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Some remarks on recent results about S-unimodal maps

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1. INTRODUCTION

One aim of this paper is to review and tie together some recent advances in the theory of unimodal maps with negative Schwarzian derivative. Our interest is focused on maps with sensitive dependence to initial conditions, so we do not mention recent work on maps with solenoidal attractors, although most of our references contain also contributions for those maps. The papers we shall concentrate on are [BL2], [Ma], [GJ], [K], and [HK], more exactly those parts of these papers which discuss the relations between attractors, transient and recurrent behaviour, and invariant densities. Additionally we show that the examples from [HK] have a nonintegrable invariant density. To the best of our konowledge these are the first examples of unimodal maps for which the existence of such a density is proved.

2. SURVEY OF RESULTS

Let $f: [0, 1] \to [0, 1]$ be a unimodal map with negative Schwarzian derivative and critical point c, e.g. $f(x) = ax(1-x)$, $(0 < a \leqq 4)$, where $c = \frac{1}{2}$. *We assume that f is sensitive to initial conditions, i.e.* that f has no stable periodic orbit and is also not infinitely renormalizable.

It was proved independently and with different methods in [BL2], [GJ], and [K] that for each such map there is a compact set A such that $\omega(x) = \mathrm{A}$ for m-a.e. x and that A is either a finite union of intervals which is cyclically permuted by f (interval attractor) or a Cantor-type set (1). Here $\omega(x)$ is the set of all accumulation points of the sequence $x, f(x), f^2(x), f^3(x), \ldots$, and m denotes Lebesgue measure on [0, 1]. If A is an interval attractor, it agrees with the topological attractor $\mathrm{A}_{\mathrm{top}}$ of f, and if f is topologically mixing, then $\mathrm{A}_{\mathrm{top}}$ is just one interval.

The following relations between A and $\omega(c)$ are known:

- $\omega(c) \subseteqq \mathrm{A}$.
- If A is of Cantor-type, then $\mathrm{A} = \omega(c)$ (2).

Blokh and Lyubich [BL2] show:

- m is ergodic for f (3).
- The restriction $f_{|\mathrm{A}}$ of f to A is conservative (4).
- If A is of Cantor-type, then $f_{|\mathrm{A}}$ is minimal and the topological entropy of $f_{|\mathrm{A}}$ is zero.
- If f has an integrable invariant density, then it has positive metric entropy with respect to this density.

The results of Martens [Ma] allow a classification in terms of $\omega(c)$ (5).

He proves:

- If $f_{|\omega(c)}$ is minimal, then $\omega(c)$ is of Cantor-type and $m(\omega(c)) = 0$.
- If $f_{|\omega(c)}$ is not minimal, then A is an interval attractor.
- If A is of Cantor type, then $\mathrm{A} = \omega(c)$ and $f_{|\mathrm{A}}$ is minimal.

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(1) We would like to remark that Blokh and Lyubich announced this result already in 1987 [BL1] (Russian) and that it is also contained in a 1988 version of Guckenheimer's and Johnson's preprint [GJ]. Working on his forthcoming thesis M. Martens gave another proof of it.

(2) This is stated explicitly in [BL2] and [K], but less explicitly also contained in [GJ].

(3) Ergodic means that $f^{-1}(\mathrm{B}) = \mathrm{B}$ implies $m(\mathrm{B}) = 0$ or $= 1$ for each measurable $\mathrm{B} \subseteqq [0, 1]$.

(4) Conservative means that $f_{|\mathrm{A}}^{-1}(\mathrm{B}) \supseteqq \mathrm{B}$ implies $m(f_{|\mathrm{A}}^{-1}(\mathrm{B}) \setminus \mathrm{B}) = 0$ for each measurable $\mathrm{B} \subseteqq \mathrm{A}$.

(5) This is the aspect of his work fitting best into our discussion. His point of departure is the classification of maps into two classes according to the lengths and distortion properties of the branches of their iterates. In particular, his classification is not topological but of metrical nature.

• If $\omega(c)$ is of Cantor type but $f_{|\omega(c)}$ is not minimal, then f has an invariant density (integrable or not).

Keller [K] studies the Markov extension $\hat{f}: \hat{X} \to \hat{X}$ associated with f. This is a piecewise smooth dynamical system with a countable Markov partition which admits f as a factor. $\hat{f}$ is either dissipative, or it has a unique absorbing subset on which it is conservative (we say "essentially conservative"). In the latter case it admits a σ-finite invariant density (integrable or not). In detail:

• If $\hat{f}$ is essentially conservative, then A is an interval attractor and $f_{|A}$ is conservative and ergodic.

• If $\hat{f}$ is dissipative, then $A = \omega(c)$.

• If $\hat{f}$ has an integrable invariant density, then also f has an integrable invariant density.

• If f has an integrable invariant density of positive entropy, then also $\hat{f}$ has an integrable invariant density of positive entropy.

The work of Guckenheimer and Johnson [GJ] does not fit so well into our discussion, because in their approach periodic orbits play a fundamental role, whereas in the above cited references the critical orbit is more important. They obtain a dichotomy similar to the one of Keller. Furthermore they obtain a topological condition on the critical orbit (they call it "critical monotonicity") which implies that A is an interval attractor, that $f_{|A}$ is conservative, and that a certain transformation induced by f has an integrable invariant density. They also show that if f is not critically monotonic, then $\omega(c)$ is of Cantor-type. In particular:

• If $\omega(c)$ is not of Cantor type, then $f_{|A}$ is conservative.

