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Characteristic initial value problem for hyperbolic systems of second order differential equations

by

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ABSTRACT. — A quasilinear hyperbolic system of second order differential equations, each having the same principal part $g^{ab}\,\partial_a\,\partial_b$ (where g^{ab} is indefinite) is considered. For example Einstein's vacuum field equation (in harmonic coordinates) are of this type. Initial data are given on two intersecting *null* (*i.e.* characteristic) hypersurfaces. At first an *existence theorem* for the corresponding linear case is proven. This theorem is so *strong* that it allowed to set up an *iteration* (for the *quasilinear* case) with the following properties:

There will be *no loss of differentiability* orders (in the sense of Sobolev spaces), when one proceeds from one iteration step to the next one. Moreover, within each iteration step the solution fulfils an energy inequality, whose energy inequality "constant" remains (¹) unchanged when one makes the domain, on which the solution is considered, as *small* as one likes. (This domain has to be made sufficiently small in order to prove a local existence theorem for the quasilinear case.) Furthermore the energy inequality "constant" is *stable* against small variations of the coefficients (those variations are induced by the iteration).

In order to obtain a solution for the quasilinear case it remains to be shown that the iteration converges; this convergence will be shown in a forth-coming paper.

(¹) At first one shows that the energy inequality "constant" c remains bounded (*see* Theorem 4.2). Then one replaces c by $\bar{c}$, where $\bar{c}$ is the upper bound of c.

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In order to obtain a s-times differentiable solution (in the sense of Sobolev spaces) one has to assume that the data are $(2\,s-1)$-times differentiable (with sufficiently large s) and that certain assumptions on the coefficients hold. This means that there is a *gap* of differentiability orders between the solution (s-times differentiable) and the data [$(2\,s-1)$-times differentiable]. It will be *proven*, that – in the generic case – this *gap cannot be reduced* by more than one half of differentiability order (in the sense of Sobolev spaces of fractional orders of differentiability). – This paper is an extension and improvement of the previous paper [3] of Müller zum Hagen and Seifert.

RÉSUMÉ. – Prenons en considération un système quasi linéaire hyperbolique d'équations différentielles du second ordre, dans lequel chaque équation a la même partie principale $g^{ab}\,\partial_a\,\partial_b$ (g^{ab} étant indéfini). Les équations d'Einstein sont, par exemple, de ce type. Nous donnerons des valeurs initiales aux deux hypersurfaces *caractéristiques* sécantes. Il est d'abord possible de faire la démonstration d'un *théorème d'existence* pour les cas linéaires associés. Le théorème est si *fort* qu'il permet une itération (au cas quasi linéaire) avec les propriétés suivantes :

Il *n'y a pas de perte d'ordre différentiel* (dans le sens des espaces de Sobolev) quand on passe d'un pas d'itération à un autre. A l'intérieur d'un pas d'itération cette solution satisfait à une inégalité d'énergie. Dans un tel cas la constante de l'inégalité énergétique ne change (²) pas si *l'on réduit, arbitrairement le domaine* pour lequel la solution est envisagée (on doit suffisamment réduire ce domaine, afin que l'on puisse démontrer un théorème d'existence locale dans les cas quasi linéaires). De plus, en ce qui concerne de faibles variations des coefficientes la constante de l'inégalité énergétique reste *stable* (de telles variations sont engendrées par l'itération).

Pour obtenir une solution du cas quasi linéaire, il est nécessaire de démontrer la convergence de l'itération; nous démontrerons cette convergence dans une publication prochaine.

Pour obtenir une solution s-fois différentiable (au sens des espaces de Sobolev), on doit supposer que les valeurs initiales sont differentiables $(2\,s-1)$-fois (avec un s suffisamment grand) et que certaines conditions sont remplies en ce qui concerne les coefficients. Ce qui signifie qu'il existe une lacune entre les ordres de différentiation de la solution (s-fois différentiable) et celle des valeurs initiales [$(2\,s-1)$-fois différentiable]. Nous *ferons la démonstration* suivante : dans les cas génériques *cette lacule*

(²) Dans un premier temps ont montrera que la constante de l'inégalité énergétique reste une borne (*voir* théorème 4.2). Ensuite on remplacera c par $\bar{c}$ où $\bar{c}$ sera la borne supérieure de c.

ne peut être réduite de plus d'un demi-ordre de différentiation (dans le sens des espaces de Sobolev avec ordre fractionnaire de différentiation). – Cette publication est une extension des travaux précédents publiés [3] par Müller zum Hagen et Seifert.

CONTENTS

I. INTRODUCTION

I.1. The problem

The following hyperbolic – *i. e.* g^{ab} is indefinite – system of differential equations

$$\sum_{a,\,b} g^{ab}\frac{\partial^2 u^{\mathrm{A}}}{\partial x^a\,\partial x^b}+\sum_{a,\,\mathrm{B}} b^{a\,\mathrm{A}}_{\mathrm{B}}\frac{\partial u^{\mathrm{B}}}{\partial x^a}+\sum_{\mathrm{B}} a^{\mathrm{A}}_{\mathrm{B}}u^{\mathrm{B}}=f^{\mathrm{A}}\quad\text{on } \mathrm{L}_T\subset\mathbb{R}^{n+1}\tag{I.1}$$

(a, b runs from $1, \ldots, n+1$; A, B runs from $1, \ldots, \mathrm{N}$)

is considered for the unknown $u := (u^{\mathrm{A}}) := (u^1, \ldots, u^{\mathrm{N}})$; moreover, the data $\underset{\omega}{u}^{\mathrm{A}}$

$$u^{\mathrm{A}}=\underset{\omega}{u}^{\mathrm{A}}\quad\text{on } \mathrm{G}^{\omega}\qquad(\omega=1,\,2),\tag{I.2}$$

are given on the two intersecting *null* (*i. e.* characteristic) hypersurfaces G^1, G^2; the set $\mathrm{G}^1\cap\mathrm{G}^2$ is a spacelike $(n-1)$-dimensional surface; furthermore $\mathrm{G}^1\cup\mathrm{G}^2$ is part of the boundary of L_{T} [*see* also Figure 1 and formula (1.3)]. We assume that L_{T} is compact.

One distinguishes the following two cases:

linear case: the coefficients g^{ab}, $b^{a\,\mathrm{A}}_{\mathrm{B}}$, $a^{\mathrm{A}}_{\mathrm{B}}$, f^{A} are functions of (x^a);

quasilinear case: g^{ab} is a function (u^{A}), f^{B} is a function of $\left(u^{\mathrm{A}},\dfrac{\partial u^{\mathrm{A}}}{\partial x^a}\right)$, and $b^{a\,\mathrm{A}}_{\mathrm{B}}=a^{\mathrm{A}}_{\mathrm{B}}=0$. *Einstein's vaccum field equation of gravity* are of this type (in harmonic coordinates).

Einstein's vaccum field equation and tis characteristic data have been rewritten bei Seifert and Müller zum Hagen in [3] in such a way that the equations (1), (2) can be used; this rewriting was done by using harmonic coordinates (*see* Dautcourt [24]) on L_{T}. This rewriting can also be done by other methods, which have been mentioned in a survey given in [3], or by using in L_{T} "source function" (*see* Friedrich [25]) and using the characteristic data (on G^{ω}) mentioned by Friedrich in [26].

It has been shown in [3] (in the introduction) that the characteristic initial value problem for Enstein's field equations can be *applied* to many interesting problems (3) of Einstein's theory of gravitation. Some further

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(3) A cosmological problem has also been mentioned in [3], where one should add that essentially all observations, which we can make of the universe, lie on the past *null cone;* this has been shown by Ellis *et al.* in [17].

applications have been made to the following problems:

(*a*) In order to find a large class of metrics, which *solve Einstein's vaccum vaccum equations* and have a prescribed *asymptotic* (4) behaviour, Friedrich reduces (*see* [18], [19]) the *global* existence theorem problem to a local one, namely to the local characteristic initial value problem. He treasts the so-called *"asymptotic characteristic initial value problem"*.

(*b*) A characterization of *"purely radiative space-times"* has been given by using the methods of (*a*), which has been shown in [19].

(*c*) *Numerical solutions of Einstein's field equations* have been calculated in [20] by solving the characteristic initial value problem, because this is simpler than solving (numerically) the Cauchy problem. This is so, because of the following reason: in the characteristic problem as well as the Cauchy problem one has at first to solve certain equations on the data-surface; these equations are ordinary differential equations in the case of the characteristic problem, whereas they are elliptic partial differential equations (the constraints equations) in the case of the Cauchy problem; these *elliptic* equations are much more difficult to solve than those ordinary differential equations.

The above remarks show, where the characteristic initial value problem has been applied (and will be applied) and why one should try to extend the theorems of [3]. (The work on the above problems of General Relativity is by no means finished.)

I.2. Aims and results

As far as the style is concerned this paper will follow the previous paper [3], which in turn followed Hawking and Ellis [9] (the chapter on the Cauchy initial value problem of Einstein's equation). So, the intention is that this paper is selfcontained and could be understood without reading a lot of other papers. The proofs will be written down so explicitly and constructively that generalizations are accessible.

This paper is an extension (*see* next paragraph) of [3] by Müller zum Hagen and Seifert and it fills a gap of the proof of [3]; this gap (5) has been filled by Christodoulou and Müller zum Hagen in [4], which was a short paper in which an existence theorem (6) for the linear as well as the

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(4) In [19] solutions are treated possessing a smooth structure at past *null infinity* (which forms the "future null cone at past timelike infinity with complete generators"). In [18] solutions are treated which possess a smooth structure at a part of past *null infinity*.

(5) This gap has been filled by deriving the so-called "stable-boundedness property" (*see* below in this subsection).

(6) The theorem of [4] is – apart from a slight modification – the same as in this paper. The norm $\| \cdot \|_{E_s' (L_T)}$ of [4] is equal to $\| \cdot \|_{L_T, s}$.

quasilinear case has been stated; however, there has been given in [4] a brief sketch of the proof, only. So the present paper will give a (simplified version) of the *proof* for the linear case. Moreover, the theorem to be proven will be so *strong* that the following two properties hold:

(I) *Iteration property:* in section 8 we shall set up an iteration for the *quasilinear* case. There will be *no loss of differentiability* orders (in the sense of Sobolev spaces) when one proceeds from one iteration step to the next one. Moreover, the solution of each iteration step fulfils an energy inequality, whereby the energy inequality constant remains *bounded* thoughout the whole of the iteration (*see* Remark B of section 8).

(II) *Stable-boundedness property* (7): The solutions of our linear equation fulfils an energy inequality, whereby the energy inequality constant c remains *bounded*, when one makes the domain D, on which the solution is considered, as small as one likes (*see* Theorem 6.2 and 8.1); in the quasilinear case *one has to make the domain D sufficiently small* (due to the possible occurrance of singularities). Moreover, the bounds for c are strictly positive and *stable* against small variations of the coefficients of our linear equation (*see* Theorem 6.2 and 8.1). Such variations occur during the iteration mentioned in (I).

Using (8) the above properties (I), (II) one can prove that the iteration [mentioned in (I)] converges against a *solution of the quasilinear* characteristic initial value problem (*see* Remark B of section 8). The details of the convergence proof will be given in a forthcoming paper.

Our existence theorem (for the linear case) extends the results of [3] into three directions:

(A) the existence theorem will be *stronger* than the existence theorem of [3] (*see* Remark A of section 8);

(B) the above *stable-boundedness* property [*see* (II)] is satisfied;

(C) our treatment can be *generalized* (*see* Remark 4.3 of section 4) to other kinds of characteristic initial value problems such as the following *"cone problem"*: $u=(u^A)$ fulfils (1) and assumes given data $\underset{0}{u}^A$ on G:

$$u^A=\underset{0}{u}^A \quad \text{on G}, \tag{I.3}$$

where G is the null (*i.e.* characteristic) cone at some point x_0. [G is generated by the (future-directed) null geodesics emanating from the point x_0].

In the following the differentiability order is considered in the sense of some kind of Sobolev classes (*see* section 5 and 7). As will be pointed

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(7) This property will also be discussed in I.5 of the introduction.

(8) One also uses the ball property (*see* Remark 3.1 of section 3).

out in section 6 and 8 the solution of the characteristic initial value problem (1), (2) is s-times differentiable, where $s > \frac{n}{2} + 2$ [with n of (1)]. However, it is most likely that this bound for s can be lowered [as has been done in a sequence of papers of Choquet *et al.* (e. g. [2])]. *Low differentiability* orders are important for the treatment of *shocks* or *singular solutions* (by similar methods as in [14]).

It one wants of obtain a s-times differentiable solution (*see* section 6 and 8), then one has to assume that the *data are* $(2\,s-1)$-*times dfferentiable* and that certain assumptions hold on g, b, a, f of (1) (in the linear as well as the quasilinear case). This means that there is a *gap of differentiability class* between the solution (s-times differentiable) and the data [$(2\,s-1)$-times differentiable]. It will be proven that this gap *cannot* be reduced by one differentiability order [in fact this gap *cannot* be reduced by more than half a differentiability order (*see* Remark C of section 6].

Furthermore the case $s = \infty$ is included; in this case data and solutions are infinitely often differentiable (in the usual sense).

I.3. Related papers

A survey on related papers has been given in [3]. Furthermore Cagnac [23] considers a quasilinear hyperbolic system of the type

$$\sum_{b,\,c} g_{bc}(x^a, u^A) \frac{\partial^2 u^B}{\partial x^b\, \partial x^c} + f^B\left(x^a, u^A, \frac{\partial u^A}{\partial x^a}\right) = 0 \qquad \text{(I.3 A)}$$

(a, b, c runs from $1, \ldots, n+1$; A, B runs from $1, \ldots,$ N) and poses data on a characteristic cone. He solves this characteristic initial value problem; however, he makes – apart from some differentialbility assumptions on the data – the following *supplementary assumption:* the data and the functions obtained from these data by solution of the propagation equations [*c.f.* (4.40), (4.44)] are such that they fulfil (3 A) (up to some order) at the tip of the cone. Rendall [15] in turn shows for the C^∞-case (by using a quite different method) that this *supplementary assumption is not necessary* for the following type of the single quasilinear hyperbolic equation

$$\sum_{b,\,c} g^{bc}(x^a, \varphi) \frac{\partial^2 \varphi}{\partial x^b\, \partial x^c} + \sum_b B^b(x^a, \varphi) \frac{\partial \varphi}{\partial x^b} + f(x^a, \varphi) = 0;$$

moreover, he conjectures that this supplementary assumption is also not necessary for the equations (3 A).

Furthermore the paper [15] of Rendall has to be mentioned. In [15] he transforms the characteristic initial value problem into a Cauchy initial value problem. However, he proves the transformation to be equivalent for the C^∞-*case only*. Furthermore – as it is discussed in [15] – there is *no proof that the transformation would be equivalent in the case of finite differentiability orders*, since in [15] occurs (after transformation) a loss of differentiability orders. Moreover, the proof in [15] (for the finite differentiability order case) *cannot in principle* be strenthend is such a way that a loss of differentiability orders would be avoided; this will be shown in Remark D of section 6. This loss of differentiability order will become relevant when one wants to treat those singular solution (e. g. shocks), which have been mentioned in the last subsection.

Dossa (9) derives in [22] an *energy inequality* for the cone problem; however, he remarks (Remark 3.3.5 of [22]) that his energy inequality does *not* fulfil the "stable-boundedness property" (mentioned in the last subsection).

I.4. The tools

Apart from using the tools of [3], we use (generalized) Sobolev norms with *weight factors* [*see* (6) below], which have been first introduced by Christodoulou and Müller zum Hagen in [4]. These norms are constructed in such a way that they induce the *stable-boundedness* property (of the last but one subsection); moreover, the corresponding function spaces of these norms are Banach algebras (or Hilbert algebras, respectively), which have the "right" denseness (10) properties.

The introduction of those weight factors has been made possible by deriving the *energy inequality* in a way, which is a generalization of the way used in [3] (*see* also Remark 4.1 of section 4).

In order to give a *typical example of a weight factor* one first defines the following (generalized) Sobolev norm, which has been used by Choquet, Christodoulou, Francaviglia in [2] for the Cauchy initial value problem:

$$|u|_{L_T, s}^{\text{Cauchy}} := \operatorname*{ess\,sup}_{t \in [o, T]} |u|_{\Lambda_t}^{(s)}, \tag{I.4}$$

whzre L_T is decomposed into a congruence of hypersurfaces $\Lambda_t := \{ x \in L_T | \tau (x = t \}$ and τ being a time-function (*i. e.* Λ_t being a Cauchy hypersurface); furthermore $| \, . \, |_{\Lambda_t}^{(s)}$ denotes the s'th order Sobolew norm on

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(9) Dossa alsso remarks (Remark 3.3.5 of [22]) that the energy inequality in [3] does not fulfil the "stable-boundedness property" (mentioned in the last subsection). However, this gap has been filled in [4] and in the present paper by using those norm with *weight factors*, which we introduce in the next subsection.

(10) The C^∞-functions are *dense* in the Hilbert space $\mathfrak{h}_s (L_T)$ of section 5. The Banach space $\mathfrak{E}_s (L_T)$ in turn is related to $\mathfrak{h}_s (L_T)$ via Theorem 7.3 and formula (7.14), (7.4), whereby the norm of $\mathfrak{E}_s (L_T)$ is $\| \, . \, \|_{L_T, s}$ which is a generalization of $| \, . \, |_{L_T, s}$ of (6).

Λ_t (inclucing derivatives non-tangential to Λ_t), namely

$$|u|^{(s)}_{\Lambda_t} := \Big(\sum_{r_1 + \ldots + r_{n+1} \leqq s} \; \sum_{A=1}^{N} |D_1^{r_1} \ldots D_{n+1}^{r_{n+1}} \, u^A|^2_{\Lambda_t} \Big)^{1/2} \qquad (I.5)$$

with $u := (u^A)$; $D_k := \dfrac{\partial}{\partial x^k}$ and $| \, . \, |_{\Lambda_t}$ being the $L^2(\Lambda_t)$ norm with respect to the canonical measure induced on Λ_t by some positive definite auxiliary metric on L_T.

We now insert into (4) the *weight factor* $t^{-1/2}$, and *define* ([11]) *one of our norms:*

$$|u|_{L_T, s} := \operatorname*{ess\,sup}_{t \in [0,\, T]} t^{-1/2} \, |u|^{(s)}_{\Lambda_t} \qquad (I.6)$$

with $|u|^{(s)}_{\Lambda_t,\, s}$ of (5). *All our norm* will be endowed with analogous weight factors.

The weigth factor in (6) is *singular* at $t = 0$ and it will be discussed in the next subsection.

I.5. The weight factor

As in many other existence proofs (*cf.* [2]) one needs inequalities of the following kind: for any u, v, with finite norm $| \, . \, |_{L_T, s}$ it holds

$$\left.\begin{array}{l} |uv|_{L_T, s} \leqq c_0 \, |u|_{L_T, s} \, |v|_{L_T, s} \quad \text{with} \quad T \in (0, T_0] \\ \qquad c_0 \text{ independent of } u, v \end{array}\right\} \qquad (I.7)$$

with $c_0 > 0$ and $s > \frac{1}{2} . (n+1)$.

Now, the *weight factor* $t^{-1/2}$ *in* (6) *has been choosen such that* there exists a c_0 fulfilling (7), where

$$\underline{c_0} \text{ is finite and } \textit{independent of } T \in [0, T_0]. \qquad (I.8)$$

In the characteristic initial value problem the property (8) cannot be achieved wthout some weight factor (or similar devices), namely

$$\left.\begin{array}{c} \textit{if one would omit the weight factor } t^{-1/2} \text{ in } | \, . \, |_{L_T, s} \\ \text{of (6), the quantity } c_0 \text{ of (7) would fulfil} \\ \underline{\lim_{T \to 0} c_0 = \infty} \end{array}\right. \qquad (I.9)$$

([11]) We shall also use weight factors, which we insert into the usual Sobolev norms [the norm of (6) is a *generalized* Sobolev norm]. – *See* (3.9), which is equivalent to Definition 2.1.

in contrast to (8). This would cause great difficulties, as one has to vary T in the *quasilinear case*, namely one has to make T sufficiently small such that there exists a solution in L_T (T cannot – in general – be prescribed; this is due to the possible occurrance of a singularity of the solution).

This difficulty also perpetuates to the *energy inequality* and it will also be settled by endowing all our norms with appropriate weight factors [e. g. (6)]. With this we obtain the *stable-boundedness* property of subsection I.2.

(9) in turn is a *consequence* of

$$\lim_{t \to 0} V(\Lambda_t) = 0 \qquad (t \in (0, T]) \tag{I.10}$$

(see *Fig.* 1), where $V(\Lambda_t)$ is the *volume* of Λ_t:

$$V(\Lambda_t) := \int_{\Lambda_i} d\Lambda_t, \tag{I.11}$$

where $d\Lambda_t$ is the volume element of the hypersurface Λ_t induced by some positive definite auxiliary metric (on L_T). The property (10) and hence (9) *follows from the geometrical shape* of the domain of dependence of the characteristic initial value problem (see *Fig.* 1). – As *opposed to the characteristic* initial value problem the difficulty (9) does *not* occur in the case of the Cauchy initial value problem [if one uses the norm (4)].

I.6. Organization of the paper

In order to make the paper accessible to generalizations, at each stage only those structures are introduced which are really used at that stage. There are three stages:

(I) Section 2, 3 and 4 deals with inequalities involving *norms*; the *function spaces*, which correspond to these norms, *are not yet introduced.* The energy inequalities of section 4 are derived (so far) for C^∞ functions only.

(II) In section 5 and 6 the *function space* H_s (*and its generalization* $\mathfrak{h}_s$) *is introduced*, to which the following norm belongs

$$|u|_s^{L_T} := T^{-1} \cdot \left(\int_0^T (|u|_{\Lambda_t}^{(s)})^2 \, dt \right)^{1/2} \tag{I.16}$$

[with $|.|_{\Lambda_t}^{(s)}$ of (5)].

It is derived an existence theorem and energy inequality expressed in terms of $\mathfrak{h}_s$-function.

(III) In section 7 and 8 the *function space* E_s (*and its generalization* $\mathfrak{E}_s$) *is introduced*, which belongs to $|.|_{T, s}$ of (6), whose topology is stronger than the topology of $|.|_s^{L_T}$ of (16). Moreover, unlike $|.|_s^{L_T}$ the norm $|.|_{L_T, s}$ fulfils the *ball property* defined in Remark 3.1. It is derived an existence theorem and energy inequality in terms of $\mathfrak{E}_s$. Also an iteration for the *quasilinear* case has been expressed in terms of $\mathfrak{E}_s$-functions (*see* Remark B of section 8).

I.7. Outline of the proof of the existence theorem

Our derivation of the *energy inequality* will be an important generalization of [3] (*see* Remark 4.1 of section 4); due to this we are able to introduce those norms with weight factors and finally obtain the *stable-boundedness* property (*see* subsection I.2).

One starts with an existence theorem for the case that the coefficients and data – and hence the solution – are C^∞-functions. Using this one then approximates the coefficients $\in \mathfrak{h}_s$ and data $\in H_{2s-1}$ (*see* (II) of the previous subsection) by C^∞-functions and thus obtains a sequence $\{u_n\}_{n\in\mathbb{N}}$ of C^∞-solutions, which converges *weakly* towards some $u \in \mathfrak{h}_s$, where *u is a solution* of our initial value problem with main coefficients $\in \mathfrak{h}_s$ and data $\in H_{2s-1}$ (*see* section 6).

