ANNALES DE L'I. H. P., SECTION A

JÜRGEN GÄRTNER
ERRICO PRESUTTI

Shock fluctuations in a particle system

Annales de l'I. H. P., section A, tome 53, nº 1 (1990), p. 1-14
<http://www.numdam.org/item?id=AIHPA_1990__53_1_1_0>

Ann. Inst. Henri Poincaré,

Vol. 53, n° 1, 1990, p. 1-14. *Physique théorique*

Shock fluctuations in a particle system

by

Jürgen GÄRTNER

Akademie der Wissensschaften der DDR,
Karl-Weierstrass Institut für Mathematik,
Moherenstrasse 39, Postfach 1304, Berlin, DDR-1086

and

Errico PRESUTTI

Dipartimento di Matematica dell'Università di Roma TorVergata,
Via Fontanile di Carcaricola 00133, Roma, Italy

ABSTRACT. — The hydrodynamical behavior of the one-dimensional nearest neighbor asymmetric simple exclusion process is described by the inviscid Burgers equation. This equation has shock wave solutions and when the density before the shock is 0, the shock, at the particle level, has stable shape and rigidly fluctuates around its average position with Brownian law, [20] and [8]. We prove here that in the hydrodynamical limit such fluctuations are determined exclusively by the initial particle configuration and are not influenced by the randomness produced by the evolution.

RÉSUMÉ. — Le comportement hydrodynamique du processus de simple exclusion entre plus proches voisins asymmétriques en dimension un est décrit par l'équation de Burgers inviscide. Cette équation a des solutions ondes de choc, et quand la densité est nulle en avant du choc, le choc au niveau particulaire a une forme stable et fluctue rigidement autour de sa

Research partially supported by CNRPSMMAIT.

Annales de l'Institut Henri Poincaré - Physique théorique - 0246-0211
Vol. 53/90/01/1/14/$3.40/© Gauthier-Villars

position moyenne avec une loi Brownienne, [20] et [8]. Nous prouvons que dans la limite hydrodynamique ces fluctuations sont déterminées uniquement par la configuration initiale des particules et qu'elles ne sont pas influencées par l'aléatoire de l'évolution.

INTRODUCTION

The hydrodynamical behavior of the one-dimensional nearest neighbor asymmetric simple exclusion process is described by the inviscid Burgers equation

$$\frac{\partial}{\partial t}\rho+(p-q)\frac{\partial}{\partial r}(\rho(1-\rho))=0 \tag{1.1}$$

where $p>\frac{1}{2}$ and $q=1-p\geqslant 0$ are parameters defining the exclusion process, see below. Since (1.1) may develop singularities (discontinuities) even when the initial conditions are smooth and uniqueness does not hold (*see* e.g. [19]), a more precise statement is needed.

We first recall the definition of the process (*cf.* [17]). The configuration space is $\{0, 1\}^{Z}$ and we denote by $\eta=(\eta_x, x\in Z)$ the configurations in $\{0, 1\}^{Z}$ ($\eta_x=1$ means that there is a particle at x). To derive (1.1) we choose the initial measure μ^ε as a product measure which depends on *a scaling parameter* ε in such a way that $\mu^\varepsilon(\{\eta_x=1\})=\rho_0(\varepsilon x)$ where ρ_0 is the initial condition in (1.1), the statements below hold also for more general μ^ε. The process is defined so that each particle, independently, waits for an exponential time of mean 1 and then it attempts to jump with probability p to its right and with probability q to its left. The jump takes place if and only if the chosen site is empty. Therefore the initial condition that there is at most one particle per site is preserved by the evolution.

Given $r\in R$ and $t\geqslant 0$, denote by $\mu^\varepsilon_{r,t}$, the measure on $\{0, 1\}^{Z}$ which is obtained from the law of the process at time t (starting with law μ^ε) by a space shift of the integer part of r. Then, [18], [17] and [4],

$$\lim_{\varepsilon\to 0}\mu^\varepsilon_{\varepsilon^{-1}r,\,\varepsilon^{-1}t}=\nu_{\rho_t}(r) \tag{1.2}$$

where ν_ρ is the Bernoulli measure with constant density ρ [*i.e.* the product measure on $\{0, 1\}^{Z}$ such that $\nu_\rho(\{\eta_x=1)\}$ ρ for all x], while ρ_t is the entropic solution (*see* below) to (1.1) with initial condition ρ_0. Furthermore the convergence in (1.2) holds for all (r, t) which are continuity points for $\rho_t(r)$ (there are some restrictions on the initial profile ρ_0, but cases where singularities develop are included). Recall that the entropic solutions are obtained by adding a viscosity term to (1.1):

$$\frac{\partial}{\partial t}\rho+(p-q)\frac{\partial}{\partial r}(\rho(1-\rho))=\frac{\lambda}{2}\frac{\partial^2}{\partial r^2}\rho. \tag{1.3}$$

Then for any fixed initial condition there is a unique solution to (1.3) which depends on λ. This solution has a limit when $\lambda \to 0$ and the limit solves (1.1) (weakly): it is the entropic solution of (1.1). Analogous results on the hydrodynamical behavior of (1.2) have been obtained for an asymmetric zero range process, very closely related to the asymmetric simple exclusion ([2], [3]).

