ANNALES DE L'I. H. P., SECTION A

M. IRAC-ASTAUD

# Perturbation around exact solutions for nonlinear dynamical systems : application to the perturbed Burgers equation

*Annales de l'I. H. P., section A*, tome 53, nº 3 (1990), p. 343-358

<http://www.numdam.org/item?id=AIHPA_1990__53_3_343_0>

*Ann. Inst. Henri Poincaré,*
Vol. 53, n° 3, 1990, p. 343-358. *Physique théorique*

# Perturbation around exact solutions for nonlinear dynamical systems: application to the perturbed Burgers equation

by

**M. IRAC-ASTAUD**

Laboratoire de Physique Théorique et Mathématique,
Université Paris-VII,
2, place Jussieu, Paris, France

ABSTRACT. — When two dynamical systems of nonlinear partial differential equations differ by a term that can be considered as a perturbation, their solutions equal at the initial time, can be related by a linear integral equation. Due to this equation, if the solution of one of these systems is known, let us call it free, the solution of the other one, the perturbed one, can be written as a perturbation expansion, the terms of which are completely explicit expressions of the free solution. This generalises the usual perturbation theories around free solutions satisfying linear equations. The perturbed Burgers equation is taken as an example.

RÉSUMÉ. — Quand deux systèmes dynamiques d'équations non-linéaires aux dérivées partielles diffèrent par un terme qui peut être considéré comme une perturbation, leurs solutions égales à l'instant initial sont reliées par une équation intégrale linéaire. Grâce à cette équation, si la solution de l'un des systèmes est connue, nous l'appelons la solution libre, la solution de l'autre, solution perturbée, peut être représentée par un développement en perturbations dont les termes sont complètement explicites en fonction de la solution libre. Ceci généralise les théories de perturbations autour de solutions libres satisfaisant des équations linéaires. L'équation de Burgers perturbée est traitée comme application.

## INTRODUCTION

A dynamical system is called perturbed when it differs by a perturbation term from a free one, the solutions of which are supposed known. Denoting by $\mathscr{F}$ the space of the smooth mappings of $\mathbb{R}^n$ in $\mathbb{R}^N$

$$y \in \mathscr{F}: \quad x \in \mathbb{R}^n \to y(x) = (y_1(x), \ldots, y_N(x) \in \mathbb{R}^N \tag{1}$$

and $\mathscr{E}$ the set of the mappings from $\mathscr{F}$ into $\mathscr{F}$

$$F \in \mathscr{E}: \quad y \in \mathscr{F} \to F[y] \in \mathscr{F}, \tag{2}$$

we consider the system

$$\frac{\partial}{\partial t} v(t) = F[t; v] \tag{3}$$

where $v(t)$ belongs to $\mathscr{F}$ and F to $\mathscr{E}$. This system is assumed to be solved for $v$ satisfying the initial condition

$$v(s) = y, \qquad y \in \mathscr{F}, \tag{4}$$

it is what we call the free system. The most general perturbed system can be written on the form

$$\frac{\partial}{\partial t} u(t) = S_\lambda[t; u]; \qquad u(t) \in \mathscr{F} \tag{5}$$

where $S_\lambda(t) = F(t) + \lambda N(t)$ belongs to $\mathscr{E}$. The functional $F(t)$ is the free part of $S_\lambda(t)$, and $N(t)$ its perturbative one. These functionals depending on $t$ can be local and contain differential operators with respect to $x$, for instance $F[t, y] = a(t) \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} y^3$. They can as well be nonlocal and associated to a kernel, for instance $F[t, y] = \int_{-\infty}^{x} K[t, x - x'] y(t, x')$. We are interested with the solution $u(t)$ of (5) satisfying the same boundary condition as $v$

$$u(s) = y, \qquad y \in \mathscr{F}. \tag{6}$$

The parameter $\lambda$ that multiplies the perturbation term can have a physical significance (for instance, $\lambda$ can be a coupling constant) or can be artfully introduced in a system that cannot be exactly solved in order to get an approximate solution. Numerical physical perturbed systems correspond to dynamical systems submitted to external forces that may be random or not, the perturbation term is then a stochastic or a classical function.

When an exact closed-form solution of the system (5) can't be found, we intend to relate it to the solution of the system (3), assuming that the solutions of the two systems (3) and (5) corresponding to the same initial conditions (4) and (6) are close, at least in a neighbourhood of $\lambda = 0$, $t = s$. Supposing analytical the solution $u$ of the perturbed system, the perturbation method consists in seeking it in the form of a power series in $\lambda$, $u=\sum_{n\geq 0} U_n \lambda^n$. The coefficients $U_n$ can be recursively determined, they satisfy a linear system [1], the coefficients of which depend on the $U_i$, $i<n$, $U_0$ being the solution $v$ of the free system. Solving this system is not always possible and when it is, not always easy [2]. We propose here an explicit expression of these coefficients in terms of the free solution $v$.

This expression is obtained by iterating a *linear* integral equation relating the solutions $u$ and $v$; we establish this integral equation in Section I.

In the case where the free equation is linear, we show on an example that this perturbation theory leads to the usual one where the perturbed solution is expressed with the Green function corresponding to the free equation.

In this paper we obtain the solution of the perturbed system (5) in terms of the solution of the free one even when this last is nonlinear, these solutions coinciding at the initial time $t=s$. This approach differs from approximate methods that look for particular solutions, solitons, nearly periodic [3]...

