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# Localisation for the spin J-boson Hamiltonian

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ABSTRACT. — We investigate the phase diagram of the ground state for a spin J coupled linearly to a Bose field. We prove, under suitable infrared conditions, that there exists a critical coupling strength, $\alpha_c(J)$, above which the left-right symmetry of the system is broken: the spin becomes localized. We establish lower and upper bounds on $\alpha_c$ (J). In particular, they imply that $\alpha_c(J=\infty)$ agrees with the critical coupling strength of the semiclassical theory.

RÉSUMÉ. — Nous étudions le diagramme de phase de l'état fondamental pour un spin J couplé linéairement à un champ de Bosons. Nous montrons que sous des conditions infrarouges appropriées, il existe une valeur critique $\alpha_c(J)$ de l'amplitude du couplage au dessus de laquelle la symétrie droite-gauche du système est brisée: le spin devient localisé. Nous donnons des bornes supérieures et inférieures pour $\alpha_c(J)$. Elles impliquent en particulier que $\alpha_c(J=\infty)$ coïncide avec la valeur critique de la théorie semi classique.

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## 1. INTRODUCTION

The spin-boson Hamiltonian models a spin 1/2 coupled to a bosonic field. It is *the* prototypical example of a dissipative quantum system. We refer to [1] for a recent review. The coupling between the spin and the environment may be so strong that the ground state of the system becomes twofold degenerate with a broken left-right symmetry. This phenomenon is necessarily associated with the generation of an infinite number of infrared bosons ([3], [4]). From a quantum mechanical point of view a natural question to ask is what happens if the spin 1/2 is replaced by a spin J. For large J one can use the semiclassical theory ([2], [13]). How does then the quantum regime (small J) link up with the semiclassical regime?

The spin J-boson Hamiltonian reads

$$\mathrm{H} = -\frac{\varepsilon}{\mathrm{J}} \mathrm{S}^x \otimes \mathbf{1} + \mathbf{1} \otimes \int dk\, \omega(k)\, a^*(k)\, a(k) + \frac{\sqrt{\alpha}}{\mathrm{J}} \mathrm{S}^z \otimes \int dk\, \lambda(k)\, [a^*(k) + a(k)] - \frac{h}{\mathrm{J}} \mathrm{S}^z \otimes \mathbf{1}. \quad (1)$$

Here $\mathbf{S} = (\mathrm{S}^x, \mathrm{S}^y, \mathrm{S}^z)$ are the spin J matrices with $[\mathrm{S}^x, \mathrm{S}^y] = i\,\mathrm{S}^x$ plus cyclic permutations and $\mathbf{S}.\mathbf{S} = \mathrm{J}(\mathrm{J}+1)$. $\{ a(k), a^*(k) \,|\, k \in \mathbb{R}^d \}$ are annihilation and creation operators in momentum space of a $d$-dimensional Bose field, $[a(k), a^*(k')] = \delta(k-k')$. Since dimension plays no particular role, we set $d=1$ for simplicity. Our results hold for any dimension. $\omega(k) \geqq 0$ is the dispersion relation of the Bose field and $\lambda(k) = \lambda(k)^*$ are the couplings. For convenience we require

$$\int dk\, \lambda(k)^2 < \infty. \quad (2)$$

$\alpha \geqq 0$ is the coupling parameter. We normalize it by setting

$$\int dk \frac{\lambda(k)^2}{\omega(k)} = \frac{1}{2}. \quad (3)$$

The integral in (3) has to be finite in order to ensure that H is bounded from below.

For $h=0$, H is invariant under the discrete symmetry, $\tau$, defined by

$$\begin{aligned} &\tau\, a(k) = -a(k), \qquad \tau\, a^*(k) = -a^*(k), \\ &\tau\, \mathrm{S}^x = \mathrm{S}^x, \qquad \tau\, \mathrm{S}^y = -\mathrm{S}^y, \qquad \tau\, \mathrm{S}^x = -\mathrm{S}^z. \end{aligned} \quad (4)$$

Clearly $\tau^2 = 1$. We want to understand under what conditions this left-right symmetry is spontaneously broken in the ground state. We approach the problem by means of an order parameter (other, equivalent, possibilities are discussed in [3], [4]), denoted by $m^*$, which may be defined through the following limit procedure: We confine the Bose field to a finite box, $\Lambda$, in physical space and impose periodic boundary conditions. Moreover we introduce a ultraviolet-cutoff $|k| \leqq k_{\max}$. The $k$-integrals in (1) become then finite sums over a momentum lattice, denoted by K. The Hamiltonian with these cutoffs has a unique ground state, denoted by $\Psi_{K, h}$. The order parameter is given by

$$m^* := \lim_{h \searrow 0} \lim_{K \to \mathbb{R}} \langle \Psi_{K, h} | \frac{1}{J} S^z | \Psi_{K, h} \rangle. \tag{5}$$

The order of limits is essential. It is part of our proof that these limits exist. If $m^* = 0$, then H has a unique ground state. The $\tau$ symmetry is unbroken. If $m^* > 0$, the $\tau$-symmetry is spontaneously broken and H has a twofold degenerate ground state. In the following $\varepsilon$ will be kept fixed and we investigate how $m^*$ depends on $\alpha$ and J. Actually, $m^*$ is increasing in $\alpha$. This allows us to define a critical coupling strength, $\alpha_c(J)$, by

$$\begin{aligned} m^* &= 0 \quad \text{for} \quad \alpha < \alpha_c(J), \\ m^* &> 0 \quad \text{for} \quad \alpha > \alpha_c(J). \end{aligned} \tag{6}$$

The two extreme cases, $J = 1/2$ and $J = \infty$, are well understood. For the spin 1/2 case the central quantity is the effective potential

$$W(t) = \int dk\, \lambda(k)^2 e^{-\omega(k)|t|} \tag{7}$$

$\left(\text{note that } W(t) \text{ is bounded because of (2) and } \int dt\, W(t) = 1 \text{ by (3)}\right)$. If

$$\lim_{t \to \infty} t^2 W(t) = 0, \tag{8}$$

then $m^* = 0$ and hence $\alpha_c(1/2) = \infty$. On the other hand, if the limit in (8) is strictly positive (or infinite), then $\alpha_c(1/2) < \infty$. For sufficiently strong coupling the $\tau$-symmetry is broken. At $\alpha = \alpha_c(1/2)$, $m^*$ either vanishes or not, depending on details ([3], [4]).

On the other hand, for large J we can use the result of Lieb [2] who proves that in the limit $J \to \infty$, $\frac{1}{J} S$ becomes a classical variable and the partition function for the Hamiltonian (1) converges to the partition function for the semiclassical Hamiltonian

$$H_{sc} = -\varepsilon \cos(\varphi) \sin(\theta) + \int dk\, \omega(k)\, a^*(k)\, a(k) + \sqrt{\alpha} \cos(\theta) \int dk\, \lambda(k) [a^*(k) + a(k)] - h \cos(\theta), \quad (9)$$

where $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi < 2\pi$. Since now $x$- and $z$-component of the spin commute, the ground state is easily determined. Computing $m^*$ through the limit $h \searrow 0$, one obtaines $\alpha_c(\infty) = \varepsilon$, *independent* of the large $t$ decay of the effective potential $W(t)$, *cf.* Appendix.

