

# ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 52, n° 3 (1990), p. 237-257

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# How small is the phase space in quantum field theory?

by

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**ABSTRACT.** — The existing compactness and nuclearity conditions characterizing phase space properties of a quantum field theory are compared and found to be particularly sensitive either in the infrared or the ultraviolet domain. A sharpened condition combining both features is proposed and shown to be satisfied in free field theory in four space-time dimensions.

**RÉSUMÉ.** — Nous comparons les conditions connues de compacité et de nucléarité qui caractérisent les propriétés d'espace de phase d'une théorie quantique des champs. Nous montrons qu'elles sont particulièrement adaptées soit au domaine ultraviolet ou infrarouge. Nous proposons une nouvelle condition qui combine ces deux aspects et qui est vérifiée pour le champ libre en dimension quatre.

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## 1. INTRODUCTION

In the structural analysis of quantum field theory it has proved to be useful to describe the size of phase space with the help of compactness or nuclearity conditions ([1]-[4]). This approach is based on the heuristic idea that the number of states of fixed total energy and limited spatial extension

should be finite. Loosely speaking, it ought to be proportional to the volume of phase space which is occupied by these states.

Since phase space is an ambiguous concept in quantum field theory the mathematical description of this idea is somewhat subtle and has led to different formulations. We briefly recall here the relevant proposals. The first class of conditions can be expressed in terms of certain specific maps  $\Theta_{\beta, \mathcal{O}}$ ,  $\beta > 0$  mapping the local  $C^*$ -algebras  $\mathfrak{A}(\mathcal{O})$  associated with bounded space-time regions  $\mathcal{O}$  into the underlying Hilbert space  $\mathcal{H}$ . These maps are given by

$$\Theta_{\beta, \mathcal{O}}(A) = e^{-\beta H} A \Omega, \quad A \in \mathfrak{A}(\mathcal{O}) \quad (1.1)$$

where  $H$  is the Hamiltonian and  $\Omega \in \mathcal{H}$  the vector representing the vacuum state. It has been argued by Haag and Swieca [1] that in theories with a sensible particle interpretation there holds

CONDITION C. – The maps  $\Theta_{\beta, \mathcal{O}}$  are compact <sup>(1)</sup> for any bounded region  $\mathcal{O}$  and any  $\beta > 0$ .

Based on thermodynamical considerations it was shown in [2] that the requirement of compactness in this condition may be strengthened to nuclearity <sup>(1)</sup>. In fact one may demand [3]

CONDITION N. – The maps  $\Theta_{\beta, \mathcal{O}}$  are  $p$ -nuclear <sup>(1)</sup> for any bounded region  $\mathcal{O}$  and any sufficiently large  $\beta > 0$ .

The second class of conditions is in a sense dual to the preceding ones: instead of starting from local excitations of the vacuum whose energy is then cut off as in (1.1) one proceeds from states of limited energy and subsequently restricts them to the local algebras.

This procedure is formalized as follows: given the space  $\mathcal{T}$  of normal linear functionals on  $\mathcal{B}(\mathcal{H})$  (which can be identified with the set of trace-class operators on  $\mathcal{H}$ ) one first selects for each  $\beta > 0$  the subset of functionals  $\varphi \in \mathcal{T}$  which are in the domain of  $e^{\beta H}$  under simultaneous left and right multiplication, *i.e.*  $e^{\beta H} \varphi e^{\beta H} \in \mathcal{T}$ . This set, equipped with the norm  $\|\varphi\|_{\beta} = \|e^{\beta H} \varphi e^{\beta H}\|$ , forms a Banach space  $\mathcal{T}_{\beta}$  of functionals of limited energy. The localization in configuration space is then accomplished by restricting these functionals to the algebras  $\mathfrak{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ . Accordingly, one considers in this approach the maps  $\Pi_{\beta, \mathcal{O}}: \mathcal{T}_{\beta} \rightarrow \mathfrak{A}(\mathcal{O})^*$  (where  $\mathfrak{A}(\mathcal{O})^*$  denotes the dual space of  $\mathfrak{A}(\mathcal{O})$ ) given by

$$\Pi_{\beta, \mathcal{O}}(\varphi) = \varphi \upharpoonright \mathfrak{A}(\mathcal{O}), \quad \varphi \in \mathcal{T}_{\beta}. \quad (1.2)$$

Making use of these notions, Fredenhagen and Hertel have studied in an unpublished manuscript [4] the following variant of Condition C.

<sup>(1)</sup> The formal definition of this notion will be recalled in Sect. 2.

CONDITION  $C_{\#}$ . — The maps  $\Pi_{\beta, \mathcal{O}}$  are compact for any bounded region  $\mathcal{O}$  and any  $\beta > 0$ .

