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Finding eigenvalues of the period-doubling operator from the characteristic equation

by

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ABSTRACT. — The infinitely period-doubled map at the transition to chaos in one-humped maps, is attracted under renormalization to the Feigenbaum fix-point, which has representation as a simple hyperbolic repelling map with two branches.

The equation for the eigenvalues of the linearized renormalization operator around that fixpoint has a particularity elegant formulation in terms of the periodic orbits of the repeller. In this note we extend earlier results [11] to show that not only the leading eigenvalue, but also the subdominant ones can be extracted in this way.

In essence the method finds an eigenvalue by solving a characteristic equation for a transfer matrix, which is not the recomended numerical scheme [10]. In addition, the subleading eigenvalues are all found by (numerical) analytical continuation of a characteristic equation beyond its original region of convergence. Except for the leading eigenvalue, the accuracy does not indeed compare favorably with other computations [14], but provides interesting information about the analytical structure of the characteristic equations.

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1. INTRODUCTION

Family of one-humped functions on the unit interval, such as,

$$F_\mu(x) = 4\mu x(1-x) \tag{1}$$

have been extensively studied for their own sake, and as models of cascades of period-doublings in real systems [1]. As μ increases, F_μ undergoes a series of bifurcations forming periodic orbits of period 1, 2, 4, etc.; each new orbit is formed by period-doubling the one before, it, which then becomes unstable. The period-doublings accumulate at a finite value of μ_∞, and the resulting aperiodic orbit has a self-similar structure encoded by the renormalization operator

$$R:\quad F_{\mu_\infty}(x) \to \alpha F_{\mu_\infty}(F_{\mu_\infty}(x/\alpha)) \tag{2}$$

where $\alpha = -2.502{,}97\ldots$ for maps with a quadratic maximum.

All the universal properties (common to a large class of families of maps F) are determined by the Cvitanović-Feigenbaum [1] fixpoint of (2).

$$g(x) = \alpha g \,.\, g(x/\alpha) \tag{3}$$

The "universal" period-doubling attractor is then the closure of the forward orbit of zero of the map g. By writing out the number of iterations in base two and repeatedly using (3), one sees that this set is also a repeller of the period doubling representation ([6], [2], [11]) function, a related map defined by

$$\left.\begin{array}{ll} F_0(x) = \alpha g(x), & g(\alpha^{-1}) \leqq x \leqq 1 \\ F_1(x) = \alpha x, & \alpha^{-1} \leqq x \leqq \alpha^{-2}. \end{array}\right\} \tag{4}$$

The map F maps the interval like a "tent" map ($x \to 4\mu x(1-x)$; $\mu > 1$), linear in the branch with index 1 and slightly non-linear in the other [11]. The points remaining on the repeller are labelled by their passages on the left or right hand branches (which we call 1 and 0). This labelling is simply the complement ($0 \to 1$) and *vice versa* of the iteration written out as a binary number.

The linearised renormalization equation is obtained by substituting $g(x) \to g(x) + h_n(g(x))$ in (3):

$$h_{n-1}(g(x)) = \alpha g'(g(x/\alpha))\, h_n g(x/\alpha) + \alpha h_n(g(x)/\alpha) \tag{5}$$

The condition that the function $h(x)$ is an eigenfunction with eigenvalue δ gives:

$$h(g(x)) = \delta^{-1}\,.\,(\alpha g'(g(x/\alpha))\, hg(x/\alpha) + \alpha h(g(x)/\alpha)) \tag{6}$$

Many iterations of the linearized operator give more and more powers of δ; on the other hand, the value of h at one point is then a sum over initial values of h at a large number of points.