The problem posed is the behaviour of $\alpha_c(J)$ inbetween these two extreme cases. To our own surprise, the spin J-boson Hamiltonian interpolates in the simplest possible way: For $h \neq 0$ and general $W(t)$, we have

$$\lim_{J \to \infty} m(J, h) = m_{sc}(h), \quad (10)$$

where $m(J, h)$ is defined in (5) but without the limit $h \searrow 0$ and $m_{sc}(J, h)$ is the corresponding quantity obtained from the semiclassical Hamiltonian (9). If $\alpha > \varepsilon$, then $m_{sc}(h)$ has a jump discontinuity at $h = 0$. For $h = 0$ and if a decay condition slightly faster than in (8) holds, then $\alpha_c(J) = \infty$ for every J. On the other hand if

$$\lim_{t \to \infty} t^2 W(t) > 0, \quad (11)$$

then $\alpha_c(J) < \infty$. Presumably $\alpha_c(J)$ is *decreasing* in J. We will prove the bounds

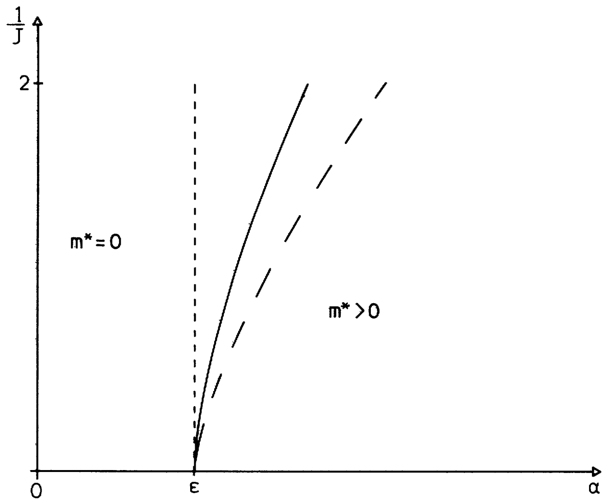
$$\varepsilon = \alpha_c(\infty) \leqq \alpha_c(J) \leqq \alpha_+(J) < \infty. \quad (12)$$

If the limit in (11) is infinite, then

$$\lim_{J \to \infty} \alpha_+(J) = \varepsilon. \quad (13)$$

We expect this property to hold whenever (11) is satisfied. In the following figure we present a schematic phasediagram.

The technique to prove results as (12), (13) is similar to the spin 1/2 case with one extra twist however. For spin 1/2 one exploits a mapping to a ferromagnetic one-dimensional continuum Ising model (spin $\sigma(t) = \pm 1/2$) with pair potential $\alpha W(t)$. $m^*$ becomes then the usual order parameter of spontaneous magnetisation. If the pair potential decays sufficiently slowly, then the Ising model orders and $m^* > 0$. It turns out




—— is the curve $\alpha_c(J)$ with lower (- - -) and upper (— —) bound.

that in the corresponding mapping for the spin J model the spin magnitude J introduces an extra dimension. The continuum model now consists of 2 J coupled Ising-lines. Let $\sigma_j(t)$ be the spin configuration in the $j$-th line, $1 \leqq j \leqq 2J$, $\sigma_j(t) = \pm 1/2$. The energy of the spin configuration in the two dimensional volume $[-\beta/2, \beta/2] \times \{1, 2, \ldots, 2J\}$ is then

$$\frac{1}{J} \sum_{i,j=1}^{2J} \int_{-\beta/2}^{\beta/2} dt \int_{-\beta/2}^{\beta/2} ds \, \alpha \, J W(J \mid t-s \mid) \sigma_i(t) \sigma_j(s). \tag{14}$$

In the $t$-direction the strength of the potential decreases, whereas in the J-direction the coupling is independent of the location of the pair of spins. As it should be, the total energy is extensive, *i. e.* proportional to $\beta J$.

The energy (14) has two mechanisms for ordering. If $W(t)$ decays slowly and if $\alpha$ is sufficiently large, then the spin system orders in the $t$-direction for fixed J. On the other hand, for fixed $\beta$, as $J \to \infty$ the energy (14) is of mean field type and the system must have a mean field phase transition. Note that as $J \to \infty$, $JW(J \mid t-s \mid)$ converges to $\delta(t-s)$ and, *a priori*, it is not quite obvious how the two mechanisms combine.

To give a short outline of the remainder of the paper: In Section 2 we establish the mapping between the spin J boson Hamiltonian and the just mentioned system of 2 J coupled Ising lines. In particular, we relate the order parameter $m^*$ to the spontaneous magnetisation. In Section 3 we prove a lower and in Section 4 an upper bound on the critical coupling strength $\alpha_c(J)$.

## 2. ORDER PARAMETER AND FUNCTIONAL INTEGRAL REPRESENTATION

To define the order parameter we first have to introduce a cutoff Hamiltonian, $H_K$. Let $\Lambda \subset \mathbb{R}$ be an interval of length $|\Lambda|$, the physical volume. We impose periodic boundary conditions. Let K be the set of modes in $\Lambda$ with ultraviolet-cutoff $|k| \leqq k_{\max}$ (if necessary, zero modes are also removed from K). Then the cutoff Hamiltonian is given by

$$H_K = -\frac{\varepsilon}{J} S^x \otimes \mathbf{1} + \mathbf{1} \otimes \sum_{k \in K} \omega_k a_k^* a_k + \frac{\sqrt{\alpha}}{J} S^z \otimes \sum_{k \in K} \lambda_k (a_k^* + a_k) - \frac{h}{J} S^z \otimes \mathbf{1}, \quad (15)$$

with a suitable choice of $\omega_k$ and $\lambda_k$, cf. the proof of Proposition 2. $\{ a_k, a_k^* \mid k \in K \}$ constitute a representation of the CCR. Since $|K| < \infty$, this representation is equivalent to the Schrödinger representation. Therefore $H_K$ can be regarded as a linear operator on $\mathcal{H}_K = \mathbb{C}^{2J+1} \otimes \mathcal{F}_K^S$, where $\mathcal{F}_K^S \cong L^2(\mathbb{R}, d\lambda)^{\vee |K|}$ is the symmetric $|K|$-particle Fock space. Here $\vee$ N, $N \in \mathbb{N}$, denotes N-fold symmetric tensor product.

$H_K$ is a finite particle Hamiltonian generating a positivity improving one parameter semigroup, $e^{-\beta H_K}$, and thus $H_K$ has a unique ground state $\Psi_{K,h} \in \mathcal{H}_K$. We define the order parameter by

$$m(h) := \lim_{K \to \mathbb{R}} \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle, \quad (16)$$

$$m^* := \lim_{h \searrow 0} m(h). \quad (17)$$

We will prove below that the sequence in (16) is monotone increasing and that $m(h)$ decreases monotonically to $m^*$.