It was pointed out in [4] that Condition  $C_{\#}$  entails Condition C, but the question of whether also the converse holds true remained unclear. We will clarify this point by giving a counter-example: the free massless scalar field in three space-time dimensions. This example makes plain that Condition  $C_{\#}$  imposes strong restrictions on the admissible number of states associated with the infrared domain of phase space. In contrast, Condition N has been designed in order to control this number in the ultraviolet region.

In view of this situation it seems natural to look out for a criterion which combines the desirable features of Conditions  $C_{\#}$  and N, and hence provides a more accurate description of the size of the phase space in quantum field theory. We propose as such a criterion

CONDITION  $N_{\#}$ . — The maps  $\Pi_{\beta, \mathcal{O}}$  are  $p$ -nuclear for any bounded region  $\mathcal{O}$  and any sufficiently large  $\beta > 0$ .

The relation between the various conditions given above can be summarized in the diagram (the lattice of constraints)

$$\begin{array}{ccc} N_{\#} & \rightarrow & N \\ \downarrow & & \downarrow \\ C_{\#} & \rightarrow & C \end{array}$$

where the arrows point from the stronger to the weaker conditions. This diagram provides a complete description of the general state of affairs inasmuch as no further arrows may be added (apart from redundant ones) and none of the arrows can be reversed. These facts will be established in Sect. 2.

In the major part of this paper (Sect. 3) we will prove that Condition  $N_{\#}$  is satisfied in the free field theory of massive and massless scalar particles in four space-time dimensions. Although these calculations are elementary we find it worthwhile to place them on record since they provide some insight into the mechanism which enforces Condition  $N_{\#}$ . In particular the rôle of the dimension of space can explicitly be traced in this example. The paper concludes with a brief outlook.

## 2. THE LATTICE OF CONSTRAINTS

In this section we study the interrelations between the various compactness and nuclearity conditions given in the Introduction. We begin by supplying the formal definition of concepts already used.

DEFINITION. — Let  $\mathfrak{E}$  and  $\mathfrak{F}$  be Banach spaces and let  $\Theta$  be a linear map from  $\mathfrak{E}$  into  $\mathfrak{F}$ .

(i) The map  $\Theta$  is said to be compact if it can be approximated by maps of finite rank in the norm topology. (This restrictive definition of compactness is sufficient since the spaces  $\mathfrak{F}$  which are of interest here have the approximation property [5].)

(ii) The map  $\Theta$  is said to be  $p$ -nuclear,  $p$  being any real positive number, if there exist functionals  $e_i \in \mathfrak{E}^*$  and elements  $F_i \in \mathfrak{F}$  such that in the sense of strong convergence

$$\Theta(\cdot) = \sum_i e_i(\cdot) \cdot F_i \quad (2.1 a)$$

and

$$\|\Theta\|_p = \inf \left[ \sum_i \|e_i\|^p \cdot \|F_i\|^p \right]^{1/p} < \infty. \quad (2.1 b)$$

Here the infimum is to be taken with respect to all possible choices of  $e_i \in \mathfrak{E}^*$  and  $F_i \in \mathfrak{F}$  in the representation (2.1 a) of  $\Theta$ . The quantity  $\|\cdot\|_p$  will be called  $p$ -norm, yet it should be noticed that it is only a quasi-norm [5] if  $p < 1$ . For  $p = 1$  one just obtains the nuclear maps.

*Remark.* — Besides the notion of  $p$ -nuclearity, the related notion of “order of a map” has proved to be very useful in our context. For a comparison of these notions cf. [3]. We only note here that if one replaces in Condition N (respectively  $N_*$ ) the term “ $p$ -nuclear” by “of order  $p$ ”, the ensuing condition is equivalent to the original one.

Let us now establish the structure of our lattice of constraints. We begin with the relations which are almost obvious.

$N_* \rightarrow C_*$ : Let  $\beta > 0$  be given. Then we define for  $E > 0$  a map  $\Lambda_E: \mathcal{T}_\beta \rightarrow \mathcal{T}_\beta$  by

$$\Lambda_E(\varphi) = P_E \cdot \varphi \cdot P_E, \quad \varphi \in \mathcal{T}_\beta \quad (2.2)$$

where  $P_E$  is the spectral projection of the Hamiltonian  $H$  corresponding to the spectrum in  $[0, E]$ . We also consider the map  $\bar{\Lambda}_E: \mathcal{T}_\beta \rightarrow \mathcal{T}_{\bar{\beta}}$  for arbitrary  $\bar{\beta} > 0$  which is likewise given by (2.2). Both maps are bounded. Moreover, given  $\mathcal{O}$ , we have  $\Pi_{\beta, \mathcal{O}} \circ \Lambda_E = \Pi_{\bar{\beta}, \mathcal{O}} \circ \bar{\Lambda}_E$ . Now according to Condition  $N_*$  there exists a  $\bar{\beta} > 0$  such that  $\Pi_{\bar{\beta}, \mathcal{O}}$  is nuclear, hence  $\Pi_{\beta, \mathcal{O}} \circ \Lambda_E$  is nuclear too for all  $E > 0$ . On the other hand

$$\|\Pi_{\beta, \mathcal{O}} - \Pi_{\bar{\beta}, \mathcal{O}} \circ \Lambda_E\| \leq \sup \{ \|\varphi - P_E \varphi P_E\| : \|\varphi\|_\beta \leq 1 \} \leq 2 \cdot e^{-\beta E}. \quad (2.3)$$