In Section II, we apply this result to the case of the perturbed Burgers equation [1] that is

$$\frac{\partial}{\partial t}u+u\frac{\partial}{\partial x}u=\mu\frac{\partial^2}{\partial x^2}u+\lambda N[t, x; u],\qquad x\in\mathbb{R},\ u(x)\in\mathbb{R}. \tag{7}$$

Here, $S_\lambda[t; u]$ is equal to $-u\frac{\partial}{\partial x}u+\mu\frac{\partial^2}{\partial x^2}u+\lambda N$, N being an arbitrary functional of $u$. The solution of equation (7) with $\lambda=0$, the Burgers equation, being known [4], the coefficients $U_n$ are completely explicit.

## I. INTEGRAL EQUATIONS RELATING THE PERTURBED AND THE FREE SOLUTIONS

First we deal with a more abstract and general problem that we will use to solve ours in the following.

Let $\mathscr{F}_0$ be any vector space and $\mathscr{E}_0$ the space of the mappings from $\mathscr{F}_0$ into $\mathscr{F}_0$. The flow $\Phi_{ts}$ belonging to $\mathscr{E}_0$ is defined by the following semi-group properties

$$\Phi_{st}\circ\Phi_{t\tau}=\Phi_{s\tau} \tag{8}$$

where $\circ$ is the composition law for the mappings in $\mathscr{E}_0$ and

$$\Phi_{ss}=I \tag{9}$$

where I is the identity in $\mathscr{E}_0$.

Let us introduce now the substitution operator $\Phi_{ts}^*$ related to the flow $\Phi_{ts}$, belonging to $\mathscr{L}(\mathscr{E}_0, \mathscr{E}_0)$ and defined by

$$\Phi_{ts}^*(F) = F \circ \Phi_{ts}, \qquad \forall F \in \mathscr{E}_0. \tag{10}$$

In particular when F is equal to the identity I, this formula reads

$$\Phi_{ts}^*(I) = \Phi_{ts}. \tag{11}$$

The semigroup properties of the flow give

$$\Phi_{t\tau}^* \Phi_{st}^* = \Phi_{s\tau}^* \tag{12}$$

where the product in the right hand side is the product of operators. From the formula (9), we have

$$\Phi_{ss}^* = \bar{I}, \tag{13}$$

where $\bar{I}$ is the identity in $\mathscr{L}(\mathscr{E}_0, \mathscr{E}_0)$.

The field $S_\lambda^*(t)$ associated to the operator $\Phi_{ts}^*$ is defined by

$$\left\{ \frac{\partial}{\partial \tau} \Phi_{\tau t}^* \right\}_{\tau = t} = S_\lambda^*(t). \tag{14}$$

From the equations (12) and (14), we deduce

$$\left\{ \frac{\partial}{\partial \tau} \Phi_{\tau s}^* \right\}_{\tau = t} = \left\{ \frac{\partial}{\partial \tau} \Phi_{ts}^* \Phi_{\tau t}^* \right\}_{\tau = t} = \Phi_{ts}^* S_\lambda^*(t) \tag{15}$$

that reads

$$\frac{\partial}{\partial t} \Phi_{ts}^* = \Phi_{ts}^* S_\lambda^*(t). \tag{16}$$

The property (13) acts as the initial condition. Equation (12) where $\tau = s$, and equation (13) lead to

$$\Phi_{ts}^* \Phi_{st}^* = \bar{I}. \tag{17}$$

The differentiation with respect to $t$ of this last equation and the using of formula (16) give

$$\frac{\partial}{\partial t} \Phi_{st}^* = -S_\lambda^*(t) \Phi_{st}^*. \tag{18}$$

Assuming that the field $S_\lambda^*(t)$ takes the form $F^*(t) + \lambda N^*(t)$, we introduce the flow $\Psi$ and the operator $\Psi^*$ reducing to $\Phi$ and $\Phi^*$ when $\lambda$ vanishes. The problem we are interested in is to relate the operator $\Phi_{ts}^*$ to $\Psi_{ts}^*$ that satisfies the equation (16) with $\lambda$ equal to zero that is

$$\frac{\partial}{\partial t} \Psi_{ts}^* = \Psi_{ts}^* F^*(t). \tag{19}$$

To this aim, let us calculate

$$\frac{\partial}{\partial \tau} \Phi_{\tau s}^* \Psi_{t\tau}^* = \Phi_{\tau s}^* \lambda N^*(\tau) \Psi_{t\tau}^* \tag{20}$$

and

$$\frac{\partial}{\partial t} \Psi^*_{\tau s} \Phi^*_{t\tau} = - \Psi^*_{\tau s} \lambda \mathrm{N}^* (\tau) \Phi^*_{t\tau}. \tag{21}$$

The integrations of these equalities on the interval $[s, t]$ lead to

$$\Phi^*_{ts} = \Psi^*_{ts} + \lambda \int_s^t d\tau \, \Phi^*_{\tau s} \mathrm{N}^* (\tau) \Psi^*_{t\tau} \tag{22}$$

and

$$\Phi^*_{ts} = \Psi^*_{ts} + \lambda \int_s^t d\tau \, \Psi^*_{\tau s} \mathrm{N}^* (\tau) \Phi^*_{t\tau}. \tag{23}$$