Of course one of the most interesting features of these models is that they develop discontinuities, shocks, giving us the chance to study in a particle model the formation, structure and stability of shocks when the hydrodynamic assumption of *small gradients* fails. The shock waves $\rho_t(r)$ are solutions to (1.1) such that $\rho_t(r)=\rho(r-vt)$, where $\rho(r)$ is any step function such that the density ρ_- to the left of the step is smaller than the density to the right, ρ_+. The velocity v of the wave is then given by $(p-q)(1-\rho_- - \rho_+)$. To produce this shock in the particle system we choose the initial measure μ^ε as above with $\rho_0(r)=\rho_-$ if $r<0$ and $=\rho_+$ if $r\geqslant 0$. Such an initial condition is among those allowed in [4]. To examine the microscopic structure of the shock at finite macroscopic times we first consider an observer which moves with the speed of the shock. We have

$$\lim_{\varepsilon \to 0} \mu^{\varepsilon}_{\varepsilon^{-1}vt,\, \varepsilon^{-1}t}=\frac{1}{2}[\nu_{\rho_-}+\nu_{\rho_+}] \tag{1.4}$$

as proven in [1] for the case $\rho_- + \rho_+ = 1$ [*i. e.* $v=0$]; in [20] for $\rho_-=0$ and $p=1$ and then extended in [8] to all $p>\frac{1}{2}$, ρ_- being still equal to 0.

Equation (1.4) shows that we cannot follow the shock by moving with its limiting speed. With equal probability it will be behind or ahead of us. The shock fluctuates randomly around its expected position and there is no deterministic way to follow it. We should therefore look around its expected position in a way that depends on the particular realizations of the process, on the particular runs in a computer simulation. The question then is whether in the limit as $\varepsilon \to 0$ we see some other densities which continuously interpolate between ρ_- and ρ_+, or we see an abrupt transition from ρ_- to ρ_+ which does not get smoother when $\varepsilon \to 0$. There is evidence pointing to this latter possibility, which allows for a microscopic definition of the shock's position.

When $\rho_-=0$ it has been proven, [11], that there is a unique invariant measure as seen from the leftmost particle. Asymptotically to the right the invariant distribution converges to $\nu_{\rho+}$, so that the wave front persists also at the particle level and it is not a byproduct of the limiting procedure used to derive the hydrodynamical equation (1.1): its shape is stable in the strongest possible sense. Furthermore in [20] and [8] it is proven that the displacements of the shock or the motion of the leftmost particle are, in a suitable scaling, Brownian fluctuations around the average drift. The corresponding diffusion coefficient D has been shown to coincide with the velocity of the wave front, in the case $\rho_-=0$.

Observe that the stability of the shape of the shock is also consistent with the behavior predicted by (1.1). In the space of all profiles, the *manifold* obtained by shifting the traveling wave is in fact stable under local perturbations, [10]: a profile which is a local perturbation of a step and which evolves according to (1.1) differs from the pure step in the limit when $t\to\infty$, only by a finite space shift.

Computer simulations indicate that the previous results hold also when $\rho_->0$, [6]. The numerical evidence indicates *that there is a random traveller who sees a stationary distribution of particles with densities* ρ_-, $[\rho_+]$, *to his left*, [*right*]. More recent theoretical results confirm this, [12]. There is one more observation in agreement with this picture. By taking $p-q=\varepsilon$ (the process is then called the *weakly asymmetric simple exclusion process*) and rescaling times as ε^{-2}, while spaces are still scaled like ε^{-1}, one gets an equation like (1.3), with $p-q$ [in (1.3)] and λ replaced by 1, ([13], [9]); a cellular automaton version of this process has been studied in [5] and [15]. Equation (1.3) has again traveling wave solutions which smoothly connect ρ_- to ρ_+. By studying the density fluctuations around such profiles it has been found, *cf.* [9], that at long times there are essentially only rigid displacements of the profile. The corresponding diffusion coefficient D, in proper units, is just the same as that computed either theoretically when $\rho_-=0$, or numerically, in the general case.

