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An explicit determination of the Petrov type D space-times on which Weyl's neutrino equation and Maxwell's equations satisfy Huygens' principle

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ABSTRACT. — We show that there exist no Petrov type D space-times on which the Weyl neutrino equation or Maxwell's equations satisfy Huygens' principle. In passing we give a new proof of the same result for the conformally invariant scalar wave equation which does not require the use of the seventh necessary condition.

RÉSUMÉ. — On démontre qu'il n'existe aucun espace-temps de type D de Petrov sur lequel l'équation de neutrino de Weyl ou les équations de Maxwell satisfait au principe d'Huygens. On donne en passant une preuve nouvelle du même résultat pour l'équation invariante conforme des ondes scalaires où on n'emploie pas la septième condition nécessaire.

1. INTRODUCTION

This paper is the fourth in a series devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation, Weyl's neutrino equation and Maxwell's equations. These equations may be written respectively as

$$\Box u + \frac{1}{6} R u = 0, \tag{1.1}$$

$$\nabla^{A\dot{A}} \varphi_A = 0, \tag{1.2}$$

$$\nabla^{A\dot{A}} \varphi_{AB} = 0. \tag{1.3}$$

The conventions and formalisms in this paper are those of Carminati and McLenaghan [3]. All considerations in this paper are entirely local.

According to Hadamard [9], Huygens' principle (in the strict sense) is valid for equation (1.1) if and only if for every Cauchy initial value problem and every $x_0 \in V_4$, the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$ where S denotes the initial surface and $C^-(x_0)$ the past null conoid from x_0. Analogous definitions of the validity of the principle for Weyl's equation (1.2) and Maxwell's equations (1.3) have been given by Wünsch [18] and Günther [7] respectively in terms of appropriate formulations of the initial value problems for these equations. Hadamard's problem for the equations (1.1), (1.2) or (1.3), originally posed only for scalar equations, is that of determining *all* space-times for which Huygens' principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens' principle, the determination may only be effected up to an arbitrary conformal transformation of the metric on V_4

$$\tilde{g}_{ab} = e^{2\varphi} g_{ab}, \tag{1.4}$$

where φ is an arbitrary function.

Huygens' principle is valid for (1.1), (1.2) and (1.3) on any conformally flat space-time and also on any space-time conformally related to the exact plane wave space-time ([6], [11], [19]), the metric of which has the form

$$ds^2 = 2\,dv\,[du + [D(v)\,z^2 + \bar{D}(v)\,\bar{z}^2 + e(v)\,z\bar{z}]\,dv] - 2\,dz\,d\bar{z}, \tag{1.5}$$

in a special co-ordinate system, where D and e are arbitrary functions. These are the only *known* space-times on which Huygens' principle is valid for these equations. Furthermore, it has been shown ([8], [12], [19]) that these are the only conformally empty space-times on which Huygens' principle is valid. In the non-conformally empty case several results are known. In particular for Petrov type N space-times Carminati and McLenaghan ([1], [2]) have proved the following result: *Every Petrov type* N *space-time on which the conformally invariant scalar wave equation* (1.1) *satisfies Huygens' principle is conformally related to an exact plane wave space-time* (1.5). This result together with Günther's [6] solves Hadamard's problem for the conformally invariant scalar wave equation on type N space-times. An analogous result has been proved for the non-self adjoint scalar wave equation on type N space-times by McLenaghan and Walton

[15]. For the case of Petrov type D space-times Carminati and McLenaghan [3] (CM in the sequel) have established the following theorem: *There exists no Petrov type* D *space-times on which the conformally invariant scalar wave equation* (1.1) *satisfies Huygens' principle.* A similar result also holds for space-times of Petrov type III under a certain mild assumption [4].