These two equations relate the operators $\Phi^*_{ts}$ and $\Psi^*_{ts}$. Their iterations can be performed and lead to analogous results, the one corresponding to (23) gives

$$\Phi^*_{ts} = \Psi^*_{ts} + \sum_{k=1}^{\infty} \lambda^k \int_s^t d\tau_k \int_{\tau_k}^t d\tau_{k-1} \times \int_{\tau_2}^t d\tau_1 \, \Psi^*_{\tau_k s} \mathrm{N}^* (\tau_k) \Psi^*_{\tau_{k-1}\tau_k} \dots \times \Psi^*_{\tau_1 \tau_2} \mathrm{N}^* (\tau_1) \Psi^*_{t\tau_1}. \tag{24}$$

In this relation the operator $\Phi^*_{ts}$ is expressed in terms of $\Psi^*$. Let us summarise this result:

*Let a substitution operator* $\Phi^*_{ts}$ *belonging to* $\mathscr{L}(\mathscr{E}_0, \mathscr{E}_0)$, $\mathscr{E}_0$ *being the space of the mappings from a vector space* $\mathscr{F}_0$ *into itself, satisfy a differential equation:* $\frac{\partial}{\partial t} \Phi^*_{ts} = \Phi^*_{ts} (\mathrm{F}^* (t) + \lambda \mathrm{N}^* (t))$ *with the condition that* $\Phi^*_{ts}$ *is the identity when* $t = s$. *The operator* $\Phi^*_{ts}$ *is related to* $\Psi^*$, *its value when* $\lambda = 0$, *by a functional equation* (23) *that can be solved by iteration giving an expression of* $\Phi^*_{ts}$ *in terms of* $\Psi^*$ (24).

This result was already established in the case where $\mathscr{F}_0$ is a real manifold of dimension N and applied to ordinary differential equations [5].

We now use this result to deal with the initial problem which we were interested in. We first formulate it in a convenient form. We consider the case where the space $\mathscr{F}_0$ is the space $\mathscr{F}$ defined in formula (1). Remarking that $u(t)$, solution of the equation (5) and (6), is a functional of $y$, depending on the initial time $s$, we can express it with the help of the flow $\Phi_{ts}$

$$y \in \mathscr{F} \xrightarrow{\Phi_{ts}} u(t) = \Phi_{ts} [y] \in \mathscr{F}, \tag{25}$$

$\Phi$ belongs to the space $\mathscr{E}$, defined in formula (2).

We introduce several linear operators acting on $\mathscr{E}$:

We will use the operator $V_x$ that associates to any element of $\mathscr{E}$ its value in $x$; for instance

$$V_x F[t; y] = F[t, x; y]; \qquad \forall F(t) \in \mathscr{E}, \quad \text{for } x \in \mathbb{R}^n. \tag{26}$$

This is the analogous to the operator "projection on a component" in a finite vector space. Here both exist:

$$P_i F[t; y] = F_i[t; y]; \qquad \forall F(t) \in \mathscr{E}, \quad \text{for } i = 1, \ldots N. \tag{27}$$

The operators $P_i$ and $V_x$ commute.

We associate to an element of $\mathscr{E}$, $F(t)$, the field, $F^*(t)$, belonging to $\mathscr{L}(\mathscr{E}, \mathscr{E})$ such as:

$$F^*(t) = \sum_{i=1}^{N} \int dx' \{ V_{x'} P_i F[t; y] \} = \frac{\delta}{\delta y_i(x')}; \tag{28}$$

let us remark that $F^*(t)$ is an operator depending on $y$.

In particular, we have

$$F^*(t) V_x y_i = \sum_{j=1}^{N} \int dx' F_j[t, x'; y] \times \frac{\delta}{\delta y_j(x')} y_i(x) = F_i[t, x; y] = V_x F_i[t; y]. \tag{29}$$

In the following we will denote

$$\sum_{i=1}^{N} V_x P_i F[t; y] \frac{\delta}{\delta y_i(x)} = \sum_{i=1}^{N} F_i[t, x; y] \frac{\delta}{\delta y_i(x)} = F[t, x; y] \frac{\delta}{\delta y(x)}. \tag{30}$$

To the two functionals $S_\lambda(t)$ and $F(t)$, appearing in the equations (5) and (3), correspond by the definition (28) the fields $S_\lambda^*(t)$ and $F^*(t)$, belonging to $\mathscr{L}(\mathscr{E}, \mathscr{E})$. The operators $\Phi_{ts}^*$ and $\Psi_{ts}^*$, related to the flows $\Phi_{ts}$ and $\Psi_{ts}$, are solutions of the formulas (16) and (19) if the functions $u(t)$ and $v(t)$ are solutions of the systems (5) and (3). The problem of relating $u(t)$ to $v(t)$, solutions of (5) and (3), reduces to that of relating $\Phi^*$ to $\Psi^*$ satisfying (16) and (19) for which we can apply the previous result. The two equations (22) and (23) acting on an arbitrary functional Q of $\mathscr{E}$ furnish two integral equations relating the functions $Q[u(t)]$ and $Q[v(t)]$

$$Q[u[t, s; y]] = Q[v[t, s; y]] + \lambda \int_s^t d\tau \int d\sigma \left\{ N[\tau, \sigma; z] \frac{\delta}{\delta z(\sigma)} Q[u[t, \tau; z]] \right\}_{z = v[\tau, s; y]} \tag{31}$$

and

$$Q[u[t, s; y]] = Q[v[t, s; y]] + \lambda \int_s^t d\tau \int d\sigma \left\{ N[\tau, \sigma; z] \frac{\delta}{\delta z(\sigma)} Q[v[t, \tau; z]] \right\}_{z = u[\tau, s; y]}. \tag{32}$$

These equations correspond to that obtained in [V] when the variable × does not appear in the dynamical system. They relate, by an integral equation, any physical quantity of the perturbed system, expressed in terms of $u$, to the corresponding one of the free system. For instance, when the dynamical systems are the Burgers equation, (7) with $\lambda=0$, for the free one and the Korteveg DeVries equation, that is $\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + \alpha\frac{\partial^3}{\partial x^3}u = 0$ for the perturbed one, the corresponding physical quantities are related through the formula (31) with $N[t, x; y] = -\left(\alpha\frac{\partial}{\partial x} + \mu\right)\frac{\partial^2}{\partial x^2}y$.