We want to express $m(h)$ as an expectation value with respect to a stochastic process on the time interval $[-\beta/2, \beta/2]$ taking values in $\{-J, \ldots, J\}$. For this purpose we construct first the measure generated by $\exp(t \varepsilon S^x/J)$. Here and in what follows we will work in the $S^z$-basis. In this basis the ground state of $S^x$ is given by

$$\Omega_0(m) = \frac{1}{2^J} \binom{2J}{J+m}^{1/2} > 0, \qquad -J \leqq m \leqq J. \quad (18)$$

Let $\Gamma^\beta$ be the set of piecewise constant paths on $[-\beta/2, \beta/2]$ taking values in $\{-J, \ldots, J\}$. Let $S(.)$ be a path in $\Gamma^\beta$ with jumps at $-\frac{\beta}{2}<t_1<\ldots<t_n<\frac{\beta}{2}$ and with the value $S(t)=m_i\in\{-J, \ldots, J\}$ for $t_i \leqq t < t_{i+1}$, $0\leqq i \leqq n$, $t_0=-\beta/2$, $t_{n+1}=\beta/2$. We assign to $S(.)$ the weight

$$\Omega_0(m_0)\,\Omega_0(m_n)\langle m_0|\frac{\varepsilon}{J}S^x|m_1\rangle\times\ldots\times\langle m_{n-1}|\frac{\varepsilon}{J}S^x|m_n\rangle\,dt_1\ldots dt_n, \tag{19}$$

where

$$\langle m|S^x|m'\rangle = \sqrt{J(J+1)-m(m+1)}\,\delta_{m,m'+1}+\sqrt{J(J+1)-m(m-1)}\,\delta_{m,m'-1}$$

are the matrix elements of $S^x$ in the $S^z$-basis. The so defined (unnormalized) measure on $\Gamma^\beta$ is denoted by $d\mu^\beta(S)$.

Let us define an action functional by

$$A_J(S)=-\frac{\alpha}{2J^2}\int_{-\beta/2}^{\beta/2}dt\int_{-\beta/2}^{\beta/2}ds\,W_K(t-s)\,S(t)\,S(s)-\frac{h}{J}\int_{-\beta/2}^{\beta/2}dt\,S(t), \tag{20}$$

where

$$W_K(t)=\frac{2\pi}{|\Lambda|}\sum_{k\in K}\lambda_k^2\,e^{-\omega_k|t|}. \tag{21}$$

This is a Riemann sum with limit

$$W(t):=\lim_{K\to\mathbb{R}}W_K(t)=\int dk\,\lambda(k)^2\,e^{-\omega(k)|t|}, \tag{22}$$

compare with (7). Expectation values with respect to the normalized measure $\frac{1}{Z}\exp[-A_J(S)]\,d\mu^\beta(S)$ are denoted by $\langle\,.\,\rangle_J(\beta, K)$.

PROPOSITION 1. – *Let* $\Psi_{K,h}$ *be the ground state of* $H_K$. *Then*

$$\langle\Psi_{K,h}|\frac{1}{J}S^z|\Psi_{K,h}\rangle=\lim_{\beta\to\infty}\langle\frac{1}{J}S(0)\rangle_J(\beta, K).$$

*Proof.* – Let $H_K^0$ be the Hamiltonian (15) with $\alpha=h=0$. This is the Hamiltonian of a spin J and $|K|$ independent harmonic oscillators. Its ground state, $\Phi_K$, is the product of $\Omega_0$ and $|K|$ harmonic oscillator ground states. Since $s\text{-}\lim_{\beta \to \infty} \exp[-\beta(H_K - E_{K,0})] = \Pr_{\Psi_{K,h}}$, the orthogonal projection on $\Psi_{K,h}$, and since $\langle \Phi_K | \Psi_{K,h} \rangle > 0$ by positivity, we have

$$\lim_{\beta \to \infty} \frac{1}{\| e^{-\beta H_K} \Phi_K \|_2^2} \langle \Phi_K | e^{-\beta H_K} \frac{1}{J} S^z e^{-\beta H_K} | \Phi_K \rangle = \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle. \quad (23)$$

$\langle \Phi_K | e^{-\beta H_K} \frac{1}{J} S^z e^{-\beta H_K} | \Phi_K \rangle$ can be rewritten as a functional integral. The free process is a product of $d\mu^\beta(S)$ and $|K|$ independent Ornstein-Uhlenbeck processes. The action is given by

$$\frac{\sqrt{\alpha}}{J} \int_{-\beta/2}^{\beta/2} dt\, S(t) \sum_{k \in K} \lambda_k q_k(t) - \frac{h}{J} \int_{-\beta/2}^{\beta/2} dt\, S(t), \quad (24)$$

where the $q_k(.)$ are Ornstein-Uhlenbeck paths on the time interval $[-\beta/2, \beta/2]$. The bosonic degrees of freedom can be integrated out, compare with [3, 5]. The net result is

$$\frac{1}{\| e^{-\beta H_K} \Phi_K \|_2^2} \langle \Phi_K | e^{-\beta H_K} \frac{1}{J} S^z e^{-\beta H_K} | \Phi_K \rangle = \langle \frac{1}{J} S(0) \rangle_J(\beta, K) \quad (^1). \quad \square \quad (25)$$

It turns out that the limit $J \to \infty$ can be better controlled in a system of 2 J coupled Ising lines, which we introduce next. As an additional bonus this system makes it easy to prove correlation inequalities. The 2 J coupled Ising lines can be viewed as a quantum version of Griffiths' method of analogue systems, [6].

For $1 \leq j \leq 2J$ let $\sigma_j(.)$ be a piecewise constant path on $[-\beta/2, \beta/2]$ with values $\pm 1/2$. By $d\nu^\beta(\sigma_j)$ we denote $d\mu^\beta(S)$ for $J = 1/2$. In particular, if $\sigma_j(.)$ flips at $-\beta/2 < t_1 < \ldots < t_n < \beta/2$, its weight is $\left(\frac{\varepsilon}{J}\right)^n dt_1 \ldots dt_n$, independent of the initial and final values of $\sigma_j(.)$.

(1) Note that due to our boundary conditions expectation values are taken in the harmonic oscillator ground states rather than over thermal states as in [5] or [3], compare with equation (5.47) in [5].

LEMMA 1. — *Let* $S(t) := \sum_{j=1}^{2J} \sigma_j(t)$. *The weight of* $S(t)$ *under* $\prod_{j=1}^{2J} d\nu^\beta(\sigma_j)$ *equals* $d\mu^\beta(S)$.