• If A is a Cantor attractor, then A is the intersection of a nested sequence of forward invariant Cantor-type sets each of which absorbs m-a. e. point.

Combining the above results, we arrive at the following classification of S-unimodal maps with sensitive dependence:

(I) $A = \omega(c)$ *is a Cantor attractor:*

In this case $f_{|A}$ is minimal with topological entropy zero, $m(A) = 0$, and $\hat{f}$ is dissipative ([6]).

(II) A *is an interval attractor and* $\omega(c)$ *is of Cantor-type:*

In this case $A = A_{top}$, $f_{|A}$ is conservative and ergodic, $m(\omega(c)) = 0$, and $\hat{f}$ is essentially conservative.

([6]) These maps are just the "non-Markov" maps in the terminology of [Ma].

(III) $A = \omega(c)$ *is an interval attractor:*

In this case $A = A_{top}$ and $f_{|A}$ is conservative and ergodic.

In the next section we furnish proofs for the following additional pieces of information on the above classification:

THEOREM 1. – *Each map of type (II) has an invariant density (integrable or not).*

THEOREM 2. – *There are maps of type (II) which have a nonintegrable invariant density.*

Remark 1. – The maps in Theorem 2 can be constructed with various additional properties. We describe some of them:

For a probability measure μ on [0, 1] let

$$\varpi_f(\mu) := \left\{ \text{all weak accumulation points of } \frac{1}{n} \sum_{k=0}^{n-1} \mu \circ f^{-k} \right\}.$$

By δ_x denote the unit mass in the point x.

1. Let $0 \leqq h_0 < h_1 < \frac{1+\sqrt{5}}{2}$. There are maps f of type (II) such that

(*a*) $\{h_\nu(f) : \nu \in \varpi_f(\delta_c),\ \nu \text{ ergodic}\} = [h_0, h_1]$,

(*b*) $\varpi_f(\delta_x) = \varpi_f(m) = \varpi_f(\delta_c)$ for m-a. e. x.

2. There are maps f of type (II) such that

(*a*) $\varpi_f(\delta_c) = \{\delta_{z_f}\}$, where z_f is the unique *unstable* fix point $> c$ of f.

(*b*) $\varpi_f(\delta_x) = \varpi_f(m) = \{\delta_{z_f}\}$ for m-a. e. x.

THEOREM 3. – *There are maps f of type (III) without integrable invariant density.*

So, even if for such maps $\hat{f}$ has a (necessarily nonintegrable) invariant density, this cannot easily be "pushed down" to an invariant density for f as for type (II) maps (*see* the proof of Theorem 1).

Remark 2. – Looking a bit closer at the proofs of [HK] one can even construct examples of type (III) maps satisfying either of the two assertions of Remark 1.

Here are some open problems:

- Are there maps of type (I) with sensitive dependence? This was already asked in [Mi], [BL2] and [GJ].
- Are there maps f of type (III) which have no invariant density (integrable or not), or for which $\hat{f}$ is dissipative?
- Are there maps f of type (III) with a nonintegrable invariant density for which $\hat{f}$ has no invariant density?

We close this section with an example of a map f which seems a reasonable candidate for a type (I) map. We want to stress that this does not express a strong believe in the existence of such maps. However, if there exist type (I) maps at all, then the following map might be one.

Example 1 (The Fibonacci map). – Developping ideas from [Ho] we proved in [HK] that the following recursive construction leads always to a (admissible) kneading sequence for a map of the type $f_a(x)=ax(1-x)$:

For $\gamma \in \mathbf{N} \cup \{+\infty\}$ let

$$\mathbf{N}_\gamma = \{n \in \mathbf{N} : 1 \leqq n < \gamma\}.$$

A map $Q : \mathbf{N}_\gamma \to \mathbf{N}_\gamma \cup \{0\}$ is a *kneading map*, if

$$Q(k) < k \quad \text{for all } k \in \mathbf{N}_\gamma \tag{1}$$

and

$$\left.\begin{array}{c}(Q(j))_{k<j<\gamma} \geqq (Q(Q(Q(k))+j-k))_{k<j<\gamma} \\ \text{for all } k \in \mathbf{N}_\gamma \quad \text{with } Q(k) \geqq 1.\end{array}\right\} \tag{2}$$

Here "$\geqq$" denotes the lexicographic ordering on sequences of integers.