The formula (31), when Q is equal to $V_x P_i$ I, reads

$$u_i[t, s, x; y] = v_i[t, s, x; y] + \lambda \int_s^t d\tau \int d\sigma \left\{ N[\tau, \sigma; z] \frac{\delta}{\delta z(\sigma)} u_i[t, \tau, x; z] \right\}_{z = v[\tau, s; y]}. \tag{33}$$

This integral equation is linear with respect to $u$ and therefore easy to iterate. The analogous one obtains by using (32) is linear with respect to $v$ and nonlinear with respect to $u$

$$u_i[t, s, x; y] = v_i[t, s, x; y] + \lambda \int_s^t d\tau \int d\sigma \left\{ N[\tau, \sigma; z] \frac{\delta}{\delta z(\sigma)} v_i[t, \tau, x; z] \right\}_{z = u[\tau, s; y]}. \tag{34}$$

This last formula is interesting because it furnishes an explicit functional equation for $u$, the integrand can be calculated as soon as $v$ is known. Let us illustrate this property on an example: the perturbed diffusion equation, that is

$$\frac{\partial}{\partial t}u(t, x) = \mu\frac{\partial^2}{\partial x^2}u(t, x) + \lambda N[t, x; u] \tag{35}$$

with

$$u(s, x) = y(x). \tag{36}$$

After introducing the diffusion kernel

$$\mathcal{G}(t, x/t', x') = \frac{1}{\sqrt{4\pi\mu(t-t')}} \exp\left\{\frac{-(x-x')^2}{4\mu(t-t')}\right\} \tag{37}$$

that satisfies

$$\mathscr{G}(t, x/t, x')=\delta(x-x'). \tag{38}$$

the solution of the free equation, linear in this case, takes the form

$$v[t, s, x; y]=\int dx' \mathscr{G}(t, x/s, x') y(x'). \tag{39}$$

The integral equation (34) reads in this case

$$u[t, s, x; y]=\int dx' \mathscr{G}(t, x/s, x') y(x') + \lambda \int_s^t dt' \int dx' \mathscr{G}(t, x/t', x') \mathrm{N}[t', x'; u]. \tag{40}$$

This integral equation furnishes the power expansion with respect to $\lambda$ of the solution $u$ of (35) in terms of the diffusion solution $v$ (39). We can compare it to the integral equation obtained directly from (35) by using the Green function $\theta(t)\,\mathscr{G}$ corresponding to the free linear equation

$$u[t, s, x; y]=\lambda \int_{-\infty}^t dt' \int dx' \mathscr{G}(t, x/t', x') \mathrm{N}[t', x'; u]. \tag{41}$$

To furnish a power series in $\lambda$, this integral equation has to be modified, that is it must be put on the form (40). This is obtained by suitably introducing in the equation (41) the initial condition (36) written as follows

$$y(x)=\lambda \int_{-\infty}^s dt' \int dx' \mathscr{G}(s, x/t', x') \mathrm{N}[t', x'; u]. \tag{42}$$

Therefore the formula (40) and the usual perturbation expansion lead to the same result in this particular case, this is the same for any free linear equation.

The iteration of (33) gives the same result as the formula (24) applied to $\mathrm{V}_x \mathrm{P}_t(y)$, it furnishes the $i$-th component of the function $u[t, s; y]$ as a power series in $\lambda$. The coefficient $\mathrm{U}_k$ of $\lambda^k$ is a triangular integral of order $k$ the integrand of which $\mathscr{K}_k[t, \tau_1, \ldots, \tau_k, s, x; y]$ is completely explicit in terms of the function $v$ and its derivatives with respect to $y$:

$$\mathscr{K}_\iota[t, \tau_1, \ldots, \tau_k, s, x; y]=\Psi^*_{\tau_k s} \mathrm{N}^*(\tau_k) \ldots \Psi^*_{\tau_1 \tau_2} \mathrm{N}^*(\tau_1) v_\iota[t, \tau_1, x; y]. \tag{43}$$

It can be recursively constructed from the preceding one

$$\mathscr{K}_\iota[t, \tau_1, \ldots, \tau_k, s, x; y] = \int d\sigma \left\{ \mathrm{N}[\tau_k, \sigma; z] \frac{\delta}{\delta z(\sigma)} \mathscr{K}_\iota[t, \tau_1, \ldots, \tau_{k-1}, \tau_k, x; z] \right\}_{z=v[\tau_k, s; y]}. \tag{44}$$

In the next section we apply these results to the case of the perturbed Burgers equation (7).