This shows that  $\Pi_{\beta, \mathcal{O}}$  can be approximated in norm by nuclear operators, and consequently also by operators of finite rank.

$N \rightarrow C$ : The argument proving this relation is completely analogous to the preceding one and is therefore omitted.

$N_* \rightarrow N$ : Let  $\Pi_{\beta, \mathcal{O}}$  be  $p$ -nuclear. Since  $p$ -nuclear maps are also  $p'$ -nuclear for any  $p' \geq p$  it suffices to consider the case  $p \leq 1$ . Then we can find

elements  $S_i \in \mathcal{T}_\beta^*$  and  $\varphi_i \in \mathfrak{A}(\mathcal{O})^*$  such that for any  $\varphi \in \mathcal{T}_\beta$  (in the sense of absolute convergence)

$$\Pi_{\beta, \mathcal{O}}(\varphi) = \sum_i S_i(\varphi) \cdot \varphi_i \tag{2.4}$$

and  $\sum_i \|S_i\|^p \cdot \|\varphi_i\|^p < \infty$ . Since  $\mathcal{T}_\beta = e^{-\beta H} \mathcal{T} e^{-\beta H}$  and since each  $S_i$  is continuous, we can represent the functionals  $\psi \in \mathcal{T} \rightarrow S_i(e^{-\beta H} \psi e^{-\beta H})$  in the form

$$S_i(e^{-\beta H} \psi e^{-\beta H}) = \psi(T_i), \tag{2.5}$$

where  $T_i \in \mathcal{T}^* = \mathcal{B}(\mathcal{H})$  are operators with  $\|T_i\| = \|S_i\|$ . Plugging this information into relation (2.4) we obtain for each  $A \in \mathfrak{A}(\mathcal{O})$  the equality in  $\mathcal{B}(\mathcal{H})$

$$e^{-\beta H} A e^{-\beta H} = \sum_i \varphi_i(A) \cdot T_i, \tag{2.6}$$

where  $\sum_i \|\varphi_i\|^p \cdot \|T_i\|^p < \infty$ . Hence the map  $\Xi_{\beta, \mathcal{O}}: \mathfrak{A}(\mathcal{O}) \rightarrow \mathcal{B}(\mathcal{H})$  given by

$$\Xi_{\beta, \mathcal{O}}(A) = e^{-\beta H} A e^{-\beta H}, \quad A \in \mathfrak{A}(\mathcal{O}) \tag{2.7}$$

is  $p$ -nuclear. In fact  $\|\Xi_{\beta, \mathcal{O}}\|_p = \|\Pi_{\beta, \mathcal{O}}\|_p$ , as is easily seen. Since  $\Theta_{\beta, \mathcal{O}}(\cdot) = \Xi_{\beta, \mathcal{O}}(\cdot) \cdot \Omega$  and since the map  $B \in \mathcal{B}(\mathcal{H}) \rightarrow B\Omega$  is bounded, we conclude that  $\Theta_{\beta, \mathcal{O}}$  is  $p$ -nuclear, too.

We note for later reference that the equality of the  $p$ -norms of  $\Pi_{\beta, \mathcal{O}}$  and  $\Xi_{\beta, \mathcal{O}}$  implies that Condition  $N_\#$  is equivalent to

CONDITION  $N^*$ . – The maps  $\Xi_{\beta, \mathcal{O}}$  are  $p$ -nuclear for any bounded region  $\mathcal{O}$  and any sufficiently large  $\beta > 0$ .

$C_\# \rightarrow C$ : Let  $\Pi_{\beta, \mathcal{O}}$  be compact. Then there exists for each  $\varepsilon > 0$  a finite number of elements  $S_i \in \mathcal{T}_\beta^*$  and  $\varphi_i \in \mathfrak{A}(\mathcal{O})^*$ ,  $i = 1, \dots, n$  such that for any  $\varphi \in \mathcal{T}_\beta$

$$\|\Pi_{\beta, \mathcal{O}}(\varphi) - \sum_{i=1}^n S_i(\varphi) \cdot \varphi_i\| \leq \varepsilon \cdot \|\varphi\|_\beta. \tag{2.8}$$

As in the preceding argument we conclude that there are operators  $T_i \in \mathcal{B}(\mathcal{H})$  such that for each  $A \in \mathfrak{A}(\mathcal{O})$

$$\|e^{-\beta H} A e^{-\beta H} - \sum_{i=1}^n \varphi_i(A) \cdot T_i\| \leq \varepsilon \cdot \|A\|, \tag{2.9}$$

showing that the map  $\Xi_{\beta, \mathcal{O}}$  is compact. This implies by the same reasoning as before that the map  $\Theta_{\beta, \mathcal{O}}$  is compact.