Moreover, there exists (*see* section 7 and 8) a sequence $\{\bar{u}_k\}_{k\in\mathbb{N}}$, which is contained in the convex hull of $\{u_n\}_{\in\mathbb{N}}$ and which converges *strongly* towards the above solution $u \in \mathfrak{h}_s(L_T)$. One evaluates this and the energy inequality [with the $\mathfrak{E}_s(L_T)$-norm] on almost every hypersurface Λ_t, where $t \in (0, T]$. Then one takes $\operatorname{ess\,sup}_{t\in[0,T]}$ and shows (in section 8) that $|u|_{L_T, s} < \infty$ and moreover $u \in \mathfrak{E}_s(L_T)$.

1. BASIC DEFINITIONS AND ASSUMPTIONS

We *define*

$$\delta^{ab} := \begin{pmatrix} 1 \text{ for } a=b \\ 0 \text{ for } a \neq b \end{pmatrix}, \text{ likewise for } \delta_{ab} \text{ and } \delta_b{}^a.$$

We use the usual summation convention, for example

$$v_{d\alpha c}{}^{aB}\, w_{Be}{}^{ad\alpha} := \sum_{d,\,B,\,\alpha} v_{d\alpha c}{}^{aB}\, w_{Be}{}^{ad\alpha} \, (\textit{no} \text{ summation over } a, c, e).$$

We *assume* the following:

$L \subseteqq \mathbb{R}^{n+1}$ is *compact* and homeomorphic to some closed $(n+1)$-dimensional sphere; the boundary of L is piecewise smooth.

$$g^{ab} D_a D_b \qquad \text{with} \quad D_a := \frac{\partial}{\partial x}$$

(indices *a*, *b*, *c*, ... *run from* 1 *to* $(n+1)$) is a *hyperbolic* differential operator.

$G^1 \cup G^2$ is going to be our *data surface*, where $G^\omega \subseteq L$ (ω runs from 1 to 2) is part of the *boundary* of L; G^ω is a *characteristic* (with respect to g^{ab}) surface; $\Gamma := G^1 \cap G^2$ is a smooth *spacelike* $(n-1)$-dimensional surface (see *Fig.* 1).

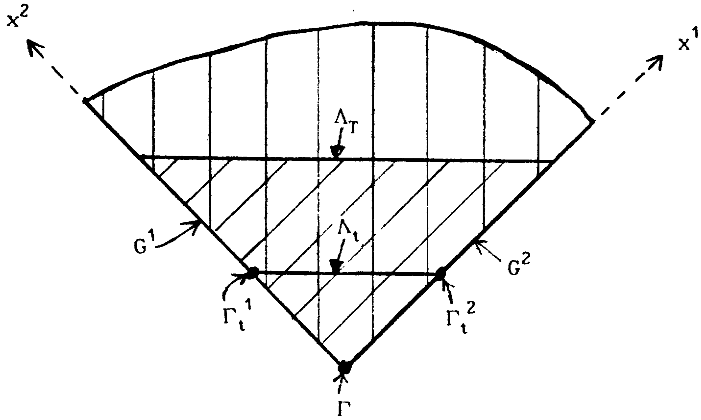

FIG. 1. – The coordinates $x^3, \ldots, x^{n+1}$ are subpressed in the figure. The domain L is shaded as |||. The domain L_T is shaded as ///. One defines the hypersurface $\Lambda_t := \{ x \in L \mid T(x) = t \}$ with $T := x^1 + x^2$ being a time function.

We assume that there exists a coordinate system $x := (x^a) := (x^1, \ldots, x^{n+1})$ such that

$$G^\omega = \{x \in L \mid x^\omega = 0\}, \qquad G^1 \cap G^2 = \{x \in L \mid x^1 = 0,\ x^2 = 0\} \tag{1.1}$$

and such that

$$\tau(x) := x^1 + x^2 \text{ is a time-function, } i.e.$$
$$g^{ab} \frac{\partial \tau}{\partial x^a} \frac{\partial \tau}{\partial x^b} < 0 \quad \text{in L.} \tag{1.2}$$

We shall always use this coordinate system, unless otherwise stated.

We *assume* that

$$g^{ab} \textit{ is regularly hyperbolic on } L \textit{ with hyperbolic constant } h > 0; \tag{1.3}$$

by this we mean:

(I) L is a *convex hyperbolic set* based on $G^1 \cup G^2$, *i.e.* any past directed null curve (in L) can be extended to a null-curve, which hits $G^1 \cup G^2$ and lies completely in L; moreover, $\partial L \setminus (G^1 \cup G^2)$ (∂L being the boundary of L) is a piecewise smooth non-timelike hypersurface;

(II) there exist constants $h, h_1, h_2, h_3 > 0$ with

$$h := \max \{h_1^{-1}, h_2^{-1}, h_3\}, \tag{1.5}$$

where h_1, h_2, h_3 fulfil

$$-g^{ab}\,t_a\,t_b > h_1 \qquad \text{with} \quad t_a := \frac{\partial\tau}{\partial x^a}, \tag{1.6}$$

$$g^{ab}\,k_a\,k_b > h_2\,|k|^2 \tag{1.7}$$

for any covector k_a with $k_a\,t_b\,g^{ab} = 0$,

$$|g| < h_3, \tag{1.8}$$

where

$$|k|^2 := k_a\,k_b\,e^{ab}, \quad |g|^2 := g_{ac}\,g_{bd}\,e^{ab}\,e^{cd} \tag{1.9}$$

where e^{ab} is some positive definite auxiliary metric on L; for simplicity we choose

$$e^{ab} = \delta^{ab} \tag{1.10}$$

in the coordinate system (x^a) of (1), (2).

We define for

$$t \leqslant \mathrm{T}_0 := \max_{x \in \mathrm{L}} \tau(x) \tag{1.10A}$$

the following point sets (see *Fig.* 1):

$$\begin{gathered} \mathrm{L}_t := \{x \in \mathrm{L} \mid \tau(x) \in [0,\, t]\}, \qquad \Lambda_t := \{x \in \mathrm{L} \mid \tau(x) = t\} \\ \mathrm{G}^\omega_t := \{x \in \mathrm{G}^\omega \mid \tau(x) \in [0,\, t]\} = \mathrm{G}^\omega \cap \mathrm{L}_t \\ \Gamma^\omega_t := \{x \in \mathrm{G}^\omega \mid \tau(x) = t\} = \mathrm{G}^\omega \cap \Lambda_t; \qquad (\omega = 1,\, 2), \\ \Gamma := \mathrm{G}^1 \cap \mathrm{G}^2 = \Gamma^1_0 = \Gamma^2_0. \end{gathered} \tag{1.11}$$

With respect to the coordinate system of (1), (2) we *define*

$$\mathrm{D}_a := \frac{\partial}{\partial x^a}; \qquad \mathrm{D}_\xi := \frac{1}{2}(\mathrm{D}_1 - \mathrm{D}_2),$$

$$\mathrm{D} := (\mathrm{D}_a) := (\mathrm{D}_1,\, \ldots,\, \mathrm{D}_{n+1}),$$

D_S means a basis of derivatives tangent to a surface $\mathrm{S} \leqslant \mathbb{R}^{n+1}$, in particular

$$\begin{gathered} \mathrm{D}_{\Gamma_t}\omega := (\mathrm{D}_3,\, \ldots,\, \mathrm{D}_{n+1}), \qquad \mathrm{D}_{\mathrm{G}^1} := (\mathrm{D}_2,\, \mathrm{D}_3,\, \ldots,\, \mathrm{D}_{n+1}), \\ \mathrm{D}_{\mathrm{G}^2} := (\mathrm{D}_1,\, \mathrm{D}_3,\, \ldots,\, \mathrm{D}_{n+1}), \qquad \mathrm{D}_{\Lambda_t} := (\mathrm{D}_\xi,\, \mathrm{D}_3,\, \ldots,\, \mathrm{D}_{n+1}); \end{gathered}$$

$$\mathrm{D}^k_\mathrm{S} = \underbrace{\mathrm{D}_\mathrm{S}\,\mathrm{D}_\mathrm{S}\,\ldots\,\mathrm{D}_\mathrm{S}}_{k\text{-times}}.$$

$\int_\mathrm{S} f\,d\mathrm{S}$ denotes the Lebesque-integral over the surface $\mathrm{S} \subset \mathrm{L}$, where $d\mathrm{S}$ is the volume element induced on S by the auxiliary metric e^{ab} [of (10)] on L. We *define*

$$\mathrm{C}^k(\mathrm{S}) := \{v \mid v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2} \text{ are } k\text{-times continuously differentiable on S}\},$$

where we consider tensor fields $v := (v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})$ over S.

2. THE DEFINITION OF THE NORMS

In (1.9), (1.10) we defined for (k_a)

$$|k|:=(k_a k_b e^{ab})^{1/2}=(k_a k_b \delta^{ab})^{1/2}=(\sum_a (\mathrm{k}_a)^2)^{1/2};$$

in this sense we define for $v:=(v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})$ with $\mathrm{A}_k=1, \ldots, \mathrm{N}_k$ $(k=1, \ldots, \mathrm{R})$

$$|v|:=|(v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})|:=\left(\sum_{k=1}^{\mathrm{R}} \sum_{\mathrm{A}_k=1}^{\mathrm{N}_k} (v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})^2\right)^{1/2},$$

$$|\mathrm{D}_{\Gamma_t} v|:=|(\mathrm{D}_\alpha v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})| \quad \text{with} \quad \alpha=3, \ldots, n+1.$$

Let S be either some surface or S = L; we now consider some tensor field v on S with $v:=(v_{\mathrm{A}_3 \ldots \mathrm{A}_\mathrm{R}}{}^{\mathrm{A}_1\mathrm{A}_2})$, where $\mathrm{A}_k=1, \ldots, \mathrm{N}_\mathrm{R}$, $k=1, \ldots, \mathrm{R}$; we define

$$|v|_s:=\left(\int_\mathrm{S} |v|^2 d\mathrm{S}\right)^{1/2}. \tag{2.1}$$

For certain point sets S_t we define

$$|\mathrm{S}_t|:=t^\alpha \text{ with } \alpha:=\begin{pmatrix} \frac{1}{2} & \text{for} \quad \mathrm{S}_t=\Lambda_t,\ \mathrm{G}_t^\omega \\ 1 & \text{for} \quad \mathrm{S}_t=\mathrm{L}_t \\ 0 & \text{for} \quad \mathrm{S}_t=\Gamma_t,\ \Gamma \end{pmatrix} \tag{2.2}$$

and with this

$$|v|_s^{\mathrm{S}_t, \mathrm{R}}:=|\mathrm{S}_t|^{-1}\left(\sum_{k=0}^{s} |\mathrm{D}_\mathrm{R}{}^k v|_{\mathrm{S}_t}^2\right)^{1/2} \quad \text{with} \quad \mathrm{S}_t \subset \mathrm{R} \subset \mathrm{L}, \tag{2.3}$$

$$|v|_s^{\mathrm{S}_t}:=|v|_s^{\mathrm{S}_t, \mathrm{S}}. \tag{2.4}$$

Remark. — $|v|_s^{\mathrm{S}_t}$ is (apart from factor $|\mathrm{S}_t|^{-1}$) the usual *Sobolev norm.* — Let $\mathrm{S}_\mathrm{t}:=\Lambda_t$, $\mathrm{G}_t{}^\omega$ or L_t. If we would have defined $|v|_s^{\mathrm{S}_t, \mathrm{R}}$ or $|v|_s^{\mathrm{S}_t}$ without the *weight factor* $|\mathrm{S}_t|^{-1}$, the c_0 of (3.1), (3.6) and (3.8) (*see* below) would fulfil $\lim\limits_{t \to 0} c_0=\infty$.

Such a t-dependance would cause great difficulties in the quasilinear case, which have been discussed around (I.9) of the introduction (12). This t-dependance has been counter balanced by the right choice of $|\mathrm{S}_t|^{-1}$ [in

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(12) V (Λ_t) of the introduction is (up to a constant) equal to $|\Lambda_t|^2$, where $|\Lambda_t|^{-1}$ is one weight factor of (3) (for $\mathrm{S}_t=\Lambda_t$).

(2)], which is such that

$$\lim_{t \to 0} \left[|S_t|^{-1} \cdot \left(\int_{S_t} 1\, ds \right)^{1/2} \right] \tag{2.5}$$

is neither ∞ nor 0.

We define the following modified Sobolev norm (*cf.* [2])

$$|v|_{S_T, s} := \operatorname*{ess\,sup}_{t \in [0, T]} |v|_s^{\Sigma_t, S_T}, \tag{2.8}$$

where we decomposed S_T into a congruence of surfaces $\Sigma_t := S_T \cap \Lambda_t$ with $t \in [0, T]$.

In Definition 2.1 (*see* below) we shall define further norms; the norms of the right-hand side (left-hand side, respectively) correspond to the norm in (8) [(4) respectively].

DEFINITION 2.1:

$$\| v \|_s^{G_T^\omega} := \left(\sum_{k=0}^{s-1} \left(|D_\omega{}^k v|_{2(s-k)-1}^{G_T^\omega} \right)^2 \right)^{1/2}$$

$$\| v \|_{s, 1}^{G_T^\omega} := \left(\sum_{k=1}^{s-1} \left(|D_\omega{}^k v|_{2(s-k)-1}^{G_T^\omega} \right)^2 \right)^{1/2}$$

$$\| v \|_s^{L_T} := \left(|v|_s^{L_T})^2 + \sum_{\omega=1}^{2} \left(\| v \|_s^{G_T{}^\omega} \right)^2 \right)^{1/2}$$

$$\| v \|_{s, 1}^{L_T} := \left((|v|_s^{L_T})^2 + \sum_{\omega=1}^{2} \left(\| v \|_{s, 1}^{G_T{}^\omega} \right)^2 \right)^{1/2}$$

$$\| v \|_{G_T^\omega, s} := \left(\sum_{k=0}^{s=1} \left(|D_\omega{}^k v|_{G_T^\omega, 2(s-k)-1} \right)^2 \right)^{1/2}$$

$$\| v \|_{G_T^\omega, s, 1} := \left(\sum_{k=1}^{s-1} \left(|D_\omega{}^k v|_{G_T^\omega, 2(s-k)-1} \right)^2 \right)^{1/2}$$

$$\| v \|_{L_T, s} := \left(|v|_{L_T, s}^2 + \sum_{\omega=1}^{2} \| v \|_{G_T^\omega, s}^2 \right)^{1/2}$$

$$\| v \|_{L_T, s, 1} := \left(|v|_{L_T, s}^2 + \sum_{\omega=1}^{1} \| v \|_{G_T^\omega, s, 1}^2 \right)^{1/2}$$

Furthermore we *define*

$$\| v \|_s^{\Gamma, G^\omega, 1} := \left(\sum_{k=0}^{s-1} \left(|D_\omega{}^k v|_{2(s-k)-1-1}^{\Gamma, G^\omega} \right)^2 \right)^{1/2}. \tag{2.9}$$

Let there be given p vector fields $v_{(1)}, \ldots, v_{(p)}$ on S then we define:

$$|v_{(1)}, \ldots, v_{(p)}|_s^{\mathrm{S}} := \left(\sum_{k=1}^{p} (| v_{(k)} |_s^{\mathrm{S}})^2 \right)^{1/2} \tag{2.10}$$

likewise for any of the above norms.

Remark on Definition 2.1 *concerning the use of these norms:*

We anticipate the following facts: the proof of the existence theorems rely on the fact that the space (*see* section 5 and 7), which corresponds to the norm $\|.\|_{\mathrm{L_T},s}$ ($\|.\|_s^{\mathrm{L_T}}$ respectively), fulfils the following *main properties* (*a*, *b*, *c*):

(*a*) it is a *Banach* space (*Hilbert* space respectively), which is shown in section 7 (5 respectively);

(*b*) it is an *algebra* under pointwise multiplication [*see* below (3.3) and (3.2)]; (this is of special importance for the quasilinear case);

(*c*) the solutions of (I. 1, 2) fulfils *energy inequalities* in terms of $\|.\|_{\mathrm{L_T},s}$ ($\|.\|_s^{\mathrm{L_T}}$ respectively), which is shown in Theorem 8.1 (6.2 respectively). Furthermore the *energy-inequality constant* fulfils the *stable-bounded-ness property* [*see* (II) of subsection I.2 of the introduction]. This in turn is – to some extend – a consequence of formula (I.8) of the introduction (*see* also section 3).

Furthermore the topology of $\|.\|_{\mathrm{L_T},s}$ is strictly *stronger* than the topology of $\|.\|_s^{\mathrm{L_T}}$.

$$\begin{array}{c}\text{The topology of } \||.|\|_s^{\mathrm{L_T}} \text{ of [3] is equivalent to}\\ \text{the topology of } \|.\|_s^{\mathrm{L_T}} \text{ (for fixed T);}\end{array} \tag{2.11}$$

though c of

$$\begin{array}{c}\||u.v|\|_s^{\mathrm{L_T}} \leqslant c \||u|\|_s^{\mathrm{L_T}} \||v|\|_s^{\mathrm{L_T}}\\ \text{with } c \text{ being independent of } u, v\end{array} \tag{2.12}$$

fulfils

$$\lim_{\mathrm{T} \to 0} c = \infty, \tag{2.13}$$

whereas $\lim\limits_{\mathrm{T} \to 0} c \neq \infty$, if one replaces $|\|.\||_s^{\mathrm{L_T}}$ of (2.12) by $\|.\|_s^{\mathrm{L_T}}$ (or $\|.\|_{\mathrm{L_T},s}$ respectively). The consequences of (2.13) are discussed around (I.9) of the introduction.

3. SOME BASIC PROPERTIES OF THE NORMS

One uses the imbedding (and Hoelder) formulas of Adams in [1] and some of the techniques of Christodoulou of [12], [13] and one considers the t-dependance of $|\,.\,|_s^{S_t}$ ($|\,.\,|_{S_t,s}$ respectively) together with (2.2). Then one obtains the following formulas (of this section):

We shall assume throughout

$$0 \leqslant t, \qquad T \leqslant \max_{x \in L} \tau(x).$$

There exists a constant $c_0 > 0$ (, which is independent of t, u, v,) such that

$$\left.\begin{aligned} &|u.v|_s^{S_t} \leqslant c_0 |u|_{s1}^{S_t} \, |v|_{s2}^{S_t} \qquad \text{for} \quad s_{1,2} \geqslant s, \\ &s_1 + s_2 > s + \frac{1}{2} \dim S_t \end{aligned}\right. \tag{3.1}$$

analogously for $\|\,.\,\|_s^{M_t}$ and $\|\,.\,\|_s^{M_t}, 1$ for $M_t = L_t$ or $G_t{}^{\omega}$,

$$\left.\begin{aligned} &|u.v|_{S_t,s} \leqslant c_0 |u|_{S_t,s_1} |v|_{S_t,s_2} \qquad \text{for} \quad s_{1,2} \geqslant s, \\ &s_1 + s_2 > s + \frac{1}{2}(-1 + \dim S_t) \end{aligned}\right. \tag{3.2}$$

analogously for $\|\,.\,\|_{M_t,s}$ and $\|\,.\,\|_{M_t,s,1}$ for $M_t = L_t$ or $G_t{}^{\omega}$,

$$\|u.v\|_s^{L_t} \leqslant c_0 \|u\|_s^{L_t} \|v\|_s^{L_t} \qquad \text{for} \quad s > \frac{n+1}{2}, \tag{3.3}$$

$$\|u \,.\, v\|_{L_t,s} \leqslant c_0 \|u\|_{L_t,s} \qquad \text{for} \quad s > \frac{n}{2}, \tag{3.3.A}$$

provided the norms on the right-hand side (of the inequality) exist.

There exists a constant $c_0 > 0$ (, which is independent of t, v) such that

$$\left.\begin{aligned} &|v|_s^{S_t} \leqslant c_0 |v|_{S_t,s} \\ &\text{likewise the pair } (\|\,.\,\|_s^{M_t}, \|\,.\,\|_{M_t,s}) \text{ and the} \\ &\text{pair } (\|\,.\,\|_{s,1}^{M_t} \, \|\,.\,\|_{M_t,s,1}) \text{ with } M^t = G_t{}^{\omega} \text{ or } L_t, \end{aligned}\right. \tag{3.4}$$

provided the norms, which occur on the right hand side of the inequality, exists.

The quantities

$$\left.\begin{aligned} &t\,|v|_s^{L_t},\ t^{1/2}\,|v|_s^{\Lambda_t},\ t^{1/2}\,|v|_s^{G_t\omega},\ |v|_s^{\Gamma_t\omega} \\ &\text{are non-decreasing functions of } t \text{ (for fixed } v\text{)} \end{aligned}\right\} \tag{3.5}$$

[because of (2.2), (2.3)].

There exists a constant $c_0 > 0$ (, which is independent of t, v,) such that

$$\max_{x \in \Gamma_t} |v(x)| \leqslant c_0 |v|_s^{\Gamma_t} \qquad \text{for} \quad s > \frac{n-1}{2}, \tag{3.6}$$

$$\max_{x \in \Lambda_t} |v(x)| \leqslant c_0 |v|_s^{\Lambda_t} \qquad \text{for} \quad s > \frac{n}{1} \tag{3.7}$$

$$\int_0^t \tau^{1/2} |v|_m^{\Lambda_\tau, L} d\tau \leqslant c_0 t^{3/2} |v|_m^{L_t}, \qquad \int_0^t |v|_m^{\Gamma_\tau{}^{\omega}, G^{\omega}} d\tau \leqslant c_0 t |v|_m^{G_t{}^{\omega}}, \tag{3.8}$$

provided the norm of the right-hand side (of the inequality) exists.

Remark 3.1.: Let $E_s(L_T)$ (at the moment) be any given space of generalized functions defined on L_T, where $E_s(L_T)$ is endowed with the norm $|.|_{L_T, s}$. Let $B^{E_s}_{r, T}$ denote the ball $\{v \in E_s(L_T) \| v|_{L_T, s} \leqslant r\}$. We shall say that the norm $|.|_{L_T, s}$ fulfils the ball *property*, if and only if for any given $T' \in (0, T_0]$, $r_1 > 0$ it holds

$$w \in B^{E_s}_{r, T'} \quad \Rightarrow \quad [w]_{L_T} \in B^{E_s}_{r, T} \tag{3.9}$$

for all $T \in (0, T']$ and $r \in (0, r_1]$. This property becomes important when one has to make T sufficiently small in order to obtain a solution (on L_T) for the *quasilinear* characteristic initial value problem (*see* Remark B of section 8). The *ball property holds analogously for the norm* $\|.\|_{L_T, s}$. The ball property does *not* hold for $|.|^{L_T}_s$, $\|.\|^{L_T}_s$ (their weight factors are differently placed than the ones of $|.|_{L_T, s}$, $\|.\|_{L_T, s}$).

