. Thus, to have (5.11), it is sufficient that $\nabla (x.\nabla \Phi) \in L^2$. On the other hand, from the previous step and (5.12), $\Delta (x.\nabla \Phi) \in L^2$. Then, $x.\nabla \Phi \in H^2$. Particularly, $\nabla (x.\nabla \Phi) \in L^2$ and (5.11) holds.

To sum up: regularity problems are reduced to proving (5.10), (5.12).

To compute $L_A \partial_z \Phi$, $L_A (x.\nabla \Phi)$ directly, by classical derivation, one needs Φ in C^3. Come back to (5.3) and remember that $\Phi \in W_{loc}^{3,q}$, q arbitrarily large; then, the RHS is in $W^{2,q}$, q arbitrarily large (we make use of $p \geqq 2$); $\Phi \in W_{loc}^{4,q}$, q arbitrarily large; Φ is C^3. Now:

$$(-\Delta + b^2/4\rho^2) \partial_z \Phi = -\omega \partial_z \Phi - \alpha p \Phi^{p-1} \partial_z \Phi,$$

$$(-\Delta + b^2/4\rho^2)(x.\nabla \Phi) = -2\omega \Phi - b^2 \rho^2 \Phi - 2\alpha \Phi^p - \omega x.\nabla \Phi - \alpha p \Phi^{p-1} x.\nabla \Phi,$$

and all the functions in the RHS's are in L^2. ••

5.8. The preceeding arguments will *possibly* generalize to $H^1_{A,k}$, $k \neq 0$.

6. PROOF OF THE MAIN THEOREM. CONCLUSION

Proof of Theorem 1.6. – Apply Theorem 5.1:

$$\Phi \in D((V'(0))^2),\ x.\nabla\Phi \in D(V'(0)).$$

From Theorem 4.1, Φ and $\Psi = 3/2\,\Phi + x.\nabla\Phi$ verify GCI, GCADS. From Theorem 3.1, there exists an ADS. Last, apply Theorem 2.1. ●

7. FINAL REMARKS

7.1. If there is no magnetic field, our variant of the GSS formalism applies as well. We work, *a priori*, in $H^1(\mathbf{R}^3, \mathbf{C})$ and, by means of spherical symmetrization, Φ is chosen, *a posteriori*, as a radial (nonnegative, nonincreasing) function. We come to the conclusion that

$$\Xi = \{ e^{-i\theta}\Phi(\widehat{x-y}) : \theta \in \mathbf{R},\ y \in \mathbf{R}^3 \}$$

is unstable by the flow, provided $p \in (1+4/3, 5)$. In this case, one has to deal with a four paremeters group, but, owing to radiality of Φ and absence of gauge transformations, construction of ADS is easy.

7.2. If there is no magnetic field, $p_0 = 1+4/3$ is *optimal:* for $p < p_0$, critical points given by the homologous of Theorem 1.3 are Ξ-stable (*see* [6]). In the case of a magnetic field, we have *different limit values for* p : $p < 1+4/3$ for $\tilde{\Sigma}$-stability (*see* 1.9); $p \geqq p_{\text{uns}}$ for Σ-instability. Let's explain why:

If there is no magnetic field, dilation and scaling yields the ω-variable trajectory, the convexity/concavity of wich separates stability from instability. In the case of a magnetic field, the ω-variable trajectory will certainly still separate stability from instability. But such trajectory is *not* obtained by scaling and dilation.

How can one get a trajectory Φ_ω with variable ω and *fixed b?*

Will such trajectory yield an optimal p somewhere in $(1+4/3, p_{\text{uns}})$, *and independent from b?*

Will the optimal *p depend on b*, rather?

Will it exist (for fixed *b*) *an optimal p at all?*

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