## II. PERTURBED BURGERS EQUATION

First let us recall that the equation (7) when $\lambda$ is equal to zero, can be linearized by introducing the function $\nu$ such as [4]:

$$\nu[t, s, x; y] = \exp\left\{ -\frac{1}{2\mu}\bar{v}[t, s, x; y] \right\} \tag{45}$$

where $\bar{v}[t, s, x; y]$ is a primitive of $v$ with respect to $x$. We can wrtie $v$ on the form

$$v[t, s, x; y] = V_x \Psi_{ts}[y] = -2\mu \frac{\partial}{\partial x} \log \nu[t, s, x; y]. \tag{46}$$

The function $\nu$ is solution of the diffusion equation

$$\frac{\partial}{\partial t}\nu - \mu \frac{\partial^2}{\partial x^2}\nu = 0 \tag{47}$$

with the initial condition deduced from equations (4) and (45)

$$\nu[s, s, x; y] = \exp\left\{ -\frac{\bar{y}(x)}{2\mu} \right\} \tag{48}$$

where $\bar{y}(x)$ is a primitive of $y$. The solution $\nu$ of (47) and (48) takes the form

$$\nu[t, s, x; y] = \int_{-\infty}^{+\infty} dx' \mathscr{G}(t, x/s, x') \exp\left\{ -\frac{\bar{y}(x')}{2\mu} \right\} \tag{49}$$

where $\mathscr{G}$ is the diffusion kernel defined in (37). This expression replaced in formula (46) gives the Burgers solution $v$; for theorems of existence see for instance [4].

Considering now the equation (7) with $\lambda$ different from zero, we try to linearize it as previously; let us define the function $\omega$ such as

$$\omega[t, s, x; y] = \exp\left\{ -\frac{1}{2\mu}\bar{u}[t, s, x; y] \right\} \tag{50}$$

and

$$u[t, s, x; y] = -2\mu \frac{\partial}{\partial x} \log \omega[t, s, x; y] \tag{51}$$

where $\bar{u}[t, s, x; y]$ is a primitive of $u$ with respect to $x$. Taking back this expression in equation (7), we get

$$\frac{\partial}{\partial t}\omega - \mu \frac{\partial^2}{\partial x^2}\omega = \frac{\lambda}{2\mu}\omega \{ \mathscr{N}[t, x; u] + C(t) \} \tag{52}$$

where $\mathscr{N}$ is the functional equal to a primitive with respect to $x$ of $N[t, x; u]$; the constant of integration with respect to $x$, $C(t)$, depends on the choice of the primitive $\bar{u}$, and can be taken off without changing $u$. The function $\omega$ fulfils the same initial condition (48) as the function $\nu$. The theory of the diffusion with an external source [2] gives for the solution of (45)

$$\omega[t, s, x; y] = \nu[t, s, x; y] - \frac{\lambda}{2\mu}\int_s^t dt' \int dx' \mathcal{G}(t, x/t', x') \omega[t', s, x'; y] \mathcal{N}[t', x'; u]. \quad (53)$$

Substituting $\omega$ and $\nu$ by their expressions (45) and (50), we obtain an integral equation relating the perturbed solution $u$ to the free one $v$. This equation is obviously nonlinear in $u$ and in $v$ and does not suit to find easily the coefficients $U_k$, when the direct application of the result of the previous section furnishes integral equations very convenient.

The straithforward application of the formula (43) gives the expression of the coefficients $U_k$

$$U_k = \int_s^t \dots \int_{\tau_2}^t d\tau_k \dots d\tau_1 \mathcal{K}[t, \tau_1, \dots, \tau_k, s, x; y]. \quad (54)$$

The kernel $\mathcal{K}$ involved in the chronological integral are given by formula (43)

$$\mathcal{K}[t, \tau_1, \dots, \tau_k, x; y] = \Psi^*_{\tau_k s} N^*(\tau_k) \dots \Psi^*_{\tau_1 \tau_2} N^*(\tau_1) v[t, s, x; y] \quad (55)$$

where the free solution $v$ is given by (46) and (49); the operators $\Psi^*_{ts}$ and $N^*$ being defined in (10) and (28) with the help of the functional $\psi_{ts}$ representing the flow associated to $v$ and of the perturbation term N involved in the equation (7).

Let us calculate the first kernel

$$\mathcal{K}[t, \tau_1, s, x; y] = \int dx' \left\{ N[\tau_1, x'; z] \frac{\delta}{\delta z(x')} v[t, \tau_1, x; z] \right\}_{z = v[\tau_1, s; y]}. \quad (56)$$

The functional derivative of $v$ is obtained by using (45) and (49)

$$\frac{\delta}{\delta y(x')} v[t, s, x; y] = -2\mu \frac{\partial}{\partial x} \frac{\delta}{\delta y(x')} \log \nu[t, s, x; y]. \quad (57)$$

Recalling the property of the functional derivative

$$\frac{\delta}{dy(x')} \bar{y}(x) = \theta(x - x') \quad (58)$$

and replacing v by its expression (49), we obtain

$$\frac{\delta v[t, s, x; y]}{\delta y(x')} = \frac{\partial}{\partial x}\int_{x'}^{+\infty} dx'' \mathcal{G}(t, x/s, x'') \times \exp\frac{1}{2\mu}\{\bar{v}[t, s, x; y] - \bar{y}(x'')\}. \quad (59)$$

Putting this result in the formula (56) and remarking that

$$\bar{v}[t, \tau_1, x, v[\tau_1, s; y]] = \bar{v}[t, s, x; y] \quad (60)$$

we finally have for the first kernel

$$\mathcal{K}[t, \tau_1, s, x; y] = \frac{\partial}{\partial x}\int dx_1 \,\mathrm{N}[\tau_1, x_1, v[\tau_1, s; y]] \times \int_{x_1}^{+\infty} dx' \mathcal{G}(t, x/\tau_1, x') \exp\frac{1}{2\mu}\{\bar{v}[t, s, x; y] - \bar{v}[\tau_1, s, x'; y]\}. \quad (61)$$