The purpose of this paper is to extend the result obtained on a type D space-time for the conformally invariant scalar wave equation, to Weyl's equation and Maxwell's equations. The precise result obtained is stated in the following theorem

THEOREM. — *There exist no Petrov type* D *space-times on which the conformally invariant scalar wave equation* (1.1), *Weyl's equation* (1.2) *or Maxwell's equations* (1.3) *satisfy Huygens' principle.*

The proof will proceed by making use of Theorem 1 and Theorem 4 of CM which apply to all three wave equations on a type D space-time. We restate them here for ease of reference:

THEOREM 1. — *The validity of Huygens' principle for the conformally invariant scalar wave equation* (1.1), *Weyl's equation* (1.2), *or Maxwell's equations* (1.3) *on a Petrov type* D *space-time implies that both principal null congruences* (*defined by the null vector fields l and n*) *of the Weyl tensor are geodesic and shear free, that is*

$$l^b \nabla_b l_a = f l_a, \qquad n^b \nabla_b n_a = g n_a, \tag{1.6}$$

$$[\nabla^{(b} l^{a)}] \nabla_b l_a = \frac{1}{2} [\nabla^a l_a]^2, \qquad [\nabla^{(b} n^{a)}] \nabla_b n_a = \frac{1}{2} [\nabla^a n_a]^2. \tag{1.7}$$

THEOREM 4. — *There are no space-times of Petrov type* D *where both principal null congruences of the Weyl tensor are hypersurface orthogonal, on which the conformally invariant scalar wave equation* (1.1), *Weyl's equation* (1.2) *or Maxwell's equations* (1.3) *satisfy Huygens' principle.*

A space-time of Petrov type D where both principal null congruences of the Weyl tensor are hypersurface orthogonal, satisfies

$$l_{[a} \nabla_b l_{c]} = 0, \qquad n_{[a} \nabla_b n_{c]} = 0. \tag{1.8}$$

In the proof of the Theorem only the third and fifth necessary conditions for the validity of Huygens' principle will be used in contrast to the earlier result obtained for the scalar wave equation in CM which made use of the seventh necessary condition. This is the new feature in the proof of the scalar wave equation result. The absence of a requirement for the seventh necessary condition is also important in obtaining the results for Weyl's equation and Maxwell's equations since the condition has not been computed for either of these equations, being only available for the conformally invariant scalar wave equation [16].

2. PROOF OF THEOREM

The spinor form of the third and fifth necessary conditions (III's and V's) are [5], [12], [13], [14], [17]and [18]

$$\nabla^{K}{}_{\dot{A}}\nabla^{L}{}_{\dot{B}}\Psi_{ABKL}+\nabla_{A}{}^{\dot{K}}\nabla_{B}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}+\Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL}+\bar{\Phi}^{\dot{K}\dot{L}}{}_{AB}\bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}=0, \quad (2.1)$$

$$\begin{aligned}
& k_1[\nabla^{K\dot{K}}\Psi_{ABCD}][\nabla_{K\dot{K}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}] \\
&\quad +k_2[\nabla_{(D(\dot{D}}\Psi^{K}{}_{ABC)}][\nabla_{K}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}]+k_2[\nabla_{(D(\dot{D}}\bar{\Psi}^{\dot{K}}{}_{\dot{A}\dot{B}\dot{C})}][\nabla^{L}{}_{\dot{K}}\Psi_{ABC)L}] \\
&\quad -2(8k_1-k_2)[\nabla^{K}{}_{(\dot{D}}\Psi_{K(ABC}][\nabla_{D)}{}^{\dot{K}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{K}}] \\
&\quad -k_2\Psi^{K}{}_{(ABC}\nabla_{D)(\dot{D}}\nabla_{K}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}-k_2\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\nabla_{(D\dot{D})}\nabla^{L}{}_{\dot{K}}\Psi_{ABC)L} \\
&\quad +4k_1\Psi^{K}{}_{(ABC}\nabla_{|K|(\dot{D}}\nabla_{D)}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})\dot{L}}+4k_1\bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\nabla_{(D|\dot{K}|}\nabla^{L}{}_{\dot{D})}\Psi_{ABC)L} \\
&\quad +2(k_2-4k_1)\Psi^{K}{}_{(ABC}\Phi_{D)K\dot{K}(\dot{D}}\bar{\Psi}_{\dot{A}\dot{B}\dot{C})}{}^{\dot{K}} \\
&\quad -2(4k_1+k_2)\Lambda\Psi_{ABCD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}=0. \qquad (2.2)
\end{aligned}$$

where k_1 takes the values 3, 8, 5 and k_2 the values 4, 13, 16 depending on whether the equation under consideration is the conformally invariant scalar wave equation, Weyl's equation or Maxwell's equations respectively.