*Proof.* — Let $S(t)$ take values $m_i$ in the intervals $[t_i, t_{i+1})$, $0 \leqq i \leqq n$, $t_0 = -\beta/2, t_{n+1} = \beta/2$. Its weight under $\prod_{j=1}^{2J} d\nu^\beta(\sigma_j)$ is of the form $u(m_0) p(m_0, m_1) \ldots p(m_{n-1}, m_n)$, where $u(m_0)$ is the number of ways $m_0$ can be realized and $p(m, m')$ is the number of ways $m'$ can be obtained given $m$, weighted by $\varepsilon/2J$ (the factor 1/2 is the proper normalisation). We have $u(m) = \frac{1}{2^{2J}} \binom{2J}{J+m} = \Omega_0(m)^2$ and

$$p(m, m') = \begin{cases} \frac{\varepsilon}{J}(J-m) & \text{if} \quad m' = m+1 \\ \frac{\varepsilon}{J}(J+m) & \text{if} \quad m' = m-1 \\ 0 & \text{else.} \end{cases}$$

Comparing with (19) the claim follows from

$$\Omega_0(m) p(m, m') \Omega_0(m')^{-1} = \langle m | \frac{\varepsilon}{J} S^x | m' \rangle. \quad \square$$

As a Consequence of Lemma 1 we have

$$\int d\mu^\beta(S) f(S) = \int \left( \prod_{j=1}^{2J} d\nu^\beta(\sigma_j) \right) f(\sigma_1 + \ldots + \sigma_{2J}) \tag{26}$$

for any (bounded) function $f$ on $\Gamma^\beta$.

The 2J coupled Ising lines have $\prod_{j=1}^{2J} d\nu^\beta(\sigma_j)$ as free measure and in terms of the $\sigma_j$ the action (20) reads

$$A(\sigma) = -\frac{\alpha}{2J^2} \int_{-\beta/2}^{\beta/2} dt \int_{-\beta/2}^{\beta/2} ds\, W_K(t-s) \sum_{i,j=1}^{2J} \sigma_i(t) \sigma_j(s) - \frac{h}{J} \int_{-\beta/2}^{\beta/2} dt \sum_{j=1}^{2J} \sigma_j(t), \tag{27}$$

where we use $\sigma$ as a short hand for $(\sigma_1, \ldots, \sigma_{2J})$. Expectations with respect to the normalized measure $\frac{1}{Z} \exp[-A(\sigma)] \prod_{j=1}^{2J} d\nu^\beta(\sigma_j)$ are denoted by $\langle . \rangle (\beta, K)$.

The functional (27) is explicitly ferromagnetic. Also each $d\nu^{\beta}(\sigma)$ can be approximated by discrete Ising spin chains with ferromagnetic interactions, *see* [3]. Therefore the 2 J coupled Ising lines is a *ferromagnetic* spin model.

PROPOSITION 2. — *The limits* (16) *and* (17) *exist and* $m^*$ *agrees with the spontaneous magnetisation of the* 2 J *coupled Ising lines. Furthermore the limits* $\beta \to \infty$ *and* $\mathrm{K} \to \mathbb{R}$ *commute,*

$$m(h) = \lim_{\mathrm{K} \to \mathbb{R}} \lim_{\beta \to \infty} \langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K})$$

$$= \lim_{\beta \to \infty} \lim_{\mathrm{K} \to \mathbb{R}} \langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K}). \quad (28)$$

*Proof.* — By Proposition 1 and Lemma 1,

$$\langle \Psi_{\mathrm{K}, h} | \frac{1}{\mathrm{J}} \mathrm{S}^z | \Psi_{\mathrm{K}, h} \rangle = \lim_{\beta \to \infty} \langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K}).$$

Let us first prove that $\langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K})$ increases monotonically as $\mathrm{K} \to \mathbb{R}$ for all $\beta > 0$.

We choose the discretisation of $\omega(k)$ and $\lambda(k)$ such that $\mathrm{W}_{\mathrm{K}}(t)$ approximates $\mathrm{W}(t)$ monotonically from below for all $t \in \mathbb{R}$. Let $k \in \mathrm{K}$ and $k_1$, $k_2$ be in the closed interval of length $\frac{2\pi}{|\Lambda|}$ with center at $k$ such that $\omega(k_1) \geqq \omega(k')$ and $|\lambda(k_2)| \leqq |\lambda(k')|$ for all $k'$ in the corresponding interval. Let $\omega_k = \omega(k_1)$ and $\lambda_k = \lambda(k_2)$ for all $k \in \mathrm{K}$. Then $\lambda_k^2 e^{-\omega_k |t|} \leqq \lambda(k')^2 e^{-\omega(k')|t|}$ for all $t$. Since (21) is a Riemann sum approximating the integral (22), this choice amounts in approximating the integral monotonically from below as $\mathrm{K} \to \mathbb{R}$ for all $t \in \mathbb{R}$. By Griffiths' second inequality, the same monotonicity property holds then for $\langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K})$ for all $\beta > 0$. Therefore $\langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K})$ is monotone increasing in K also in the limit $\beta \to \infty$ and $m(h)$ is well defined.

The limits $\mathrm{K} \to \mathbb{R}$ and $\beta \to \infty$ commute since $\langle \frac{1}{\mathrm{J}} \sum_{j=1}^{2\mathrm{J}} \sigma_j(0) \rangle (\beta, \mathrm{K})$ increases monotonocally with $\beta$ for all K because each Ising line has free boundary conditions at $t = \pm \beta/2$.

Again by Griffiths' second inequality, $m(h)$ decreases with $h$. Therefore, $m^* = \lim_{h \quad 0} m(h)$ is well defined. It is known that this $m^*$ agrees with the spontaneous magnetisation defined by taking the infinite volume limit with "+" boundary conditions ([3], [4]). □

We have shown that ground state expectations of the spin J-boson Hamiltonian agree with the expectations for the 2 J coupled Ising lines. Because they are ferromagnetic, the "infinite volume" limit $\beta \to \infty$ and the limit $K \to \mathbb{R}$ exist. From now on we study the 2 J coupled Ising lines and adapt our notation accordingly. $\langle . \rangle (\alpha)$ denotes infinite volume expectations, where the brackets indicate the coupling parameter.

Finally we note that $A(\sigma)$ originates in the Euclidean action of a Hamiltonian. Integrating out the bosonic degrees of freedom in a system of 2 J *independent* spins coupled linearly to a harmonic lattice yields the effective action $A(\sigma)$.