In this expression appears a modified diffusion kernel

$$\mathcal{G}[t, x/t', x', s; y] = \mathcal{G}(t, x/t', x') \exp\frac{1}{2\mu}\{\bar{v}[t, s, x; y] - \bar{v}[t', s, x'; y]\}. \quad (62)$$

Its primitive with respect to $x'$

$$\bar{\mathcal{G}}[t, x/t', x', s; y] = \int_{x'}^{+\infty} dx'' \mathcal{G}[t, x/t', x'', s; y] \quad (63)$$

can be written on the form

$$\bar{\mathcal{G}}[t, x/t', x', s; y] = \frac{\int_{x'}^{+\infty} dx'' \mathcal{G}(t, x/t', x'') \exp(-1/2\mu)\bar{v}[t', s, x''; y]}{\int_{-\infty}^{+\infty} dx'' \bar{\mathcal{G}}(t, x/t', x'') \exp(-1/2\mu)\bar{v}[t', s, x''; y]}. \quad (64)$$

We remark that $\bar{\mathcal{G}}$ trivially belongs to [0, 1].

The modified kernel satisfies a condition deduced from (38)

$$\mathcal{G}[t, x/t, x', s; y] = \delta(x - x'). \quad (65)$$

Let us note that

$$\int dx_1 \mathcal{G}[t, x/\tau_1, x_1, s; y] \mathcal{G}[\tau_1, x_1/\tau_2, x_2, s; y] = \mathcal{G}[t, x/\tau_2, x_2, s; y] \quad (66)$$

and

$$\mathcal{G}[t, x/\tau_1, x_1, \tau_2; v[\tau_2, s; y]] = \mathcal{G}[t, x/\tau_1, x_1, s; y]. \quad (67)$$

A relation analogous to (67) holds for $\bar{\mathcal{G}}$.

Introducing (62) in (61), the first coefficient $U_1$ can be written on the form

$$U_1(t, x) = \int_s^t d\tau_1 \mathscr{K}[t, \tau_1, s, x; y]$$
$$= \int_s^t d\tau_1 \frac{\partial}{\partial x} \int dx_1 \, N[\tau_1, x_1; v] \mathscr{G}[t, x/\tau_1, x_1, s; y]. \quad (68)$$

After performing an integration by parts with respect to $x_1$, it reads

$$U_1(t, x) = \hat{U}_1(t, x) + \tilde{U}_1(t, x) \quad (69)$$

where

$$\tilde{U}_1(t, x) = \int_s^t d\tau_1 \tilde{\mathscr{K}}[t, \tau_1, s, x; y]$$
$$= \int_s^t d\tau_1 \frac{\partial}{\partial x} \int dx_1 \mathscr{N}[\tau_1, x_1; v] \mathscr{G}[t, x/\tau_1, x_1, s; y] \quad (70)$$

$\mathscr{N}$ being a primitive of N with respect to $x_1$ already introduced in (52). The expression (70) does not depend on the primitive $\mathscr{N}$ choosen, for

$$\int_s^t d\tau_1 \frac{\partial}{\partial x} \int dx_1 \mathscr{N}[\tau_1, x_0; v] \mathscr{G}[t, x/\tau_1, x_1, s; y]$$
$$= \int_s^t d\tau_1 \mathscr{N}[\tau_1, x_0; v] \frac{\partial}{\partial x} 1 = 0. \quad (71)$$

The term $\hat{U}_1$ is the boundary term

$$\hat{U}_1(t, x) = \left[ \int_s^t d\tau_1 \frac{\partial}{\partial x} \mathscr{N}[\tau_1, x_1; v] \mathscr{G}[t, x/\tau_1, x_1, s; y] \right]_{x_1 = -\infty}^{+\infty}. \quad (72)$$

We can check that $\tilde{U}_1$ satisfies the same linear partial differential equation as $U_1$; this equation is obtained by sustituting $u$ by $v + \lambda U_1$ in (5) and taking the first order in $\lambda$, it reads

$$\frac{\partial}{\partial t} U_1(t, x) + \frac{\partial}{\partial x}[v[t, s, x; y] U_1(t, x)] = \mu \frac{\partial^2}{\partial x^2} U_1(t, x) + N[t, x; v]. \quad (73)$$

Moreover the functions $U_1$ and $\tilde{U}_1$ fulfil the same initial condition

$$U_1(s, x) = \tilde{U}_1(s, x) = 0. \quad (74)$$

Therefore if they exist, these two functions are equal, we say that they are formally equal. In practice we have two expressions for the first coefficient (68) and (70), we have to verify their existence, the one that exists is the coefficient we were looking for; we give an example at the end of the section.

By an analogous calculus we can find the first order in $\lambda$ of the cinetic energy $E[u] \equiv 1/2\, mu^2$ when the velocity $u$ satisfies the perturbed Burgers equation (7); using (31), we have

$$\mathrm{E}[u[t, s, x; y]] = \mathrm{E}[v[t, s, x; y]] + \lambda \int_s^t d\tau_1 \int dx_1 \left\{ \mathrm{N}[\tau_1, x_1; y] \frac{\delta}{\delta z(x_1)} \mathrm{E}[v[t, \tau_1, x; y]] \right\}_{z=v[\tau_1, s; y]} + \dots \quad (75)$$

The term in $\lambda$ can be easily calculated and is equal to the expected result $mv\,\mathrm{U}_1$.