A type D space-time is characterized by the Weyl spinor having a pair of two-fold principal spinors, that is there exist two linearly independent spinors o_A and ι_A satisfying $o_A\iota^A=1$ such that

$$\Psi_{ABCD}=6\Psi\, o_{(A}o_B\iota_C\iota_{D)} \qquad (2.3)$$

If $\{o_A,\iota_A\}$ is chosen as the spin frame, it follows immediately that the only non-vanishing component of the Weyl spinor is $\Psi_2=\Psi$.

The conformal invariance of Huygens' principle is particularly useful in that we can employ a conformal transformation to simplify the form of the Weyl spinor. In particular the conformal transformation (1.4) induces the transformation

$$\tilde{\Psi}=e^{-2\varphi}\Psi. \qquad (2.4)$$

Before proceeding with the proof we will obtain a stronger form of condition III's. From the Ricci identity we have

$$[\nabla_{A}{}^{\dot{K}},\nabla^{K}{}_{\dot{A}}]\Phi_{BK\dot{B}\dot{K}}=\Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL}-\bar{\Phi}^{\dot{K}\dot{L}}{}_{AB}\Psi_{\dot{A}\dot{B}\dot{K}\dot{L}}. \qquad (2.5)$$

Applying the Bianchi identity

$$\nabla_{A}{}^{\dot{K}}\Phi_{BC\dot{A}\dot{K}}=\nabla^{K}{}_{\dot{A}}\Psi_{ABCK}+2\varepsilon_{A(B}\nabla_{C)\dot{A}}\Lambda, \qquad (2.6)$$

to the left hand side of (2.5) gives [10]

$$-\nabla^{K}{}_{\dot{A}}\nabla^{L}{}_{\dot{B}}\Psi_{ABKL}+\nabla_{A}{}^{\dot{K}}\nabla_{B}{}^{\dot{L}}\bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}=\Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL}-\bar{\Phi}^{\dot{K}\dot{L}}{}_{AB}\bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}. \qquad (2.7)$$

Together with condition III's (2.1) this implies that

$$\nabla^{K}{}_{\dot{A}}\nabla^{L}{}_{\dot{B}}\Psi_{ABKL}+\Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL}=0 \qquad (2.8)$$

For the remainder of this paper we will refer to this equation as condition III's. In order to refer to the individual components of conditions III's and V's we shall subscript the roman numerals III and V in a manner analogous to that used to refer to the components of the Ricci spinor.

The result of Carminati and McLenaghan (1.6), (1.7) implies that in the canonically chosen spin frame we have

$$\kappa=\nu=\sigma=\lambda=0. \tag{2.9}$$

Taking the condition III_{00} and removing the term in $D\rho$ using the NP equations we have

$$D^2\Psi-(\varepsilon+\bar{\varepsilon}+6\rho)D\Psi+2(3\rho^2-\Phi_{00})\Psi=0. \tag{2.10}$$

If we use the conformal transformation (2.4) to set

$$\Psi\bar{\Psi}=1 \tag{2.11}$$

we obtain on differentiating

$$\Psi D\bar{\Psi}+\bar{\Psi}D\Psi=0, \tag{2.12}$$

and

$$\bar{\Psi}D^2\Psi+\Psi D^2\bar{\Psi}+2(D\Psi)(D\bar{\Psi})=0. \tag{2.13}$$

We may eliminate the second derivative terms appearing in equation (2.13) using equation (2.10) and it's complex conjugate. Simplifying the result with (2.11) and (2.12), we obtain

$$(D\Psi)(D\bar{\Psi})+3(\rho-\bar{\rho})\bar{\Psi}D\Psi+2\Phi_{00}-3\rho^2-3\bar{\rho}^2=0. \tag{2.14}$$