The question which has motivated this paper is the following: what is the origin of the shock fluctuations? in particular, are they determined by the same randomness present in the dynamics? are they going to disappear in a deterministic system which simulates the Burgers equation?

In Section 2 we present our results while proofs are given in Section 3.

2. RESULTS

One can apply *the fluctuating hydrodynamic theory to compute* the diffusion coefficient D of the Brownian fluctuations of the shock, as noticed by Herbert Spohn, *cf.* [16]. The computation consists of three steps.

1. According to (1.1) a local perturbation of a step profile decays when $t \to \infty$ giving rise to a rigid displacement of the step. The value of this space shift is given by the integral of the difference between the perturbed and unperturbed initial data divided by the quantity $(\rho_+ - \rho_-)$, as it follows from [10].

2. The initial density fluctuation field is defined as

$$Y^\varepsilon(\varphi) \equiv \sqrt{\varepsilon} \sum_x \varphi(\varepsilon x) [\eta_x - \rho_x] \tag{2.1}$$

where $\varphi(r)$, $r \in \mathbb{R}$, is a smooth test function and $\rho_x = \rho_-$, respectively ρ_+, if $x < 0$, respectively $x \geqslant 0$. The distribution of $Y^\varepsilon(\varphi)$ is determined by μ^ε, the product measure such that $\mu^\varepsilon(\{\eta_x = 1\}) = \rho_x$. In the limit as $\varepsilon \to 0$ the fluctuation field becomes Gaussian with covariance

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(Y^\varepsilon(\varphi) Y^\varepsilon(\psi)) = \int dr\, \varphi(r) \psi(r) \rho_r (1 - \rho_r) \tag{2.2}$$

This can be interpreted by saying that the next order correction in ε to the initial profile ρ_r is given $\sqrt{\varepsilon}\, \delta\rho_r$, and

$$\langle \delta\rho_r, \delta\rho_{r'} \rangle = \rho_r (1 - \rho_r) \delta(r - r')$$

3. The density fluctuations evolve according to the linearization of (1.1).

Il we fix a macroscopic time t the only fluctuations, roughly speaking, which can reach the front of the shock [which at time 0 is at 0] are, by statement 3, those which at time 0 are in the macroscopic interval $[r_-(t), r_+(t)]$, where

$$r_-(t) = -(p-q)[1 - 2\rho_- - (1 - \rho_- - \rho_+)]\, t$$

and

$$r_+(t) = (p-q)[1 - \rho_- - \rho_+) - (1 - 2\rho_+)]\, t.$$

Hence by statement 1 the macroscopic displacement of the shock produced by these fluctuations is

$$X \equiv \frac{1}{\rho_+ - \rho_-} \sqrt{\varepsilon} \int_{r_-(t)}^{r_+(t)} dr\, \delta\rho_r$$

The diffusion coefficient D, properly normalized, is given by

$$D = \frac{1}{\varepsilon t} \langle X^2 \rangle$$

which by statement 2 gives

$$D=(p-q)\frac{1}{\rho_+-\rho_-}[\rho_-(1-\rho_-)+\rho_+(1-\rho_+)]. \tag{2.3}$$

Notice that $D=v$ if $\rho_-=0$. Hence the fluctuating hydrodynamic theory gives the correct result for the diffusion coefficient in this case. Furthermore when $\rho_->0$ (2.3) agrees with the value obtained for the diffusion in the weakly asymmetric simple exclusion process. Since the above arguments are based only on the analysis of the final hydrodynamic equation (1.1) and the fluctuations are only the initial ones, this suggests that asymptotically the shock displacements might be *measurable* hence determined by the initial particles' configuration. To check the validity of this conjecture we have considered the simplest case, $\rho_-=0$ and $p=1$ and, indeed, in this case the conjecture is true, *cf.* Theorem 2.1 below for a mathematically precise statement.

In the following we restrict ourselves to the completely asymmetric case ($p=1$). We denote by μ the product measure such that $\mu(\{\eta_x=1\})=0$ if $x<0$; $\mu(\eta_x)=\rho>0$ if $x>0$ [and $\rho<1$ to have a non trivial case] and finally $\mu(\{\eta_0=1\})=1$. We then denote by x_t the position at time t of the leftmost particle. Its average position is vt, $v=1-\rho$, while the particles distribution shifted by x_t is again given by μ, all this is contained in [20] and [8]. We have the following result:

THEOREM 2.1. — *Denote by* E_η *the expectation with respect to the totally asymmetric process starting from* η. *Then*

$$\lim_{t\to\infty} t^{-1}\int \mu(d\eta)\,E_\eta[(x_t-E_\eta(x_t))^2]=0. \tag{2.4}$$