Now, and this is the main advantage of our method, the upper orders are obtained by a calculus quite analogous to the preceding one, that is very easy, though the direct calculation of these terms by using the expressions (51) and (53) is quite difficult. Let us apply the formula (44)

$$\mathscr{K}[t, \tau_1, \tau_2, s, x; y] = \int dx_2 \left\{ \mathrm{N}[\tau_2, x_2; z] \frac{\delta}{\delta z(x_2)} \mathscr{K}[t, \tau_1, \tau_2, x; z] \right\}_{z=v(\tau_2, s; y]} \quad (76)$$

Let us first calculate by using (59), (60), (62) and (63)

$$\left\{ \frac{\delta}{\delta z(x_2)} \mathscr{G}[t, x/\tau_1, x_1, \tau_2, z] \right\}_{z=v[\tau_2, s; y]} = \frac{1}{2\mu} \mathscr{G}[t, x/\tau_1, x_1, s; y] \{ \overline{\mathscr{G}}[t, x/\tau_2, x_2, s; y] - \overline{\mathscr{G}}[\tau_1, x_1/\tau_2, x_2, s; y] \}. \quad (77)$$

The straighforward calculus of (76), using (59), (61), and (77), leads to

$$\begin{aligned}\mathscr{K}[t, \tau_1, \tau_2, s, x; y] &= \frac{1}{2\mu} \frac{\partial}{\partial x} \int dx_1\, dx_2 \int_{x_1}^{+\infty} dx' \mathscr{G}[t, x/\tau_1, x', s; y] \\ &\times \{ \overline{\mathscr{G}}[t, x/\tau_2, x_2, s; y] - \overline{\mathscr{G}}[\tau_1, x'/\tau_2, x_2, s; y] \} \\ &\times \mathrm{N}[\tau_1, x_1; v[\tau_1, s; y]] \times \mathrm{N}[\tau_2, x_2; v[\tau_2, s; y]] \\ &+ \frac{\partial}{\partial x} \int dx_1 \mathscr{G}[t, x/\tau_1, x_1, s; y] \\ &\times \int dx_2 \left\{ \mathrm{N}[\tau_2, x_2; z] \times \frac{\delta}{\delta z(x_2)} \mathrm{N}[\tau_1, x_1; v[\tau_1, \tau_2; z]] \right\}_{z=v[\tau_2, s; y]}.\end{aligned} \quad (78)$$

As for the first order, an integration by parts can be performed on this kernel. It replaces the first term of the right member by the following one, formally equal to it

$$\mathcal{K}_1[t,\tau_1,\tau_2, s, x; y] = \frac{1}{2\mu}\frac{\partial}{\partial x}\int dx_1\, dx_2\, \mathcal{G}[t, x/\tau_1, x_1, s; y] \times \{\mathcal{G}[t, x/\tau_2, x_2, s; y] - \mathcal{G}[\tau_1, x_1/\tau_2, x_2, s; y]\} \times \mathcal{N}[\tau_1, x_1; v[\tau_1, s; y]] \times \mathcal{N}[\tau_2, x_2; v[\tau_2, s; y]]. \quad (79)$$

Let us denote

$$\overline{\tilde{\mathrm{U}}}_1[t, x; \mathcal{N}] = \int_s^t d\tau_1 \int dx_1\, \mathcal{N}[\tau_1, x_1; y]\, \mathcal{G}[t, x/\tau_1, x_1, s; y] \quad (80)$$

the primitive of $\tilde{\mathrm{U}}_1$ in which we specify the functional dependence with respect to $\mathcal{N}$. The contribution of the first term to $\mathrm{U}_2$ is equal to

$$\frac{1}{2\mu}\left\{\frac{\partial}{\partial x}\frac{1}{2}\{\overline{\tilde{\mathrm{U}}}_1[t, x; \mathcal{N}]\}^2 - \tilde{\mathrm{U}}_1[t, x; \mathcal{N}\,\overline{\tilde{\mathrm{U}}}_1]\right\}.$$

The last term is just

$$\mathcal{K}_2[t,\tau_1,\tau_2, s, x; y] = \Psi^*_{\tau_2 s}\mathrm{N}^*(\tau_2)\Psi^*_{\tau_1\tau_2}\mathrm{N}[\tau_1, x_1; y]. \quad (81)$$

It corresponds to the first order kernel of

$$\mathrm{N}[\tau_1, x_1; u] \equiv \Phi^*_{\tau_1 s}\mathrm{N}[\tau_1, x_1; y]$$

as we can see by writing $\Phi^*_{\tau_1 s}$ with the help of the formula (24). This term happens too in the calculation performed from the expression (53) of $\omega$. If we don't use the result of the preceding section to calculate this kernel, we have to do the Taylor expansion of composed functionals and this is not very easy. Therefore, already for the second order kernel we can see the advantage of the method proposed in this paper. For the following terms, the straighforward application of this method leads to calculations that can be performed without difficulties because they use always the same intermediate result (77).