In the presentation function form, the linearized renormalization equations are [11]

$$h_{n-1}(x_m) = \frac{dF_0}{dx}(x_{0m})\, h_n(x_{0m}) + \frac{dF_1}{dx}(x_{1m})\, h_n(x_{1m}) \tag{7}$$

where we use

$$x_{\varepsilon m} = F_\varepsilon^{-1}(x_m)$$

Assuming now that the perturbation $h(x)$ is smooth, we replace it by $h_n(x_i)$, where x_i is the periodic point with symbol sequence $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n$. We can then define the iterated perturbation by

$$h_{n-1}(x_m) = F_0'(x_{0m})\, h_n(x_{0\,m}) + F_1'(x_{1m})\, h_n(x_{1m}) \tag{8}$$

After n iterations we expect that

$$\Gamma_n = \sum_{\varepsilon}^{(n)} F^{n'}(x_i)\, h_n(x_i) \propto \delta^n \tag{9}$$

It is now clear that we can solve for the eigenvalues of the linearized equation either from (6) or from (9), which has the form of a partition sum of a one-dimensional spin system, as other averages (fractal dimensions, lyapunov exponents, etc.) in hyperbolic dynamical systems ([3], [11], [5]).

The asymptotic in n growth of Γ_n can be monitored by the generating function

$$\Omega(z) = \sum z^n \Gamma_n \tag{10}$$

which diverges when $z = 1/\delta$. By combinatorial rearrangement ([4], [11]), ones sees that Ω is the logarithmic derivative of the zeta function [4] that has a nice development as an infinite product over non-repeating (primitive) orbits p of the representation function:

$$\zeta^{-1}(z) = \prod_p (1 - z^p D_p) \tag{11}$$

D_p is here a shorthand for $F^{p'}(x_p)$ the derivative of p iterations staring from any one of the points x_p on the cycle p. The smallest zero in z of (11) is hence $1/\delta$ [11], where δ is the leading eigenvalue, 4.669... of (6). This note is about the higher zeros of (11) and related functions, and their closeness to the higher eigenvalues of (6). It is not quite (11) that behaves as a characteristic equation of (6) (having its zeros at the location of the eigenvalues), but a product of such functions. Before we state the numerical results, we will therefore interprete (7) as an integral equation with a singular kernel, and write down the formal expression for its characteristic equation (the Fredholm determinant) [12].

2. FREDHOLM DETERMINANT

We may write (7) as an integral equation by simply taking the kernels to be appropriate delta functions:

$$\mathrm{H}(h(x))=\int \sum_{y=\mathrm{F}^{-1}(x)} \delta(\omega-y)\frac{d\mathrm{F}}{d\omega}\,.\,h(\omega)\,d\omega. \tag{12}$$

The kernel of this integral operator is not bounded, but suppose for a moment that it is, for instance by broadening the delta function to $\delta_\sigma(z-y)$. Then it would have a discrete spectrum, and one can form the resolvent function G, that satisfies a twinning relation with the kernel [12]:

$$\mathrm{G}(x,y,z)=\mathrm{H}(x,y)+z\int \mathrm{H}(x,s)\,\mathrm{G}(s,y,z)\,ds \tag{13}$$

and has the power series expansion

$$\mathrm{G}(x,y,z)=\sum_{n=0}^{\infty} z^n \int ds_1\,ds_2\ldots ds_n\,\mathrm{H}(x,s_1)\,\mathrm{H}(s_1,s_2)\ldots\mathrm{H}(s_n,y) \tag{14}$$

convergent for small enough z. The singularities of G are only poles, and they are simultaneously the zeros of the Fredholm determinant $\mathrm{D}(z)$, that is a characteristic function of the operator H. D is connected to the trace of G in a very similar way as the zeta function (11) is connected to the generating function (10):

$$-\frac{d}{dz}\mathrm{D}(z)=\int ds\,\mathrm{G}(s,s,z) \tag{15}$$

If one now makes delta dunction in the kernel sharp ([9], [11]), the traces of powers of H are simply

$$\mathrm{tr}(\mathrm{H})^n=\sum_{x\in \mathrm{Fix}\,\mathrm{F}^n}\frac{\mathrm{F}'^n(x)}{\det(1-\mathrm{F}'^{-n}(x))} \tag{16}$$

where the denominator, which arises from the integration over the delta function, can be written out as an infinite sum:

$$\frac{1}{\det(1-\mathrm{F}'^{-n}(x))}=\sum_{l=0}^{\infty}(\mathrm{F}'^{-n}(x))^l \tag{17}$$