4. ENERGY INEQUALITIES, THE C^∞-CASE

4a. Without reduction to data

From formula (4.11) onwards we shall use the following

DEFINITION 4.1. – With c_k $(k = 1, 2, \ldots)$ we denote a *continuous* function

$$c_k : \mathbb{R}_+{}^M \to \mathbb{R}, \quad \mathbb{R}_+{}^M := \{(z_1, \ldots, z_M) \in \mathbb{R}^M \,|\, z_l \geqslant 0 \text{ for } l = 1, \ldots, M\},$$

which is *increasing* in the following sense

$$\left.\begin{array}{c} c_k(z_1, z_2, \ldots, z_M) \leqslant c_k(z'_1, z'_2, \ldots, z'_M) \quad \text{for} \quad z_l \leqslant z'_l, \\ \text{with } 1 = 1, \ldots, M; \end{array}\right\} \tag{4.1}$$

for any $z_1^{(0)} > 0$ there exists a $c_k^{(0)} > 0$ with

$$\left.\begin{array}{c} c_k(z_1, \ldots, z_M) \geqslant c_k^{(0)} \quad \text{for all } (z_1, \ldots, z_M) \in \mathbb{R}_+{}^M \\ \text{with } z_1 \geqslant z_1^{(0)}. \end{array}\right\} \tag{4.2}$$

Furthermore by $c_{k, 0} > 0$, and c_0 we denote constants with

$$c_{k, 0} > 0, \qquad c_0 > 0. \tag{4.3}$$

Remark. – The above c_k will be used in the following context

$$c_k(h) \quad \text{or} \quad c_k(h, |g|^{L_t}_1, |b|^{L_t}_1, t) \quad \text{or} \ \ldots \tag{4.4}$$

[with h of (1.5)]; these quantities will be *constants*, when the coefficients g, b, ... of our differential equation (18) (*see* below) are fixed [, which will not always be the case as can be seen for instance in (5.25)]; in particular the quantities of (4) will *always be independent of the solution* u of our differential equation (18).

We introduce the following *"energy-momentum vector"*

$$P^m := \mathrm{P}^m[u] := [(g^{mr} g^{ks} + g^{ms} g^{kr} - g^{mk} g^{rs}) \mathrm{D}_r u . \mathrm{D}_s u] t_k, \tag{4.5}$$

where

$$t_k := \mathrm{D}_k \tau \,(\text{time function } \tau),\ \mathrm{D}_r u . \mathrm{D}_s u := \mathrm{D}_t u^{\mathrm{A}} \mathrm{D}_s u^{\mathrm{B}} \delta_{\mathrm{AB}} \tag{4.6}$$

and where u is – for the time being [up to formula (4.17)] – *not* necessarily a solution of our differential equation.

We shall derive properties of P^m, which we shall use for the proof of Lemma 4.1. We assume all quantities to be C^∞ (*i.e.* infinitely often differentiable).

We use the Gauss-integration formula

$$\int_{\mathrm{L}_t} \mathrm{D}_m \mathrm{P}^m \, d\mathrm{L} = \int_{\Lambda_t} \mathrm{P}^m (\sqrt{2}\, t_m) \, d\Lambda - \sum_{\omega=1}^{2} \int_{\mathrm{G}_t{}^\omega} \mathrm{P}^m \delta_m{}^\omega \, d\mathrm{G}^\omega + \int_{\mathrm{R}_t} \mathrm{P}^m n_m \, d\mathrm{R} \tag{4.7}$$

$$\mathrm{R}_t := [\text{boundary } \mathrm{L}_t] \setminus (\Lambda_t \cup \mathrm{G}_t{}^1 \cup \mathrm{G}_t{}^2), \tag{4.8}$$

where

$$n_a \text{ is the future-directed normal of } \mathrm{R}_t \text{ with } n_a n_b \delta^{ab} = 1. \tag{4.9}$$

For any covector m_a we define

$$\left.\begin{aligned} \gamma_m{}^{ab} &:= -g^{ab} m_c t^c + t^a m^b + t^b m^a, \\ m^a &:= g^{ab} m_b,\ t^a := g^{ab} t_b. \end{aligned}\right\} \tag{4.10}$$

We assume that

$$g^{ab} \text{ is regularly hyperbolic on L with hyperbolicity constant } h > 0, \tag{4.11}$$

(*cf.* (1.3).

With (10) and (11) we gain

$$\begin{aligned} \sqrt{2}\,\mathrm{P}^a t_a &= \sqrt{2}\,\gamma_t{}^{ab} \mathrm{D}_a u . \mathrm{D}_b u \\ &\geqq \sqrt{2}\,(1 + h_2^{-1})^{-1} (2 h_1^{-1/2} + 2 h_1^{-3/2})^{-1} |\mathrm{D}u|^2 \geqq c_1 (h)^{-1} |\mathrm{D}u|^2 \end{aligned} \tag{4.12}$$

for some c_1 (of Definition 4.1).

$\delta_a{}^\omega$ is a normal covector of G^ω, which is a characteristic surface; hence $\delta_a{}^\omega \delta_b{}^\omega g^{ab} = 0$ and thus $\gamma^{a\omega} = \gamma_\omega{}^{ab} \delta_b{}^\omega = 0$ (with $\gamma_\omega := \gamma_{\delta^\omega}$) and thus

$$\mathrm{P}^a \delta_a{}^\omega = \sum_{a,\, b \neq \omega} \gamma_\omega{}^{ab} \mathrm{D}_a u . \mathrm{D}_b u \leqq |\gamma_\omega| \, |\mathrm{D}_{\mathrm{G}^\omega} u|^2 \leqq c_2 (h) |\mathrm{D}_{\mathrm{G}^\omega} u|^2 \tag{4.13}$$

using (11) and (1.8).

(11) implies that R_t is non-timelike hence for its normal n_a

$$-n_a n_b g^{ab} \geqq 0; \qquad \beta := -t^a n_a > 0 \tag{4.14}$$

as n_a is future-directed. Furthermore the metric

$$h^{ab} := g^{ab} + t^a t^b . (-t_c t^c)^{-1} \text{ is positive definite} \tag{4.15}$$

since t^a is timelike. With $w_a{}^{\mathrm{A}} := n_a \beta^{-1} (t^c \mathrm{D}_c u^{\mathrm{A}}) + \mathrm{D}_a u^{\mathrm{A}}$ it follows from (5, 6)

$$\mathrm{P}^a n_a = \frac{\beta}{2} h^{ab} w_a{}^{\mathrm{A}} w_b{}^{\mathrm{B}} \delta_{\mathrm{AB}} + \frac{1}{2\beta} (-g^{ab} n_a n_b) [t^c \mathrm{D}_c \mathrm{u}^{\mathrm{A}}] [t^c \mathrm{D}_c u^{\mathrm{B}}] \delta_{\mathrm{AB}} \geqq 0 \tag{4.16}$$

the latter because of (14, 15).
(12, 16) and (7) imply

$$\begin{aligned} c_1 (h)^{-1} \int_{\Lambda_t} |\mathrm{D} u|^2 d\Lambda &\leqq \int_{\Lambda_t} \mathrm{P}^a (\sqrt{2}\, t_a) \, d\Lambda + \int_{\mathrm{R}_t} \mathrm{P}^a n_a \, d\mathrm{R} \\ &= \sum_{\omega=1}^{2} \int_{\mathrm{G}^\omega} \mathrm{P}^m \delta_m{}^\omega \, d\mathrm{G}_\omega + \int_{\mathrm{L}_t} \mathrm{D}_m \mathrm{P}^m \, d\mathrm{L} \\ &\leqq c_2 (h) \sum_{\omega=1}^{2} \int |\mathrm{D}_{\mathrm{G}^\omega} u|^2 + \int_{\mathrm{L}_t} \mathrm{D}_m \mathrm{P}^{\mathrm{m}} \, d\mathrm{L} \end{aligned} \tag{4.17}$$

the latter because of (13). We shall use (17) in the proof of Lemma 4.1. We now consider the following linear *characteristic* initial value problem

$$\mathbb{L} u := g^{ab} \mathrm{D}_a \mathrm{D}_b u + b^a . \mathrm{D}_a u + a . u = f \quad \text{on } \mathrm{L}_{\mathrm{T}} \tag{4.18}$$

$$u = \underset{\omega}{u} \quad \text{on } \mathrm{G}_{\mathrm{T}}{}^\omega, \qquad (\omega = 1, 2) \tag{4.19}$$

(with given data $\underset{\omega}{u}$) or using index notation

$$g^{ab} \mathrm{D}_a \mathrm{D}_b u^{\mathrm{A}} + b^a{}_{\mathrm{B}}{}^{A} \mathrm{D}_a u^{\mathrm{B}} + a^{\mathrm{A}}{}_{\mathrm{B}} u^{\mathrm{B}} = f^{\mathrm{A}} \quad \text{on } \mathrm{L}_{\mathrm{T}} \tag{4.20}$$

$$u^{\mathrm{A}} = \underset{\omega}{u}{}^{\mathrm{A}} \quad \text{on } \mathrm{G}_{\mathrm{T}}{}^\omega. \tag{4.21}$$

We now state the *non-reduced energy inequalities:*

LEMMA 4.1. — *We assume*

$$\left.\begin{aligned} &\underset{1}{u} = \underset{2}{u} \quad \textit{on } \mathrm{G}^1 \cap \mathrm{G}^2, \qquad u, g, b, a, f \in \mathrm{C}^\infty (\mathrm{L}_{\mathrm{T}}), \\ &\underset{\omega}{u} \in \mathrm{C}^\infty (\mathrm{G}_{\mathrm{T}}{}^\omega), \qquad (\omega = 1, 2) \end{aligned}\right. \tag{4.22}$$

and we assume that g^{ab} is regularly hyperbolic on L_{T} with a hyperbolicity constant $h > 0$ [cf. (1.3)]. *For any solution u of* (18, 19) *the following energy* *inequalities hold*

$$(t^{1/2}\,|u|_{1}^{\Lambda_t,\,\mathrm{L}})^2 \leqq c_{10}(h)\Bigg[\sum_{\omega=1}^{2}(t^{1/2}\,|\underset{\omega}{u}|_{1}^{\mathrm{G}_t\omega})^2$$
$$+\int_0^t (\tau^{1/2}\,|u|_{1}^{\Lambda_\tau,\,\mathrm{L}}(\tau^{1/2}\,|\,\mathrm{f}\,|_{0}^{\Lambda_\tau}) + (\tau^{1/2}\,|u|_{1}^{\Lambda_\tau,\,\mathrm{L}})^2(\gamma_1^{\Lambda_\tau}))\,d\tau\Bigg] \tag{4.23}$$

$$|u|_{1}^{\Lambda_t,\,\mathrm{L}} \leqq c_{12}\,.\Bigg[\sum_{\omega=1}^{2}|\underset{\omega}{u}|_{1}^{\mathrm{G}_\omega t} + t^{-1/2}\int_0^t \tau^{1/2}\,|f|_{0}^{\Lambda_\tau}\,d\tau\Bigg], \tag{4.24}$$

for $t\in[0,\,\mathrm{T}]$, *where*

$$c_{12} := c_{12}\left(h,\,\int_0^t \gamma_1^{\Lambda_\tau}\,d\tau,\,t\right)$$
$$:= c_{11}(h)(\exp c_{10}(h)\,t^2)\left[c(h)\int_0^t \gamma_1^{\Lambda_\tau}\,d\tau\right], \tag{4.24 A}$$

$$\gamma_s^{\Lambda_t} := |a|_{s_0}^{\Lambda_t,\,\mathrm{L}} + |b|_{s_1}^{\Lambda_t,\,\mathrm{L}} + |b|_{s_2}^{\Lambda_t,\,\mathrm{L}},$$
$$s_k \geqq s-1, \qquad s_k \geqq \frac{n}{2}+k-1. \tag{4.25}$$

Remark 4.1. – The energy inequality constant c_{12} fulfils the *stable-boundedness property* (*see* subsection I.2 of the introduction). One aspect of this is the following *boundedness property:* for any $\mathrm{T}_1\in(0,\,\mathrm{T}_0]$ it holds

$$c_{12}\left(h,\,\int_0^t \gamma_1^{\Lambda_\tau}\,d\tau,\,t\right)\in[c_{12,1};\,c_{12,2}] \quad \text{for all } \mathrm{t}\in[0,\,\mathrm{T}_1] \tag{4.25 A}$$

with $c_{12,1} := c_{12}(h,0,0) > 0$ and $c_{12,2} := c^{12}\left(h,\,\int_0^{\mathrm{T}_1}\gamma_1^{\Lambda_\tau}\,d\tau,\,\mathrm{T}_1\right)$. The quantity c_{12} is an increasing function [*see* (1)]. This implies (25 A). – The derivation of our energy inequality fulfilling the boundedness property is an *important generalization* of the method of [3], namely: when deriving an energy inequality there occur products of the unknown u with the coefficients of our differential equation. One has to separate u from the coefficient. Unlike [3] this sepatation will *not* be done *before* applying the so-called Gronwal's Lemma. Instead we apply some generalized Gronwal's Lemma (*see* Appendix I), which deals with the above products; then u gets separated in such a way that the weight factors can be introduced, which induces the boundedness property (25 A).

Proof. – In order to prove (23) it remains to estimate $\int \mathrm{D}_m \mathrm{P}^m\,d\mathrm{L}$ of (17); let u be a solution of (18, 19); we differentiate $\mathrm{P}^m := \mathrm{P}^m[u]$ of (5)

$$\begin{aligned}\mathrm{D}_m\mathrm{P}^m &= 2\,t^c(\mathrm{D}_c u)\,.\,[g^{ab}\mathrm{D}_a\,\mathrm{D}_b\,u] + \mathrm{G}^{rs}\,\mathrm{D}_r u\,.\,\mathrm{D}_s u\\ &= 2\,t^c(\mathrm{D}_c u)\,.\,[f - b\,.\,\mathrm{D}u - a\,.\,u] + \mathrm{G}^{rs}\,\mathrm{D}_r u\,.\,\mathrm{D}_s u\end{aligned} \tag{4.26}$$

with

$$G^{rs} := [D_m (g^{mr} g^{kr} + g^{ms} g^{kr} - g^{mk} g^{rs})] t_k. \tag{4.27}$$

We estimate the above quantities by using (3.7), (1.8) and (3.1)

$$\max_{x \in \Lambda_t} |G| \leqq c_0 (\max_{x \in L} |g|) . \max_{x \in \Lambda_t} |Dg| \leqq c_3 (h) |g|^{\Lambda_t, L}_{s_2} \tag{4.28}$$

$$|b . Du + a . u|^{\Lambda_t}_0 \leqq c_0 (|b|^{\Lambda_t, L}_{s_1} + |a|^{\Lambda_t, L}_{s_0}) |u|^{\Lambda_t, L}_1 \tag{4.29}$$

with s_k of (25).

Using (26) we get (30) (*see* below), then we get (31 32) using the Hölder-inequality and (28, 29):

$$\int_{\Lambda_\tau} |D_m P^m| \, d\Lambda$$

$$\leqq \int_{\Lambda_\tau} c_4 (h) (|Du| [|f| + |b . Du + a . u|] + (\max_{x \in \Lambda_t} |G|) |Du|^2) \, d\Lambda \tag{4.30}$$

$$\leqq c_5 (h) (\tau |Du|^{\Lambda_\tau}_0 . [|f|^{\Lambda_\tau}_0 + |b . Du + a . u|^{\Lambda_\tau}_0] \\ + |g|^{\Lambda_\tau, L}_{s_2} . (\tau^{1/2} |u|^{\Lambda_\tau, L}_1)^2) \tag{4.31}$$

$$\leqq c_6 (h) ((\tau^{1/2} |u|^{\Lambda_\tau, L}_1) (\tau^{1/2} |f|^{\Lambda_\tau}_0) \\ + (\tau^{1/2} |u|^{\Lambda_\tau, L}_1)^2 (|b|^{\Lambda_\tau, L}_{s_1} + |a|^{\Lambda_\tau, L}_{s_0} + |g|^{\Lambda_\tau, L}_{s_2})) \tag{4.32}$$

$$\leqq c_7 (h) (y (\tau) f_0 (\tau) + y (\tau)^2 \gamma^{\Lambda_\tau}_1) \tag{4.32 A}$$

with $\gamma^{\Lambda_\tau}_1$ of (25) and

$$y (\tau) := \tau^{1/2} |u|^{\Lambda_\tau, L}_1, \qquad f_0 (\tau) := \tau^{1/2} |f|^{\Lambda_\tau}_0. \tag{4.32 B}$$

We rewrite (17) by using (32 A) and inserting (32 B) we gain

$$(t^{1/2} |Du|^{\Lambda_t}_0)^2 \leqq c_8 (h) \left(\sum_\omega (t^{1/2} |\underset{\omega}{u}|^{G_t^\omega}_1)^2 + \int_0^t \left[\int_{\Lambda_\tau} |D_m P^m| \, d\Lambda \right] d\tau \right)$$

$$\leqq c_g (h) \left(d (t)^2 + \int_0^t [y (\tau) f_0 (\tau) + y (\tau)^2 . \gamma^{\Lambda_\tau}_1] \, d\tau \right) \tag{4.33}$$

with

$$d (t) := \sum_{\omega = 1}^{2} (t^{1/2} |\underset{\omega}{u}|^{G_t^\omega}_1)^2. \tag{4.33 A}$$

For any $u \in C^\infty (L)$, which is not necessarily a solution, it holds for some constant c_0

$$(t^{1/2} |u|^{\Lambda_t}_0)^2 \leqq c_0 (\sum_\omega (t^{1/2} |u|^{G_t^\omega}_0)^2 + \left[\int_0^t (\tau^{1/2} |u|^{\Lambda_\tau, L}_1)^2 \, d\tau \right] . t); \tag{4.33 B}$$

because of Lemma A.2 of the apendix. We add this to (33) and use (33 A)

$$y(t)^2 \leqq c_{10}(h) \left[d(t)^2 + \int_0^t y(\tau) f_0(\tau)\, d\tau + \int_0^t y(\tau)^2 [\gamma_1^{\Lambda_\tau} + t]\, d\tau \right]. \qquad (4.33\,\mathrm{C})$$

This *proves* (23).

For $t \leqq t'$ $(t' \geqq 0)$ it holds $d(t) \leqq d(t')$ and thus by (33 C)

$$y(t)^2 \leqq c_{10}(h)\, d(t')^2 + \int_0^t y(\tau)(c_{10} f_0(\tau))\, d\tau + \int_0^t y(\tau)^2 (c_{10}(h) [\gamma_1^{\Lambda_\tau} + t']\, d\tau;$$

from this it follows by a generalization of Gronwal's Lemma (*see* Lemma A.1 of the appendix)

$$y(t) \leqq \sqrt{2(c_{10}(h)}\, d(t') \left(\exp \int_0^t c_{10}(h) [\gamma_1^{\Lambda_\tau} + t']\, d\tau \right)$$
$$+ \int_0^t c_{10}(h) f_0(\tau) \left(\exp \int_0^t c_{10}(h) [\gamma_1^{\Lambda_s} + t']\, ds \right) d\tau. \qquad (4.33\,\mathrm{D})$$

From this for $t = t'$ we deduce

$$y(t') \leqq c_{11}(h).(\exp c_{10}(h)\, t'^2)$$
$$\times \left[\exp \int_0^{t'} (c_{10}(h)\, \gamma_1^{\Lambda_\tau}\, d\tau \right] \left[d(t') + \int_0^{t'} f_0(\tau)\, d\tau \right].$$

We multiply this by $t'^{-1/2}$ and thus *prove* (24).

Some further *energy inequalities* hold

THEOREM 4.2. — *Let the assumptions of Lemma* 4.1 *hold and let*

$$\gamma_r^{\mathrm{L}_t} := t\, |g, a, b|_r^{\mathrm{L}_t} \qquad (4.33\,\mathrm{E})$$

then it holds for any solution u of (18, 19) *and for* $m > \frac{n}{2} + 2$ *and* $t \in [0, \mathrm{T}]$

$$|u|_1^{\Lambda_t, \mathrm{L}} \leqq c_{15}(h, \gamma_m{}^{\mathrm{L}_i}, t) \left[\left(\sum_{\omega=1}^{2} |u|_1^{\mathrm{G}_t{}^\omega} \right) + t\, |f|_0^{\mathrm{L}_t} \right]. \qquad (4.33\,\mathrm{F})$$

Moreover, for $s > \frac{n}{2} + 2$ *and* $t \in [0, \mathrm{T}]$

$$|u|_s^{\Lambda_t . \mathrm{L}} \leqq c_{19}(h, \gamma_s{}^{\mathrm{L}_t}, t) \left[\sum_{\omega=1}^{2} (|u|_s^{\mathrm{G}_t{}^\omega} + \sum_{\substack{k=1 \\ \omega}}^{s-1} |\mathrm{D}_\omega{}^k u|_{s-k}^{\mathrm{G}_t{}^\omega}) + t\, |f|_{s-1}^{\mathrm{L}_t} \right], \qquad (4.34)$$

$$|u|_{\mathrm{L}_\mathrm{T}, s} \leqq c_{20}(h, \gamma_s{}^{\mathrm{L}_\mathrm{T}}, \mathrm{T})$$
$$\times \left[\sum_{\omega=1}^{2} (|u|_{s+1}^{\mathrm{G}_\mathrm{T}{}^\omega} + \sum_{\substack{k=1 \\ \omega}}^{s-1} |\mathrm{D}_\omega{}^k u|_{\mathrm{G}_T{}^\omega, s-k}) + \mathrm{T}\, |f|_{s-1}^{\mathrm{L}_T} \right]. \qquad (4.34\,\mathrm{A})$$

Furthermore the quantities c_{15}, c_{19}, c_{20} *fulfil the* stable-boundedness property, i. e. *for any* $T_1 \in [0, T_0]$ *it holds:*

$$c_{20}(h, \gamma_s^{L_T}, T) \in [c_{20,1}; c_{20,2}] \quad \textit{for all} \ T \in [0, T_1] \qquad (4.34\,B)$$

with $c_{20,1} := c_{20}(h, 0, 0)$ *and* $c_{20,2} := c_{20}(h, T_1 \,|\, g, a, b \,|_s^{L_{T_1}}, T_1)$, *and the intervall* $[c_{20,1}; c_{20,2}]$ *being* stable (13) *against small variations of the coefficients* (g, a, b),

$$(4.34\,C)$$

where "stability" is defined in Definition 4.2 (*below*). *The analogous properties hold for* c_{15} *and* c_{10}.

Remark 4.2. – We call (34) [and likewise (34 A)] a "*non-reduced*" energy inequality, since the term $\| D_\omega{}^k u \|_{..}^{G_t^\omega}$ still have to be reduced to the data $\underset{\omega}{u}$. – The relevance of the *stable-boundedness* property is discussed in the introduction (subsection I.2) and in Remark B of Theorem 8.1. *All* the quantities c_k $(k = 1, 2, \ldots)$ occurring in our inequalities fulfil this property. The proof of the above energy inequalities fulfilling this property relies on Lemma 4.1.