Finally let us test our result in the case where the forced Burgers equation can be explicitely solved, that is in the case where $\mathrm{N}[t, x; y]$ is equal to a time dependent function $\eta(t)$

$$\frac{\partial}{\partial t}u[t, s, x; y] + u[t, s, x; y]\frac{\partial}{\partial x}u[t, s, x; y] = \mu\frac{\partial^2}{\partial x^2}u[t, s, x; y] + \lambda\,\eta(t) \quad (82)$$

Let us consider the Miura transformation [6]

$$\left.\begin{array}{l} t' = t \\ x' = x - \lambda\,\bar{\bar{\eta}}(t) \end{array}\right\} \quad (83)$$

and

$$u[t, s, x; y] = u'[t', s, x'; y] + \lambda\,\bar{\eta}(t') \quad (84)$$

where $\bar{\eta}(t)$ and $\bar{\bar{\eta}}(t)$ are respectively the primitives of $\eta$ and $\bar{\eta}$ vanishing for $t=s$. The substitution of (83) and (84) into (82) shows that $u'$ fulfils the free Burgers equation with the same initial condition as $u$, we therefore have

$$u[t, s, x; y]=v[t, s, x-\lambda\bar{\bar{\eta}}(t); y]+\lambda\bar{\eta}(t). \tag{85}$$

This explicit solution developped with respect to $\lambda$ gives for the two first terms

$$U_0=v[t, s, x; y] \tag{86}$$

and

$$U_1=\bar{\eta}(t)-\bar{\bar{\eta}}(t)\frac{\partial}{\partial x}v[t, s, x; y] \tag{87}$$

Let us check that this last term coincides with (70) when $N[t, x; y]=\eta(t)$. We choose the primitive of N, $\mathcal{N}[t, x; y]$ to be equal to $\eta(t)x$ and we replace it in (70)

$$\tilde{U}_1=\frac{\partial}{\partial x}\int_s^t d\tau_1\int dx_1\,\eta(\tau_1)\,x_1\,\mathcal{G}(t, x/\tau_1, x_1)\frac{\nu[\tau_1, s, x_1; y]}{\nu[t, s, x; y]} \tag{88}$$

After replacing $x_1\,\mathcal{G}(t, x/\tau_1, x_1)$ by $\left\{2\mu(t-\tau_1)\frac{\partial}{\partial x}+x\right\}\mathcal{G}(t, x/\tau_1, x_1)$, and using the equality

$$\nu[t, s, x; y]=\int\mathcal{G}(t, x/\tau_1, x_1)\,\nu[\tau_1, s, x_1; y] \tag{89}$$

we obtain

$$\tilde{U}_1=\frac{\partial}{\partial x}\left[-v[t, s, x; y]\int_s^t d\tau_1\,\eta(\tau_1)(t-\tau_1)+x\int_s^t d\tau_1\,\eta(\tau_1)\right] \tag{90}$$

that is equal to (87). Let us remark that contrarily to $\tilde{U}_1$ that is well defined, the expression (68) giving $U_1$ in this case is ambiguous, it is why we have to choose the expression (70) to test our result. By an analogous calculation, the expression (79) leads to the second order expected result $\frac{1}{2}\bar{\bar{\eta}}(t)\frac{\partial^2}{\partial x^2}v[t, s, x; y]$.

To end let us give the integral equation deduced from (34) for the present example

$$u[t, s, x; y]=v[t, s, x; y]+\lambda\frac{\partial}{\partial x}\int_s^t d\tau \times\frac{\int dx'\,\mathcal{N}[\tau, x'; u]\,\mathcal{G}(t, x/\tau, x')\exp-(\bar{u}[\tau, s, x'; y]/2\mu)}{\int dx'\,\mathcal{G}(t, x/\tau, x')\exp-(\bar{u}[\tau, s, x'; y]/2\mu)} \tag{91}$$

after a integration by parts. In particular, when

$$\mathcal{N}[t, x; y] = -\left(\alpha \frac{\partial}{\partial x} + \mu\right)\frac{\partial}{\partial x} y,$$

this integral equation stands for the solution $u$ of the Korteveg DeVries equation.

The expression (91) does not trivially reduce to the already found equation (53). However by stressing that the equation (53) can be obtained by applying the relation (31) to the functional $Q[u] = \omega = \exp\left(-\frac{\bar{u}}{2\mu}\right)$, we establish that both these equation result from the same equation (31) or (22) and therefore we relate them though it is difficult to directly connect them.

In conclusion, the fact that the perturbation theory developped in this paper furnishes completely explicit expressions for the terms of the perturbation expansion may be of course very useful for the study of the convergence properties of this expansion or for the study of the stability of the solution, for which we have to examine each particular problem.

## REFERENCES


[1] W. F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Academic Press, Inc., Vol. **18**, 1965, p. 208.

[2] I. N. Sneddon, *Elements of Partial Differential Equations*, McGraw-Hill, New York, 1985.

[3] D. J. Kaup, A Perturbation Expansion for the Zakharov-Shabat Inverse Scattering Transform, *Siam J. Appl. Math.*, Vol. **31**, 1, 1976, p. 121; R. M. Miura, The Korteweg-DeVries Equation: A Survey of Results. *Siam. Rev.*, Vol. **18**, 3, 1976, p. 412.

[4] E. Hopf, *Commun. Pure Appl. Math.*, Vol. **3**, 201, 1950; J. D. Cole, *Quart. Appl. Math.*, Vol. **9**, 225, 1951.

[5] M. Irac-Astaud, *Proceedings du XVII^e Colloque International sur la Théorie des Groupes en Physique*, 1988, Sainte Adèle, Québec.

[6] R. M. Miura, *J. Math. Phys.*, Vol. **9**, 1968, p. 1202.


(*Manuscript received July* 6, 1989.)