If one thus forms the generating function by summing over all orders, one obtains a double sum over prime orbits and l:

$$\begin{aligned}\sum_{n=1}^{\infty} z^n \operatorname{tr}(\mathrm{H})^n &= \sum_{l=0}^{\infty} \sum_{p} p\,(z^{|p|}\mathrm{D}_p^{1-l} + z^{2.|p|}\mathrm{D}_p^{2-2l} + \ldots) \\ &= -z\frac{d}{dz}\sum_{l=0}^{\infty} \log \prod_{p} (1 - z^p \mathrm{D}_p^{1-l}) \\ &= -z\frac{d}{dz}\log \prod_{l=0}^{\infty} \prod_{p} (1 - z^p \mathrm{D}_p^{1-l}) \\ &= -z\frac{d}{dz}\log \mathrm{Z}(z)\end{aligned} \tag{18}$$

where we by analogy with the Selberg zeta function in number theory [7], call the infinite product of functions like (11) $\mathrm{Z}(z)$. It is this product which behaves as a characteristic equation of the operator (5), and one sees that by numerically finding the poles and zeros of each of the factors.

The poles of one zeta function cancel with zeros of other ones, and the remaining zeros match the eigenvalues of (6) ([14], [1]) as computed by other means.

In fact, one can not find very many poles and zeros: if all computations are carries out to extended precision on a SUN 3/50 workstation (18 decimal digits), one is limited to three poles for (11). It is fortunate that all the cancellations we describe occur among the low-lying poles and zeros: otherwise they would not be seen.

3. NUMERICAL RESULTS

The numerical results are rather straight-forward to present. I have used an approximation of g by Lanford [13], and found the periodic orbits and their stabilities as in [11]. Stabilities of periodic orbits up to length 4, with their symbolic labels are given in Table I. Similar results up to length 6 have been given in [11]; here I have sometimes used cycles up to length 12, which are too many in number (and not very interesting in themselves) to be quoted here.

One then multiplies out the product defining each zeta function, where the stabilities are taken to power n, writing it as a taylor series in the complex variable z ([4], [11]). The resulting power series for $n=1$ and $n=-1$ are written out in table 2.

From the numerics, these power series grow or decay geometrically, *i.e.* have a finite radius of convergence limited by a pole. Inside this pole they have zero, which can be found as an ordinary zero of the truncated

TABLE I. — *The first periodic orbits with their cycle derivatives.*

p	D_p
0.	62645478312170368
1.	−2.5029078750958931
01.	−13.369876879005480
001.	−81.008962761421344
011.	35.886950967061139
0001.	−504.63646849768725
0011.	222.89901268839910
0111.	−87.409121049762746

polynomial. Thereafter the zeros and poles roughly alternate, so that one has divide out the first singularity to find the second zero, etc. Estimates of converged poles and zeros are found in table III and IV (actually written are the inverses of the poles and zeros to facilitate the comparison with the eigenvalues).

Before I discuss by what methods these numbers have been obtained, it is useful to consider how many one could hope to get with finite accuracy of the cycle eigenvalues. Let us take the case $n=1$ as an example and suppose that the cycle eigenvalues have been computed to 18 decimal digits. Formally one can compute as many orders as one wants of the zeta function, but as the number of terms multiplying z^m grow as 2^m, and each of them is large between α^n and α^{2m}, there will be cancellations of $(\alpha^2)^m$. One thus arrives at a taylor coefficient that grows as 2^m. When one subtracts off this leading divergence, the resulting power series decays as $(-0.214,4)^m$, which is an order of magnitude less. If one only wants two poles, one can not use more than 10 orders before all significance is lost: the values in table II show that in fact already at order 8 errors begin to creep in. In eight taylor coefficients there is not more information than to indicate the location of at most eight poles *and* zeros, and since the zeros are generally closer, that shows that it is not reasonable to try more than three poles. Actually I don't get reliable estimates of the third pole: presumably it lies another order of magnitude away.