We shall now consider, how the quantities $c_{20,1}$, $c_{20,2}$ depend on (g, a, b) and on h; the hyperbolicity constant h in turn depends on g and we write

$$h = h_g. \qquad (4.34\,D)$$

In this sense we state

DEFINITION 4.2. – Let $C_2(.,.,.)$ [or $C_1(.)$ respectively] be a real-valued function on $\{(z_1, z_2, z_3) \in \mathbb{R}^3 \,|\, z_1, z_2, z_3 \geqq 0\}$ (or $\{z \in \mathbb{R} \,|\, z \geqq 0\}$ respectively) and let g, a, b be the coefficients of four differential equation. Then we denote

$$C_1 := C_1(h_g), \qquad C_1^{(0)} := C_1(h_{\underset{0}{g}})$$
$$C_2 := C_2(h_g, \| g, a, b \|_1, \| g, a, b \|_2),$$
$$C_2^{(0)}(h_{\underset{0}{g}}, \| \underset{0}{g}, \underset{0}{a}, \underset{0}{b} \|_1, \| \underset{0}{g}, \underset{0}{a}, \underset{0}{b} \|_2)$$

with $\| g, a, b \|_k := [\| g \|_k^2 + \| a \|_k^2 + \| b \|_k^2]^{1/2}$ with $k = 1, 2$, where $\| . \|_k$ denotes some norm. We say that $[C_1, C_2]$ is *stable* (13) against small variations of (g, a, b), if and only if for any given $(\underset{0}{g}, \underset{0}{a}, \underset{0}{b})$ and $\varepsilon > 0$ there exists a $\delta > 0$ with

$$[C_1, C_2] \subseteqq [C_1^{(0)}, C_2^{(0)} + \varepsilon] \qquad (4.34\,E)$$

---

(13) Stability with respect to that space, which has the norm $| . |_s^{L_{T_1}}$. This is the space $H_s(L_{T_1})$ introducted in section 5.

for all (g, a, b) with

$$\|g-\underset{0}{g}, a-\underset{0}{a}, b-\underset{0}{b}\|_k<\delta \qquad \text{with} \quad k=1, 2. \tag{4.34 F}$$

In the case that C_2 depends on $\|g, a, b\|_1$ only one has to cancel every term with $\|.\|_2$. In the case of Theorem 4.2 we have $C_2(h_g, \|g, a, b\|_1):=c_{20,2}(h, \|g, a, b\|_1, T_1)$ with $h=h_g$ and $\|g, a, b\|_1:=T_1|g, a, b|_s^{L_{T_1}}$ [because of (33 E)].

Proof of Theorem 4.2. — There exists a constant c_0 (independent of t, w) such that for any $w\in C^\infty(L)$

$$\int_0^t |w|_l^{\Lambda_\tau, L}\,d\tau \leqq c_0\, t\,|w|_{l+1}^{L_t} \tag{4.35 A}$$

With (25), (35 A) and (1) we gain

$$\exp\left(\int_0^t c_{10}(h)\,\gamma_1{}^{\Lambda_\tau}\,d\tau\right)\leqq \exp(c_{13}(h)\,t\,|a, b, g|_m^{L_t})\leqq c_{14}(h, \gamma_m{}^{L_t}) \tag{4.35 A1}$$

for $m>\frac{n}{2}+2$ [with $\gamma_m{}^{L_T}$ of (33 E)]. (24) proves with (35 A1), (3.8) the *inequality* (33 F).

We now scetch the proof of (34); differentiating our differential equation, we gain

$$\bar{g}^{ab}D_aD_b\bar{u}+\bar{b}.D\bar{u}+\bar{a}.\bar{u}=\bar{f} \quad \text{on } L, \qquad \bar{u}=\underset{\omega}{\bar{u}} \quad \text{on } G^\omega \tag{4.35 B}$$

with

$$\left.\begin{array}{c}\bar{g}:=g, \qquad \bar{u}:=Du, \qquad \bar{b}:=b+Dg,\\ \bar{a}:=a+Db, \bar{f}:=Df-Da.u\end{array}\right\} \tag{4.35 C}$$

$$\underset{1}{\bar{u}}=\underset{2}{\bar{u}} \quad \text{on } G^1\cap G^2. \tag{4.35 D}$$

We apply (23) of Lemma 4.1 to the initial value problem (35 B), (35 D) and obtain

$$t^{1/2}|\bar{u}|_1^{\Lambda_t, L}\leqq c_{15}(h)\Bigg[\sum_\omega(t^{1/2}|\underset{\omega}{\bar{u}}|_1^{G_t{}^\omega})^2 + \int_0^t(\tau^{1/2}|\bar{u}|_1^{\Lambda_\tau, L}\,\tau^{1/2}|\bar{f}|_0^{\Lambda_\tau}+(\tau^{1/2}|\bar{u}|_1^{\Lambda_\tau, L})^2(\bar{\gamma}_1{}^{\Lambda_\tau}+t))\,d\tau\Bigg]. \tag{4.36}$$

We insert $\bar{u}:=Du$, $\underset{\omega}{\bar{u}}=[\bar{u}]_{G^\omega}=[Du]_{G^\omega}$ and (35 C) into (36) and we use the inequality (23), then we gain

$$t^{1/2}|u|_2^{\Lambda_\tau, L}\leqq c_{16}(h)\Big(\sum_\omega[t^{1/2}|\underset{\omega}{u}|_2^{G_t{}^\omega}+t^{1/2}|D_\omega u|_1^{G_t{}^\omega}] + \int_0^t[\tau^{1/2}|u|_2^{\Lambda_\tau, L}\,\tau^{1/2}|f|_1^{\Lambda_\tau, L}+(\tau^{1/2}|u|_2^{\Lambda_\tau, L})^2(\gamma_2{}^{\Lambda_\tau}+t)]\,d\tau\Big) \tag{4.36 A}$$

[with $\gamma_2 \Lambda t$ of (25)]. Thus we proceeded from estimate of $|u|_1^{\Lambda_t, \mathrm{L}}$ [*see* (23)] to the estimate of $|u|_2^{\Lambda_t, \mathrm{L}}$ [*see* (36 A)]; analogously we can proceed finally (by induction) to the estimate of $|u|_s^{\Lambda_t, \mathrm{L}}$. To this we apply the generalized Gronwal's Lemma A.1 of the appendix [this is the same procedure as in the derivation of (24)]; thus we obtain

$$|u|_s^{\Lambda_t, \mathrm{L}} \leqq c_{17}(h).(\exp c_{18}(h)t^2).\left(\exp \int_0^t c_{18}(h)\gamma_s^{\Lambda_\tau} d\tau\right)$$
$$\times\left[\left(\sum_{\omega=1}^{2}\left(|u|_s^{\mathrm{G}_t^\omega} + \sum_{k=0}^{s-1} |\mathrm{D}_\omega{}^k u|_{s-k}^{\mathrm{G}_t^\omega}\right)\right) + t^{-1/2}\int_0^t \tau^{1/2}|f|_{s-1}^{\Lambda_\tau, \mathrm{L}} d\tau\right] \quad (4.36\,\mathrm{B})$$

with $\gamma_s^{\Lambda_t}$ of (25) for any $s \geqq 1$. From this one deduces (34) *of Theorem* 4.2 in the same way as we deduced (33 F) from (24).

There exists a constant c_0 (independent of t, T, w) such that for any $w \in \mathrm{C}^\infty(\mathrm{G_T}^\omega)$ (and any integer $r \geqq 0$)

$$|w|_r^{\mathrm{G}_t^\omega} \leqq c_0 |w|_{r+1}^{\mathrm{G_T}^\omega} \quad \text{for} \quad t \leqq \mathrm{T}. \quad (4.37)$$

From the monotony properties (4.1), (3.4, 5, 8) and (37) it follows that the right-hand side of (36 B) is smaller (or equal) than

$$c_{19}(h, \gamma_s{}^{\mathrm{L_T}}, \mathrm{T})\left[\left(\sum_{\omega}\left[c_0 |\underset{\omega}{u}|_{s+1}^{\mathrm{G_T}^\omega} + \sum_{k=1}^{s-1} c_0 |\mathrm{D}_\omega{}^k u|_{\mathrm{G_T}^\omega, s-k}\right]\right) + \mathrm{T}|f|_{s-1}^{\mathrm{L_T}}\right]$$

for any $t \leqq \mathrm{T}$; this together with (34) *proves* (34 *A*).

(34 B) can be proved analogously to the proof of (25 A) by using (33 E). We shall now prove (34 C). Stability is meant with respect to that function space, which has the norm $\mathrm{T}_1 . |.|_s^{\mathrm{L_{T_1}}}$, which is equivalent to $|.|_s^{\mathrm{L_{T_1}}}$ (for fixed T_1). This function space is the Sobolev space $\mathrm{H}_s(\mathrm{L_{T_1}})$ of section 5. Now, let there be given $(\underset{0}{g}, \underset{0}{a}, \underset{0}{b}) \in \mathrm{H}_s(\mathrm{L_{T_1}}) \times \mathrm{H}_s(\mathrm{L_{T_1}}) \times \mathrm{H}_s(\mathrm{L_{T_1}})$. Let $\underset{0}{g}$ satisfy the *inequalities* (1.5)-(1.8) for $h = h_{g_0}$. *Also* g with

$$\max_{x \in \mathrm{L_{T_1}}} |g - \underset{0}{g}| \leqq \varepsilon_1 \quad (4.37\,\mathrm{A})$$

satisfies those inequalities for $h = g_{g_0}$, provided $\varepsilon_1 > 0$ issufficiently small. Thus

$$h_g = h_{\underset{0}{g}} \text{ for } g \text{ fulfilling (37 A)} \quad (4.37\,\mathrm{B})$$

with sufficiently small ε_1. Furthermore there exists a constant c_0 with

$$\max_{x \in \mathrm{L_{T_1}}} |g - \underset{0}{g}| \leqq c_0 |g - \underset{0}{g}|_s^{\mathrm{L_{T_1}}} \leqq c_0 |g - \underset{0}{g}, a - \underset{0}{a}, b - \underset{0}{b}|_s^{\mathrm{L_{T_1}}} \quad (4.37\,\mathrm{C})$$

for any $g, a, b, \underset{0}{g}, \underset{0}{a}, \underset{0}{b} \in \mathrm{H}_s(\mathrm{L_{T_1}})$.

Due to (37 B), (33 E) along with (37 C) there exists $\delta>0$ such that

$$c_{20,1}=c_{20}(\underset{0}{h_g},0,0),\qquad c_{20,2}=c_{20}(\underset{0}{h_g},|g, a, b|_s^{\mathrm{L}_{\mathrm{T}_1}}.\mathrm{T}_1, \mathrm{T}_1)\quad(4.37\,\mathrm{D})$$

for all (g, a, b) with

$$|g-\underset{0}{g}, a-\underset{0}{a}, b-\underset{0}{b}|_s^{\mathrm{L}_{\mathrm{T}_1}}\leqq\delta.\qquad(4.37\,\mathrm{E})$$

Furthermore $c_{20}(z_1, z_2, z_3)$ depends continuously on $(z_1, z_2, z_3)\in\mathbb{R}3$ with $z_1, z_2, z_3\geqq 0$ (*see* Definition 4.1). Thus the last quantity of (37 D) depends *continuously* on $(g, a, b)\in \mathrm{H}_s(\mathrm{L}_{\mathrm{T}_1})\times \mathrm{H}_s(\mathrm{L}_{\mathrm{T}_1})\times \mathrm{H}_s(\mathrm{L}_{\mathrm{T}_1})$. This and (37 D) imply that for given $\varepsilon>0$:

$$[c_{20,1}; c_{20,2}]\subseteqq[c_{20}(\underset{0}{h_g}, 0, 0); \varepsilon+c_{20}(\underset{0}{h_g}, |\underset{0}{g}, \underset{0}{a}, \underset{0}{b}|_s^{\mathrm{L}_{\mathrm{T}_1}}.\mathrm{T}_1, \mathrm{T}_1)]\quad(4.37\,\mathrm{F})$$

for (g, a, b) fulfilling (37 E) with sufficiently small δ.

Remark 4.3. – In order to obtain for the *cone problem* (I.1), (I.3) an *energy inequality*, which fulfils the *stable-boundedness property* (as in Theorem 4.6), one should proceed as follows: using the notation of section 2 we define analogously to (2.8) the following generalized Sobolev norms

$$|u|_{\mathrm{S}_{\mathrm{T}}, s}^{\mathrm{cone}}:=\operatorname*{ess\,sup}_{t\in[0,\mathrm{T}]}|u|_{s,\,\mathrm{cone}}^{\Sigma_t,\,\mathrm{S}_{\mathrm{T}}}\qquad\text{with}\quad\Sigma_t:=\mathrm{S}_{\mathrm{T}}\cap\Lambda_t\qquad(4.37\,\mathrm{G})$$

for $\mathrm{S}_{\mathrm{T}}=\mathrm{L}_{\mathrm{T}}$ or G_{T} (G_{T} is the data surface), where [*cf.* (2.3)]

$$|u|_{s,\,\mathrm{cone}}^{\Pi_t,\,\mathrm{R}}:=t^{-\delta_\pi}\left(\sum_{k=0}^{s}|\mathrm{D}_{\mathrm{R}}{}^k u|_{\Sigma_t}^2\right)^{1/2},$$

$$\delta_\pi:=[\text{dimension of }\Pi_t]\quad(4.37\,\mathrm{H})$$

for $\Pi_t=\Lambda_t$ or Γ_t or G_t or L_t (with $\mathrm{R}\supseteq\Pi_t$); furthermore we define [analogously to (2.4)]

$$|u|_{s,\,\mathrm{cone}}^{\Pi_t}:=|u|_{s,\,\mathrm{cone}}^{\Pi_t,\,\Pi_t}\qquad(4.37\,\mathrm{I})$$

with these norms one can derive *non-reduced energy inequalities*, which are analogous to (33 F), (34) (34 A) and which fulfil the *stable-boundedness property* (*see* Theorem 4.2).

The non-reduced energy inequality [*cf.* (34 A)] contains on the right-hand side terms like $[\mathrm{D}^k u]_{\mathrm{G}}$ (with $k=1, 2, \ldots$), where G is the cone on which the datum is given (D contains deritatives non-tangential to G). These terms have to be reduced to data (*cf.* section 4 *b*). This reduction should be done by the methods of [22], because during the reduction occur *singularities*, which are due to the singular tip of the cone.

We should add that one should use *our* method of deriving *non-reduced* energy inequalities and not the method of [22], since the non-reduced energy inequalities of [22] do *not* (^{14}) fulfil the stable-boundedness property.

4.b. Reduction to data

We shall now estimate the terms $|D_\omega{}^l u|_{G^\omega \dots}$ ($\omega = 1, 2$; $l = 0, 1, 2, \dots$) of our non-reduced energy inequality (34 A) by terms involving the *data only*. These estimates are obtained from energy inequalities of certain differential equations [*see* (40), (44)], which are fulfilled by the $[D_\omega{}^l u]_{G^\omega}$. These differential equations do not hold on L but just on the hypersurface G^ω.

We define $\overset{l}{w} := D_1{}^l w$, $g^\alpha := 2 g^{\alpha 1}$ (α, β, ... runs from 2 to $n+1$) and rewrite our differential (18) as follows

$$g^\alpha D_\alpha \overset{1}{u} + b^1 . \overset{1}{u} + g^{11} \overset{2}{u}$$

$$= f^1 := f - g^{\alpha\beta} D_\alpha D_\beta u - b^\alpha . D_\alpha u - a . u \quad \text{on L.} \qquad (4.38)$$

As G^1 *is a characteristic surface* and the covector $\delta_a{}^1$ is a normal of G^1, we gain

$$g^{11} = g^{ab} \delta_a{}^1 \delta_b{}^1 = 0 \quad \text{on } G^1. \qquad (4.39)$$

We evaluate (38) on G^1 and use (39), thus

$$g^\alpha D_\alpha \overset{1}{u} + b^1 . \overset{1}{u} = f_1 \quad \text{on } G^1. \qquad (4.40)$$

$$\overset{1}{u} = D_1 \underset{2}{u} \quad \text{on } \Gamma, \qquad (\Gamma \subset G^1) \qquad (4.41)$$

with

$$f_1 := f - g^{\alpha\beta} D_\alpha D_\beta \underset{1}{u} - b^\alpha . D_\alpha \underset{1}{u} - a . \underset{1}{u} \quad \text{on } G^1 \qquad (4.42)$$

where we used $u = \underset{\omega}{u}$ on G^ω and

$$[\overset{1}{u}]_\Gamma = [D_1 u]_\Gamma = [[D_1 u]_{G^2}]_\Gamma \qquad (\Gamma \subset G^2). \qquad (4.43)$$

We apply $D_1{}^{l-1}$ to (38) and obtain [analogously to (40-42)] for $l \geqq 2$

$$g^\alpha D_\alpha \overset{l}{u} + b^1 . \overset{l}{u} + \overset{1}{g}{}^{11} \overset{l}{u} = f_1 \quad \text{on } G^1 \qquad (4.44)$$

(^{14}) Dossa mentions this in section 3-3-5 of [22]. He also mentions that the energy inequality of [3] does not fulfil the stable-boundedness property. However, this gap has been filled in [4] as well as in the present paper in Theorem 4.2 and 4.6.

$$\overset{l}{u}=\mathrm{D}_1{}^{l}\underset{2}{u} \quad \text{on } \Gamma \qquad (\Gamma\subset \mathrm{G}^1) \tag{4.45}$$

with

$$f_1:=\overset{l-1}{f}+\sum_{k=1}^{l-1} c_{k,0}(\overset{k}{g}{}^{\alpha}\mathrm{D}_{\alpha}\overset{l-k}{u}+\overset{k\,1}{b}.\overset{l-k}{u})+\sum_{k=2}^{l-1}\bar{c}_{k,0}\overset{k}{g}{}^{11}\overset{l+1-k}{u}$$
$$+\sum_{k=0}^{l-1}\hat{c}_{k,0}(g^{\alpha\beta}\mathrm{D}_{\alpha\beta}\overset{l-1-k}{u}+\overset{k}{b}{}^{\alpha}.\mathrm{D}_{\alpha}\overset{l-1-k}{u}+\overset{k}{a}.\overset{l-1-k}{u})$$

$$\text{with } \overset{0}{u}=\underset{1}{u} \text{ on } \mathrm{G}^1 \tag{4.46}$$

with some constants $c_{k,0}$, $\tilde{c}_{k,0}$, $\hat{c}_{k,0}$.

At first one determines $\overset{1}{u}$ (on G^1) by the intial value problem (40, 41), where the coefficients and datum are given, since $\underset{1}{u}$, $\underset{2}{u}$ are given [$\underset{1}{u}$, $\underset{2}{u}$ are our data of (19)]. Then one determines $\overset{2}{u}$ by (44, 45) for $l=2$ (inserting the known $\overset{1}{u}$). Then finally (by induction) one determines $\overset{l}{u}$.

$\overset{1}{u}$ (and similarly $\overset{l}{u}$) fulfils and intitial value problem of the following type

$$g^{\alpha}\mathrm{D}_{\alpha}v+\beta.v=\gamma \quad \text{on } \mathrm{G}^1 \tag{4.47}$$
$$v=v_0 \quad \text{on } \Gamma \qquad (\Gamma\subset\mathrm{G}^1) \tag{4.48}$$

with v being the unknown.

(47) is a *hyperbolic* differential equation with

$$\hat{\tau}:=x^2=[\tau]_{\mathrm{G}^1} \quad \text{on } \mathrm{G}^1 \tag{4.49}$$

being a time function, fulfilling the following hyperbolicity condition

$$g^{\alpha}\mathrm{D}_{\alpha}\hat{\tau}\geqq[(h_2 h_1)^{-1}+2\,h_1{}^{-2}]^{-1/2}\geqq c_{20}(h) \quad \text{on } \mathrm{G}^1 \tag{4.50}$$

which follows from (49), (1.5-1.8).

(47, 48) is a *Cauchy initial value problem* (for a first-order differential equation on G^1); thus the usual energy inequalities of [2] hold, which we exhibit below in Lemma 4.3 As [2] deals with second-order differential equations [instead of first-order as in (47)] we shall give in the appendix a scetch of the proof of Lemma 4.3 (the proof is analogous to [2]).

LEMMA 4.3. – *Let* $g^{\alpha}\mathrm{D}_{\alpha}$ *fulfil the hyperbolicity conditions* (50, 49) *and let be* $g^{\alpha}:=2\,g^{\alpha 1}$ *with* g^{ab} *of section* 1. *Let* (g^{α}), β, γ, $v\in\mathrm{C}^{\infty}(\mathrm{G_T}^1)$. *Then* *for any solution* v *of* (47, 48) *the following energy inequality holds for* $m \geqq 0$

$$|v|_{G_T{}^1,\, m} \leqq c_{21}\,(h,\, \varepsilon_m^{G_T 1},\, \varepsilon_m^{\Gamma,\,1}).\left[\,|\underset{0}{v}|_m^{\Gamma} + \int_0^T |\gamma|_m^{\Gamma_\tau{}^1,\, G^1}\, d\tau + |\gamma|_{m-1}^{\Gamma,\, G^1}\right] \tag{4.51}$$

with

$$\varepsilon_m^{G_T{}^1} := T.(|\beta|_{m_0}^{G_T{}^1} + |(g^\alpha)|_{m_1}^{G_T{}^1}), \tag{4.52}$$

$$\varepsilon_m{}^{\Gamma,\,1} := |\beta|_{m_0-1}^{\Gamma,\, G^1} + |(g^\alpha)|_{m_1-1}^{\Gamma,\, G^1}, \qquad m_k \geqq m, \quad m_k > \frac{n-1}{2} + k. \tag{4.53}$$

On the right-hand side of our *final energy inequality* (68) *occur the data* $\underset{1}{u}$, $\underset{2}{u}$ *only* (apart from f, c...). In order to achieve this, we have to estimate $|D_\omega{}^k u|_{G^\omega}...$ of (35) by terms involving the data $\underset{1}{u}$, $\underset{2}{u}$ only (apart from $f.c$...).

One of these estimates, which we shall proove by using the above Lemma 4.3, is the following *reduction formula* (reduction to the data $\underset{1}{u}$, $\underset{2}{u}$):

LEMMA 4.4. — *Let the assumptions of Lemma* 4.1 *be fulfilled. Then for any solution* u *of* (18, 19) *it holds for* $m \geqq 0$

$$|D_1 u|_{G^1,\, m} \leqq c_{22}.[\,|D_1 \underset{2}{u}|_m^{\Gamma} + |\underset{1}{u}|_{m+2}^{G_T{}^1} + T\,|f|_m^{G_T{}^1} + |f|_{m-1}^{\Gamma,\, G_T{}^1}] \tag{4.54}$$

with

$$\left.\begin{aligned} c_{22} &:= c_{22}\,(h,\, \gamma_m{}^{G_T{}^1},\, \gamma_m{}^{\Gamma,\,1},\, \bar\varepsilon_m{}^{G_T{}^1},\, \bar\varepsilon_m{}^{\Gamma,\,1}),\\ \gamma_m{}^{G_T{}^1} &:= (|a|_{m-1}^{G_T{}^1} + |b|_{m_0}^{G_T{}^1} + |g|_{m_1}^{G_T{}^1}).T, \end{aligned}\right\} \tag{4.55}$$

$$\left.\begin{aligned} \gamma_m{}^{\Gamma,\,1} &:= |a|_{m_{-1}-1}^{\Gamma,\, G_T{}^1} + |b|_{m_0-1}^{\Gamma,\, G^1} + |g|_{m_1-1}^{\Gamma,\, G^1},\\ \bar\varepsilon_m{}^{G_T{}^1} &:= (|b^1|_{m_0}^{G_T{}^1} + |(g^{\alpha\,1})|_{m_1}^{G_T{}^1}).T, \end{aligned}\right\} \tag{4.56}$$

$$\begin{aligned} \bar\varepsilon_m{}^{\Gamma,\,1} &:= |b^1|_{m_0-1}^{\Gamma,\, G^1} + |(g^{\alpha\,1})|_{m_1-1}^{\Gamma,\, G^1},\\ m_k \geqq m, &\qquad m_k > \frac{n-1}{2} + k \qquad (k = -1,\, 0,\, 1), \end{aligned} \tag{4.57}$$

$$m'_k \geqq m,\ m'_k > \frac{n-1}{2} + k - 1 \qquad (k = -1,\, 0,\, 1). \tag{4.58}$$

Proof. — We shall use the following formula: There exists a constant c_0 (independent of T, w, e) such that for any w, $e \in C^\infty(G_T{}^1)$

$$\begin{aligned} &\int_0^T |ew|_r^{\Gamma_t,\, G^1}\, dt \leqq c_0 T\, |e|_{r_1}^{G_T{}^1}\, |w|_{r_2}^{G_T{}^1},\\ &r_{1,\,2} \geqq r, \qquad r_1 + r_2 > r + \frac{n-1}{2}, \end{aligned} \tag{4.59}$$

$$|w|_l^{\Gamma,\, G^\omega} \leqq c_0 |w|_{l+1}^{G_T^\omega}. \tag{4.60}$$

Now, setting $v=\overset{1}{u}=D_1 u$, we apply (51) to (40, 41) and again

$$|\overset{1}{u}|_{G_T^1,\, m} \leqq c_{21}\,(h,\, \bar{\varepsilon}_m^{\,G_T 1},\, \bar{\varepsilon}_m^{\,\Gamma,\, 1}) \times \left[|D_1 \underset{2}{u}|_m^\Gamma + \int_0^T |f_1|_m^{\Gamma_t^1,\, G^1}\, dt + |f_1|_{m-1}^{\Gamma,\, G^1} \right] \tag{4.61}$$

[$\bar{\varepsilon}$ of (56, 57)]. We insert f_1 of (42) into (61) and again

$$|\overset{1}{u}|_{G_T^1,\, m} \leqq c_{23}\,(h,\, \bar{e}_m^{G_T^1},\, \bar{\varepsilon}_m^{\Gamma,\, 1})[\,|D_1 \underset{2}{u}|_m^\Gamma + T\, |f|_m^{G_T^1} + |\underset{1}{u}|_{m+2}^{G_T^1} \cdot \gamma_m^{G_T^1} + |f|_{m-1}^{\Gamma,\, G^1} + |\underset{1}{u}|_{m+1}^{\Gamma,\, G^1} \cdot \gamma_m^{\Gamma,\, 1}] \tag{4.62}$$

[γ of (55, 56)], where we used (59). With (62) and (60) one proves (54).

