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Non-hyperbolicity and invariant measures for unimodal maps

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There are several results dealing with the existence of an invariant probability measure, which is absolutely continuous with respect to the Lebesgue measure (acim in short), *see* for example [C-E], [K], [N-S]. In this note we want to describe a weak condition which guarantees the existence of such a measure. We believe this condition is even equivalent to the existence of acim's. We say that the critical point c of a smooth map f has order l if there are constants O_1, O_2 so that

$$O_1 \,|x-c|^{l-1} \leqq |Df(x)| \leqq O_2 \,|x-c|^{l-1} \qquad \text{(NF)}.$$

As usual let f^n be the n-th iterate of f and let $c_1 = f(c)$. Furthermore denote the Lebesgue measure of a measurable set I by $|I|$.

MAIN THEOREM. — *Suppose that f is unimodal,* C^3, *has negative Schwarzian derivative and that the critical point of f is of order $l \geqq 1$. Moreover assume that the growth-rate of $|Df^n(c_1)|$ is so fast that*

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds, Then f has a unique absolutely continuous invariant probability measure μ which is ergodic and of positive entropy. Furthermore there exists a positive constant K *such that*

$$\mu(A) \leqq K\,|A|^{1/l},$$

for any measurable set $A \subset (0, 1)$.

M. Benedicks and L. S. Young announced the existence of acim's for maps for which $|Df^n(c_1)|$ grows at least polynomially.

Of course the estimate $\mu(A) \leqq K|A|^{1/l}$ shows that the poles of the invariant measure μ are at most of the form $|x-x_0|^{1/l-1}$. It is not hard to show that any absolutely continuous invariant probability measure has a pole of this order at the critical values $f^n(c)$, $n \geqq 1$, and therefore this estimate is optimal. Even for maps for which $|Df^n(c_1)|$ grows exponentially this result is new (the results in [C-E] and [N-S] only give some bounds for the order of the poles).

A REFORMULATION OF THE MAIN THEOREM AND AN OUTLINE OF ITS PROOF

In [BL] it is shown that any unimodal map with negative Schwarzian derivative is ergodic (w.r.t. to the Lebesgue measure) and that any absolutely continuous invariant probability measure μ has positive metric entropy. Therefore, in order to prove the Main Theorem it is enough to establish the existence of an absolutely continuous invariant probability measure μ.

In order to prove the existence of this invariant measure we will use the strategy of [N-S]. Using general arguments one can show that f has an absolutely continuous invariant probability measure provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set A with $|A| < \delta$ one has that $|f^{-n}(A)| < \varepsilon$ for all $n > 0$. In fact in this paper we will prove the following more precise statement: there exists a constant K such that for every n and every measurable set A,

$$|f^{-n}(A)| < K|A|^{1/l}. \tag{1}$$

One of the main results in [N-S] was to show that (1) can be deduced from the following: there exists a constant K′ such that for any n and every $\varepsilon > 0$,

$$|f^{-n}(c_1 - \varepsilon, c_1)| < K' \varepsilon^{1/l} \tag{2}$$

where l is the order of the critical point of f. Because of the non-flatness condition at the critical point this is equivalent to: there exists a constant K″ such that for every $n > 0$ and every $\varepsilon > 0$

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K'' \varepsilon.$$

From all this it follows that the Main Theorem can be deduced from

THEOREM. — *Suppose that f is unimodal,* C^3, *has negative Schwarzian derivative and that the critical point of f is of order* $l \geqq 1$. *Moreover assume* *that*

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds. Then there exists a constant $K < \infty$ *such that for each* $\varepsilon > 0$,

$$|f^{-n}(c-\varepsilon, c+\varepsilon)| < K\varepsilon. \qquad (3)$$

Let us say a few words about the proof of inequality (3). The main idea is to show that each component of $f^{-n}(c-\varepsilon, c+\varepsilon)$ is either contained in or at least can be compared in size (this process we will call 'sliding') with a set of the form

$$f^{-(n-k)}\left(c - \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}, c + \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}\right).$$

Using this and the summability condition, inequaltiy (3) will then be proved by induction.

Details of the proof can be found in a paper which will appear in Inventiones Mathematicae.

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