## 3. LOWER BOUNDS ON THE CRITICAL COUPLING

Let us differentiate the pair correlation for $h=0$ with respect to $\alpha$. Using the Lebowitz inequality we obtain for the infinite volume expectations

$$\begin{aligned}
\frac{d}{d\alpha}&\langle \sigma_j(0)\,\sigma_k(t)\rangle(\alpha) \\
&= \frac{1}{2J^2}\int ds \int ds'\, W(s-s') \sum_{l,n=1}^{2J} [\langle \sigma_j(0)\,\sigma_k(t)\,\sigma_l(s)\,\sigma_n(s')\rangle(\alpha) \\
&\qquad - \langle \sigma_j(0)\,\sigma_k(t)\rangle(\alpha)\langle \sigma_l(s)\,\sigma_n(s')\rangle(\alpha)] \\
&\leqq \frac{1}{J^2}\int ds \int ds'\, W(s-s') \\
&\qquad \times \sum_{l,n=1}^{2J} \langle \sigma_j(0)\,\sigma_l(s)\rangle(\alpha)\langle \sigma_k(t)\,\sigma_n(s')\rangle(\alpha) \qquad (29)
\end{aligned}$$

for $1 \leqq j, k \leqq 2J$. $\langle \sigma_j(0)\,\sigma_k(t)\rangle(\alpha)$ is bounded by the solution of the differential equation corresponding to (29) with initial condition

$$\langle \sigma_j(0)\,\sigma_k(t)\rangle(\alpha=0) = \int dv^\infty(\sigma_j)\,\sigma_j(0)\,\sigma_j(t)\,\delta_{jk} = \frac{1}{4}e^{-\varepsilon|t|/J}\delta_{jk} \quad ([3], [8]).$$

Thus we have

$$\begin{aligned}
&\langle \sigma_j(0)\,\sigma_k(t)\rangle(\alpha) \\
&\quad \leqq \frac{1}{\sqrt{2\pi}\,2J}\int d\omega\, e^{i\omega t} \sum_{l=1}^{2J} e^{i\pi l(j-k)/J} \frac{\hat{G}(\omega)}{1-(4\pi/J)\,\delta_{l0}\,\alpha\,\hat{W}(\omega)\,\hat{G}(\omega)}, \qquad (30)
\end{aligned}$$

where $\hat{W}(\omega)$ and $\hat{G}(\omega)$ are the Fourier transforms of $W(t)$ and $\frac{1}{4}e^{-\varepsilon|t|/J}$, respectively. (30) is valid as long as $1 > (4\pi/J)\,\alpha\,\hat{W}(\omega)\,\hat{G}(\omega)$ for all $\omega$. Since

$\hat{W}(\omega)$ and $\hat{G}(\omega)$ take their maximum at $\omega=0$, this means

$$1 > \frac{4\pi}{J}\alpha\hat{W}(0)\hat{G}(0) = \frac{\alpha}{\varepsilon}. \tag{31}$$

$\left(\text{Note that } \hat{W}(0) = \int dt\, W(t) = 1 \text{ by (3)}\right)$. As in [3] and [8] we thus arrive at the mean field bound

PROPOSITION 3. — *If* $\alpha < \varepsilon$, *then* $m^* = 0$.

If the interaction decays faster than $t^{-2}$ for $t \to \infty$, we can use the energy-entropy argument of [3] and [7] to prove

PROPOSITION 4. — *Let* $\int dt\, t\, W(t) < \infty$. *Then* $m^* = 0$ *for all* $\varepsilon > 0$, $\alpha \geqq 0$ *and all* J.

## 4. UPPER BOUNDS ON THE CRITICAL COUPLING

We state the main result of our investigation.

THEOREM 1. — *Let* $\lim_{t \to \infty} t^2 W(t) > 0$. *Then for any* $J \geqq 1/2$ *there exists a* $\alpha_+(J)$ *such that*

$$\varepsilon \leqq \alpha_c(J) \leqq \alpha_+(J) < \infty. \tag{32}$$

*Furthermore, if* $\lim_{|t| \to \infty} t^2 W(t) = \infty$, *then*

$$\lim_{J \to \infty} \alpha_+(J) = \varepsilon. \tag{33}$$

The bound $\varepsilon \leqq \alpha_c(J)$ is an obvious consequence of Proposition 3.

Our proof of $\alpha_+(J) < \infty$ and (33) is divided into two steps. We first partition the system into blocks of length $\delta$ and decouple the free measure (this yields a lower bound on $m^*$). The magnetisations per block form then a standard spin model over the one dimensional lattice. Applying Wells' inequality, its magnetisation is bounded below by the magnetisation of a $\pm 1$ Ising spin system — a well understood model [12]. To obtain useful bounds we have to control the *a priori* distribution of the magnetisation in a single block, in particular its behavior for large J. This is carried through in step two. The crucial point there is that for sufficiently large coupling the single block has a mean field phase transition as $J \to \infty$. Therefore the single site measure cannot concentrate at zero as $J \to \infty$.

STEP 1. – We change the time-scale in the action (27) by setting $t' = t/J$. Let $\beta' = \beta/J$, then the free process, $\prod_{j=1}^{2J} d\nu^{\beta'}(\sigma_j)$, refers to paths on the time interval $[-\beta'/2, \beta'/2]$ and the action is given by

$$A(\sigma) = -\frac{\alpha}{2}\frac{1}{J}\int_{-\beta'/2}^{\beta'/2} dt \int_{-\beta'/2}^{\beta'/2} ds\, JW(J\,|\,t-s\,|) \times \sum_{i,j=1}^{2J} \sigma_i(t)\,\sigma_j(s) - h\int_{-\beta'/2}^{\beta'/2} dt \sum_{j=1}^{2J} \sigma_j(t). \quad (34)$$

With the new scale the action is explicitly extensive, *i.e.* proportional to $\beta' J$. (34) has a mean field interaction in the "spatial" direction, $\{-J, \ldots, J\}$. In the time direction, $[-\beta'/2, \beta'/2]$, the interaction strength, $JW(J\,|\,t\,|)$, becomes strong and shortranged as J increases with total (integrated) strength independent of J.

We partition the interval $[-\beta'/2, \beta'/2]$ into intervals of length $\delta$, independent of J. For notational convenience we set $\beta' = N\delta$ with $N \in \mathbb{N}$. For $-N \leqq l, n \leqq N$ let

$$\tilde{W}(t, s) = \tilde{W}(n-l) = \min\left\{ W(t-s) \,\middle|\, \left(l - \frac{1}{2}\right)\delta \leqq s \leqq \left(l + \frac{1}{2}\right)\delta, \left(n - \frac{1}{2}\right)\delta \leqq t \leqq \left(n + \frac{1}{2}\right)\delta \right\}. \quad (35)$$

Then $W(t-s) \geqq \tilde{W}(t, s)$. As in Section 2 let $S(t) = \sum_{j=1}^{2J} \sigma_j(t)$ and define the magnetisation per volume in the block $l$ by

$$M_l = \frac{1}{\delta J}\int_{(l-1/2)\delta}^{(l+1/2)\delta} dt\, S(t). \quad (36)$$

Clearly, $|M_l| \leqq 1$.