The fluctuations of the shock are given by the fluctuations of the leftmost particle, and

$$\lim_{t\to\infty} t^{-1}\int \mu(d\eta)\,E_\eta[(x_t-E_\mu(x_t))^2]=D \tag{2.5}$$

where E_μ denotes the expectation with respect to the initial measure μ, *cf.* [8] and [20]. It then follows from (2.4) and (2.5) that the shock diffusion coefficient D equals

$$D=\lim_{t\to\infty} t^{-1}\int \mu(d\eta)\,(E_\eta(x_t)-vt)^2 \tag{2.6}$$

D is therefore also the diffusion coefficient associated to the process $E_\eta(x_t)$, which, from this point of view, cannot be distinguished from the process x_t. This shows that the overwhelming part of the shock wave fluctuations originates in the randomness of the initial configuration and is not caused by the random character of the dynamics.

While (2.6) proves that the diffusion coefficient only depends on the initial configuration, yet it does not confirm completely the predictions of the fluctuating hydrodynamic theory. In particular we do not know if $E_\eta(x_t)$ only depends on the *density fluctuations* present in η. We would be surprised if this were not the case, we conjecture that the same phenomena occur when $p<1$ and for the diffusion of a tagged particle in equilibrium in the asymmetric simple exclusion process, *cf.* [7] and [14]. More generally the fluctuating hydrodynamic theory predicts a similar behavior whenever (1) the fluctuations refer to a *collective variable*, like for instance the density in a system where particles are conserved, and (2) the hydrodynamic equations are of the Euler type, that is they are invariant when space and time are scaled in the same way. Notice that our model fits into this class, since the fluctuations of the leftmost particle are the same as the fluctuations of the shock profile, *i. e.* a macroscopic quantity. For another, more elementary, example consider a system of non interacting Brownian motions with drift, *i. e.* the particles move like independent diffusions with generator

$$\frac{1}{2}\frac{d^2}{dx^2}-v.\frac{d}{dx}$$

$v>0$. The hydrodynamical equation for the model is

$$\frac{\partial}{\partial t}\rho+v.\frac{d}{dr}\rho=0$$

so that condition (2) stated above is fulfilled. Let then the initial particle distribution μ be Poissonian on the positive half-axis with intensity 1, while no particle is present on the other half. Denote by x_t the number of particles which are on the negative half-axis at time t. This becomes a macroscopic variable for large t, so that also the condition (1) above is fulfilled. Finally a relatively simple computation shows that (2.4) and (2.5) are satisfied with $D=v$.

The proof of Theorem 2.1, given in the next section, is based, quoting David Wick, on a *tour de force of couplings*.

3. PROOFS

We start by making explicit the isomorphism between the asymmetric simple exclusion process as seen from the leftmost particle and the zero range process. We set some notation and recall statements already proven which we shall use in the following.

Notation and known results

1. Let η_t, $t \geqq 0$, be the totally asymmetric simple exclusion process and assume that the law of η_0 is given by the product measure μ defined in the previous section (before Theorem 2.1). We denote by x_t the position of the leftmost particle in the configuration η_t. This particle will be called *zero particle*, the next will be called *first particle*, and so on. We denote by $\eta_t - x_t$ the configuration η_t shifted to the left by x_t. It is known that the law of $\eta_t - x_t$ is still μ, *cf.* for instance [11] where a measure with this invariance property is constructed for all $p > \frac{1}{2}$.

2. Using the process η_t, $t \geqq 0$, we construct a new process ξ_t, $t \geqq 0$. Its configurations $\xi = \xi(x)$, $x \in \mathrm{N}\,[\mathrm{N} = \{0, 1, 2 \dots\}]$ are obtained in the following way: $\xi_t(x)$ is the number of empty sites between the x-th and the $(x+1)$-th particle in the configuration η_t. The process defined in this way is a Markov process with state space N^{N}. Its generator L acts on the bounded cylindrical functions f according to

$$(\mathrm{L} f)(\xi) = \sum_{x \geqq 0} 1_{\{(\xi(x) > 0)\}} [f(\xi^{x, x-1}) - f(\xi)] \tag{3.1}$$

Thereby $\xi^{x, y}$, x, $y \in \mathrm{N}$, denotes the configuration which is obtained from ξ by removing a particle from site x and adding it to y; $\xi^{0, -1}$ is obtained from ξ by removing a particle from 0. Hence the ξ particles keep jumping to the left and finally they disappear after jumping from 0.

3. The displacement x_t of the leftmost particle in the exclusion process equals the number N_t of ξ-particles which have jumped from 0 in the time interval $[0, t]$. N_t may be considered as the *current through* 0 in the ξ process.