The following *final version of the reduction formula* (reduction to the data $\underset{1}{u}$, $\underset{2}{u}$) holds:

LEMMA 4.5. – *Let the assumptions of Lemma* 4.1 *hold. Then for any solution of* (18, 19) *it holds*

$$\|u\|_{G_T^\omega,\, s,\, 1} \leqq c_{24}\,(h,\, \bar{\gamma}_s^{\,G_T^\omega},\, \bar{\gamma}_s^{\,\Gamma,\, \omega}) \times [\,|\underset{1}{u}|_{2s-1}^{G_T^1} + |\underset{2}{u}|_{2s-1}^{G_T^2} + T\, \|f\|_{s-1}^{G_T^\omega} + \|f\|_{s-1}^{\Gamma,\, G^\omega,\, 1}] \tag{4.63}$$

for $s \geqq \dfrac{n-1}{4} + \dfrac{5}{2}$,

$$\sum_{k=1}^{s-1} |D_\omega^{\ k} u|_{G_T^\omega,\, s-k} \leqq c_0 \|u\|_{G_T^\omega,\, s,\, 1} \tag{4.64}$$

for some constant c_0 (independent of T, u), where

$$\bar{\gamma}_s^{G_T^\omega} := \|a,\, b,\, g\|_{s-1}^{G_T^\omega} . T, \qquad \bar{\gamma}_s^{\Gamma,\, \omega} = \|a,\, b,\, g\|_{s-1}^{\Gamma,\, G^\omega,\, 1}. \tag{4.65}$$

Proof. – We shall prove (63) for $\omega=1$: Analogously to the derivation of the estimate of $|D^1 u|_{G_T^1,\, m}$ [*see* (54)] we now set $v=\overset{2}{u} := D_1^{\ 2} u$ and apply (51) to (44, 45) for $l=2$; then we gain an estimate for $|D_1^{\ 2} u|_{G_T^1,\, m}$ analogously to (54), except for the fact that terms like $|D_1\, u|_{G_T^1,\, \dots}$ occur, which we estimate by (54).

Analogously we gain an estimate of $|D_1^{\ 3} u|_{G_T^1,\, \dots}$ and finally (by induction) of $|D_1^{\ k} u|_{G_T^1,\, \dots}$; with this we obtain [with $\bar{\gamma}$ of (65)]

$$\sum_{k=1}^{s-1} |D_1^{\ k} u|_{G_T^1,\, 2\,(s-k)-1} \leqq c_{25}\,(h,\, \bar{\gamma}_s^{\,G_T^1},\, \bar{\gamma}_s^{\,\Gamma,\, 1}) \Bigg[\left(\sum_{k=1}^{s-1} |D_1^{\ k} \underset{2}{u}|_{2\,(s-k)-1}^{\Gamma} \right)$$

$$+|\underset{1}{u}|^{G_T{}^1}_{2\,s-1}+T\,\|f\|^{G_T{}^1}_{s-1}+\sum_{k=0}^{s-2}|D_1{}^k f|^{\Gamma,\,G^1}_{2\,(s-1-k)-2}\Bigg].\quad(4.67)$$

By using (60) we gain

$$|D_1{}^k\,\underset{2}{u}|^{\Gamma}_{2\,(s-k)-1}\leqq c_0\,|\underset{2}{u}|^{G_T{}^2}_{2\,s-k}\leqq c_0\,|\underset{2}{u}|^{G_T{}^2}_{2\,s-1}\qquad\text{for}\quad k=1,\ldots,s-1.$$

We insert this into (67) and thus *prove* (63).

(64) is *proven* by the definition of $\|\,.\,\|_{G_T{}^\omega,\,s,\,1}$ and $s-k\leqq 2\,(s-k)-1$ for $k=1,\ldots,s-1$.

Finally, we are able to state the *reduced energy inequality:*

THEOREM 4.6. – *Let* g^{ab} *of our initial value problem* (18, 19) *be regularly* hyperbolic *on* L_T *with hyperbolicity constant* $h>0$ [cf. (1.3)]. *Let* g, b, a, $f\in C^\infty(L_T)$, $\underset{\omega}{u}\in C^\infty(G_T{}^\omega)$ $(\omega=1,2)$, $\underset{1}{u}=\underset{2}{u}$ *on* Γ. *Then for any solution* $u\in C^\infty(L_T)$ *of* (18, 19) *the following* reduced energy inequalities *hold for* $s>\frac{n}{2}+2$ *and* $n\geqq 1$

$$\|u\|^{L_T}_s\leqq R_s{}^T:=c_{32}\,(h,\bar\gamma^{L_T}_s,\bar\gamma^{\Gamma}_s,T)$$
$$\times\left[\left(\sum_{\omega=1}^{2}|\underset{\omega}{u}|^{G_T{}^\omega}_{2\,s-1}\right)+T\,\|f\|^{L_T}_{s-1}+\sum_{\omega=1}^{2}\|f\|^{\Gamma,\,G^\omega,\,1}_{s-1}\right],\quad(4.68)$$

$$\|u\|_{L_T,\,s,\,1}\leqq R_s{}^T,\quad(4.69)$$

$$\|u\|_{L_T,\,s}\leqq c_{33}\,(h,\bar\gamma^{L_T}_s,\bar\gamma_s{}^\Gamma,T)$$
$$\times\left[\left(\sum_{\omega=1}^{2}|\underset{\omega}{u}|_{G_T{}^\omega,\,2\,s-1}\right)+T\,\|f\|^{L_T}_{s-1}+\sum_{\omega=1}^{2}\|f\|^{\Gamma,\,G^\omega,\,1}_{s-1}\right],\quad(4.70)$$

where

$$\bar\gamma^{L_T}_s:=\|a,b,g\|^{L_T}_s.T,\qquad\gamma_s{}^\Gamma:=\sum_{\omega=1}^{2}\|a,b,g\|^{\Gamma,\,G^\omega,\,1}.\quad(4.71)$$

Moreover, the energy inequality "constants" c_{32}, c_{33} fulfil the *stable-boundedness property*, i.e. for any $T'\in[0,T_0]$ it holds

$$c_{33}\,(h,\bar\gamma^{L_T}_s,\gamma^{\Gamma}_s,T)\in[c_{33,1},c_{33,2}]\qquad\text{for all}\quad T\in[0,T']\quad(4.71\ A)$$

with $c_{33,1}:=c_{33}\,(h,0,0,0)$ and

$$c_{33,2}:=c_{33}\left(h,\|a,b,g\|^{L_T}_s.T',\sum_{\omega=1}^{2}\|a,b,g\|^{\Gamma,\,G^\omega,\,1},T'\right),\quad(4.71\ B)$$

where the intervalle

$[c_{33,1}, c_{33,2}]$ is *stable* against small variations of the coefficients (a, b, g), (4.71 C) where stability is defined in Definition 4.2. The same properties hold for c_{32}.

Remark 4.4. – In the quasilinear case one can majorize the right-hand side of (71 B) by

$$c_{32}\,(h,\ \| a,\ b,\ g\|_s^{L_{T'}}.\mathrm{T}',\ d,\ \mathrm{T}') \tag{4.71 D}$$

where d depends (¹⁵) on the value of the *given data* (and its derivatives) taken *on* Γ *only*. Thus d is independent of (a, b, g) and T′. The stability (¹⁵) of (71 C) then refers [according to (71 D)] to the terms $(h,\ \| a,\ b,\ g\|_s^{L_{T'}}.\mathrm{T}')$ only.

Proof. – We insert (64) into (34 A) and then insert (63); thus we obtain

$$|u|_{L_T,\,s}\leqq[c_{20}\,(h,\ \bar{\gamma}_s{}^{L_T},\ \gamma_s{}^{\Gamma},\ \mathrm{T})]\,.\,\bar{\mathrm{R}}_s{}^{\mathrm{T}} \tag{4.72}$$

with

$$\bar{\mathrm{R}}_s{}^{\mathrm{T}}:=\left(\sum_{\omega=1}^{2}|\underset{\omega}{u}|_{2\ s-1}^{G_T^{\omega}}\right)+\mathrm{T}\,\|f\|_{s-1}^{L_T}+\sum_{\omega=1}^{2}\|f\|_{s-1}^{\Gamma,\ G^{\omega},\ 1}$$

To this we add (63):

$$\|u\|_{L_T,\ s,\ 1}\leqq c_0\left(|u|_{L_T,\ s}+\sum_{\omega=1}^{2}\|u\|_{G_T^{\omega},\ s,\ 1}\right)$$
$$\leqq c_0\,(c_{20}+c_{24})\,\bar{\mathrm{R}}_s{}^{\mathrm{T}}\leqq c_{29}\,\bar{\mathrm{R}}_s{}^{\mathrm{T}} \tag{4.73}$$

for some constant $c_0>0$ and $c_{29}:=c_{29}\,(h,\ \bar{\gamma}_s{}^{L_T},\ \gamma_s{}^{\Gamma},\ \mathrm{T})$ [with $\bar{\gamma}$, γ of (71)].

With (3.4) and (73) we again

$$\|u\|_{s,\ 1}^{L_T}\leqq[c_{30}\,(h,\ \bar{\gamma}_s{}^{L_T},\ \gamma_s{}^{\Gamma},\ \mathrm{T})]\,.\,\bar{\mathrm{R}}_s{}^{\mathrm{T}}.$$

To this we add $\sum_{\omega}|u|_{2\ s-1}^{G_T^{\omega}}$ (using $u=\underset{\omega}{u}$ on $G_T{}^{\omega}$) and gain

$$\|u\|_s{}^{L_T}\leqq[c_{31}\,(h,\ \bar{\gamma}_s{}^{L_T},\ \gamma_s{}^{\Gamma},\ \mathrm{T})\,.\,\bar{\mathrm{R}}_s{}^{\mathrm{T}}. \tag{4.74}$$

For $c_{32}:=c_{31}+c_{29}$ it follows from (74) [(73) respectively]

$$\|u\|_s{}^{L_T}\leqq c_{32}\,\bar{\mathrm{R}}_s{}^{\mathrm{T}},$$
$$\|u\|_{L_T,\ s,\ 1}\leqq c_{32}\,\bar{\mathrm{R}}_s{}^{\mathrm{T}}.$$

(¹⁵) In the quasilinear case (*see* subsection I.1) is $a=b=0$, $g=\mathrm{F}\,(u^{\mathrm{A}})$ where $u:=(u^{\mathrm{A}}):=(u^1,\ \dots,\ u^{\mathrm{N}})$ is the unknown and *F some given* function on some subset of $\mathbb{R}^{\mathrm{N}}$. The *variation* (with respect to stability) of g is caused by a variation of u, whereby the following quantities remain *fixed*: $[u]_{G^{\omega}}=\underset{\omega}{u}$ ($\underset{\omega}{u}$ being the *given data*), $[\mathbb{F}\,(u)]_{\Gamma}=[\mathrm{F}\,(\underset{\omega}{u})]_{\Gamma}$, $[\mathrm{D}_{G^{\omega}}\,\mathrm{F}\,(u)]_{\Gamma}=\left[\left(\dfrac{\partial \mathrm{F}}{\partial u^{\mathrm{A}}}\,(\underset{\omega}{u})\right)\mathrm{D}_{G^{\omega}}\,\underset{\omega}{u}^{\mathrm{A}}\right]_{\Gamma}$, ... One can express $\|g\|_s^{\Gamma,\ G^{\omega},\ 1}$ by these *fixed* quantities.

This proves (68, 69).

With (73) and (3.4) it follows

$$\| u \|_{L_T,\, s,\, 1} \leqq c_{33}\,(h,\, \bar{\gamma}_s^{L_T},\, \gamma_a^{\Gamma},\, T) \times \left[\left(\sum_{\omega=1}^{2} | \underset{\omega}{u} |_{G_\omega^T,\, 2\, s-1} \right) + T\, \| f \|_{s-1}^{L_T} + \sum_{\omega=1}^{2} \| f \|_{s-1}^{\Gamma,\, G^\omega,\, 1} \right].$$

To this we add $\sum_{\omega=1}^{2} | u |_{G_T^\omega,\, 2\, s-1}$ (using $u = \underset{\omega}{u}$ on G_T^ω) and thus *prove* (70).

The proof of (71 A), (71 B) is analogous to the proof of (34 B), (34 C).

5. THE SPACES $\mathfrak{h}_s$, H_s; UNIQUENESS OF THE SOLUTION

So far the Lemmas and Theorems used C^∞ functions. Now we shall use Sobolev-class functions. At first we shall give a few definitions:

Let $S = L_T$ or let $S \subset L_T$ be a smooth compact surface. We define $L^2(S) := \left\{ v \,|\, v \text{ is measurable on S and } \int_S v^2\, dS < \infty \right\}$, where we shall write $v = w$ (for $v, w \in L$), if and only if *v differs from w only on a set of measure zero.*

We define the *Sobolev-space*

$$H_s(S) := \{ v \in L^2(S) \,|\, D_s^k v \in L^2(S),\, k = 1, \ldots, s \}$$

with its *norm* $|\,.\,|_s^S$ ($D_s^k v$ in the distributional sense). $H_s(S)$ is a *Hilbert-space* and according to [1] is

$$C^\infty(S) \textit{ dense in } H_s(S). \tag{5.1}$$

Furthermore we *denote* by $\mathfrak{h}_s(L_T)$ *the completion of* $C^\infty(L_T)$ *with respect to* the norm $\|\,.\,\|_s^{L_T}$; we endowe $\mathfrak{h}_s(L_T)$ with the *norm* $\|\,.\,\|_s^{L_T}$; some properties of $\mathfrak{h}_s(L_T)$ are listed below in Lemma 5.2.

For some Banach space B we *define* B* as the dual of B, *i.e.*

$$B^* := \{ F \,|\, F : B \to \mathbb{R},\, F \text{ linear and continuous} \};$$

furthermore,

$$v_n \Rightarrow v \in B$$

means convergence in B (also $B = \mathbb{R}$ is allowed), whereas

$$b_n \to v \in B$$

means weak convergence, *i.e.* $\lim_{n \to \infty} F(v_n) = F(v)$ for all $F \in B^*$.

For the proof of the uniqueness and existence theorem we shall need the following four Lemmas:

LEMMA 5.1. – *Let* B_k $(k=1, 2, 3)$ *denote a real Banach-space with its norm* $\|\,.\,\|_{\mathrm{B}_k}$; *let*

$$\beta\colon \left\{\begin{matrix} (h, v) \to \beta\,(h, v) \\ \mathrm{B}_1 \times \mathrm{B}_2 \to \mathrm{B}_3 \end{matrix}\right\} \text{ be bilinear with}$$

$$\|\beta\,(h, v)\|_{\mathrm{B}_3} \leqq c\,\|h\|_{\mathrm{B}_1} \cdot \|v\|_{\mathrm{B}_2} \tag{5.1 A}$$

for some constant c independent of h, v. Then

$$\beta\,(h_n, v_n) \to \beta\,(h, v) \in \mathrm{B}_3, \tag{5.2}$$

provided

$$h_n \;\Rightarrow\; h \in \mathrm{B}_1, \qquad v_n \to v \in \mathrm{B}_2, \qquad \|v_n\|_{\mathrm{B}_2} \leqq c' \tag{5.3}$$

for some constant c′ independent of n. Furthermore,

$$\beta\,(h_n, v_n) \;\Rightarrow\; \beta\,(h, v) \in \mathrm{B}_3 \tag{5.4}$$

provided

$$h_n \;\Rightarrow\; h \in \mathrm{B}_1, \qquad v_n \;\Rightarrow\; v \in \mathrm{B}_2. \tag{5.5}$$

Proof. – β is bilinear, thus

$$\beta\,(h_n, v_n) - \beta\,(h, v) = \beta\,(h_n - h, v_n) + \beta\,(h, v_n - v). \tag{5.6}$$

Furthermore,

$$\beta\,(h_n - h, v_n) \;\Rightarrow\; 0 \in \mathrm{B}_3, \tag{5.7}$$

which follows from (1 A), (3). We insert

$$\beta\,(h, v_n - v) \to 0 \in \mathrm{B}_3 \tag{5.8}$$

and (7) into (6) and thus *prove* (2).

It remains to show (8): Because of (1 A) is $\beta\,(h, .) : \mathrm{B}_2 \to \mathrm{B}_3$ continuous and linear (for fixed h); for any $\mathrm{F} \in \mathrm{B}_3{}^*$ is $\mathrm{F} : \mathrm{B}_3 \to \mathbb{R}$ continuous and linear; hence $\mathrm{F}\,(\beta\,(h, .))$: $\mathrm{B}_2 \to \mathbb{R}$ is continuous and linear, *i.e.* $\mathrm{F}\,(\beta\,(h, .) \in \mathrm{B}_2{}^*$. This and (3) implies $\lim\limits_{n \to \infty} \mathrm{F}\,(\beta\,(h, v_n - v)) = 0$ (for any $\mathrm{F} \in \mathrm{B}_3{}^*$); this implies (8).

We now prove (4). Due to (5) there exists a constant c'' (independent of n) with $\|v_n\|_{\mathrm{B}_2} \leqq c''$; thus one can use the proof above in order to obtain $\beta\,(h_n - h, v_n) \Rightarrow 0 \in \mathrm{B}_3$ [*see* (7)] and analogously $\beta\,(h, v_n - v) \Rightarrow 0 \in \mathrm{B}_3$; this implies (4) by using (6).

We summarize the following obvious fact on $\mathfrak{h}_s\,(\mathrm{L_T})$:

LEMMA 5.2. – $\mathfrak{h}_s\,(\mathrm{L_T})$ *is a* Hilbert space *with respect to the inner product*

$$(u, v)_{\mathfrak{h}_s} := (u, v)_{\mathrm{H}_s(\mathrm{L_T})} + \sum_{\omega=1}^{2} \sum_{k=0}^{s-1} (\mathrm{D}_\omega{}^k\, u, \mathrm{D}_\omega{}^k\, v)_{\mathrm{H}_{2\,(s-k)-1}(\mathrm{G}_T{}^\omega)} \tag{5.9}$$

with

$$(u, v)_{\mathrm{H}_m(\mathrm{S})} := \sum_{l=0}^{m} \int_{\mathrm{S}} (\mathrm{D}_s{}^l u)(\mathrm{D}_s{}^l v)\, d\mathrm{S}. \tag{5.10}$$

Furthermore,

$$\mathfrak{h}_s(\mathrm{L_T}) \hookrightarrow \mathrm{H}_s(\mathrm{L_T}), \tag{5.11}$$

i. e. $\mathfrak{h}_s(\mathrm{L_T})$ *is* continuously imbedded *in* $\mathrm{H}_s(\mathrm{L_T})$. *By definition is*

$$\mathrm{C}^\infty(\mathrm{L_T}) \text{ dense } \mathit{in}\ \mathfrak{h}_s(\mathrm{L_T}). \tag{5.12}$$

Furthermore let $\{v_n\}$ *be a bounded sequence:*

$$\| v_n \|_s^{\mathrm{L_T}} \leqq c \qquad (\textit{for any } n), \tag{5.13}$$

then there exists a subsequence $\{v_{\bar{n}}\}$ *and* $v \in \mathfrak{h}_s(\mathrm{L_T})$ *with*

$$v_{\bar{n}} \rightarrow v \in \mathfrak{h}_s(\mathrm{L_T}), \qquad \| v \|_{\mathrm{L_T}} \leqq c. \tag{5.14}$$

Remark. — $\mathfrak{h}_s(\mathrm{L_T})$ is a reflexive Banach space; thus $\{ v \in \mathfrak{h}_s(\mathrm{L_T}) \mid \| v \|_s^{\mathrm{L_T}} \leqq c \}$ is weakly sequentially compact and thus (13) implies (14).

LEMMA 5.3. — *Let* $\mathrm{S} = \mathrm{L_T}$ *or* $\mathrm{S} \subset \mathrm{L_T}$ *a smooth surface (fulfilling the cone-property of* [1]). *Let* $\hat{\mathrm{S}} = \{ x \in \mathrm{S} \mid \tau(x) = \mathrm{Const.} \}$ *or* $\hat{\mathrm{S}} = \{ x \in \mathrm{S} \mid x^k = \mathrm{Const.} \}$ *for some given* k $(k \in \{1, \ldots, n+1\})$. *Let* $\dim \hat{\mathrm{S}} = (\dim \mathrm{S}) - 1$. *Then* $v_n \rightarrow v \in \mathrm{H}_m(\mathrm{S})$ *implies* $[v_n]_{\hat{\mathrm{S}}} \rightarrow [v]_{\hat{\mathrm{S}}} \in \mathrm{H}_{m-1}(\hat{\mathrm{S}})$ *for* $m \geqq 1$.

Proof. — The map

$$\rho : \left\{ \begin{array}{c} w \rightarrow [w]_{\hat{\mathrm{S}}} \\ \mathrm{H}_m(\mathrm{S}) \rightarrow \mathrm{H}_{m-1}(\hat{\mathrm{S}}) \end{array} \right\} \text{ is continuous for any } m \geqq 1 \tag{5.15}$$

(*see* [1]). Given any $\mathrm{F} \in \mathrm{H}_{m-1}(\hat{\mathrm{S}})^*$ then $\mathrm{F}(\rho(.))$ is a continuous linear map $\mathrm{H}_m(\mathrm{S}) \rightarrow \mathbb{R}$, which implies with $v_n \rightarrow v \in \mathrm{H}_m(\mathrm{S})$ the following:

$$\mathrm{F}([v_n]_{\hat{\mathrm{S}}}) = \mathrm{F}(\rho(v_n)) \quad \Rightarrow \quad \mathrm{F}(\rho(v)) = \mathrm{F}([v]_{\hat{\mathrm{S}}}) \in \mathbb{R};$$

this *proves* $[v_n]_{\hat{\mathrm{S}}} \rightarrow [v]_{\hat{\mathrm{S}}} \in \mathrm{H}_{m-1}(\hat{\mathrm{S}})$.