By $d\varphi_J(M_l)$ we denote the distribution of $M_l$ under

$$\frac{1}{Z}\exp\left[\frac{\alpha}{2}\int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds\, W(J\,|\,t-s\,|)\, S(t)\, S(s)\right] d\mu^{\delta}(S). \quad (37)$$

Here $d\mu^{\delta}(S)$ is the measure on $\Gamma^{\delta}$ generated by $\exp(\varepsilon\delta S^x)$ with free boundary conditions as defined in Section 2 and Z is the normalisation constant. If obvious from the context we will supress the J dependence of $d\varphi_J$. Let $\langle\,.\,\rangle_{\varphi}(\alpha)$ denote expectations with respect to the normalized measure

$$\frac{1}{Z}\exp\left[\frac{\alpha}{2}\delta^2 J^2 \sum_{l\neq n=-N}^{N} \tilde{W}(J|n-l|)M_l M_n + h\delta J \sum_{l=-N}^{N} M_l\right] \prod_{l=-N}^{N} d\varphi(M_l). \quad (38)$$

Since compared to $\langle . \rangle(\alpha)$ ferromagnetic interactions have been decreased, $m^* \geqq \lim_{h \searrow 0} \langle M_0 \rangle_\varphi (\alpha)$.

To control the width of the single site measure in the limit $J \to \infty$ we use the following property.

PROPOSITION 5. — *For each* $\alpha > \varepsilon$ *there exists a* $v > 0$, *independent of* J, *and a* $\delta_1 > 0$ *such that for all* $\delta > \delta_1$

$$\int d\varphi_J(M_0)\, M_0^2 \geqq v^2. \quad (39)$$

This proposition will be proved in step two.

Let $\langle . \rangle_1(\alpha')$ denote expectations with respect to the normalized Ising measure

$$\frac{1}{Z}\exp\left[\frac{\alpha'}{2} \sum_{l\neq n=-N}^{N} \tilde{W}(J|n-l|)M_l M_n + h' \sum_{l=-N}^{N} M_l\right] \prod_{l=-N}^{N} \frac{1}{2}(\delta_{-1}(M_l)+\delta_1(M_l)). \quad (40)$$

We apply Wells' inequality [3, 9] to (38). By Proposition 5 there exists then a $0 < u \leqq v$ independent of J, such that

$$\langle M_0 \rangle_{\varphi J}(\alpha) \geqq \langle M_0 \rangle_1 (\alpha J^2 \delta^2 u^2). \quad (41)$$

The phase diagram of the Ising model (40) for $N \to \infty$, equivalent $\beta' \to \infty$, with coupling $\alpha' = \alpha J^2 \delta^2 u^2$ is discussed in [12]. If $\lim_{t \to \infty} t^2 W(t) > 0$, then the Ising model orders provided $\alpha'$, equivalently $\alpha$, is large enough. This proves (32). Let us chose an arbitrary $\alpha > \varepsilon$ and let $\lim_{t \to \infty} t^2 W(t) = \infty$. Then the nearest neighbor coupling, $J^2 W(J t)$, diverges as $J \to \infty$. Furthermore, for J sufficiently large,

$$\lim_{n \to \infty} n^2 \alpha \delta^2 J^2 u^2 \tilde{W}(Jn) > 1. \quad (42)$$

Therefore, $\varepsilon < \alpha_+(J) < \alpha$ provided J is large enough. □

STEP 2 (Proof of Proposition 5). — We have to investigate the single block measures $d\varphi_J$ in the limit $J \to \infty$. Substituting $JW(Jt)$ by $\delta(t)$ (which gives a negligible error) we obtain the mean field problem

$$\frac{1}{Z}\exp\left[\frac{\alpha}{2J}\sum_{i\neq j=1}^{2J}\int_{-\delta/2}^{\delta/2} dt\,\sigma_i(t)\,\sigma_j(t)\right]\prod_{j=1}^{2J} d\nu^\delta(\sigma_j). \tag{43}$$

In more familiar cases the single site space consists only of two points, say $\pm 1$. Here we must deal with the *a priori* measure $d\nu^\delta$. Fortunately such general mean field systems have been studied before. In [10] the single site space is a bounded volume in $\mathbb{R}^d$ equipped with the Lebesgue measure. The proof in [10] has to be modified only slightly in order to apply to (43). Before doing so let us explain the main result of [10].

Let $\rho$ be a bounded density relative to $d\nu^\delta$, $0\leqq\rho\leqq a$, with normalisation $\int d\nu^\delta(\sigma)\,\rho(\sigma)=1$. For such a "state" $\rho$ we define the energy

$$E(\rho)=\alpha\int d\nu^\delta(\sigma)\int d\nu^\delta(\sigma')\,\rho(\sigma)\,\rho(\sigma')\int_{-\delta/2}^{\delta/2} dt\,\sigma(t)\,\sigma'(t), \tag{44}$$

the entropy

$$S(\rho)=-\int d\nu^\delta(\sigma)\,\rho(\sigma)\ln\rho(\sigma), \tag{45}$$

and the free energy

$$F(\rho)=E(\rho)-S(\rho). \tag{46}$$

$F(\rho)$ is bounded from below. Let $\mathcal{M}_f$ be the set of $\rho$'s minimizing F.

For each $\rho$ we can build the product measure

$$d\nu_\rho=\prod_{j=1}^{\infty}\rho(\sigma_j)\,d\nu^\delta(\sigma_j). \tag{47}$$

Now let us choose a subsequence $J\to\infty$ such that $\varphi_J$ converges weakly to $\bar\varphi$. Since $\bar\varphi$ must be permutation invariant, the theorem of Hewitt and Savage ensures that $\bar\varphi$ can be decomposed into product measures as

$$\bar\varphi=\int\psi(d\rho;\bar\varphi)\,\nu_\rho. \tag{48}$$

The main result of [10] is that the decomposition measure, $\psi(d\rho;\bar\varphi)$, is concentrated on $\mathcal{M}_f$. In particular, along the chosen subsequence,

$$\lim_{J\to\infty}\int d\varphi_J(M_0)\,M_0^2=\int_{\mathcal{M}_f}\psi(d\rho,\bar\varphi)\int d\nu^\delta(\sigma)\,\rho(\sigma)\left[\frac{1}{\delta}\int_{-\delta/2}^{\delta/2} dt\,\sigma(t)\right]^2. \tag{49}$$

Thus the proof of Proposition 5 is accomplished by studying the minima of the free energy functional (46).