4. The image λ of the measure μ induced on N^{N} by the above construction of the ξ-process is a product of geometric distributions with parameter $\bar{\rho} = 1 - \rho$, namely

$$\lambda(\xi(x) = k) = (1 - \bar{\rho})\, \bar{\rho}^k$$

which, of course, is an invariant measure for the process with generator L.

5. Since λ is an invariant measure for the Markov process with generator L the dual process obtained by letting the time run backwards in the original process, is a Markov process whose generator L* acts on the bounded cylindrical functions f as

$$(\mathrm{L}^* f) = \sum_{x \geqq 0} 1_{\{\xi(x) > 0\}} [f(\xi^{x, x+1}) - f(\xi)] + \bar{\rho}\,[f(\xi^0) - f(\xi)] \tag{3.2}$$

where ξ^0 is obtained from ξ by adding an extra particle in 0. Therefore this is the process where particles jump to the right and there is an infinite source of particles in -1 which sends in particles in 0 with intensity $\bar{\rho}$. This whole statement can be easily proven, in any case we refer to [8].

6. Both the original and the dual processes can be extended to processes on the whole lattice, namely with state space N^Z. This is trivial for the original process since particles move to the left hence what happens after they cross 0 does not influence the process in $[0, \infty)$. To make this precise we define the generator L as in (3.1) dropping the requirement that $x \geqq 0$, now x runs over the whole Z. This is the *zero range process* which will be considered in the sequel. The products of (identical) geometric distributions are again invariant. We will use the letter λ also to denote the product on N^Z of geometric distributions with parameter $\bar{\rho}$. To recover the previous process on N^N we simply have to take the marginal of the zero range process on N^N. It is less obvious but still true that the zero range process on the whole Z with jumps only to the right and invariant measure λ has a marginal on N^N whose law is the same as that of the dual process defined in statement 5 above, for a proof we refer to [8]. We shall denote again by L^* the generator of this last zero range process and usually we shall insert $*$ superscripts when referring to it. In the sequel we shall not distinguish whether we are realizing the original and the dual processes as in 1 and 5 above or as the associated zero range processes on the whole Z. We shall switch from one interpretation to the other according to local convenience.

7. Given a zero range process, let us call its particles *first class particles*. A *second class particle* is then defined as an extra particle which is allowed to jump only when there is no first class particle at the same site. When alone, it jumps with the same intensities on the left and right as if it were a first class particle. Analogously one defines processes with several second class particles and, in a similar fashion, one introduces the notion of third class particles and so forth, *cf.* [2] for more details. We denote the law of the zero range process by P_λ [and the law of its dual by P^*_λ. If we add at time zero a second class particle at x, then we will denote its position at time t by $z_t\,[z^*_t]$ and the underlying probability law by $P_{\lambda, x}\,[P^*_{\lambda, x}]$. Finally if the initial distribution is supported by a configuration ξ we will use the subscript ξ instead of λ.

At last we have all the notation for stating the following.

PROPOSITION 3.1. — *Equation* (2.4) *is a consequence of*

$$\lim_{t \to \infty} t^{-1} \int_0^t ds\, E^*_{\lambda, 0}\,(P_{\xi^*_s, z^*_s}(z_t \geqq 0)) = 0 \tag{3.3}$$

Remarks. — The expectation in (3.3) has the following meaning. We start considering the dual process in equilibrium and, at time 0, we add a second class particle at 0. We let the process run for a time s and at this time we have a random configuration ξ^*_s and a random position z^*_s for the second class particle. We switch to the original process starting from this situation, then particles move to the left and we consider the event that the second class particle after a time t has not yet crossed the origin. The probability of such an event is what we need to evaluate. The contribution to this probability comes from trajectories where the second class particle travels a longer distance in a time s with the dual process than, subsequently, in a time t, with the original one. Since $t \geqq s$ and because of the factor t^{-1} in (3.3) the condition (3.3) will be a consequence of a law of large numbers for the motion of a second class particle (Proposition 3.2 below). By using statement 6 in the beginning of this section such an estimate on the motion of the second class particle is only needed in equilibrium.