We can write

$$\zeta_n(z)=\prod_p (1-z^p D_p^n)=\frac{Z_n(z)}{Z_{n-1}(z)} \tag{19}$$

where

$$Z_n(z)=\prod_{l=0}^{\infty}\prod_p (1-z^p D_p^{n-l}) \tag{20}$$

Each of the functions Z_n is formally a Fredholm determinant just like Z, and I assume that it is entire analytic in z, which makes the zeta functions

TABLE II.1. – *The taylor series* $c1$ $[i]$ for $n=1$. The subtracted series are $c2[i]=c1[i]-c1[i-1]/0.500000$, and $c3[i]=c3[i]+c2[i-1]/4.592$. The asymptotic ratios are estimated by generalized Shanks transformations from the most convergent diagonal entries in the Padé table [15].

i	$c1$ $[i]$	$c1$ $[i-1]/c1$ $[i]$	$c2$ $[i]$	$c2$ $[i-1]/c2$ $[i]$	$c3$ $[i]$
0.. .	1.0000000000	0.0000000	1.000 e+00	0.0000000	1.000 e+00
1.. .	−3.7616399561	−0.2658415	−5.762 e+00	−0.1735617	−5.544 e+00
2.. .	−2.3097092217	1.6286206	5.214 e+00	−1.1051236	3.959 e+00
3.. .	−5.1706512821	0.4466960	−5.512 e−01	−9.4580176	5.840 e−01
4.. .	−10.2203212869	0.5059187	1.210 e−01	−4.5563552	9.547 e−04
5.. .	−20.4670501559	0.4993549	−2.640 e−02	−4.5812452	−6.528 e−05
6.. .	−40.9283406461	0.5000704	5.759 e−03	−4.55854951	8.906 e−06
7.. .	−81.8579373098	0.4999923	−1.257 e−03	−4.5803485	−3.353 e−06
8.. .	−163.7155964382	0.5000008	2.756 e−04	−4.5628720	1.784 e−06
9.. .	−327.4314671376	0.4999996	− − −	− − −	− − −
10.. .	−654.8658550014	0.4999978	− − −	− − −	− − −
11.. .	−1313.3739899965	0.4986134	− − −	− − −	− − −
∞.. .	−	0.50000000 (1)	−	−4.592 (5)	−

TABLE II.2. – The taylor series $c1[i]$ for $n=-1$. The subtracted series are $c2[i]=c1[i]-c1[i-1]/5.240863$, and $c3[i]=c2[i]+c2[i-1]/15.386718$. The asymptotic ratios are estimated by generalized Shanks transformations from the most convergent diagonal entries in the Padé table [15].

i	$c1$ $[i]$	$c1$ $[i-1]/c1$ $[i]$	$c2$ $[i]$	$c2$ $[i-1]/c2$ $[i]$	$c3$ $[i]$
0.. .	1.0000000000	0.0000000	1.000 e+00	0.0000000	1.000 e+00
1.. .	0.2399068401	4.1682847	4.910 e−02	20.3671846	1.141 e−01
2.. .	0.0110178107	21.7744565	−3.476 e−02	−1.4125678	−3.157 e−02
3.. .	0.0024228645	4.5474316	3.206 e−04	−108.4250532	−1.938 e−03
4.. .	0.0004419396	5.4823436	−2.036 e−05	−15.7430410	4.716 e−07
5.. .	0.0000856447	5.1601479	1.319 e−06	−15.4378307	−4.382 e−09
6.. .	0.0000162560	5.2684872	−8.568 e−08	−15.3943202	4.234 e−11
7.. .	0.0000031074	5.2314722	5.568 e−09	−15.3880323	−4.757 e−13
8.. .	0.0000005925	5.2440642	−3.619 e−10	−15.3863682	− − −
9.. .	0.0000001131	5.2397736	2.352 e−11	−15.3883711	− − −
10.. .	0.0000000216	5.2412348	−1.529 e−12	−15.3822283	− − −
11.. .	0.0000000041	5.2407400	9.697 e−14	−15.7662669	− − −
∞.. .	−	5.2408634 (1)	−	−15.3867178 (1)	−

(19) the quotient of two analytic functions. One notes that ζ_0 equals $(1-2.z)$ [11] so its only zero is known exactly.