Let us now introduce some notation. We write $\Gamma_{1/2}^\delta$ for $\Gamma^\delta$ if $J=1/2$. Let $\mathcal{S}$ be the set of all probability measures on $(\Gamma_{1/2}^\delta)^{\mathbb{N}}$ which are invariant under permutations. This means, $\mu \in \mathcal{S}$ if $\mu(A_1 \times \ldots \times A_n) = \mu(A_{\pi(1)} \times \ldots \times A_{\pi(n)})$ for any measurable sets $A_1, \ldots, A_n \subset \Gamma^\delta_{1/2}$, all $n \in \mathbb{N}$ and all permutations $\pi$ of $\{1, \ldots, n\}$. Let $\mathcal{S}_a \subset \mathcal{S}$ be the set of all permutation invariant measures $d\mu$ on $(\Gamma^\delta_{1/2})^{\mathbb{N}}$ such that there exist densities $f_k(\sigma_1, \ldots, \sigma_k)$, bounded above by $a^k$ for some $a > 0$, and satisfying

$$d\mu_k(\sigma_1, \ldots, \sigma_k) \equiv d\mu\,|_{(\Gamma^\delta_{1/2})^k} = f_k(\sigma_1, \ldots, \sigma_k)\, d\nu^\delta(\sigma_1) \ldots d\nu^\delta(\sigma_k). \quad (50)$$

LEMMA 2. — *The sequence of measures, $d\varphi_J$, has weak limit points in $\mathcal{S}_a$ as $J \to \infty$. Each limit point, $\bar{\varphi}$, can be decomposed into extremal measures such that the decomposition measure is concentrated on $\mathcal{M}_f$: If $\psi(d\rho; \bar{\varphi})$ denotes the decomposition measure, then $\bar{\varphi} = \int_{\mathcal{M}_f} \psi(d\rho; \bar{\varphi})\, \nu_\rho$.*

*Proof.* — We first prove that the sequence of measures $\varphi_J$ has weak limit points in $\mathcal{S}$. We cannot adopt the argument of [10] since $\Gamma^\delta_{1/2}$ is not compact. Instead we apply results of [11], chapter 4, in particular Proposition 4.7 and Example 1. We have to check that for all $1 \leqq j \leqq 2J$

$$\frac{1}{J}\left|\sum_{i=1}^{2J} \int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds\, JW(J\,|t-s|)\, \sigma_i(t)\, \sigma_j(s)\right| \quad (51)$$

is bounded uniformly in J. This is obvious since (51) is bounded by $\delta/2$ (in the terminology of [11] this means that the interaction is absolutly summable).

The Lipschitz continuity used in [10] is replaced by

$$\left|\int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds\, JW(J\,|t-s|)\, \sigma_i(t)\, \sigma_j(s) - \int_{-\delta/2}^{\delta/2} dt \int_{-\delta/2}^{\delta/2} ds\, JW(J\,|t-s|)\, \sigma'_i(t)\, \sigma'_j(s)\right| \leqq \frac{1}{2}\int_{-\delta/2}^{\delta/2} dt\,[|\sigma_i(t)\,\sigma'_i(t)| + |\sigma_j(t)\,\sigma'_j(t)|]. \quad (52)$$

Here we have used that $xy - x'y' = \frac{1}{2}(x+x')(y-y') + \frac{1}{2}(x-x')(y+y')$.

For $k \leqq 2J$ we set

$$f_k^{2J}(\sigma_1, \ldots, \sigma_k) = \frac{1}{Z}\int d\nu^\delta(\sigma_{k+1}) \ldots \int d\nu^\delta(\sigma_{2J})\, e^{-A(\sigma)}, \quad (53)$$

where Z is the normalisation constant. Let $Z_0 = \int d\nu^\delta(\sigma)$. Then we have

$$0 \leqq f_k^{2J}(\sigma_1, \ldots, \sigma_k) \leqq \left(\frac{e}{Z_0}\right)^k. \quad (54)$$

This replaces the corresponding estimate (2.6) in [10]. Furthermore there exist constants C, $a>0$, independent of J and $k$, such that for all $k \leqq 2J$

$$|f_k^{2J}(\sigma_1, \ldots, \sigma_k) - f_k^{2J}(\sigma'_1, \ldots, \sigma'_k)| \leqq C a^k \sum_{j=1}^{k} \int_{-\delta/2}^{\delta/2} dt\, |\sigma_j(t) - \sigma'_j(t)| \leqq \frac{1}{2} C a^k k \delta. \quad (55)$$

This replaces Lemma 2 in [10].

Along the given subsequence, $f_k^{2J}$ converges weakly to a limit $f_k$ which is the marginal of $\bar{\varphi}$ on the sites $\{1, \ldots, k\}$. The main technical tool in [10] is to make sure that also the entropy of $f_k^{2J}$ converges to the entropy of $f_k$. For this weak convergence is not enough. In [10] the uniform Lipschitz continuity of the densities $f_k^{2J}$ was used. This is substituted here by (55). By the theorem of Arzela Ascoli it implies the existence of pointwise convergent subsequences of $f_k^{2J}$ as $J \to \infty$ on compact sets. Since by weak convergence the limit is unique, $f_k^{2J} \to f_k$ almost surely. Because of (54) this implies the convergence of entropies. The energy of the "state" $\rho$ is given by (44) since $JW(Jt) \to \delta(t)$ as $J \to \infty$. The remainder of the proof is identical to [10]. □

Let us write $\langle . \rangle_\rho$ for expectations with respect to the measure $\rho(\sigma)\, d\nu^\delta(\sigma)$ and let $m(t) = \langle \sigma(t) \rangle_\rho$. $\rho$ is a stationary point of the free energy functional $F(\rho)$ iff

$$\rho(\sigma) = \frac{\exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt\, \sigma(t)\, m(t)\right]}{\int d\nu^\delta(\sigma) \exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt\, \sigma(t)\, m(t)\right]}. \quad (56)$$

Clearly, the weak coupling solution is $\rho_0 = Z_0^{-1} = \left[\int d\nu^\delta(\sigma)\right]^{-1}$ with $m(t) = 0$ for all $t$. To prove Proposition 5 we have to show that there are absolute minima of $F(\rho)$ with $m(t) \neq 0$.

Lemma 3. – For $\alpha < \varepsilon$ there exists a $\delta_0 > 0$ such that for all $\delta > \delta_0$ $\rho_0$ is the unique minimum of $F(\rho)$.

For $\alpha > \varepsilon$ there exists a $\delta_1 > 0$ such that for all $\delta > \delta_1$ $\rho_0$ is an unstable stationary point of $F(\rho)$.

*Proof.* – By inserting (56) into $F(\rho)$ we obtain the functional

$$\tilde{F}(m(.)) = \alpha \int_{-\delta/2}^{\delta/2} dt\, m(t)^2 - \ln \int d\nu^\delta(\sigma) \exp\left[2\alpha \int_{-\delta/2}^{\delta/2} dt\, \sigma(t)\, m(t)\right]. \quad (57)$$

Since the stationary points of $F(\rho)$ and $\tilde{F}(m(.))$ with $m(t) = \langle \sigma(t) \rangle_\rho$ agree, we only have to investigate the absolute minima of $\tilde{F}(m(.))$.