We now invoke the condition V_{11}, in view of (2.9) it reads

$$\begin{aligned}(4k_1-k_2)[&\bar{\Psi}D^2\Psi+\Psi D^2\bar{\Psi}-(\varepsilon+\bar{\varepsilon}+3\rho+3\bar{\rho})(\Psi D\bar{\Psi}+\bar{\Psi}D\Psi)\\&+(\rho-\bar{\rho})(\bar{\Psi}D\Psi-\Psi D\bar{\Psi})+2\Phi_{00}\Psi\bar{\Psi}\\&-3\Psi\bar{\Psi}((D-\varepsilon-\bar{\varepsilon}+\rho-4\bar{\rho})\rho+(D-\varepsilon-\bar{\varepsilon}+\bar{\rho}-4\rho)\bar{\rho})]\\&+16k_1[(D\Psi)(D\bar{\Psi})-2\bar{\rho}\bar{\Psi}D\Psi-2\rho\Psi D\bar{\Psi}+4\rho\bar{\rho}\Psi\bar{\Psi}]=0.\end{aligned} \tag{2.15}$$

Removing the second derivative terms in the above using equation (2.10) and it's complex conjugate and using as before equations (2.11) and (2.12) we obtain with the aid of the NP equations

$$\begin{aligned}4k_1(D\Psi)(D\bar{\Psi})+2(8k_1-k_2)(\rho-\bar{\rho})\bar{\Psi}D\Psi&\\-3(4k_1-k_2)(\rho-\bar{\rho})^2+16k_1\rho\bar{\rho}&=0.\end{aligned} \tag{2.16}$$

Elimination of the term in $(D\Psi)(D\bar{\Psi})$ between equations (2.14) and (2.16) gives

$$(2k_1-k_2)(\rho-\bar{\rho})\bar{\Psi}D\Psi+20k_1\rho\bar{\rho}+\frac{3}{2}k_2(\rho-\bar{\rho})^2-4k_1\Phi_{00}=0. \tag{2.17}$$

Under the assumption that $\rho\neq\bar{\rho}$, we may solve for $D\Psi$ in the above obtaining

$$D\Psi=\frac{(8k_1\Phi_{00}-3k_2(\rho-\bar{\rho})^2-40k_1\rho\bar{\rho})\Psi}{2(2k_1-k_2)(\rho-\bar{\rho})}. \tag{2.18}$$

Returning to equation (2.16) we now substitute for $D\Psi$ and $D\bar{\Psi}$. Collecting the terms in Φ_{00} on one side of the equation and then completing the square we find

$$[16 k_1^2 \Phi_{00} - (16 k_1^2 - 4 k_1 k_2 + k_2^2)(\rho + \bar{\rho})^2 - 4(4 k_1^2 + 4 k_1 k_2 - k_2^2)\rho\bar{\rho}]^2 \\ = (2 k_1 - k_2)^2 (\rho - \bar{\rho})^2 [(16 k_1^2 - 4 k_1 k_2 + k_2^2) \times (\rho + \bar{\rho})^2 \\ + 4 k_2 (4 k_1 - k_2)\rho\bar{\rho}]. \quad (2.19)$$

We first note that if $\rho \neq \bar{\rho}$, then $\rho \neq 0$, and therefore the right hand side of the above equation is real and strictly negative provided $4 k_1 > k_2 > 0$ and $2 k_1 \neq k_2$, which is the case for all three of the wave equations under consideration. However, the left hand side of the above equation is clearly real and positive, we therefore must have

$$\rho = \bar{\rho}, \quad (2.20)$$

in contradiction to our assumption that $\rho \neq \bar{\rho}$. Placing $\rho = \bar{\rho}$ in equation (2.16) we immediately obtain

$$(D\Psi)(D\bar{\Psi}) + 4\rho^2 = 0. \quad (2.21)$$

Both the terms appearing here are real and positive and so we must have

$$\rho = 0, \quad (2.22)$$

$$D\Psi = 0. \quad (2.23)$$

By an exactly similar procedure one can show that

$$\mu = 0, \quad (2.24)$$

$$\Delta\Psi = 0. \quad (2.25)$$

The equations (2.22) and (2.24) imply that the two principal null congruences of the Weyl tensor are hypersurface orthogonal. It follows that the conditions of Theorem 4 of CM are satisfied and hence the Theorem is proved.

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Note added in proof: After this paper was completed the authors became aware of a related work by V. Münsch [20].

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