The standard way to express functions with both poles and zeros is by Padé approximants [15]. We have already seen (table II) that the first and second pole can be determined just be the geometric growth (or decay) of the coefficients in the power series. Estimates of the poles and zeros can

TABLE III.3. — *The (inverse) poles and zeros extracted from the Padé table* [16].

power	1 st pole	2 nd pole	1 st zero	2 nd zero	3 rd zero
1. . . .	1.9999966	−0.2184067	4.6691715	0.9999982	−0.1259680
0. . . .	–	–	2.0	–	–
−1. . . .	0.1906918	−0.0650270	−0.2144860	0.1621178	−0.0560080
−2. . . .	−0.0579539	0.0242862	0.1908119	−0.0649001	0.0254959
−3. . . .	0.0264230	−0.0104594	−0.0580815	0.0241904	−0.0097971
−4. . . .	−0.0099734	0.0039700	0.0264804	−0.0104543	0.0041238
−5. . . .	0.0040855	−0.0016332	−0.0100015	0.0039739	−0.0015878
−6. . . .	−0.0016132	0.0006404	0.0040993	−0.0016365	0.0006510
−7. . . .	0.0006474	−0.0002578	−0.0016195	0.0006426	−0.0002552

TABLE IV. — The (inverse) poles extracted from the Padé table [16], and the zeros computed by zero of truncated polynomial.

power	1 st pole	2 nd pole	1 st zero	2 nd zero	3 rd zero
1. . . .	1.9999966016	−0.2184066677	4.6692016090	1.0000000387	−0.1236122206
0. . . .	–	–	2.0	–	–
−1. . . .	0.1906917903	−0.0650270401	−0.2180455971	0.1596284403	−0.0559597220
−2. . . .	−0.579538696	0.0242862497	0.1908082496	−0.0649912083	0.0254812390
−3. . . .	0.0264229519	−0.0104593575	−0.580863978	0.0242558993	0.0098238421
−4. . . .	−0.0099733600	0.0039700422	0.0264796967	−0.0010480692	0.0041397512
−5. . . .	0.0040854814	−0.0016331636	−0.01000017102	0.0039878987	−0.0015871135
−6. . . .	−0.0016132480	0.0006403580	0.0040992792	−0.0016370406	0.0006512105
−7. . . .	0.0006474262	−0.0002578325	−0.0016194642	0.0006424084	−0.0002552061

TABLE V.1. — Leading poles are compared with leading zeros.

n	1 st pole	n	1 st zero (table III)	1 st zero (table IV)
1.	1.9999966	0	2.0	2.0
−1.	0.1906918	−2	0.1908119	0.1908082496
−2.	−0.0579539	−3	−0.0580815	−0.0580863978
−3.	0.0264230	−4	0.0264804	0.0264796967
−4.	−0.0099734	−5	−0.0100015	−0.0100017102
−5.	0.0040855	−6	0.0040993	0.0040992792
−6.	−0.0016132	−7	−0.0016195	−0.0016194642

TABLE V.2. — Second poles are compared with zeros.

n	2 nd pole (table III)	n	Zero (table III)	Zero (table IV)
1.	−0.2184067	−1	−0.2144860	−0.2180455971
−1.	−0.0650270	−2	−0.0649001	−0.0649912083
−2.	0.0242862	−3	0.0241904	0.0242558993
−3.	−0.0104594	−4	−0.0104543	−0.0010480692
−4.	0.0039700	−5	0.0039739	0.0039878987
−5.	−0.0016332	−6	−0.0016365	−0.0016370406
−6.	0.0006404	−7	0.0006426	0.0006424084

also be read off directly from the Padé table by an algorithm due to Rutishauser ([16], [15]). In table III I present the values of the poles and zeros obtained in this way. It is sometimes better to extract the zeros by solving the truncated polynomial, not direclty but after one or two geometric transients have been divided out. Especially this is so if one takes seriously the connection between the poles and zeros and uses the leading zeros of one another zeta function, which is known with higher accuracy, as a best guess for the location of the poles [11]. In both cases I have applied convergence acceleration by generalized Shanks transformations on the partial sums [15]. The zeros obtained this way are listed in table IV. The number of decimal places quoted have somewhat arbitrarily been taken to 7 in table III and 10 in table IV, which is always more than the convergence without acceleration, but often less than what seems the most convergent accelerated estimates. These accelerated estimates are in a sense treated twice, and it is therefore a delicate matter to separate errors from spurious convergence, which I have not attempted to do.

TABLE VI. – Comparing (inverse) zeros that do not match any (inverse) poles with eigenvalues of (7) taken from ([14]). The zeros are identified by the power, to which the cycle derivatives are raised in the zeta function, and their order; the eigenvalues are identified by their order in decreasing size.