The quadratic variation of $\tilde{F}$ with respect to $m(.)$ at $m(.)=0$ is given by

$$\left.\frac{\delta^2 \tilde{F}}{\delta m(t)\, \delta m(s)}\right|_{m(.)=0} = 2\alpha[\delta(t-s) - 2\alpha\langle\sigma(t)\sigma(s)\rangle_{\rho 0}]. \tag{58}$$

For $J=1/2$ we have $\Omega_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and thus

$$\langle\sigma(t)\sigma(s)\rangle_{\rho 0} = \langle\Omega_0|\sigma^z e^{-\varepsilon|t-s|\sigma^x}\sigma^z|\Omega_0\rangle = \frac{1}{4}e^{-\varepsilon|t-s|}.$$

The Fourier coefficients of $\delta(t) - \frac{\alpha}{2}e^{-\varepsilon|t|}$, $-\delta/2 \leqq t \leqq \delta/2$, are given by

$$\frac{\omega_n^2 + \varepsilon(\varepsilon-\alpha)}{\omega_n^2+\varepsilon^2} + (-1)^{n+1}\frac{\varepsilon\alpha}{\omega_n^2+\varepsilon^2}e^{-\delta\varepsilon/2} \tag{59}$$

with $\omega_n = \pi n/\delta$, $n \in \mathbb{Z}$. (59) is positive for all $\omega_n$ if $\alpha < \varepsilon$ and $\delta$ is large enough. This implies stability.

Uniqueness follows by a contraction argument analogous to the one given in [10] in the proof of Theorem 3. We remark that nonuniqueness also contradicts Proposition 3 because the argument in step 1 would yield $m^* > 0$ for $\alpha < \varepsilon$.

If $\alpha > \varepsilon$, then (59) is negative for $|\omega_n|$ small enough and $\delta$ sufficiently large. From this we conclude that the quadratic variation of $\tilde{F}$ at $m(.)=0$ is not positive definite. □

*Proof of Proposition* 5. — Let $\alpha > \varepsilon$ and let $\bar{\varphi}$ be any weak limit point of $\varphi_J$ as $J \to \infty$. Then, along the given subsequence, $\lim_{J\to\infty} \varphi_J(M_0^2)$ is given by (49). Suppose that $\bar{\varphi}(M_0^2) = 0$. Then $\nu_\rho(M_0^2) = 0$ and hence $\nu_\rho(M_0) = 0$ for almost all $\rho \in \mathcal{M}_f$ which contradicts Lemma 3. Hence $\varphi_J(M_0^2)$ has to be bounded away from zero uniformly in J. □

The proof of Proposition 5 has the

Corollary. — *For* $\alpha > \varepsilon$ *we have*

$$\lim_{h \searrow 0} \lim_{J\to\infty} m(h) > 0, \tag{60}$$

*independent of the choice of* $W(t)$.

## APPENDIX

As an example we explain how to calculate ground state expectations of $\frac{1}{J} S^z$ in the semiclassical limit $J \to \infty$. We introduce the cutoff Hamiltonian corresponding to the semiclassical Hamiltonian (9),

$$H_{sc}^{-} = -\varepsilon \cos \varphi \sin \theta + \sum_{k \in K} \omega_k a_k^* a_k + \sqrt{\alpha} \cos \theta \sum_{k \in K} \lambda_k (a_k^* + a_k) - h \cos \theta, \quad (61)$$

where we suppress the K dependence of $H_{sc}^{-}$ in our notation. This Hamiltonian is defined on $\mathscr{H}_{K, sc} = \mathscr{S}^2 \otimes \mathscr{F}_K^s$, where $\mathscr{S}^2$ is the two sphere. Diagonalizing $H_{sc}^{-}$ one finds that its ground state energy for fixed $\theta$ and $\varphi$ is given by

$$g^{-}(\theta, \varphi) = \varepsilon \cos \varphi \sin \theta - \frac{\alpha}{2} \cos^2 \theta - h \cos \theta. \quad (62)$$

By $H_{sc}^{+}$ we denote the Hamiltonian (61) with all terms except $\sum_{k \in K} \omega_k a_k^* a_k$ multiplied by $(J+1)/J$. Its ground state energy for fixed $\theta$ and $\varphi$ is given by

$$g^{+}(\theta, \varphi) = \varepsilon \frac{J+1}{J} \cos \varphi \sin \theta - \frac{\alpha}{2} \left( \frac{J+1}{J} \right)^2 \cos^2 \theta - h \frac{J+1}{J} \cos \theta. \quad (63)$$

Thus the ground state energies of $H_{sc}^{\pm}$ are determined by

$$e_J^{\pm}(h) = \min_{\theta, \varphi} g^{\pm}(\theta, \varphi). \quad (64)$$

Taking the limit $\beta \to \infty$ in equation (5.4) of [2] yields then the bounds

$$\frac{e_J^{-}(h+\eta) - e_J^{+}(h)}{\eta} \leqq \langle \Psi_{K, h} | \frac{1}{J} S^z | \Psi_{K, h} \rangle \leqq \frac{e_J^{-}(h) - e_J^{+}(h-\eta)}{\eta} \quad (65)$$

for all $\eta \geqq 0$. In (65) one can take the limit $K \to \mathbb{R}$. We have $\lim_{J \to \infty} g^{+}(\theta, \varphi) = g^{-}(\theta, \varphi) \equiv g(\theta, \varphi)$. Thus

$$e(h) := \lim_{J \to \infty} e_J^{\pm}(\theta, \varphi) = \min_{\theta, \varphi} g(\theta, \varphi). \quad (66)$$

Taking the limit $J \to \infty$ and then $\eta \nearrow 0$ for the lower and $\eta \searrow 0$ for the upper bound in (65) yields

$$\frac{d}{dh'} e(h') |_{h'=h^-} \leqq \lim_{J \to \infty} \langle \Psi_{K, h} | \frac{1}{J} S^z | \Psi_{K, h} \rangle \leqq \frac{d}{dh'} e(h') |_{h'=h^+}. \quad (67)$$

If $h \neq 0$, then $g(\theta, \varphi)$ has a unique minimum $(\theta_0(h), \varphi_0(h))$ and (67) yields

$$\lim_{J \to \infty} \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle = \cos\theta_0(h). \tag{68}$$

Let $h=0$. If $\alpha < \varepsilon$, then $g(\theta, \varphi)$ has the unique minimum $(\theta_0, \varphi_0) = \left(\frac{\pi}{2}, \pi\right)$ and

$$\lim_{h \to 0} \lim_{J \to \infty} \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle = \cos\theta_0 = 0.$$

If $\alpha > \varepsilon$, $g(\theta, \varphi)$ has the two minima $(\theta_0^-, \varphi_0^-) = \left(\pi - \arcsin\frac{\alpha}{\varepsilon}, 0\right)$ and $(\theta_0^+, \varphi_0^+) = \left(\arcsin\frac{\alpha}{\varepsilon}, 0\right)$. Then

$$\lim_{h \searrow 0} \lim_{J \to \infty} \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle = \cos\theta_0^- = -\sqrt{1 - \left(\frac{\varepsilon}{\alpha}\right)^2} \tag{70}$$

and

$$\lim_{h \searrow 0} \lim_{J \to \infty} \langle \Psi_{K,h} | \frac{1}{J} S^z | \Psi_{K,h} \rangle = \cos\theta_0^+ = \sqrt{1 - \left(\frac{\varepsilon}{\alpha}\right)^2}. \tag{71}$$

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