The following Lemma is concerned with some kind of *continuity of our operator* $\mathbb{L}$ of (4.18):

LEMMA 5.4. — *Let* $s > \dfrac{n+1}{2}$ *and*

$$\begin{gathered} (g_n, b_n, a_n) \ \Rightarrow\ (g, b, a) \in \mathfrak{h}_s(\mathrm{L_T}) \times \mathfrak{h}_s(\mathrm{L_T}) \times \mathfrak{h}_s(\mathrm{L_T}) \\ u_n \rightarrow u \in \mathfrak{h}_s(\mathrm{L_T}), \qquad \| u_n \|_s \leqq c_{\mathrm{T}}, \end{gathered} \tag{5.16}$$

then

$$\mathbb{L}_n u_n := [g_n{}^{ab} \mathrm{D}^a \mathrm{D}_b u_n + b_n{}^a . \mathrm{D}_a u_n + a_n . u_n] \rightarrow \mathbb{L} u \in \mathfrak{h}_{s-2}(\mathrm{L_T}). \tag{5.17}$$

On the other hand, if $m > \frac{n}{2} + 2$ and

$$(g_n, b_n, a_n) \;\Rightarrow\; (g, b, a) \in \mathrm{H}_m(\mathrm{L_T}) \times \mathrm{H}_m(\mathrm{L_T}) \times \mathrm{H}_m(\mathrm{L_T}) \tag{5.18}$$

$$u_n \;\Rightarrow\; u \in \mathrm{H}_2(\mathrm{L_T}), \tag{5.19}$$

then

$$\mathbb{L}_n u_n \;\Rightarrow\; \mathbb{L} u \in \mathrm{H}_0(\mathrm{L_T}). \tag{5.20}$$

Proof. – Each part of $\mathbb{L} u$ is of the type $h . \mathrm{D}^k u =: \beta_k(h, u)$ (with $k = 0, 1, 2$) and it holds for $s > \frac{n+1}{2}$

$$\left\{ \begin{array}{c} \beta_k(h_n, u_n) \to \beta_k(h, u) \in \mathfrak{h}_{s-2}(\mathrm{L_T}), \\ \text{if } h_n \Rightarrow h \in \mathfrak{h}_s(\mathrm{L_T}) \text{ and } u_n \to u \in \mathfrak{h}_s(\mathrm{L_T}) \text{ and } \| u_n \| \leqslant c_\mathrm{T} \end{array} \right\}, \tag{5.21}$$

which can be proven as follows: because of (3.1) we gain

$$\| \beta_k(h, u) \|_{s-2}^{\mathrm{L_T}} = \| h . \mathrm{D}^k u \|_{s-2}^{\mathrm{L_T}} \leqslant c_0 \| h \|_s^{\mathrm{L_T}} \| u \|_s^{\mathrm{L_T}} \qquad (k = 0, 1, 2) \tag{5.22}$$

moreover, $\beta_k : \mathfrak{h}_s(\mathrm{L_T}) \times \mathfrak{h}_s(\mathrm{L_T}) \to \mathfrak{h}_{s-2}(\mathrm{L_T})$ is bi-linear; with this and (22) we can apply Lemma 5.1 and gain (21).

(21) implies (17). Analogously one proves (20).

We can now prove the following Theorem, which is concerned with an energy inequality (for Sobolev-class coefficients) and *uniqueness* of solutions:

THEOREM 5.5. – *We assume that g^{ab} of our differential equation* (4.18) *is regularly* hyperbolic *with a hyperbolicity constant $h > 0$* [cf. (1.3)]. *Furthermore, we assume*

$$\begin{array}{ll} g, b, a \in \mathrm{H}_m(\mathrm{L_T}) & \text{with } m > \frac{n}{2} + 2, \\ f \in \mathrm{H}_0(\mathrm{L_T}), & \underset{\omega}{u} \in \mathrm{H}_1(\mathrm{G_T}^{\omega}). \end{array} \tag{5.23}$$

Then the following holds:

(*I*) *Any solution $u \in \mathrm{H}_2(\mathrm{L_T})$ of* (4.18, 19) *fulfils the following* energy inequality (*for any $t \in (0, \mathrm{T}]$*)

$$| u |_1^{\Lambda_t, L} \leqslant c_{15}(h, \gamma_m{}^{\mathrm{L}_t}, t) \left[\sum_{\omega=1}^{2} | \underset{\omega}{u} |_1^{\mathrm{G}_t{}^{\omega}} + t | f |_0^{\mathrm{L}_t} \right] \tag{5.24}$$

with $\gamma_m{}^{L_t} := t | g, b, a |_1^{\mathrm{L}_t}$.

(*II*) (4.18) *has* at most one solution $u \in \mathrm{H}_2(\mathrm{L_T})$, *which assumes the given data $\underset{\omega}{u}$ on* $\mathrm{G_T}^{\omega}$ $(\omega = 1, 2)$.

Proof. – We shall prove (24) by C^∞-approximation: Because of (23) there exists g_n, b_n, $a_n \in C^\infty(L_T)$ with

$$(g_n, b_n, a_n) \Rightarrow (g, b, a) \in H_m(L_T) \times H_m(L_T) \times H_m(L_T). \tag{5.25}$$

As $u \in H_2(L_T)$ then $\exists\, u_n \in C^\infty(L_T)$ with

$$u_n \to u \in H_2(L_T). \tag{5.26}$$

This implies by using (15)

$$[u_n]_{G_T{}^\omega} \Rightarrow [u]_{G_T{}^\omega} \in H_1(G_T{}^\omega) \tag{5.27}$$

and for any $t \in (0, T]$

$$[D^l u_n]_{\Lambda_t} \Rightarrow [D^l u]_{\Lambda_t} \in H_0(\Lambda_t) \qquad \text{for} \quad l \in \{0; 1\}. \tag{5.28}$$

We define $f_n := \mathbb{L}_n u_n \in C^\infty(L_T)$ and $\underset{\omega}{u_n} := [u_n]_{G_T{}^\omega}$; the latter implies

$$\underset{1}{u_n} = \underset{2}{u_n} \quad \text{on } \Gamma, \tag{5.29}$$

since $u_n \in C^\infty(L_T)$. Now we can consider u_n as a *solution* of

$$\mathbb{L}_n u_n = f_n \quad \text{on } L_T, \qquad u_n = \underset{\omega}{u_n} \quad \text{on } G_T{}^\omega. \tag{5.30}$$

We apply Lemma (4.2) to (30, 29) and gain

$$|u_n|_1^{\Lambda_t, L} \leqslant [c_{15}(h, \gamma_m{}^{L_t, n}, t]\left[\sum_{\omega=1}^{2} |u_n|_1^{G_t\,\omega} + t\,|f_n|_0^{L_t}\right] \tag{5.31}$$

$$\text{with } \gamma_m{}^{L_t, n} := |g_n, b_n, a_n|_m^{L_t} \tag{5.32}$$

for sufficiently big n [we also used (1.5-8), (25), compactness of L_T).

(25) implies by Lemma 5.4

$$f_n := \mathbb{L}_n u_n \Rightarrow \mathbb{L}\, u \in H_0(L_t). \tag{5.33}$$

For any $t \in (0, T]$ holds: (28) and (25, 32) imply

$$|u_n|_1^{\Lambda_t, T} \Rightarrow |u|_1^{\Lambda_t, L} \in \mathbb{R}, \qquad \gamma_m{}^{L_t, n} \Rightarrow \gamma_m{}^{L_t} \in \mathbb{R}; \tag{5.34}$$

from (27) and (33) follows

$$|\underset{\omega}{u_n}|_1^{G_t\,\omega} \Rightarrow |\underset{\omega}{u}|_1^{G_t\,\omega} \in \mathbb{R}, \qquad |f_n|_0^{L_t} \Rightarrow |f|_0^{L_t} \in \mathbb{R}; \tag{5.35}$$

(34) and Definition (4.1) imply

$$c_{15}(h, \gamma_m{}^{L_t, n}, t) \to c_{15}(h, \gamma_m{}^{L_t}, t) \in \mathbb{R}. \tag{5.36}$$

We insert (34, 35, 36) into (31) and obtain (24).

It remains to prove *uniqueness*: Let us assume that (4.18, 19) has a further solution $\bar{u} \in H_2(L_T)$. Then

$$\mathbb{L}[u - \bar{u}] = \mathbb{L}\, u - \mathbb{L}\,\bar{u} = f - f = 0 \quad \text{on } L_T, \tag{5.37}$$

$$[u - \bar{u}] = u - u = 0 \quad \text{on } G_T{}^\omega. \tag{5.38}$$

To the initial value problem (37, 38) we apply part I of Theorem 5.5 and gain for all $t \in (0, \mathrm{T}]$

$$\| [u-\bar{u}] \|_1^{\Lambda_t, \mathrm{L}} \leqslant c_{15} . 0$$

and hence $[u-\bar{u}]=0$ on Λ_t, thus $u=\bar{u}$ on L_T.

6. EXISTENCE OF A SOLUTION IN $\mathfrak{h}^s$ AND ITS ENERGY INEQUALITY

6.a Theorem and remarks

At first we shall show the unique existence of a C^∞-*solution* in the case that the coefficients and data are C^∞. Then we treat coefficients and data, which are Sobolev-class functions, by *approximating* these coefficients and data by a sequence of C^∞-coefficients and data. With this sequence of C^∞-coefficients and data we form a sequence of initial value problems, which have (*see* belowe) C^∞-*solutions*. This sequence of C^∞-solution converges against a *Sobolev-class solution* of our original initial value problem (with Sobolev-class coefficients and data).

Now we begin with the c^∞-case:

LEMMA 6.1. — *We assume that* g^{ab} *of* (4.18) *is regularly* hyperbolic *on* L_T *with a hyperbolic constant* $h>0$. *Let*

$$g,\ b,\ a,\ f \in \mathrm{C}^\infty(\mathrm{L}_t), \qquad \underset{\omega}{u} \in \mathrm{C}^\infty(\mathrm{G}_\mathrm{T}{}^\omega) \quad (\omega=1, 2), \\ \underset{1}{u} = \underset{2}{u} \quad \text{on } \Gamma. \tag{6.1}$$

Then there exists a solution $u \in \mathrm{C}^\infty(\mathrm{L}_\mathrm{T})$ *of* (4.18), *which assumes the given data* $\underset{\omega}{u}$ *on* $\mathrm{G}_\mathrm{T}{}^\omega$. *The solution u is unique.*

Remarks. — The uniqueness follows from Theorem 5.5. The existence of a solution has been shown in [6], [7] and [3].

We now state the *existence* (and uniqueness) Theorem:

THEOREM 6.2. — *We assume that* g^{ab} *of our differential equation* (4.18) *is regularly* hyperbolic *on* L_T *with a hyperbolic constant* $h>0$. *Let for the coefficients hold*

$$g,\ b,\ a \in \mathfrak{h}_s(\mathrm{L}_\mathrm{T}), \qquad f \in \mathfrak{h}_{s-1}(\mathrm{L}_\mathrm{T}), \tag{6.2}$$

and for the data

$$\underset{\omega}{u} \in \mathrm{H}_{\mathrm{S}-1}(\mathrm{G}_\mathrm{T}{}^\omega), \qquad [\underset{\omega}{u}]_\Gamma \in \mathrm{H}_{2s-1}(\Gamma), \qquad \underset{1}{u} = \underset{2}{u} \quad \text{on } \Gamma \tag{6.3}$$

with $s > \frac{n}{2} + 2$ *and* $n \geqslant 1$. *Then there exists a unique* solution $u \in \mathfrak{h}_s(L_T)$ *of* (4.18), *where u assumes the given data* $\underset{\omega}{u}$ *on* $G_T{}^\omega$. *Moreover, u fulfils the* energy inequality (4.68), *where the energy inequality "constant"* c_{32} [*of* (4.68)] *fulfils the* stable-boundedness property *as pointed out in Theorem* 4.6; *the quantity* $c_{32}(.,\ .,\ .,\ .)$ *is* continuous *and* increasing (see *Definition* 4.1). *Furthermore* c_0 *of* (3.1)-(3.8) *does not depend on* $t \in [0, T_0]$.

Remark A. — One can apply the above Theorem to an iteration for treating the *quasilinear* characteristic initial value problem. This iteration is described in Remark B of section 8, provided one *replaces*

$$\mathfrak{C}_s(L_T) \text{ by } \mathfrak{h}_s(L_T),$$
$$\|\cdot\|_{L_T, s} \text{ by } \|\cdot\|_s^{L_T},$$

"Theorem 8.1" by "Theorem 6.2".

The iteration fulfils the properties mentioned in Remark B of section 8 with two exceptions: $\|.\|_s^{L_T}$ does not fulfil the ball property (*see* Remark 3.1) and we are (so far?) not able to prove the convergence of this iteration. However, we are able to prove the convergence of the iteration of Remark B of section 8 (using the norm $\|.\|_{L_T, s}$).

Remark B. — Also the case $\underline{s = \infty}$ is included in Theorem 6.2: the data, coefficients and the solution are $\overline{C^\infty}$-functions (due to $H^\infty = C^\infty$).

Remark C. — The *gap of differentiability class* between solution ($\in \mathfrak{h}_s$) and data ($\in H_{2s-1}$) amount to $(s-1)$ differentiability orders. It is in principle *not* possible to reduce this gap by *more* than half (16) a differentiability order. This can be seen by the following simple *example*: Let there be given any (small) $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$) and any $s > \frac{n}{2} + 2$ ($s \in \mathbb{N}$) and let us consider the following characteristic initial value problem for $u = u(x^1, x^2, y)$

$$D_2 D_1 u = \frac{1}{2} D_y D_y u \quad \text{on } L \tag{6.3 A}$$

$$u = \underset{\omega}{u} \quad \text{on } G^\omega. \tag{6.3 B}$$

The data are choosen such that

$$\underset{1}{u} = \underset{2}{u} \quad \text{on } \Gamma, \qquad \underset{\omega}{u} \in H_{2s-1}(G^\omega), \qquad \underset{\omega}{u} = \underset{\omega}{u}(y), \tag{6.3 C}$$

$$D_y{}^{2s-2} u \notin H_{1+\varepsilon}(G^1). \tag{6.3 D}$$

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(16) Non-integer differentiability orders are treated in [8].

[(3D) is compatible with (3C)]. Then u is a unique solution $\in \mathrm{H}_s(\mathrm{L})$ *but*

$$u \notin \mathrm{H}_{s+1/2+\varepsilon}(\mathrm{L}), \qquad (6.3\,\mathrm{E})$$

moreover it holds

$$[\mathrm{D}_1{}^{s-1}u]_{\mathrm{G}^1} \notin \mathrm{H}_{1+\varepsilon}(\mathrm{G}^1). \qquad (6.3\,\mathrm{F})$$

The proof of (3 E), (3 F) will be given below formula (3 J).

Remark D. – In the introduction I.3 we discussed the relevance of Randall's (*see* [15]) method of treating the characteristic initial value problem. In order to compare this method with our method it suffices (^{17}) to treat the following simple characteristic initial value problem for the unknown $\varphi(x^1, \ldots, x^m)$:

$$\mathrm{A}^{ij}\frac{\partial^2 \varphi}{\partial x^i \partial x^j} = 0 \qquad (i, j = 1, \ldots, m) \qquad (6.3\,\mathrm{G})$$

$$\varphi = \varphi_0 \quad \text{on } \mathrm{N}_1 \cup \mathrm{N}_2, \qquad (6.3\,\mathrm{H})$$

where N_1, N_2 denote two intersecting characteristic hypersurfaces; A^{ij} are constants. Let there be given any datum φ_0 with

$$\varphi_0 \in \mathrm{C}^0(\mathrm{N}_1 \cup \mathrm{N}_2), \qquad \varphi_0 \in \mathrm{H}_{2s-1}(\mathrm{N}_\omega) \quad (\omega = 1, 2) \qquad (6.3\,\mathrm{I})$$

with $s > \dfrac{m-1}{2} + 2$ ($s \in \mathbb{N}$). Now, the method of [15] yields (for the case of finite differentiability orders) a result, which is weaker than our result (*see* Theorems 6.2 and 8.1); moreover, we shall prove the following:

(*) *In the best case* it may be possible to strengthen the method of [15] such that the following two statements hold:

(*a*) assumption (3 I) implies (^{18}) $\underline{\varphi \in \mathrm{H}_{s-1/2}}$, and

(*b*) assumption (3 I) implies (^{18}) $\underline{\varphi \in \mathrm{H}_{s-1}}$, if one uses (^{18}) only those Cauchy problem existence theorems, which have been derived *so far* (*see* e. g. [2]).

Now, on the other hand by using our method one obtains:

$$\text{assumption (3 I) implies } \underline{\varphi \in \mathrm{H}_s}. \qquad (6.3\,\mathrm{J})$$

Comparing this with (*) (*b*) [or (*) (*a*) respectively] one sees that the method described in [15] leads (in principle) to a *loss of at least* 1 (*or* 1/2 *respectively*) *differentiability order*.

(^{17}) The loss of differentiability orders (which we shall show below) would *not become smaller*, if we would treat a characteristic intial value problem which would be *more complicated* than (3 G), (3 H).

(^{18}) In order to obtain (*) (*a*) one would have to employ a Cauchy problem existence theorem, which also treats successfully the case of non-integer differentiability orders (*see* also [8]). Such theorems do *not exist so far*. Thus one is left with (*) (*b*) so far.

The proof of (*) will be given in appendix IV.

Proof of (3 E), (3 F). – We prove (3 E) by contradiction and thus assume

$$u \in H_{s+1/2+\varepsilon}(L). \tag{6.3 S}$$

This implies (by [8])

$$\overset{k}{u} := [D_1{}^k u]_{G^1} \in H_{s-k+\varepsilon}(G^1) \quad \text{with} \quad k=1, \ldots, s-1. \tag{6.3 T}$$

$\overset{k}{u}$ fulfils the equation (4.40) or (4.44), which simplify [using (3 C) and (3 A)] to

$$D_2 \overset{1}{u} = \frac{1}{2} D_y{}^2 \underset{1}{u} \quad \text{on } G^1, \qquad \overset{1}{u}=0 \quad \text{on } \Gamma \tag{6.3 U}$$

$$D_2 \overset{l}{u} = \frac{1}{2} D_y{}^2 \overset{l-1}{u} \quad \text{on } G^1, \qquad \overset{l}{u}=0 \quad \text{on } \Gamma \tag{6.3 V}$$

for $l=2, \ldots, s-1$. The Cauchy initial value problem (3 U) has at most one solution $\in H_{s-1+\varepsilon}(G^1)$, namely $\overset{1}{u} = (D_y{}^2 \underset{1}{u}) . \frac{x^2}{2}$ [due to (3 C), (3 U)].

Inserting this into (3 U) one obtains analogously $\overset{2}{u} = (D_y{}^4 \underset{1}{u}) . \left(\frac{x^2}{2}\right)^2 . \frac{1}{2}$; then one gets $\overset{3}{u}$ and so on, and finally

$$\overset{s-1}{u} = (D^{2s-1} \underset{1}{u}) . \left(\frac{x^2}{2}\right)^{s-1} . \frac{1}{(s-1)!}. \tag{6.3 W}$$

Due to this and (3 D) one obtains $\overset{s-1}{u} \notin H_{1+\varepsilon}(G^1)$, which contradicts (3 T) for $k=s-1$.

Analogously one proves (3 F) by using especialle (3 W) together with (3 D).

6.b The proof of Theorem 6.2

The uniqueness follows from Theorem 5.5. Due to (2) there exists $g_n, b_n, f_n \in c^\infty(L_T)$ with

$$(g_n, b_n, a_n, f_n) \Rightarrow (g, b, a, f) \in \mathfrak{h}_s(L_T) \times \mathfrak{h}_s(L_T) \times \mathfrak{h}_s(L_T) \times \mathfrak{h}_{s-1}(L_T). \tag{6.4}$$

We shall show at the end of the main proof that (3) implies the existence of $\underset{\omega}{u_n} \in C^\infty(G_T{}^\omega)$ with

$$\underset{\omega}{u_n} \;\Rightarrow\; \underset{\omega}{u} \in H_{2s-1}(G_T{}^\omega), \qquad \underset{1}{u_n} = \underset{2}{u_n} \quad \text{on } \Gamma. \tag{6.5}$$

The initial value problem

$$\begin{gathered} \mathbb{L}_n u_n := g_n{}^{ab} D_a D_b u_n + b_n{}^a . D_a u_n + a_n . u_n = f_n \quad \text{on } L_T, \\ u_n = \underset{\omega}{u_n} \quad \text{on } G^\omega \end{gathered} \tag{6.6}$$

(with $\underset{1}{u_n} = \underset{2}{u_n}$ on Γ) has a unique solution $u_n \in C^\infty(L_T)$ because of Lemma 6.1.

It is our aim to show that these solutions u_n converge against a solution u of (4.17, 18).

(6) implies by Theorem 4.6.

$$\| u_n \|_s^{L_T} \leqslant R_{s,n}{}^T := [c_{32}(h, \bar{\gamma}^{L_T,n}, \gamma_s^{\Gamma,n}, T)]$$

$$\times \left[\left(\sum_{\omega=1}^{2} \left| \underset{\omega}{u_n} \right|_{2s-1}^{G_T{}^\omega} \right) + T \| f_n \|_{s-1}^{L_T} + \sum_{\omega=1}^{2} \| f_n \|_{s-1}^{\Gamma,G^\omega,1} \right], \tag{6.7}$$

$$\| u_n \|_{L_T,s,1} \leqslant R_{s,n}{}^T \tag{6.7 A}$$

with

$$\bar{\gamma}_s{}^{L_T,n} := \| a_n, b_n, g_n \|_s^{L_T} . T, \qquad \gamma_s{}^{\Gamma,n} := \sum_{\omega=1}^{2} \| a_n, b_n, g_n \|_{s-1}^{\Gamma,G^\omega,1},$$

provided $n > N$ with sufficiently big N [we used also (1.5-8), (4), compactness of L_T]. (6.7 A) will not be used within this proof but in section 8. (7) implies that for any $\varepsilon > 0$ there exists a $N_\varepsilon (N_\varepsilon > N)$ with

$$\| u_n \|_s^{L_T} \leqslant R_s{}^T + \varepsilon \qquad \text{for} \quad n > N_\varepsilon \quad [R_s{}^T \text{ of } (4.68)], \tag{6.8}$$

which we shall prove now (analogously to the proof of Theorem 5.5, part I):

By using the imbedding theorems of [1], Definition 1.1 and (4) we gain

$$\| f_n - n \|_{s-1}^{\Gamma,G^\omega,1} \leqslant c_T \| f_n - f \|_{s-1}^{G_T{}^\omega} \leqslant \bar{c}_T \| f_n - f \|_{s-1}^{L_T} \Rightarrow 0 \in \mathbb{R}, \tag{6.9}$$

where c, $\bar{c}_T$ are some constants depending on T. (9) implies

$$\| f_n \|_{s-1}^{\Gamma,G^\omega,1} \;\Rightarrow\; \| f \|_{s-1}^{\Gamma,G^\omega,1} \in \mathbb{R}$$

and likewise for $\gamma_s{}^{\Gamma,n}$ and similarly for $\bar{\gamma}_s{}^{L_T,n}$, $\| f_n \|_{s-1}^{L_T}$ and $\| \underset{\omega}{u_n} \|_{2s-1}^{G_T{}^\omega}$, where we used (4), (5). Due to this and the continuity of c_{32} [of (7)], we obtain $\lim_{n \to \infty} R_{s,n}{}^T = R_s{}^T$ [with $R_s{}^T$ of (4.68)]; thus for any $\varepsilon > 0$ there exists

$N_\varepsilon \, (N_\varepsilon > N)$ with

$$R_{s,n}{}^{T} \leqslant R_s{}^{T} + \varepsilon \qquad \text{for} \quad n > N_\varepsilon. \tag{6.9 A}$$

This *proves* (8).