Power	Order	Table III	Table IV	Eigenvalue	Order
1. . . .	1	4.6691715	4.6692016090	4.66920160910299	0
1. . . .	2	0.9999982	1.0000000387	1.00000000000000	1
1. . . .	3	−0.1259680	−0.1236122206	−0.12365271255269	3
−1. . . .	2	0.1621178	0.1596284403	0.15962844038270	2
−1. . . .	3	−0.0560080	−0.0559597220	−0.05730702106705	4
−2. . . .	3	0.0254959	0.0254812390	0.02548123897923	5

In table V the poles are paired to zeros, and in table VI the remaining zeros are compared with the best known eigenvalues of (7) [14].

Even if the quality of the fit is not perfect, it is clear that the pairing is natural. The leading pole is in all cases matched by the leading zero of the zeta function with one index less, while the second pole is in one case matched by a zero of a zeta functions with two indices less. It seems that the "unmatched" zeros all lie in the upper right corner of tables III and IV.

4. CONCLUSION

I have presented numerical results that one the one hand make probable that the formal Fredholm determinant of a singular operator is an entire analytic function, as are usual Fredholm determinants. This extends earlier considerations ([11], [9]) and accords with recent rigorous results by Ruelle [8].

On the other hand one sees that this formal Fredholm determinant has the zeros, as far as we can determine these, at the eigenvalues of the operator, whose characteristic function it is.

Except for the leading one, the accuracy of the eigenvalues so determined is no match for other methods, and if the unmatched zeros lie in the upper right corner of tables III and IV, it would be increasingly difficult to extract more of them. The last sentence should be taken as a correction to the overly optimistic conclusion in [9].

It is to be noted that the major numerical difficulty in extracting higher eigenvalues by this method, in contrast to functional iterational methods (7) [3] or (6) [1], is simply cancellation errors that I do not know how to cure.

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REFERENCES

[1] M. J. FEIGENBAUM, *J. Stat. Phys.*, Vol. **21**, 1979, p. 669.
[2] M. J. FEIGENBAUM, *J. Stat. Phys.*, Vol. **52**, 1988, p. 527.
[3] M. J. FEIGENBAUM, I. PROCACCIA and T. TÉL, *Phys. Rev. A*, Vol. **39**, 1989, p. 5359.
[4] P. CVITANOVIĆ, *Phys. Rev. Lett.*, Vol. **61**, 1988, p. 2729.
[5] P. GRASSBERGER and I. PROCACCIA, *Phys. Rev. A*, Vol. **31**, 1985, p. 1872.
[6] D. SULLIVAN, *in* P. ZWEIFEL, G. GALLAVOTTI and M. ANILE Eds., *Non-linear Evolution and Chaotic Phenomena*, Plenum, New York 1987.
[7] A. SELBERG, *J. Ind. Math. Soc.*, Vol. **20**, 1956, p. 47.
[8] D. RUELLE, *Inventiones math.*, Vol. **34**, 1976, p. 231, and D. RUELLE, *The Thermodynamic Formalism for Expanding Maps*, I.H.E.S. P/89/08, *An Extension of the Theory of Fredholm Determinants*, I.H.E.S. P/89/38.
[9] E. AURELL, Convergence of Dynamical Zeta Functions, *J. Stat. Phys.*, Vol. **58**, 1990, pp. 967-995.
[10] W. PRESS, B. FLANNERY, S. TEUKOLSKY and W. VETTERLING, *Numerical Recipies*, Cambridge University Press, 1988.

[11] R. ARTUSO, E. AURELL and P. CVITANOVIĆ, *Recycling Strange Sets I, II*, Non-linearity (in press).
[12] D. HILBERT and R. COURANT, *Methods of Mathematical Physics*.
[13] O. LANFORD, in *Universality in Chaos*, P. CVITANOVIĆ Ed., Adam Hilger, Bristol (private communication).
[14] J.-P. ECKMANN and H. EPSTEIN (private communication).
[15] BAKER, *Essentials of Padé Approximants*, Academic Press, 1975.
[16] H. RUTISHAUSER, *Z. Angew. Math. Phys.*, Vol. **5**, 1954, p. 233.

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