Given a kneading map $Q : \mathbf{N}_\gamma \to \mathbf{N}_\gamma \cup \{0\}$ we construct a 0-1-sequence $\mathbf{e}=e_1 e_2 e_3 \ldots$ as follows: Let

$$S_0 = 1 \qquad \text{and} \qquad S_k = S_{k-1} + S_{Q(k)} \quad \text{for } 1 \leqq k < \gamma. \tag{3}$$

Then, starting with $e_1 = 1$, define $\mathbf{e}$ recursively by

$$e_{S_{k-1}+1}\, e_{S_{k-1}+2} \ldots e_{S_k} = e_1\, e_2 \ldots e_{S_{Q(k)}-1}\, e'_{S_{Q(k)}} \qquad \text{for} \quad 1 \leqq k < \gamma, \tag{4}$$

(where $e' = 1 - e$) and, if $\gamma < \infty$, also by

$$e_{S_{\gamma-1}+1}\, e_{S_{\gamma-1}+2} \ldots = e_1\, e_2 \ldots \tag{5}$$

In [HK], Theorem 4, we proved that for the sequence $\mathbf{e}$ thus produced there exists a parameter a such that the critical point of f_a is nonperiodic and f_a has $\mathbf{e}$ as its kneading sequence. The fact that each kneading sequence of a map f_a arises from a kneading map was already observed in [Ho]. It is not hard to show that the kneading sequence $\mathbf{e}$ is indecomposable (*i.e.* the associated map has no restricted central point or, equivalently, is topologically mixing on its topological attractor) if and only if $\gamma = \infty$ and there is no $k \in \mathbf{N}_\gamma = \mathbf{N}$ such that $Q(j) \geqq k$ for all $j > k$.

The kneading sequence of the "Feigenbaum map", which is the simplest example of an infinitely decomposable kneading sequence, is generated by the kneading map

$$Q: \quad \mathbf{N} \to \mathbf{N} \cup \{0\}, \qquad Q(k) = k - 1.$$

Similarly, the kneading map

$$Q: \quad \mathbf{N} \to \mathbf{N} \cup \{0\}, \qquad Q(1) = 0, \qquad Q(k) = k - 2 \quad \text{for } k \geqq 2$$

generates in a very simple manner an indecomposable kneading sequence which, from the point of view of kneading maps, is as close as possible to that of the Feigenbaum map. So a corresponding map f_a seems a natural candidate for a type (I) map. We call it a *Fibonacci map*, because the sequence $(S_n)_{n \geqq 0}$ generated by Q [*see* (3)] is just the Fibonacci sequence.

As the kneading sequence of a Fibonacci map f is indecomposable, f is topologically mixing on its topological attractor A_{top}, and A_{top} is just one interval which contains in particular the unique fix point of f to the right of the critical point whose itinerary is 1^∞. It is easily seen that a neighbourhood of this point is disjoint from $\omega(c)$, *cf.* the proof of Lemma 2. Hence $\omega(c)$ is different from A_{top} and must be of Cantor type. It is also an easy exercise in symbolic dynamics to show that $f_{|\omega(c)}$ is minimal and has topological entropy zero.

Using Proposition 1 of [HK] (cited as (11) below) one can modify the Fibonacci example such that $\omega(c)$ is of Cantor type, $f_{|\omega(c)}$ is minimal and has topological entropy zero, and f has no finite invariant density but instead at least two different measures in $\varpi_f(m) = \varpi_f(\delta_c)$.

3. PROOFS

For the proofs we use heavily properties of the Markov extension $\hat{f}: \hat{X} \to \hat{X}$. For facts about Markov extensions we refer to [K], section 3. We recall that $\hat{X} = \bigcup_{k \geqq 0} \hat{D}_k$, where the $\hat{D}_k$ are disjoint copies of subintervals D_k of [0, 1]. Indeed,

$$\left.\begin{array}{c}\text{if Z is a maximal monotonicity interval of } f^n, \\ \text{then } f^n(Z) \in \{D_0, \ldots, D_n\}.\end{array}\right\} \tag{6}$$

In particular, both endpoints of each D_k belong to the forward orbit of c. $\pi: \hat{X} \to [0, 1]$ denotes the canonical embedding (identifying $\hat{D}_k$ with $D_k \subseteqq [0, 1]$).

A simple, but important, observation is:

$$\left.\begin{array}{c}\text{If Z is a maximal monotonicity interval of } f^n \text{ and if } Z \subseteqq D_k, \\ \text{then } \hat{f}^n(\pi^{-1} Z \cap \hat{D}_k) = \hat{D}_i \quad \text{for some } i \leqq n.\end{array}\right\} \tag{7}$$

LEMMA 1. – *Let f be of type (II), and denote by $\hat{h}: \hat{X} \to \mathbf{R}$ the invariant density of $\hat{f}$. If $x \in [0, 1] \setminus \omega(c)$ and if $x \neq f^k(c)$ for all $k \geqq 0$, then there are a neighbourhood U of x and $C < \infty$ such that for all $y \in U$*

$$h(y) := \sum_{\hat{y} \in \pi^{-1}\{y\}} \hat{h}(\hat{y}) < C. \tag{8}$$

Proof. – As f has no stable periodic orbit, the preimages of c under f are dense in [0, 1]. Hence, for $x \in [0, 1] \setminus \omega(c)$ with $x \neq f^k(c)$ for all $k > 0$ there is $n > 0$ such that $Z_n[x]$, the maximal monotonicity interval of f^n

that contains x, has empty intersection with the forward orbit of c. Let $U=Z=Z_n[x]$. (If x is a preimage of c, we can take for Z either of the two monotonicity intervals with endpoint x and for U the union of these two intervals.)