We apply Lemma (5.2) to $\{u_n\}$ fulfilling (8); thus there exists a subsequence $\{u_{\bar n}\}$ and $u \in \mathfrak{h}_s(L_T)$ with

$$u_{\bar n} \to u \in \mathfrak{h}_s(L_T) \qquad \text{and} \qquad \| u \|_s^{L_T} \leqslant R_s{}^{T} + \varepsilon \tag{6.10}$$

for any $\varepsilon > 0$; thus $\| u \|_s^{L_T} \leqslant R_s{}^{T}$, *i.e.* u fulfils the energy inequality (4.68).

Furthermore, we shall show that u is a solution of (4.18, 19): We apply Lemma 5.4 to (4), (10) and gain [using (6)]

$$f_{\bar n} = \mathbb{L}_{\bar n} u_{\bar n} \to \mathbb{L}\, u \in \mathfrak{h}_{s-2}(L_T).$$

This and $f_{\bar n} \to f \in \mathfrak{h}_{s-2}(L_T)$ [*see* (4)] implies

$$\mathbb{L}\, u = f. \tag{6.11}$$

Furthermore, $\mathfrak{h}_s(L_T) \subsetneqq H_s(L_T)$ implies $H_s^*(L_T) \subset \mathfrak{h}_s^*(L_T)$; this together with (10) implies $u_{\bar n} \to u \in H_s(L_T)$, and hence by applying Lemma 5.3 we gain

$$[u_{\bar n}]_{G_T{}^{\omega}} \to [u]_{G_T{}^{\omega}} \in H_{s-1}(G_T{}^{\omega}).$$

This implies $\underset{\omega}{u_{\bar n}} \to [u]_{G_T{}^{\omega}} \in H_{s-1}(G_T{}^{\omega})$. This together with (5) implies $[u]_{G_T{}^{\omega}} = \underset{\omega}{u}$; this and (11) *proves* that u is a *solution* of (4.18, 19).

It remains to show that for given $\underset{\omega}{u}$ [fulfilling (3)] there exists $\underset{\omega}{u_n} \in C^\infty(G_T{}^{\omega})$ with $\omega \in \{1;\, 2\}$, which satisfies (5). We define

$$\begin{aligned} w &:= [\underset{1}{u}]_\Gamma = [\underset{2}{u}]_\Gamma \in H_{2s-1}(\Gamma), \\ \underset{\omega}{w}(x^{\omega}, x^3, \ldots, x^{n+1}) &:= w(x^3, \ldots, x^{n+1}). \end{aligned} \tag{6.12}$$

This implies

$$\underset{\omega}{w} \in H_{2s-1}(G_T{}^{\omega}), \qquad [\underset{\omega}{u} - \underset{\omega}{w}] \in H_{2s-1}(G_T{}^{\omega}) \tag{6.13}$$

(12) implies

$$[\underset{\omega}{u} - \underset{\omega}{w}]_\Gamma = [\underset{\omega}{u}]_\Gamma - [\underset{\omega}{w}]_\Gamma = w - w = 0. \tag{6.14}$$

Because of (13, 14) there exists (*see* [8]) a sequence $\underset{\omega}{\{v_n\}} \leqslant C^\infty(G_T{}^{\omega})$ with

$$\underset{\omega}{v_n} \;\Rightarrow\; [\underset{\omega}{u} - \underset{\omega}{w}] \in H_{2s-1}(G_T{}^{\omega}), \quad (\omega = 1,\, 2), \qquad \underset{\omega}{v_n} = 0 \quad \text{on } \Gamma. \tag{6.15}$$

On the other hand—due to $w \in H_{2s-1}(\Gamma)$—there $\exists\, w_n \in C^\infty(\Gamma)$ with $w_n \Rightarrow w \in H_{2s-1}(\Gamma)$ and thus for

$$\underset{\omega}{w_n}(x^\omega, x^3, \dots, x^{n+1}) := w_n(x^3, \dots, x^{n+1})$$

it holds

$$\underset{\omega}{w_n} \;\Rightarrow\; \underset{\omega}{w} \in H_{2s-1}(G_T{}^\omega), \qquad \underset{\omega}{w_n} \in C^\infty(G_T{}^\omega); \qquad [\underset{\omega}{w_n}]_\Gamma = w_n. \tag{6.16}$$

(15, 16) implies

$$\underset{\omega}{u_n} := (\underset{\omega}{v_n} + \underset{\omega}{w_n}) \;\Rightarrow\; [\underset{\omega}{u} - \underset{\omega}{w}] + \underset{\omega}{w}] + \underset{\omega}{w} = \underset{\omega}{u} \in H_{2s-1}(G_T{}^\omega). \tag{6.17}$$

Furthermore, $[\underset{\omega}{u_n}]_\Gamma = [\underset{\omega}{v_n}]_\Gamma + [\underset{\omega}{w_n}] = 0 + w_n$ (because of (15, 16); thus $[\underset{1}{u_n}]_\Gamma = [\underset{2}{u_n}]_\Gamma$; this and (17) *proves* (5).

7. THE SPACES $\mathfrak{E}_s$, E_s

So far, we obtained a solution u of the characteristics initial value problem with $u \in \mathfrak{h}_s(L_T)$. In chapter 8 we shall improve this by showing $u \in \mathfrak{E}_s(L_T)$, where $\mathfrak{E}_s(L_T)$ with its norm $\|\,.\,\|_{L_T, s}$ is a *modified Sobolov space* (*see* below). Whereas $\|\,.\,\|_s^{L_T}$ [of $\mathfrak{h}_s(L_T)$] involves $\int_0^T \dots dt$, the norm $\|\cdot\|_{L_T, s}$ [of $\mathfrak{E}_s(L_T)$] involves $\operatorname*{ess\,sup}_{t\in[0,T]}$ *instead* (*cf.* [2]).

We define the following normed vector spaces:

$$\mathfrak{E}_s(L_T) := \{\, v \in \mathfrak{h}_s(L_T) \mid \|v\|_{L_T, s} < \infty \,\} \text{ with norm } \|\,.\,\|_{L_T, s} \tag{7.1}$$

$$E_s(M_T) := \{\, v \in H_s(M_T) \mid |v|_{M_T, s} < \infty \,\} \text{ with norm } |\,.\,|_{M_T, s}, \tag{7.2}$$

where $M_T = L_T$ or G_T^ω.

LEMMA 7.1. — $\mathfrak{E}_s(L_T)$, $E_s(L_T)$, ($E_m(G_T{}^\omega)$ *respectively*) *are* Banach spaces; *moreover, they form an* algebra (*under pointwise multiplication*) *provided* $s > \dfrac{n+1}{2}$ $\left(m > \dfrac{n}{2} \text{ *respectively*}\right)$.

Furthermore,

$$\mathfrak{E}_s(L_T) \hookrightarrow \mathfrak{h}_s(L_T), \qquad E_s(L_T) \hookrightarrow H_s(L_T), \qquad E_m(G_T^\omega) \hookrightarrow H_m(G_T{}^\omega) \tag{7.3}$$

for $s \geqq 0$ ($m \geqq 0$ *respectively*).

Remark. — For the *quasilinear* characteristic initial value problem one can use the $\mathfrak{E}_s$-space for the construction of a solution $u \in \mathfrak{E}_s(L_T)$. Some further remark on this are made in the Remarks on Theorem 8.1.

In order to prove the above Lemma (and also Theorem 8.1) we write

$$\| v \|_{\mathrm{L_T}, s} = \operatorname*{ess\,sup}_{t \in [0, \mathrm{T}]} \| v \|_{\Lambda_t, \mathrm{L}, s}, \tag{7.4}$$

where we defined

$$\| v \|_{\Lambda_t, \mathrm{L}, s} := \left[(| v |_s^{\Lambda_t, \mathrm{L}})^2 + \sum_{\omega=1}^{2} \sum_{k=0}^{s-1} (| \mathrm{D}_\omega^k v |_{2(s-k)-1}^{\Gamma_t^\omega, \mathrm{G}^\omega})^2 \right]^{1/2}. \tag{7.5}$$

— By "almost all $t \in [0, \mathrm{T}]$" we mean (as usual) "all $t \in ([0, \mathrm{T}] \setminus \mathrm{M})$", where M is some set of measure zero (our Lebesque measure of section 1).

At first we shall prove the following Lemma, which exhibits some relation between $\| \, . \, \|_{\mathrm{L_T}, s}$ (of $\mathfrak{E}_s$) and $\| \, . \, \|_s^{\mathrm{L_T}}$ (of $\mathfrak{h}_s$):

LEMMA 7.2. — *Let*

$$u_n \quad \Rightarrow \quad u \in \mathfrak{h}_s(\mathrm{L_T}),$$

then there exists a subsequence $\{ u_{\bar{n}} \} \subset \{ u_n \}$ *such, that for almost all* $t \in [0, \mathrm{T}]$

$$\| u_{\bar{n}} - u \|_{\Lambda_t, \mathrm{L}, s} \quad \Rightarrow \quad 0 \in \mathbb{R} \tag{7.6}$$

with $\| \, . \, \|_{\Lambda_t, \mathrm{L}, s}$ *of* (5).

Proof of Lemma 7.2. — Let $\mathrm{F}_n := u = u_n$, then

$$0 = \lim_{n \to \infty} (\| \mathrm{F}_n \|_s^{\mathrm{L_T}})^2 = \lim_{n \to \infty} \int_0^{\mathrm{T}} f_n(t)^2 dt \tag{7.7}$$

with

$$f_n(t) := \left[\mathrm{T}^{-2} t (| \mathrm{F}_n |_s^{\Lambda_t, \mathrm{L}})^2 + \mathrm{T}^{-1} \sum_{\omega=1}^{2} \sum_{k=0}^{s-1} (| \mathrm{D}_\omega{}^k \mathrm{F}_n |_{2(s-k)-1}^{\Gamma_t, \mathrm{G}^\omega})^2 \right]^{1/2}. \tag{7.8}$$

(7) implies $f_n \Rightarrow 0 \in \mathrm{L}^2([0, \mathrm{T}])$. This implies (*see* [9]) that there exists a subsequence $\{ f_{\bar{n}} \} \subset \{ f_n \}$ such that for almost all $t \in [0, \mathrm{T}]$ it holds $f_{\bar{n}}(t) \Rightarrow 0 \in \mathbb{R}$; hence

$$| \mathrm{F}_{\bar{n}} |_s^{\Lambda_t, \mathrm{L}} \quad \Rightarrow \quad 0 \in \mathbb{R} \qquad \text{and} \qquad \sum_{\omega=1}^{2} \sum_{k=0}^{s-1} (| \mathrm{D}_\omega{}^k \mathrm{F}_{\bar{n}} |_{2(s-k)-1}^{\Gamma_t, \mathrm{G}^\omega}) \quad \Rightarrow \quad 0 \in \mathbb{R};$$

thus (6) is proved.

Proof of Lemma 7.1. — (3) follows from (3.4). — We shall now prove the *completeness* of $\mathfrak{E}_s(\mathrm{L_T})$. Let $\{ u_n \}$ be a Cauchy sequence of $\mathfrak{E}_s(\mathrm{L_T})$, *i. e.* for any given $\varepsilon > 0$, there exists N_ε with

$$\| u_n - u_m \|_{\mathrm{L_T}, s} < \varepsilon \qquad \text{for} \quad n, m > \mathrm{N}_\varepsilon; \tag{7.9}$$

this implies by (4)

$$\left.\begin{aligned} &\|u_n-u_m\|_{\Lambda_t,\,L,\,s}<\varepsilon \qquad \text{for} \quad n, m>N_\varepsilon, \\ &\text{for almost all } t\in[0, T]. \end{aligned}\right\} \tag{7.10}$$

(9) and $\mathfrak{h}_s(L_T) \supsetneq \mathfrak{E}_s(L_T)$ implies that $\{ u_n \}$ is also a Cauchy sequence in $\mathfrak{h}_s(L_T)$ (which is complete) and thus there exists $u\in\mathfrak{h}_s(L_T)$ with

$$u_n \quad \Rightarrow \quad u\in\mathfrak{h}_s(L_T).$$

This implies by Lemma 7.2 that there exists a subsequence $\{ u_{\bar n} \} \subset \{ u_n \}$ such that $\|u-u_{\bar n}\|_{\Lambda_t,\,L,\,s} \Rightarrow 0\in\mathbb{R}$ for almost all $t\in[0, T]$; thus for the above ε there exists $N_{\varepsilon,\,t}$ (with $N_{\varepsilon\,t}>N_\varepsilon$) such that for almost all $t\in[0, T]$

$$\|u-u_{\bar n}\|_{\Lambda_t,\,L,\,s}<\varepsilon \qquad \text{for} \quad n>N_{\varepsilon\,t}.$$

This and (10) implies that for almost all $t\in[0, T]$ holds

$$\|u-u_m\|_{\Lambda_t,\,L,\,s} \leqq \|u-u_{\bar n}\|_{\Lambda_t,\,L,\,s} + \|u_{\bar n}-u_m\|_{\Lambda_t,\,L,\,s} < 2\,\varepsilon$$

for $\bar n>N_{\varepsilon,\,t}>N_\varepsilon$, $m>N_\varepsilon$; hence

$$\|u-u_m\|_{\Lambda_t,\,L,\,s}<2\,\varepsilon \text{ for } m>N_\varepsilon, \quad \text{for almost all } t\in[0,T] \tag{7.11}$$

(N_ε independent of t). (11) implies with (4)

$$\|u-u_m\|_{L_T,\,s}<\varepsilon \qquad \text{for} \quad m>N_\varepsilon. \tag{7.12}$$

This [and $u_m\in\mathfrak{E}_s(L_T)$] implies $u\in\mathfrak{E}_s(L_T)$; moreover, (12) implies $u_n \Rightarrow u\in\mathfrak{E}_s(L_T)$. A further relation between $\mathfrak{E}_s$ and $\mathfrak{h}_s$ is exhibited in the following

THEOREM 7.3. – *Let*

$$u_n \rightarrow u\in\mathfrak{h}_s(L_T). \tag{7.13}$$

Let $\langle u_n \rangle$ *denote the* convex hull *of* $\{ u_n \}$, i. e.

$$\langle u_n \rangle := \left\{ v\in\mathfrak{h}_s(L_T) \,\middle|\, v=\sum_{\alpha=1}^{m} c_\alpha u_{n_\alpha}, c_\alpha \geqq 0, \sum_{\alpha=1}^{m} c_\alpha=1, m\in\mathbb{N} \right\}.$$

Then there exists a sequence with $\{ \overline{u_k} \} \subset \langle u_n \rangle$ *with*

$$\overline{u_k} \quad \Rightarrow \quad u\in\mathfrak{h}_s(L_T);$$

moreover, there exists a subsequence $\{ \overline{u_{\bar k}} \} \subset \{ \overline{u_k} \}$ *such that*

$$\|\overline{u_{\bar k}}-u\|_{\Lambda_t,\,L,\,s} \quad \Rightarrow \quad 0\in\mathbb{R} \quad \textit{for almost all } t\in[0, T]. \tag{7.14}$$

Proof of Theorem 7.3. – (13) implies

$$u\in[\text{weak closure of } \langle u_n \rangle]=[\text{strong closure of } \langle u_n \rangle]; \tag{7.15}$$

the equality sign holds, since $\langle u_n \rangle$ is a *convex* subset of $\mathfrak{h}_s(L_T)$ (*see* [10]). Due to (15) there exists a sequence

$$\{ \overline{u_k} \} \subset \langle u_n \rangle \qquad \text{with} \quad \overline{u_k} \quad \Rightarrow \quad u\in\mathfrak{h}_s(L_T). \tag{7.16}$$

This and Lemma (7.2) imply (14).

8. THE SOLUTION IS IN FACT IN $\mathfrak{E}_s$; ENERGY INEQUALITY IN $\mathfrak{E}_s$

The solution u of Theorem 6.2 turns out to be in $\mathfrak{E}_s$ provided the data $\underset{\omega}{u} \in \mathrm{E}_{2s-1}(\mathrm{G}_\mathrm{T}{}^{\omega})$:

THEOREM 8.1. — *We assume that* g^{ab} *of our differential equation* (4.18) *is regularly hyperbolic on* L_T *with hyperbolicity constant* $h>0$. *Let for the coefficients*

$$g, b, a \in \mathfrak{h}_s(\mathrm{L}_\mathrm{T}), \qquad f \in \mathfrak{h}_{s-1}(\mathrm{L}_\mathrm{T}) \tag{8.1}$$

and for the data

$$\underset{\omega}{u} \in \mathrm{E}_{2s-1}(\mathrm{G}_\mathrm{T}{}^{\omega}), \qquad [\underset{\omega}{u}]_\Gamma \in \mathrm{H}_{2s-1}(\Gamma), \qquad \underset{1}{u}=\underset{2}{u} \quad on\ \Gamma \tag{8.2}$$

with $s>\dfrac{n}{2}+2$ *and* $n \geqq 1$. *Then there exists a unique* solution $u \in \mathfrak{E}_s(\mathrm{L}_\mathrm{T})$ *of* (4.18), *where u assumes the given data* $\underset{\omega}{u}$ *on* $\mathrm{G}_\mathrm{T}{}^{\omega}$. *Moreover, u fulfils the* energy inequality (4.70), *where the energy inequality "constant"* c_{33} *fulfils the* stable-boundedness property *as pointed out in Theorem* 4.6; *the quantity* $c_{33}(.,.,.,.)$ *is* continuous *and increasing* (see *Definition* 4.1). *Furthermore* c_0 *of* (3.1)-(3.8) *does not depend on* $t \in [0, \mathrm{T}_0]$. *The norm* $\| \,.\, \|_{\mathrm{L}_\mathrm{T}, s}$ *fulfils the* ball property *mentioned in Remark* 3.1.

Remark A. — Since $\mathfrak{E}_s(\mathrm{L}_\mathrm{T}) \subsetneq \mathfrak{h}_s(\mathrm{L}_\mathrm{T})$, the above Theorem is *stronger* than Theorem 6.2: the assumptions are the same except for the assumption

$$\underset{\omega}{u} \in \mathrm{E}_{2s-1}(\mathrm{G}_\mathrm{T}{}^{\omega});$$

but this assumption is a necessary one, since $u \in \mathfrak{E}_s(\mathrm{L}_\mathrm{T})$ implies $[u]_{\mathrm{G}_\mathrm{T}{}^{\omega}} \in \mathrm{E}_{2s-1}(\mathrm{G}_\mathrm{T}{}^{\omega})$.

Remark B. — Theorem 8.1 can be *applied* to the construction of a solution of the following *quasilinear* characteristic initial value problem

$$g^{ab}(u)\,\mathrm{D}_a \mathrm{D}_b u = f(u, \mathrm{D}u) \quad \text{on } \mathrm{L}_\mathrm{T}, \qquad u=\underset{\omega}{u} \quad \text{on } \mathrm{G}_\mathrm{T}{}^{\omega} \qquad (\omega=1,2) \tag{8.5}$$

(for sufficiently small $\mathrm{T}>0$), where g, f are rational (or even C^{∞}) functions of $u\,[(u, \mathrm{D}u)$ respectively]; furthermore it is assumed that the *data* $\underset{\omega}{u}$ are such that $g^{ab}(\underset{\omega}{u})$ is a "regular hyperbolic metric on $\mathrm{G}_\mathrm{T}{}^{\omega}$". For instance, the Einstein's gravitational vacuum equations, for which an application is planed, are of the type (5). – Equation (5) can be solved by an *iteration*, which runs basically like this: one can construct some

$$u_1 \in \mathfrak{E}_s(\mathrm{L_T}), \qquad u_1 \underset{\omega}{=} u \quad \text{on } \mathrm{G_T}^{\omega} \qquad (\omega = 1, 2).$$

According to the algebra-property (3.3 A) of $\mathfrak{E}_s(\mathrm{L_T})$ $\left(\text{for } s > \frac{n+1}{2}\right)$ and due to

$$|v^{-1}|_{\mathrm{L_T}, s} \leqq c_0 . [(\min_{x \in \mathrm{L_T}} |v(x)|)^{-1} + (\min_{x \in \mathrm{L_T}} |v(x)|)^{-s}] . [|v|_{\mathrm{L_T}, s} + |v|^{s}_{\mathrm{L_T}, s}],$$

$$v \in \mathfrak{E}_s(\mathrm{L_T}) \subsetneq \mathrm{C}^0(\mathrm{L_T}),$$

[because of (3.2)] it follows that

$$f(u_1, \mathrm{D}u_1) \in \mathfrak{E}_{s-1}(\mathrm{L_T}), \qquad g(u_1) \in \mathfrak{E}_s(\mathrm{L_T}).$$

Thus by Theorem 8.1 and by using $\mathfrak{E}_m(\mathrm{L_T}) \subsetneq \mathfrak{h}_m(\mathrm{L_T})$ there exists a unique $u_2 \in \mathfrak{E}_s(\mathrm{L_T})$ with

$$g^{ab}(u_1)\,\mathrm{D}_a \mathrm{D}_b u_2 = f(u^1, \mathrm{D}u_1) \quad \text{on } \mathrm{L_T}, \qquad u_2 \underset{\omega}{=} u \quad \text{on } \mathrm{G_T}^{\omega}. \qquad (8.6)$$

Thus u_1 is determined by u_2. In the same way u_2 determines a unique $u_3 \in \mathfrak{E}_s(\mathrm{L_T})$ and finally by induction one gains $u_n \in \mathfrak{E}_s(\mathrm{L_T})$. Thus the iteration is such that there is *no loss of differentiability* classes. Moreover, u_n fulfils the energy-inequality (4.70), where its energy inequality constant c_{33} remains *bounded* (19) (the lower bound being strictly positive) as n goes to infinity; this follows from the *stable* boundedness property (4.71 C, A). Furthermore, c_{33} also remains *bounded* (the lower bound being strictly positive), when one makes *T as small* as one likes (*see* stable-boundedness property (4.71 A, C). Due to this and the *ball property* (20) (*see* Remark 3.1) one can make T so small that the *iteration converges*. The details of the convergence proof will be given in a forthcoming paper. Within the iteration one also uses

(I) $\mathfrak{E}_s(\mathrm{L_T})$ forms an *algebra* for $s > \dfrac{n+1}{2}$ under pointwise multiplication [*see* (3.3 A)],

(II) $\mathfrak{E}_s(\mathrm{L_T})$ is *complete*.

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(19) u_n fulfilss the energy inequality (4.70); its energy inequality "constant" c_{33} *depends on* $g(u_{n-1})$ and hence *on n*.

(20) At first one makes T so small such that all u_n lay in a ball $\mathrm{B}_{r,\,\mathrm{T}}(u_o) := \{ u \in \mathfrak{E}_s(\mathrm{L_T}) \,|\, \| u - u_0 \|_{\mathrm{L_T}, s} \leqq r \}$ for small r and some $u_0 \in \mathfrak{E}_s(\mathrm{L_T})$.