Proof. – In [8] it has been proven that

$$D = \lim_{t \to \infty} t^{-1} \int \mu(d\eta) E_\eta [(x_t - vt)^2] = v. \tag{3.4}$$

By adding and subtracting $E_\eta(x_t)$ and expanding the square we see that (2.4) is then a consequence of

$$\lim_{t \to \infty} t^{-1} \int \mu(d\eta)(E_\eta(x_t) - vt)^2 = v. \tag{3.5}$$

Writing (3.5) in terms of the zero range process and recalling that $v = 1 - \rho = \bar{\rho}$, we then conclude that (2.4) is implied by

$$\lim_{t \to \infty} t^{-1} E_\lambda([E_{\xi_0}(N_t - \bar{\rho}\, t)]^2) = \bar{\rho} \tag{3.6}$$

We are going to prove that

$$E_\lambda([E_{\xi_0}(N_t - \bar{\rho}\, t)]^2) = \bar{\rho} \int_0^t ds\, E^*_{\lambda,0}(P_{\xi^*_s, z^*_s}(z_t < 0)) \tag{3.7}$$

and this will conclude the proof of Proposition 3.1.

Proof of (3.7). By (3.1)

$$N_t - \int_0^t ds\, 1_{\{\xi_s(0) > 0\}}$$

is a martingale [for P_ξ and any ξ], so that

$$E_\xi(N_t - \bar{\rho}\, t) = \int_0^t ds [P_\xi(\xi_s(0) > 0) - \bar{\rho}]$$

Hence

$E_\lambda([E_{\xi_0}(N_t - \bar{\rho}\, t)]^2)$

$$= \int_0^t dr \int_0^t ds \{ E_\lambda (1_{\{\xi_r(0)>0\}} P_{\xi_0}(\xi_s(0)>0)) - \bar{\rho}^2 \} \quad (3.8)$$

By using the dual process we get for any bounded measurable function $f(\xi)$

$$E_\lambda (1_{\{\xi_r(0)>0\}} f(\xi_0)) = E_\lambda^* (1_{\{\xi_0^*(0)>0\}} f(\xi_r^*))$$

Furthermore, since λ is a product of geometric distributions, we have

$$E_\lambda^* (f(\xi_r^*) \mid \xi_0^*(0)>0) = E_{\lambda,0}^* (f(\xi_r^* + \delta_{z_r^*}))$$

δ_x denoting the configuration which consists of only one particle placed at x, therefore the argument of f on the right hand side is the configuration obtained by adding to ξ_r^* the second class particle whose position is denoted by z_r^*.

By choosing $f(\xi) = P_\xi(\xi_s(0)>0)$ and using the last two equations we get

$$\begin{aligned} E_\lambda (1_{\{\xi_r(0)>0\}} P_{\xi_0}(\xi_s(0)>0)) &= \bar{\rho}\, E_{\lambda,0}^* (P_{\xi_r^*, z_r^*}(\xi_s(0) + \delta_{z_s}(0)>0)) \\ &= \bar{\rho}\, E_{\lambda,0}^* (P_{\xi_r^*}(\xi_s(0)>0) + P_{\xi_r^*, z_r^*}(\xi_s(0)=0, z_s=0)) \\ &= \bar{\rho}^2 + \bar{\rho}\, E_{\lambda,0}^* (P_{\xi_r^*, z_r^*}(\xi_s(0)=0, z_s=0)) \quad (3.9) \end{aligned}$$

From (3.8) and (3.9) we get

$$E_\lambda ([E_{\xi_0}(N_t - \bar{\rho}\, t)]^2) = \bar{\rho} \int_0^t dr \int_0^t ds\, E_{\lambda,0}^* (P_{\xi_r^*, z_r^*}(\xi_s(0)=0, z_s=0)) \quad (3.10)$$

Applying the generator of the process (ξ_t, z_t) to the indicator function $1_{\{z<0\}}$, we find that

$$1_{\{z_t<0\}} - \int_0^t ds\, 1_{\{\xi_s(0)=0,\, z_s=0\}}$$

is a $P_{\xi,z}$-martingale for all ξ and z. Therefore

$$P_{\xi,z}(z_t<0) = \int_0^t ds\, P_{\xi,z}(\xi_s(0)=0, z_s=0) \quad (3.11)$$

Substituting (3.11) in (3.10), we finally arrive at (3.7). ■

To complete the proof of Theorem 2.1 we use the following law of large numbers for a second class particle.