As the endpoints of the intervals D_k belong to the orbit of c, $U \cap D_k = U$ or $U \cap D_k = \emptyset$ for all $k \geqq 0$. Indeed, there is $\delta > 0$ independent of k such that the distance of U to the endpoints of D_k is $> \delta$. Hence the "local" densities $\hat{h}_{|\pi^{-1} U \cap \hat{D}_k}$ have uniformly bounded distortion (7), and it suffices to prove (8) for just one $y \in U$.

Fix a subinterval $V \subseteqq Z$. If $Z \cap D_k \neq \emptyset$, then $\hat{f}^n$ maps $\pi^{-1} Z \cap \hat{D}_k = \pi^{-1} Z_n[x] \cap \hat{D}_k$ diffeomorphically onto $\hat{D}_i$ for some $i \leqq n$, *see* (7), and $\pi^{-1} V \cap \hat{D}_k$ diffeomorphically onto $\pi^{-1} W \cap \hat{D}_i$, where $W = f^n(V)$. This implies

$$\hat{f}^{-n}(\pi^{-1} W \cap \hat{D}_i) = \pi^{-1} V.$$

Therefore, if $\bar{V} \subseteqq \operatorname{int}(Z)$ and hence $\bar{W} \subseteqq \operatorname{int}(D_i)$,

$$\int_V h(y)\, dy = \int_{\pi^{-1} V} \hat{h}(\hat{y})\, d\hat{y} = \int_{\hat{f}^{-n}(\pi^{-1} W \cap \hat{D}_i)} \hat{h}(\hat{y})\, d\hat{y} = \int_{\pi^{-1} W \cap \hat{D}_i} \hat{h}(\hat{y})\, d\hat{y} < \infty.$$

This proves that $h(y) < \infty$ *for some* $y \in V \subseteqq Z \subseteqq U$. □

Proof of Theorem 1. – As for maps of type (II) $m(\omega(c)) = 0$, Lemma 1 applies to m-a.e. $x \in [0, 1]$, such that

$$h(y) := \sum_{\hat{y} \in \pi^{-1}\{y\}} \hat{h}(\hat{y}) < \infty \quad m\text{-a.e.}$$

As $\hat{h}$ is $\hat{f}$-invariant, h is f-invariant, *cf.* [K], Lemma 2. □

Proofs of Theorems 2 *and* 3 *together with Remark* 1. – In [HK] we constructed various examples of S-unimodal maps f without integrable invariant density, in particular maps with the additional properties described in Remark 1. In order to prove Theorem 2 together with Remark 1, we show that these examples have $\omega(c)$ of Cantor-type (Lemma 2 below). As the restrictions of such maps to $\omega(c)$ are of positive entropy or have a fix point (i.e. are nonminimal) they cannot be of type (I). Theorem 3 will follow if we can show that the construction from [HK] can be modified such that $\omega(c)$ contains a nontrivial interval and that $\varpi_f(\delta_x)$ contains more than one measure for Lebesgue-a.e. x (Lemma 3 below). □

In order to show Lemmas 2 and 3, we consider the family of maps $f_a: [0, 1] \to [0, 1]$, where $f_a = ax(1-x)$ and $2 < a \leqq 4$. We remark

(7) This follows from applying Theorem 1 of [K] to the Markov extension as it is done in section 3 of [K].

that the itineraries of points $x \in \left(\frac{1}{2}, f_a\left(\frac{1}{2}\right) \right]$ under f_a are exactly those 0-1-sequences $x_1 x_2 \ldots$ which satisfy $x_1 = 1$ and $x_i x_{i+1} \ldots \quad$ **e** for all $i \geqq 1$, where $\mathbf{e} = e_1 e_2 \ldots$ is the kneading sequence of f_a. Here denotes the following order relation on $\{0, 1\}^{\mathbf{N}}$:

If $y_1 y_2 \ldots$ and $z_1 z_2 \ldots$ are two elements of $\{0, 1\}^{\mathbf{N}}$, then $y_1 y_2 \ldots \quad z_1 z_2 \ldots$, if they are equal, or if there is $i > 0$ such that either $y_1 y_2 \ldots y_i = z_1 z_2 \ldots z_i$ contains an even number of 1 and $y_i < z_i$, or such that $y_1 y_2 \ldots y_i = z_1 z_2 \ldots z_i$ contains an odd number of 1 and $y_i > z_i$.

The same notation can be used for finite 0-1-sequences. (*See* [Ho] and [HK] for more details.)

We have the following results: Suppose **e** is indecomposable (*cf.* Example 1). If there is a block $x_1 x_2 \ldots x_l$ with $x_1 = 1$ satisfying

$$x_{i+1} x_{i+2} \ldots x_l \quad e_1 e_2 \ldots e_{l-i} \qquad \text{for} \quad 0 \leqq i \leqq l-1 \tag{9}$$

which does not occur in **e**, then there is a nontrivial interval in $\left(\frac{1}{2}, f_a\left(\frac{1}{2}\right) \right]$ which is disjoint from the orbit of $c = \frac{1}{2}$ and hence $\omega(c)$ is a Cantor set. On the other hand, if all blocks $x_1 x_2 \ldots x_l$ with $l \geqq 1$ satisfying $x_1 = 1$ and (9) are contained in **e** and if f_a has no stable periodic orbit, then the orbit of $c = \frac{1}{2}$ is dense in $\left(\frac{1}{2}, f_a\left(\frac{1}{2}\right) \right]$ and hence in $\left[f_a^2\left(\frac{1}{2}\right), f_a\left(\frac{1}{2}\right) \right]$, which means $\omega(c) = \left[f_a^2\left(\frac{1}{2}\right), f_a\left(\frac{1}{2}\right) \right]$.