Remark C (*on Theorem* 8.1). — The *gap of differentiability class* between solution ($\varepsilon\,\mathfrak{E}_s$) and data ($\varepsilon\, E_{2s-1}$) amounts to ($s-1$) differentiability orders. This gap has been clarified in Remark C on Theorem 6.2.

Remark D (*on Theorem* 8.1). — Also the case $\underline{s=\infty}$ is included in Theorem 8.1: the data, coefficients and solutions are C^∞-functions (due to $C^\infty = H^\infty = E^\infty$).

Proof of Theorem 8.1. — The uniqueness follows from Theorem 5.5. — Due to $H_{2s-1} \hookleftarrow E_{2s-1}$ the assumptions of Theorem 6.2 are fulfilled, hence we have a solution $u \in \mathfrak{h}_s(L_T)$. We shall show that $u \in \mathfrak{E}_s(L_T)$:

We use the sequence $\{ u_n \}$ and $\{ u_{\bar{n}} \} \subset \{ u_n \}$ of the proof of Theorem 6.2; according to (6.10, 7.A, 9.A) we gain

$$u_{\bar{n}} \to u \in \mathfrak{h}_s(L_T), \tag{8.7}$$

$$\| u_n \|_{L_T, s, 1} \leqq R_{s,n}{}^T \quad \text{for} \quad n > N \tag{8.8}$$

for some N; moreover, for any $\varepsilon > 0$ there exist $N_\varepsilon > N$ with

$$R_{s,n}{}^T \leqq R_s{}^T + \varepsilon. \tag{8.9}$$

(8), (9) implies

$$\| u_{\bar{n}} \|_{L_T, s, 1} \leqq R_s^T + \varepsilon \quad \text{for} \quad \bar{n} > N_\varepsilon. \tag{8.10}$$

We apply Theorem 7.3 to $\{ u_{\bar{n}} \}_{\bar{n} > N_\varepsilon}$ of (7) and hence there exists a sequence $\{ \overline{u_k} \} \subset \langle u_{\bar{n}} \rangle_{\bar{n} > N_\varepsilon}$ with

$$\lim \| \overline{u_k} - u \|_{\Lambda_t, L, s} = 0 \tag{8.11}$$

for almost all $t \in [0, T]$ [*cf.* 7.5)]; thus

$$\lim_{k \to \infty} \| \overline{u_k} \|_{\Lambda_t, L, s} = \| u \|_{\Lambda_t, L, s} \tag{8.12}$$

for almost all $t \in [0, T]$.

Since, on the other hand, $\overline{u_k}$ lays in the convex hull $\langle u_{\bar{n}} \rangle_{\bar{n} > N_\varepsilon}$, we gain for any given k

$$\| \overline{u_k} \|_{L_T, s, 1} = \left\| \sum_{\alpha=1}^{m} c_\alpha u_{\bar{n}_\alpha} \right\|_{L_T, s, 1} \leqq \sum_{\alpha=1}^{m} c_\alpha (R_s{}^T + \varepsilon) \leqq R_s{}^T + \varepsilon \tag{8.13}$$

(for some c_α with $\sum c_\alpha = 1, c_\alpha > 0$), where we also used (10). Thus by Definition 2.1

$$\left[(| \overline{u_k} |_s^{\Lambda_t, L})^2 + \sum_{\omega=1}^{2} \sum_{l=1}^{s-1} (| D_\omega{}^l \overline{u_k} |_{2(s-l)-1}^{\Gamma_t{}^\omega, G^\omega})^2 \right]^{1/2} \leqq R_s{}^T + \varepsilon \tag{8.14}$$

for almost all $t \in [0, T]$.

We take $\lim_{k \to \infty}$ of (14) and use (12) [beforehand we rewrite (12) by using (7.5)]; thus

$$\left[(|u|_s^{\Lambda_t, \mathrm{L}})^2 + \sum_{\omega=1}^{2} \sum_{l=1}^{s-1} (|\mathrm{D}_\omega{}^l \underset{\omega}{u}|_{2(s-l)-1}^{\Gamma_t{}^\omega, \mathrm{G}^\omega})^2 \right]^{1/2} \leqq \mathrm{R}_s{}^{\mathrm{T}} + \varepsilon$$

for almost all $t \in [0, \mathrm{T}]$. This implies $\| u \|_{\mathrm{L}_\mathrm{T}, s, 1} \leqq \mathrm{R}_s{}^\mathrm{T} + \varepsilon$ for any $\varepsilon > 0$; hence

$$\| u \|_{\mathrm{L}_\mathrm{T}, s, 1} \leqq \mathrm{R}_s{}^\mathrm{T}. \tag{8.15}$$

This and Definition 2.1 implies with $u = \underset{\omega}{u}$ on $\mathrm{G_T}^\omega$

$$\| u \|_{\mathrm{L}_\mathrm{T}, s} \leqq \| u \|_{\mathrm{L}_\mathrm{T}, s, 1} + \sum_{\omega=1}^{2} | u |_{\mathrm{G_T}^\omega, 2s-1} \leqq \mathrm{R}_s{}^\mathrm{T} + \sum_{\omega=1}^{2} | \underset{\omega}{u} |_{\mathrm{G_T}^\omega, 2s-1}. \tag{8.16}$$

This and $|\underset{\omega}{u}|_{2s-1}^{\mathrm{G_T}^\omega} \leqq c_0 |\underset{\omega}{u}|_{\mathrm{G_T}^\omega, 2s-1}$ [due to (3.4)] *proves* the energy inequality (4.70). Since $u \in \mathfrak{h}_s(\mathrm{L_T})$ we obtain with (16) that $u \in \mathfrak{E}_s(\mathrm{L_T})$.

A. APPENDIX

APPENDIX I

The following generalization of Gronwal's Lemma holds:

LEMMA A.1. – *Let* α, β *be continuous non-negative functions on* $[0, \mathrm{T}]$. *We define for* $c \geqq 0$

$$\mathrm{F}_t[w] := -\frac{c}{2} + \int_0^t w(t')\,\beta(t')\,dt' + \int_0^t w(t')^2\,\alpha(t')\,dt' \tag{A.1}$$

for any $w \in \mathrm{C}^0([0, \mathrm{T}])$. *Let* $t \in [0, \mathrm{T}]$ *and let* $y \in \mathrm{C}^0([0, \mathrm{T}])$ *fulfil the following integral-inequality*

$$\frac{1}{2} y(t)^2 \leqq \mathrm{F}_t[y], \qquad y(t) \geqq 0. \tag{A.2}$$

Then there exists a unique $z \in \mathrm{C}^0([0, \mathrm{T}])$ *with*

$$\frac{1}{2} z(t)^2 = \mathrm{F}_t[z], \qquad z(t) \geqq 0; \tag{A.3}$$

moreover,

$$y(t) \leqq z(t), \tag{A.4}$$

$$y(t) \leqq c \left[\exp \int_0^t \alpha(t')\,dt' \right] + \int_0^t \left[\exp \int_{t'}^t \alpha(r)\,dr \right] . \beta(t')\,dt'. \qquad (\mathrm{A}.5)$$

Remark. — Because of $z \in \mathrm{C}^0[0, t]$ and $z(t) \geqq 0$ is (A.3) equivalent to the following linear ordinary differential equation

$$\frac{dz}{dt} = \beta + \alpha z,$$

provided $c > 0$. From (A.2, 3) it follows for $\Delta := y - z$ and $\bar{\Delta} := y^2 - z^2$

$$\frac{1}{2}\bar{\Delta}(t) \leqq \int_0^t \Delta(t').\beta(t')\,dt' + \int_0^t \bar{\Delta}(t').\alpha(t')\,dt';$$

with this one shows $\Delta \leqq 0 \,(\Leftrightarrow \bar{\Delta} \leqq 0)$.

APPENDIX II

LEMMA A.2. — *There exists a constant* c_0 (*independent of* t, u) *such that for any* $u \in \mathrm{C}^\infty(\mathrm{L_T})$

$$(t^{1/2} | u |_0^{\Lambda_t})^2 \leqq c_0 \left[\left(\sum_{\omega=1}^{2} (t^{1/2} | u |_0^{G_t^\omega})^2 \right) + \left(\int_0^t (s^{1/2} | u |_1^{\Lambda_s, \mathrm{L}})^2\,ds \right) . t \right] \qquad (\mathrm{A}.6)$$

for any $t \in [0, \mathrm{T}]$.

Proof. — We introduce new coordinates

$$\tau := x^1 + x^2, \qquad \xi := x^1 - x^2,$$

which is equivalente to

$$x^1 = \frac{1}{2}(\tau + \xi), \qquad x^2 = \frac{1}{2}(\tau - \xi);$$

we express $u(x^a)$ in these new coordinates

$$\bar{u}(\tau, \xi, x^{\mathrm{A}}) := u\left(\frac{1}{2}(\tau + \xi), \frac{1}{2}(\tau - \xi), x^{\mathrm{A}} \right) \qquad (\mathrm{A} = 3, \ldots, n+1).$$

We square the identity

$$\bar{u}(t, \xi, x^{\mathrm{A}}) = \bar{u}(|\xi|, \xi, x^{\mathrm{A}}) + \int_{|\xi|}^t \left[\frac{\partial \bar{u}}{\partial \tau}(\tau, \xi, x^{\mathrm{A}}) \right] d\tau \qquad (\mathrm{A}.7)$$

and use

$$\left(\int_{|\xi|}^t \frac{\partial u}{\partial \tau}\,d\tau \right)^2 \leqq t . \int_{|\xi|}^t \left(\frac{\partial u}{\partial \tau} \right)^2 d\tau \qquad (\text{since } \tau \geqq |\xi| \text{ in L})$$

and integrate over ξ; then we gain

$$\int_{-t}^{t} u(t,\xi,x^{A})^2\,d\xi \leqq 2\int_{-t}^{t} \bar{u}(|\xi|,\xi,x^{A})^2\,d\xi + 2t.\int_{-t}^{t}\left(\int_{|\xi|}^{t}\left(\frac{\partial\bar{u}}{\partial\tau}\right)^2 d\tau\right)d\xi.$$

Integrating this over (x^{A}) it becomes (expressed in the old coordinates)

$$\int_{\Lambda_t} u^2\,d\Lambda \leqq c_0\left[\left(\sum_{\omega=1}^{2}\int_{G_T{}^{\omega}} u^2\,dG^{\omega}\right)+\int_{L_t}\left(\frac{1}{2}D_1 u+\frac{1}{2}D_2 u\right)^2 dL\right]$$

for some Const. c_0. This and

$$\int_{L_t}\left(\frac{1}{2}D_1 u+\frac{1}{2}D_2 u\right)^2 dL \leqq \bar{c}_0\int_0^t (t'^{1/2}\,|u|_1^{\Lambda_{t'},L})^2\,dt'$$

(some constant $\bar{c}$) *proves* (A.6).

APPENDIX III

Sketch of the proof of Lemma 4.3

The proof is similar to the proof of Lemma 4.1 and Theorem 4.2. We define $M_t := (\text{boundary of } G_t{}^1)\diagdown(\Gamma_t{}^1\cup\Gamma) = R_t \cap G_t{}^1$ (with R_t of (4.8)]; the covector $\bar{n}_\alpha$ ($\alpha, \beta, \ldots$ runs from 2 to $n+1$) denotes the unit normal of M_t $(n_\alpha n_\beta \delta^{\alpha\beta}=1)$; we use the coordinates of (1.1, 2). Again, we introduce an energy-momentum vector

$$Q^{\alpha} := Q^{\alpha}[v] = g^{\alpha}\,v.v \qquad (v.v := v^{A}v^{B}\delta_{AB}) \tag{A.8}$$

which fulfils [analogously to (4.26, 12, 16)]:

$$D_\alpha Q^{\alpha} = g^{\alpha}(D_\alpha v).v + (D_\alpha g^{\alpha})\,v.v = 2\,[\gamma-\beta.v].v + (D_\alpha g^{\alpha})\,v.v, \tag{A.9}$$

$$Q^{\alpha}\,\delta_\alpha{}^2 = g^{\alpha}\,(D_\alpha\hat{\tau})\,v.v \geqq (c_{22}\,(h))^{-1}\,v.v \quad \text{on} \quad G^1 \tag{A.10}$$

$$Q^{\alpha}\bar{n}_\alpha \geqq 0 \quad \text{on } M_t, \tag{A.11}$$

where (9) follows from (8) and (4.47); the inequality (10) follows from (4.49, 50).

(11) can be proven as follows: n_α of (4.9) was a future-directed *normal* of R_t (non-timelike surface), and $\bar{n}_\alpha$ is a future-directed *normal* of $M_t = R_t \cap G^1$; this implies

$$n_\alpha\,\bar{n}_\beta - n_\beta\,\bar{n}_\alpha = 0 \quad \text{on } M_t \subset G^1, \tag{A.12}$$

$$\left.\begin{array}{c}\text{the covector fields } n_\alpha \text{ (on } G^1) \text{ and } \bar{n}_\alpha \text{ (on } G^1)\\ \text{point into the same direction.}\end{array}\right\} \tag{A.13}$$

On the other hand, as n_a is future-directed (on L), it follows

$$0 \leqq g^{a\,1}\,n_a = g^{\alpha}\,n_\alpha \quad \text{on } G^1\ (G^1 \subset L).$$

This and (12, 13) implies $g^\alpha \bar{n}_\alpha \geqq 0$ on $\mathrm{M}_t \subset \mathrm{G}^1$, which *prove* (A.11). (8, 10, 11) imply (14) (*see* below)

$$
\begin{aligned}
(c_{25}(h))^{-1} (|v|_0^{\Gamma_t^1})^2 = (c_{25}(h))^{-1} & \int_{\Gamma_t^1} v.v\, d\Gamma \\
& \leqq \int_{\Gamma_t^1} \mathrm{Q}^\alpha \delta_\alpha{}^2\, d\Gamma + \int_{\mathrm{M}_t} \mathrm{Q}^\alpha \bar{n}_\alpha\, d\mathrm{M} \quad (\mathrm{A}.14)\\
& = \int_{\Gamma} \mathrm{Q}^\alpha \delta_\alpha{}^2\, d\Gamma + \int_{\mathrm{G}_t{}^1} \mathrm{D}_\alpha \mathrm{Q}^\alpha\, d\mathrm{G} \quad (\mathrm{A}.15)\\
\leqq c_{26}(h) \left((|v|_0^{\Gamma})^2 + \int_0^t [|\gamma|_0^{\Gamma_s} . |v|_0^{\Gamma_s} + \varepsilon_0^{\Gamma_s} . (|v|_0^{\Gamma_s})^2]\, ds \right) & \quad (\mathrm{A}.16)
\end{aligned}
$$

with

$$
\varepsilon_m^{\Gamma_t{}^1} := |\beta|_{m_0}^{\Gamma_t{}^1,\ \mathrm{G}^1} + |(g^\alpha)|_{m_1}^{\Gamma_t{}^1,\ \mathrm{G}^1}, \quad m_k \geqq m, \qquad m_k > \frac{n-1}{2} + k; \quad (\mathrm{A}.17)
$$

where (15) follows from Gauss-integration formula; (16) follows by inserting (8, 9) into (15) and estimating similarly as in the proof of (4.23).

Rewriting (14, 16), one gains the formula (18) (*see* below) for $m=0$:

$$
\begin{aligned}
|v|_m^{\Gamma_t{}^1,\ \mathrm{G}^1} \leqq c_{27}(h) & ((|v|_m^{\Gamma,\ \mathrm{G}^1})^2 \\
& + \int_0^t [|v|_m^{\Gamma_s{}^1,\ \mathrm{G}^1} . |\gamma|_m^{\Gamma_s{}^1,\ \mathrm{G}^1} + (|v|_m^{\Gamma_s{}^1,\ \mathrm{G}^1})^2 . \varepsilon_m{}^{\Gamma_s{}^1}]\, ds \Big). \quad (\mathrm{A}.18)
\end{aligned}
$$

Using (18) for $m=0$, one can derive (18) for $m=1$ in a similar manner as in the derivation of (4.36 A) by using (4.23). This way one can proceed to (18) for $m=2$ and finally (by induction) to (18) for arbitrary m.

As in the proof of (4.24) [or (4.36 B)] we apply the generalized *Gronwal-Lemma* (Lemma A.1) to (18) and thus gain

$$
|v|_m^{\Gamma_t{}^1,\ \mathrm{G}^1} \leqq c_{28}(h) \left(\exp \int_0^t \varepsilon_m^{\Gamma_s{}^1}\, ds \right) \left[|v|_m^{\Gamma,\ \mathrm{G}^1} + \int_0^t |\gamma|_m^{\Gamma_s{}^1,\ \mathrm{G}^1}\, ds \right], \quad (\mathrm{A}.19)
$$

and then using (4.1), (3.8) we obtain

$$
|v|_{\mathrm{G_T}{}^1,\ m} \leqq c_{29}(h,\ \varepsilon_m{}^{\mathrm{G_T}{}^1}) \left[|v|_m^{\Gamma,\ \mathrm{G}^1} + \int_0^t |\gamma|^{\Gamma_s{}^1,\ \mathrm{G}^1}\, ds \right], \qquad (\mathrm{A}.20)
$$

with

$$
\varepsilon_m{}^{\mathrm{G_T}{}^1} := \mathrm{T} . (|\beta|_{m_0}^{\mathrm{G_T}{}^1} + |(g^\alpha)|_{m_1}^{\mathrm{G_T}{}^1}), \qquad m_k \geqq m, \quad m_k > \frac{n-1}{2} + k. \quad (\mathrm{A}.21)
$$

$[\mathrm{D}^k_{\mathrm{G}^1}\, v]_\Gamma$ can be *algebraically expressed* by the *Cauchy datum* $\underset{0}{v}$ [of (4.48)] and its derivatives $[\mathrm{D}_\Gamma{}^1\, v_0]$ by using our differential equation (4.47); this and (4.50) yields

$$|v|_m^{\Gamma,\ G^1} \leqq c_{30}\,(h,\ \varepsilon_m{}^{\Gamma})\,[\,|\underset{0}{v}|_m^{\Gamma} + |\gamma|_{m-1}^{\Gamma,\ G^1}], \tag{A.22}$$

with

$$\varepsilon_m{}^{\Gamma,\ 1} := |\beta|_{m_0-1}^{\Gamma,\ G^1} + |(g^{\alpha})|_{m_1-1}^{\Gamma,\ G^1}.$$

We insert (22) into (20) and thus *prove* (4.51) *of Lemma* 4.3.

APPENDIX IV

Proof of (*) of Remark D of section 6

In order to give the proof we have to go through each step of the proof of [15] and to see up to what extent those *steps could be strenthened:* in [15] one transforms the characteristic initial value problem (6.3 G), (6.3 H) for φ into a *Cauchy initial value problem* for χ, where

$$\chi := \varphi - \varphi_1 \tag{A.26}$$

where φ_1 (being not necessarily a solution) is defined on some open neighbourhood $U \subseteqq \mathbb{R}^m$ of $N_1 \cup N_2$ with

$$[\varphi_1]_{N_\omega} = [\varphi]_{N_\omega} = \varphi_0, \tag{A.27}$$

$$[\hat{D}^k_{N_\omega}\ \varphi_1]_{N_\omega} = [\hat{D}^k_{N_\omega}\ \varphi]_{N_\omega} \qquad (k = 1, \ldots, s-1) \tag{A.28}$$

($\hat{D}_{N_\omega}$ being some derivative *non*-tangential to N_ω), where $[\hat{D}^k_\omega\ \varphi]_{N_\omega}$ is determined by the so-called "propagation equations" [*i.e.* the equations (4.40), (4.41), (4.44), (4.45)], which are induced by (6.3 G), (6.3 H).

The new unknown χ fulfils—according to [15]—the following Cauchy problem

$$A^{ij}\frac{\partial^2 \chi}{\partial x^i\,\partial x^j} + F_1 = 0 \qquad (i, j = 1, \ldots, m), \tag{A.29}$$

χ has null data on a certain Cauchy hypersurface (*see* [15]); (A.30)

where

$$F_1 := A^{ij}\frac{\partial^2 \varphi_1}{\partial x^i\,\partial x^j} \quad \text{on } U^+, \qquad F_1 := 0 \quad \text{on } U \diagdown U^+ \tag{A.31}$$

with $U^+ := U \cap$ [domain of dependance of $N_1 \cup N_2$]. [*On* U^+ is (29) equivalent to (6.3 G) in the C^∞-case.]

We shall show [below formula (36)] that for each $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$) there exists

$$\varphi_0, \text{ which fulfils (6.3 I), such that } \varphi_1 \notin H_{s+1/2+\varepsilon}\,(U). \tag{A.32}$$

Now, the datum φ_0 could be *any* function fulfilling (6.3 I), thus (32) implies that in general

$$\varphi_1 \text{ is at most } H_{s+1/2}\,(U), \tag{A.33}$$

$$F_1 \text{ is at most } H_{s-3/2}\,(U), \tag{A.34}$$

whereby (34) follows from (33), (31).

For the solution χ of the Cauchy problem (29) (30) it holds:

$$\text{if } \ F_1 \in H_{s-3/2} \ \text{ then } \chi \in H_{s-1}. \ (^{21}) \tag{A.35}$$

There is at present no Cauchy problem existence theorem which is *stronger* than statement (35). This and (35), (34) and $\varphi = \chi + \varphi_1$ *imply* ($*$) (*b*) *of Remark D* (on Theorem 6.2), which was to be proven. Furthermore it holds:

$$\text{if } F_1 \in H_{s-3/2} \ \text{ then } \chi \in H_{s-1/2}, \tag{A.36}$$

provided there *whould* be a Cauchy problem existence theorem, which also treats successfully the case of non-integer differentiability orders. (So far there in *no* such theorem.) The statement (36) could *not* be made stronger (22). This and (36), (34) and $\varphi = \chi + \varphi_1$ *imply* ($*$) (*a*) *of Remark D* (on Theorem 6.2).

Proof of (32). – In contradiction to (32) we assume that there exists $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$) such that for every φ_0, which fulfils (6.3 I), it holds $\varphi_1 \in H_{s+1/2+\varepsilon}$ (U). Using this together with [8] and (28) one obtains that for every φ_0, which fulfils (6.3 I), it holds:

$$[\hat{D}^{s-1}_{N_\omega}\ \varphi]_{N_\omega} = [\hat{D}^{s-1}_{N_\omega}\ \varphi_1]_{N_\omega} \in H_{1+\varepsilon}\,(N_\omega) \qquad (\omega = 1, 2). \tag{A.37}$$

Now, according to (6.3 F) it holds: for any $\varepsilon > 0$ there exists a datum (23) φ_0, which fulfils (6.3 I), such that $[\hat{D}^{s-1}_{N_1}\ \varphi]_{N_1} \notin H_{1+\varepsilon}\,(N_1)$. This contradicts (37)!

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(21) $F_1 \in H_{s-3/2}$ implies $F_1 \in H_{s-2}$, which implies $\chi \in H_{s-1}$ (*see* e. g. [2]).

(22) There exist counter examples.

(23) The datum equation $\varphi = \varphi_0$ on $N_1 \cup N_2$ can be rewritten as $\varphi = \underset{\omega}{\varphi}$ on N_ω ($\underset{\omega}{\varphi} := [\varphi_0]_{N_\omega}, \omega = 1, 2$). This is of the form used in (6.3 F).

contain the details of the calculations which led to our paper [4]. Fourthermore I would like to thank the Max-Planck-Institut für Physik und Astrophysik for its hospitality during which part of this work had been carried out. I would like to thank Horst Beyer, Allan Rendall and Hans-Jürgen Seifert for helpful discussions and critical remarks.

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