PROPOSITION 3.2 [1]. – *For any* $0<\bar{\rho}<1$ *let* λ *be the measure on* $\mathbb{N}^{\mathbb{Z}}$ *which is the product of geometric distributions with parameter* $\bar{\rho}$. *Let* $P_{\lambda,0}^*$

[1] We have not been able to find in the literature the proof of Proposition 3.2, which looks very much like a corollary to [2].

be the law of the zero range process with jumps to the right, with initial distribution λ *and with a second class particle initially at the origin. Denoting by* z_t^* *the position at time* t *of the second class particle, we have for all* $\delta>0$

$$\lim_{t\to\infty} P_{\lambda,0}^*([(1-\bar{\rho})^2-\delta]\,t\leqq z_t^*\leqq[(1-\bar{\rho})^2+\delta]\,t)=1. \qquad (3.12)$$

Before proving the proposition we complete the proof of Theorem 2.1. Let $0<\alpha<\bar{\rho}$. Then

$$\begin{aligned}\frac{1}{t}\int_0^t ds\, E_{\lambda,0}^*(P_{\xi_s^*,z_s^*}(z_t\geqq 0))&\leqq\frac{1}{t}\int_0^t ds\, P_{\lambda,0}^*(z_s^*\geqq(1-\alpha)^2 s)\\&\quad+\frac{1}{t}\int_0^t ds\, E_{\lambda,0}^*(1_{z_s^*<(1-\alpha)^2 s}\,P_{\xi_s^*,z_s^*}(z_t\geqq 0))\\&\leqq\frac{1}{t}\int_0^t ds\, P_{\lambda,0}^*(z_s^*\geqq(1-\alpha)^2 s)+\frac{1}{t}\int_0^t ds\, P_{\lambda,[(1-\alpha)^2 s]}(z_t\geqq 0)\qquad(3.13)\end{aligned}$$

where $P_{\lambda,x}$ is the law of the zero range process with jumps on the left, initially distributed according to λ and having a second class particle initially at x. To write the last inequality in (3.13) we have used that $P_{\lambda,x}(z_t\geqq 0)$ is a non decreasing function of x. We now apply Proposition 3.2 (and its analogue for the process with jumps to the left) and we get that the first term on the right of (3.13) vanishes when $t\to\infty$. The only times s which contribute to the second integral are such that $(1-\alpha)^2 s\geqq(1-\bar{\rho})^2 t$, asymptotically for $t\to\infty$. By letting $t\to\infty$ the second term on the right of (3.13) converges to $1-(1-\bar{\rho})^2/(1-\alpha)^2$. By letting $\alpha\to\bar{\rho}$ this proves (3.3) and, by Proposition 3.1, Theorem 2.1.

Proof of Proposition 3.2. — We change notation to simplify the formulae below by dropping the superscript $*$ which referred to the process with jumps to the right and writing ρ instead of $\bar{\rho}$. We prove upper and lower bounds on $t^{-1} z_t$ which imply (3.12).

Fix $0<\alpha<\beta<1$ arbitrarily. Let $\lambda_\alpha\,[\lambda_{\alpha,\beta}]$ denote the measure on $\mathbf{N}^{\mathbf{Z}}$ which is the product of geometric distributions with parameter α [with parameter α for $x<0$ and β for $x\geqq 0$]. Since $\lambda_\alpha\leqq\lambda_{\alpha,\beta}\leqq\lambda_\beta$ stochastically, we can consider the following coupling of three zero range processes having initial distributions λ_α, $\lambda_{\alpha,\beta}$ and λ_β, respectively. The first process consists of first class particles, the second is obtained by adding initially on $x\geqq 0$ second class particles. From this the third process is obtained by adding initially on $x<0$ third class particles. Let $z_t^{(2)}\,[z_t^{(3)}]$ denote the position at time t of the leftmost second class [rightmost third class] particle. The key for the proof of the proposition is the following fact which can be extracted from the proof of Theorem 2.4 in [2]:

$$\lim_{t \to \infty} \frac{z_t^{(2)}}{t} = \lim_{t \to \infty} \frac{z_t^{(3)}}{t} = (1-\alpha)(1-\beta) \tag{3.14}$$

where the convergence is in probability.

To get an upper bound for z_t, we choose $0<\alpha<\rho$ and $\beta=\rho$. Then $z_t \leqq z_t^{(2)}$ stochastically. (To see this, consider z_t as the position of an extra fourth class particle initially located at the origin). Thus, applying the first half of (3.14) and choosing thereby α arbitrarily close to ρ, we find that

$$\limsup_{t \to \infty} t^{-1} z_t \leqq (1-\rho)^2$$

in probability.

For the lower bound we put $\alpha=\rho$ and $\rho<\beta<1$ which implies that $z_t \geqq z_t^{(3)}$ stochastically. To see this, one first checks the validity of the stochastic order in the case when no second class particles are present. Because of the second half of (3.14) this yields

$$\liminf_{t \to \infty} t^{-1} z_t \geqq (1-\rho)^2$$

in probability. This concludes the proof. ■

ACKNOWLEDGEMENTS

We are grateful to Carlo Boldrighini, Pablo Ferrari, Joel Lebowitz and Herbert Spohn for useful comments. One of us (J.G.) acknowledges very kind hospitality at the Mathematical Departments of the Universities of Roma La Sapienza and Roma Tor Vergata.