We recall some definitions and facts from [HK]. A sequence

$$\mathscr{F} = (0 = \mathrm{V}_0 < \mathrm{U}_1 < \mathrm{V}_1 < \mathrm{U}_2 < \mathrm{V}_2 \ldots)$$

of integers is called a *frame*, if it satisfies

$$\mathrm{U}_{n+1} \geqq n\, 2^{n + \mathrm{V}_n} \qquad \text{and} \qquad \mathrm{V}_n \geqq n^2\, 2^{\mathrm{U}_n} \quad \text{for } n \geqq 1. \tag{10}$$

For a frame $\mathscr{F}$ we define the *skeleton* $\mathscr{S}(\mathscr{F})$ as the set of all kneading maps $\mathrm{Q}: \mathbf{N} \to \mathbf{N} \cup \{0\}$ with

$$\mathrm{U}_k < i \leqq \mathrm{V}_k \Rightarrow \mathrm{Q}(i) = \mathrm{U}_k \qquad \text{and} \qquad \mathrm{Q}(\mathrm{U}_{k+1}) < \mathrm{U}_k.$$

In Proposition 1 of [HK] we proved:

For each $\mathrm{N} \geqq 1$ there are uncountably many different frames $\mathscr{F}$ with $\mathrm{U}_1 \geqq \mathrm{N} + 1$ such that for each $\mathrm{Q} \in \mathscr{S}(\mathscr{F})$ and each f_a with associated kneading map Q holds:

1. f_a has no integrable ergodic invariant density of positive entropy. (11)
2. $\nu_{x, a, \mathrm{S}(\mathrm{V}_n)} - \nu_{c, a, \mathrm{S}(\mathrm{U}_n)} \to 0$ as $n \to \infty$ for Lebesgue-a. e. x.

Here $\nu_{x, a, k} = \frac{1}{k} \sum_{i=1}^{k} \delta_{f_a^i x}$, and $\mathrm{S}(k)$ is just another notation for S_k.

Given a frame $\mathscr{F}$, we constructed in [HK] special kneading maps $Q \in \mathscr{S}(\mathscr{F})$ satisfying

$$Q(j)=0 \qquad \text{for} \quad 1 \leqq j \leqq U_1 \tag{12}$$

and for $n \geqq 1$

$$\begin{gathered} Q(U_n+j)=U_n \qquad \text{for} \quad 1 \leqq j \leqq V_n - U_n \\ Q(V_n+j)=U_{n-j} \qquad \text{for} \quad 1 \leqq j \leqq n-1 \\ Q(V_n+n)=1 \\ Q(V_n+n+j) \text{ is either 0 or 1 for } 1 \leqq j \leqq U_{n+1}-V_n-n \end{gathered} \tag{13}$$

The values in the last line of (13) are determined by prescribed generic points of measures. The corresponding kneading sequences are clearly indecomposable.

LEMMA 2. — *For each of the assertions of Remark* 1 *there are* S-*unimodal maps f with $\omega(c)$ of Cantor-type satisfying the assertion.*

Proof. — Let Q be one of the maps defined by (12) and (13) with $U_1 \geqq 4$ and with a frame satisfying (10). As Q is a kneading map, it determines a 0-1-sequence $\mathbf{e}$ by (3) and (4), which is the kneading sequence of a map f_a. It is shown in [HK], that such maps f_a satisfy 1 or 2 of Remark 1, if the values of the last line in (13) are chosen suitably. Hence it remains to show that $\omega(c)$ is of Cantor-type. To this end we prove that the block $B := 1010101$ is not contained in $\mathbf{e}$, but B satisfies (9), since $S_{U_1} = U_1 + 1 \geqq 5$ implies that $\mathbf{e}$ begins with 100 [*see* (14) below].

By (3) the kneading map Q determines the sequence $(S_k)_{k \geqq 0}$ and by (4) the kneading sequence $\mathbf{e}$ determined by Q satisfies

$$e_1 e_2 \ldots e_{S(U_1)} = 100\ldots0 \tag{14}$$

and for $n \geqq 1$

$$e_{S(U_n+j-1)+1} \ldots e_{S(U_n+j)} = e_1 e_2 \ldots e_{S(U_n)-1} e'_{S(U_n)} \quad \text{for } 1 \leqq j \leqq V_n - U_n \tag{15}$$

$$e_{S(V_n+j-1)+1} \ldots e_{S(V_n+j)} = e_1 e_2 \ldots e_{S(U_n-j)-1} e'_{S(U_n-j)} \quad \text{for } 1 \leqq j \leqq n-1 \tag{16}$$

$$e_{S(V_n+n-1)+1} \ldots e_{S(U_{n+1})} \text{ consists of blocks 11 and 0 beginning with 11.} \tag{17}$$

We get (17), since $e'_1 = 0$ and $e_1 e'_2 = 11$.

First we show for $n \geqq 0$

$$e_{S(U_{n+1})-4} \ldots e_{S(U_{n+1})-1} \text{ contains either 00 or 11.} \tag{18}$$

As $S_{U_1} = U_1 + 1 \geqq 5$, we get $e_{S(U_1)-2} = e_{S(U_1)-1} = 0$ by (14), and (18) holds for $n=0$. For $n \geqq 1$ we have $S_{U_{n+1}} - S_{V_n+n-1} \geqq U_{n+1} - (V_n + n - 1) \geqq 5$ by (10), since $V_n \geqq U_1 \geqq 4$. Hence the block in (17) has at least length 5 and therefore contains $e_{S(U_{n+1})-4} \dots e_{S(U_{n+1})-1}$. As the block in (17) is built up by blocks 11 and 0, (18) follows for $n \geqq 1$.

Now we can show that **e** does not contain B. To this end we show by induction on n, that $e_1 e_2 \dots e_{S(U_n)}$ does not contain B. By (14) this holds for $n=1$. Suppose it is shown that $e_1 e_2 \dots e_{S(U_n)}$ does not contain B. Adding the blocks of (15), (16) and (17) to this block, we get $e_1 e_2 \dots e_{S(U_{n+1})}$. By induction hypothesis and (18) the blocks $e_1 e_2 \dots e_{S(U_n-j)-1} e'_{S(U_n-j)}$ for $0 \leqq j \leqq n-1$ of (15) and (16) do not contain B and contain either 00 or 11 in its last five coordinates. If we add these blocks to $e_1 e_2 \dots e_{S(U_n)}$ according to (15) and (16), we get $e_1 e_2 \dots e_{S(V_n+n-1)}$, which does not contain B. This could happen only, if the first part of B is an endsegment of $e_1 e_2 \dots e_{S(U_n)}$ or of some $e_1 e_2 \dots e_{S(U_n-j)-1} e'_{S(U_n-j)}$ and the remaining part of B is an initial segment of some $e_1 e_2 \dots e_{S(U_l)-1} e'_{S(U_l)}$. But this is impossible, since the last five coordinates of $e_1 e_2 \dots e_{S(U_n)}$ and of each $e_1 e_2 \dots e_{S(U_n-j)-1} e'_{S(U_n-j)}$ contain either 00 or 11 and since $e_1 e_2 \dots e_{S(U_l)-1} e'_{S(U_l)}$ begins with 100 (remember that $S(U_l) \geqq S(U_1) \geqq 5$ for all l). Finally we add $e_{S(V_n+n-1)+1} e_{S(V_n+n-1)+2} \dots e_{S(U_{n+1})}$, the block in (17). This block is built up of the two blocks 11 and 0. Hence it cannot contain B. This block in (17) begins with 11, hence the same arguments as above apply and we get that $e_1 e_2 \dots e_{S(U_{n+1})}$ does not contain B, finishing the induction. □

LEMMA 3. — *There are* S-*unimodal maps* f *with*

$$\omega(c) = \left[f^2\left(\frac{1}{2}\right), f\left(\frac{1}{2}\right) \right]$$

and such that $\{h_\nu(f) : \nu \in \varpi_f(\delta_x),\ \nu$ *ergodic*$\}$ *contains a nontrivial interval for Lebesgue-a. e.* x.

Proof. — Let Q be a kneading map defined by (12) and (13). We construct inductively a kneading map $\tilde{Q} : \mathbf{N} \to \mathbf{N} \cup \{0\}$ together with a sequence $\mathscr{B}$ of blocks. We begin with $\tilde{Q}(j) = Q(j)$ for $1 \leqq j \leqq U_2$, which determines $\tilde{S}_j$ for $j \leqq U_2$ by (3) and the initial segment $c_1 c_2 \dots c_{\tilde{S}(U_2)}$ of the corresponding kneading sequence by (4). Let $\mathscr{B}$ contain all blocks $x_1 x_2 \dots x_l$ with $x_1 = 1$ and $l = \tilde{S}_{U_1} - 1$ satisfying

$$x_{i+1} x_{i+2} \dots x_l \lhd c_1 c_2 \dots c_{l-i} \qquad \text{for} \quad 0 \leqq i \leqq l-1. \tag{19}$$

After the $(n-1)$-th step $\tilde{Q}(j)$ for $j \leqq U_n$ is defined, which determines $\tilde{S}_j$ for $j \leqq U_n$ by (3) and $c_1 c_2 \dots c_{\tilde{S}(U_n)}$ by (4). We add to $\mathscr{B}$ all blocks $x_1 x_2 \dots x_l$ with $x_1 = 1$ and $l = \tilde{S}_{U_{n-1}} - 1$ satisfying (19). We describe the n-th step. Let $x_1 x_2 \dots x_m$ be the $(n-1)$-th block in $\mathscr{B}$. As in each step at least one block, namely an initial segment of $c_1 c_2 \dots$, is added to $\mathscr{B}$, we have $m \leqq \tilde{S}_{U_{n-1}} - 1$. Define $s = s_n \geqq 1$ and $t_1, t_2, \dots, t_{s-1} \geqq 1$ such that