Notes added in proof. – Some recent computer simulations on the Boghosian Levermore cellular automaton which simulates the Burgers equation, cf. [5] and [15], (this is a time-discrete version of the simple exclusion process) give new evidence that the shock fluctuations are determined only by the initial density fluctuations. Such conclusions are reported in

Z. Cheng, J. L. Lebowitz and E. R. Speer, *Microscopic shock structure in model particle systems: the Boghosian Levermore cellular automaton revisited*, preprint 1990.

A theoretical proof that the shock fluctuations are determined by the initial conditions has been obtained recently for the weakly asymmetric simple exclusion process:

P. Dittrich, *Travelling waves and long time behaviour of the weakly asymmetric exclusion process*, preprint 1990.

REFERENCES

[1] E. ANDJEL, M. BRAMSON and T. M. LIGGETT, Shocks in the asymmetric exclusion process, *Prob. Theor. Rel. Fields*, Vol. **78**, 1988, pp. 231-247.

[2] E. ANDJEL and C. KIPNIS, Derivation of the hydrodynamical equation for the zero range interaction process, *Ann. Prob.*, Vol. **12**, 1984, pp. 325-334.

[3] E. ANDJEL and M. E. VARES, Hydrodynamic equations for attractive particle systems in Z, *J. Stat. Phys.*, Vol. **47**, 1987, pp. 265-288.

[4] A. BENASSI and J. P. FOUQUE, Hydrodynamic limit for the asymmetric simple exclusion process, *Ann. Prob.*, Vol. **15**, 1987, pp. 546-560.

[5] B. BOGHOSIAN and D. LEVERMORE, A cellular automaton for Burgers equation, *Complex Systems*, Vol. **1**, 1987, pp. 17-30.

[6] C. BOLDRIGHINI, G. COSIMI, A. FRIGIO and M. GRASSO-NUMES, Computer simulations of shock waves in the completely asymmetric simple exclusion, *J. Stat. Phys.*, Vol. **55**, 1985, pp. 611-624.

[7] A. DE MASI and P. A. FERRARI, Self-diffusion in one dimensional lattice gases in the presence of an external field, *J. Stat. Phys.*, Vol. **38**, 1985, pp. 603-613.

[8] A. DE MASI, C. KIPNIS, E. PRESUTTI and E. SAADA, Microscopic structure at the shock in the asymmetric simple exclusion, *Stochastics* (to appear).

[9] A. DE MASI, E. PRESUTTI and E. SCACCIATELLI, The weakly asymmetric simple exclusion process, *Ann. Inst. H. Poincaré*, Vol. **25**, 1989, pp. 1-38.

[10] R. J. DIPERNA and A. MAJDA, The validity of nonlinear geometric optics for weak solutions of conservation laws, *Commun. Math. Phys.*, Vol. **98**, 1985, pp. 313-347.

[11] P. A. FERRARI, The simple exclusion process as seen from a tagged particle, *Ann. Prob.*, Vol. **14**, 1986, pp. 1277-1290.

[12] P. A. FERRARI, C. KIPNIS and E. SAADA, *Microscopic structure of traveling waves in the asymmetric simple exclusion*, CARR preprints, n.4/89, February 1989.

[13] J. GÄRTNER, Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric simple exclusion processes, *Stoch. proc. Appl.*, Vol. **27**, 1988, pp. 233-260.

[14] C. KIPNIS, Central limit theorems for infinite series of queues and applications to simple exclusion, *Ann. Prob.*, Vol. **14**, 1986, pp. 397-408.

[15] J. L. LEBOWITZ, E. ORLANDI and E. PRESUTTI, Convergence of stochastic cellular automaton to Burgers' equation: fluctuations and stability, *Physica*, D **33**, 1988, pp. 165-188.

[16] J. L. LEBOWITZ, E. PRESUTTI and H. SPOHN, Microscopic models of hydrodynamic behavior, *J. Stat. Phys.* Vol. **51**, 1988, pp. 841-962.

[17] T. M. LIGGETT, *Interacting particle systems*, Springer-Verlag, 1985.

[18] H. ROST, Non equilibrium behavior of many particle process: density profiles and local equilibria, *Z. Wahrsch. Verw. Gebiete*, Vol. **58**, 1981, pp. 41-53.

[19] J. SMOLLER, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1983.

[20] D. W. WICK, A dynamical phase transition in an infinite particle system, *J. Stat. Phys.*, Vol. **38**, 1985, pp. 1015-1025.

(*Manuscript received August* 17, 1989;
revised November 18, 1989.)