$$x_{T_{i-1}+1} x_{T_{i-1}+2} \dots x_{T_i} = c_1 c_2 \dots c_{t_i - 1} c'_{t_i} \qquad \text{for} \quad 1 \leqq i \leqq s-1$$

where $T_0=0$ and $T_j=t_1+t_2+\ldots+t_j$, and such that

$$x_{T_{s-1}+1}\,x_{T_{s-1}+2}\ldots x_m=c_1\,c_2\ldots c_{m-T_{s-1}}.$$

It follows from (19) and Lemma 1 (iii) of [Ho] that $t_i=\tilde{S}_{P(i)}$ for some $P(i)\geqq 0$. Let u be minimal, such that $m-T_{s-1}<\tilde{S}_u$. Because of $m<\tilde{S}_{U_{n-1}}$, we have $u\leqq U_{n-1}$. Set $t_s=\tilde{S}_u$. Then $m<T_s$ and we set

$$x_{m+1}\,x_{m+2}\ldots x_{T_s}=c_{m-T_{s-1}+1}\ldots c_{t_s-1}\,c'_{t_s}.$$

This means that $x_1\,x_2\ldots x_{T_s}$ has been split up into blocks $c_1\,c_2\ldots c_{t_i-1}\,c'_{t_i}$ with $t_i=\tilde{S}_{P(i)}$ for $1\leqq i\leqq s$. We get

$$P(i)\leqq U_{n-1}\qquad\text{for}\quad 1\leqq i\leqq s=s_n. \tag{20}$$

For $i=s$ this follows from the definition of t_s. For $i<s$ this follows from $t_i\leqq m<\tilde{S}_{U_{n-1}}$. The proof of Lemma 2 in [Ho] implies for $1\leqq i<s$ that

$$(\tilde{Q}(\tilde{Q}(P(i))+j))_{1\leqq j\leqq s-i}\leqq(P(i+j))_{1\leqq j\leqq s-i}\qquad\text{if}\quad P(i)>0. \tag{21}$$

Furthermore $s_n=s\leqq T_s\leqq m+t_s\leqq\tilde{S}_{U_{n-1}}+\tilde{S}_u\leqq 2\,\tilde{S}_{U_{n-1}}$. Together with (1.9) of [HK] this implies

$$s_n\leqq T_{s_n}\leqq 2^{U_{n-1}+1}. \tag{22}$$

Now we define $\tilde{Q}(j)$ for $U_n<j\leqq U_{n+1}$ by

$$\tilde{Q}(j)=Q(j)\qquad\text{for}\quad U_n<j\leqq U_{n+1}-s_n \tag{23}$$

$$\tilde{Q}(j)=P(j+s_n-U_{n+1})\qquad\text{for}\quad U_{n+1}-s_n+1\leqq j\leqq U_{n+1} \tag{24}$$

finishing the n-th step in the construction of $\tilde{Q}$.

We have to check that $\tilde{Q}$ is a kneading map. From the fact that Q is a kneading map and from (20) we get (1). As $\tilde{Q}(j)=Q(j)$ for $j\leqq U_2$ and as all blocks in $\mathscr{B}$ begin with 1, which implies $\tilde{Q}(U_{n+1}-s_n+1)=P(1)\geqq 1$, we get (2) for $k\in\{V_n+n-2,\,V_n+n-1,\,\ldots,\,U_{n+1}-s_n\}$ by the same proof given in [HK] for Q. Furthermore, $\tilde{Q}(U_{n+1}+1)=U_{n+1}>\tilde{Q}(j)$ for all $j\leqq U_{n+1}$ by (1). Hence for $k\in\{U_{n+1}-s_n+1,\,U_{n+1}-s_n+2,\,\ldots,\,U_{n+1}\}$ we get (2) from (21). For $k\in\{U_n+1, U_n+2,\,\ldots,\,V_n+n-3\}$, we have $\tilde{Q}(k)=U_l$ for some $l\in\{3,4,\ldots,n\}$ and $\tilde{Q}(k+1)\geqq U_{l-1}$ since $V_n+n<U_{n+1}-s_n$ by (22) and (10). As $\tilde{Q}(\tilde{Q}(k))=\tilde{Q}(U_l)\leqq U_{l-2}$ by (24) and (20), we get $\tilde{Q}(\tilde{Q}(\tilde{Q}(k))+1)\leqq U_{l-2}$ by (1). Hence $\tilde{Q}(k+1)>\tilde{Q}(\tilde{Q}(\tilde{Q}(k))+1)$ and (2) follows, finishing the proof that $\tilde{Q}$ is a kneading map.

Finally let $\mathbf{c}$ be the kneading sequence determined by $\tilde{Q}$ using (3) and (4). Every block $x_1\,x_2\ldots x_l$ satisfying $x_1=1$ and (19) is a subblock of some block, which was added to $\mathscr{B}$ in a step n with $\tilde{S}_{U_{n-1}}>l$. Hence it is contained in $\mathbf{c}$, since (24) is chosen in such a way, that all blocks of $\mathscr{B}$ are contained in $\mathbf{c}$. This implies that a map $f_{\tilde{a}}$ having $\mathbf{c}$ as its kneading sequence has dense critical orbit (as $\mathbf{c}$ is not eventually periodic by (2.10) of [HK], $f_{\tilde{a}}$ has no stable periodic orbit). This proves the first part of the lemma.

Let $\mathbf{e}$ be the kneading sequence determined by Q. For $n \geqq 1$ it follows from (23) and (13) that

$$e_{S(V_n+n)+1}\, e_{S(V_n+n)+2} \cdots e_{S(U_{n+1}-s_n)} = c_{\tilde{S}(V_n+n)+1}\, c_{\tilde{S}(V_n+n)+2} \cdots c_{\tilde{S}(U_{n+1}-s_n)} \qquad (25)$$

since $e_1 e_2 = c_1 c_2 = 10$. For $n \geqq 1$ we show that

$$\frac{S(U_{n+1}-s_n)-S(V_n+n)}{S(U_{n+1})} \geqq 1-\frac{2}{n}$$
$$\text{and} \qquad (26)$$
$$\frac{\tilde{S}(U_{n+1}-s_n)-\tilde{S}(V_n+n)}{\tilde{S}(U_{n+1})} \geqq 1-\frac{2}{n}.$$

For the first inequality consider $S(U_{n+1})-S(U_{n+1}-s_n) \leqq 2s_n \leqq 2^{U_{n-1}+2}$, which follows from the last line of (13) and from (22). Furthermore $S(V_n+n) \leqq 2^{V_n+n}$ by (1.9) of [HK] and $S(U_{n+1}) \geqq U_{n+1}$. Now the first part of (26) follows from (10). The second part follows in the same way, since $\tilde{S}(U_{n+1})-\tilde{S}(U_{n+1}-s_n)=T_{s_n}$ by (24) and (3) and since $T_{s_n} \leqq 2^{U_{n-1}+1}$ by (22). Recall that $\nu_{x,a,n}=\frac{1}{n}\sum_{k=1}^{n}\delta_{f_a^k(x)}$. If a is such that f_a has $\mathbf{e}$ as its kneading sequence, and if $c=\frac{1}{2}$ is the critical point of both f_a and $f_{\tilde{a}}$, then (25) and (26) imply

$$\varphi_a^*(\nu_{c,a,S(U_n)})-\varphi_{\tilde{a}}^*(\nu_{c,\tilde{a},\tilde{S}(U_n)}) \to 0 \qquad \text{for} \quad n\to\infty \qquad (27)$$

where $\varphi_a: [0,1] \to \{0,1\}^{\mathbb{N}}$ associates with each point x its itinerary under f_a (*see* [HK] for more details) and $\varphi_a^*(\nu)=\nu\circ\varphi_a^{-1}$.

Let L_a and $L_{\tilde{a}}$ denote the sets of weak accumulation points of the sequences of measures $(\nu_{c,a,S(U_n)})_{n>0}$ and $(\nu_{c,\tilde{a},\tilde{S}(U_n)})_{n>0}$, respectively. (27) implies

$$\varphi_a^*(L_a)=\varphi_{\tilde{a}}^*(L_{\tilde{a}}).$$

Write $\varpi_a(\delta_x)$ for $\varpi_{f_a}(\delta_x)$. From (11) we conclude that for Lebesgue-a. e. x the set $\varpi_{\tilde{a}}(\delta_x)$ contains the set $L_{\tilde{a}}$. Hence

$$\varphi_{\tilde{a}}^*(\varpi_{\tilde{a}}(\delta_x)) \supseteqq \varphi_a^*(L_a) \qquad \text{for} \quad \text{Lebesgue-a. e. } x.$$

But Q (equivalently $\mathbf{e}$ or a) can be chosen such that $\{h_\nu(f_a): \nu\in L_a,\ \nu \text{ ergodic}\}$ contains a nontrivial interval, *cf.* [HK], Proposition 2 and Theorem 1. This finishes the proof of the lemma. □

REFERENCES

[BL1] A. M. BLOKH and M. Yu. LYUBICH, Attractors of maps of the interval, *Funct. Anal. Appl.*, Vol. **21**, 1987, pp. 70-71 (Russian).

[BL2] A. M. BLOKH and M. Yu. LYUBICH, *Measurable Dynamics of* S-*Unimodal Maps of the Interval*, Preprint, Stony Brook, 1990.

[GJ] J. GUCKENHEIMER and S. JOHNSON, Distortion of S-Unimodal Maps, *Ann. Math.*, Vol. **132**, 1990, pp. 71-130.

[Ho] F. HOFBAUER, The Topological Entropy of the Transformation $x \mapsto ax(1-x)$, *Monatsh. Math.*, Vol. **90**, 1980, pp. 117-141.

[HK] F. HOFBAUER and G. KELLER, Quadratic maps without asymptotic measure, *Commun. Math. Phys.*, Vol. **127**, 1990, pp. 319-337.

[K] G. KELLER, Exponents, Attractors, and Hopf Decompositions for Interval Maps, Preprint, 1988, to appear in *Erg. Th. Dynam. Syst.*

[Ma] M. MARTENS, *Thesis*, Delft, 1990.

[Mi] J. MILNOR, On the Concept of Attractor, *Commun. Math. Phys.*, Vol. **99**, 1985, pp. 177-195.

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