In order to prove that there are no further relations in our lattice of constraints we only have to show that Condition  $C_\#$  and  $N$  are independent. We will do this by providing examples from free field theory. The

notation and concepts used are standard, but a brief account of the relevant definitions may also be found at the beginning of Sect. 3.1.

$N \rightarrow C_*$ : An example of a quantum field theory satisfying Condition N, but not Condition  $C_*$ , is the theory of a free massless scalar particle in three space-time dimensions. That this theory complies with Condition N was shown in the Appendix of [3]. In fact, the maps  $\Theta_{\beta, \varrho}$  are (for fixed  $\beta > 0$ )  $p$ -nuclear for any  $p > 0$ , *i. e.* they are of order 0.

For the proof that this theory violates Condition  $C_*$  we exhibit a sequence of functionals in the unit ball of  $\mathcal{F}_\beta$  whose restriction to the local algebras fails to be precompact in the norm topology. To this end we pick a sequence of single-particle wave functions  $h_n \in \mathcal{K} = L^2(\mathbb{R}^2)$  which in momentum space is given by

$$\tilde{h}_n(\mathbf{p}) = [i|\mathbf{p}|^{3/2} \ln |\mathbf{p}|]^{-1} \quad \text{for } 1/n \leq |\mathbf{p}| \leq 1/2, \quad (2.10)$$

and  $\tilde{h}_n(\mathbf{p}) = 0$  for all other momenta  $\mathbf{p}$ . Setting  $\Phi_n = e^{-\beta H} W(h_n) \Omega$ , where  $W(f)$ ,  $f \in \mathcal{K}$  are the unitary Weyl operators acting on the Fock space  $\mathcal{H}$ ,  $\Omega$  is the Fock vacuum, and  $H$  is the Hamiltonian, we obtain functionals  $\varphi_n(\cdot) = (\Phi_n, \cdot \Phi_n)$  on  $\mathcal{B}(\mathcal{H})$  for which  $\|\varphi_n\|_\beta = 1$ .

Now by its very definition each local algebra  $\mathfrak{A}(\mathcal{O})$  contains Weyl operators of the form  $W(\omega^{-1/2} g)$ , where  $\omega^{-1/2} g$  are elements of a certain specific real-linear subspace  $\mathcal{L} \subset \mathcal{K}$ . Here  $\omega$  denotes the restriction of the Hamiltonian to  $\mathcal{K}$  (which acts in momentum space by multiplication with  $|\mathbf{p}|$ ). The functions  $g$  are real, have compact support, and, most important, not all of them vanish at the origin in momentum space. For an estimate of the norm-differences of the functionals  $\varphi_n$  on  $\mathfrak{A}(\mathcal{O})$  we proceed to the lower bound

$$\|(\varphi_n - \varphi_m) \upharpoonright \mathfrak{A}(\mathcal{O})\| \geq \sup \{ |(\varphi_n - \varphi_m)(W(\omega^{-1/2} g))| : \omega^{-1/2} g \in \mathcal{L} \}. \quad (2.11)$$

The right-hand side of this inequality can be evaluated with the help of the explicit formula

$$\begin{aligned} \varphi_n(W(\omega^{-1/2} g)) = \exp \left( \|e^{-\beta\omega} h_n\|^2 - \|h_n\|^2 - \frac{1}{2} \|\omega^{-1/2} g\|^2 \right. \\ \left. - \langle \omega^{-1/2} g, e^{-\beta\omega} h_n \rangle + \langle e^{-\beta\omega} h_n, \omega^{-1/2} g \rangle \right), \quad (2.12) \end{aligned}$$

which follows from the Weyl relations and the fact that the vectors  $W(h_n)\Omega$  are coherent states. Now the crucial point is that with our choice of  $h_n$  the sequence  $(\|e^{-\beta\omega} h_n\|^2 - \|h_n\|^2)$  converges, whereas the scalar product  $\langle \omega^{-1/2} g, e^{-\beta\omega} h_n \rangle$  behaves for large  $n$  like  $2\pi i \tilde{g}(0) \cdot \ln \ln n$ , apart from convergent terms. From relation (2.11) it therefore follows that there exists a  $\delta > 0$  such that  $\|(\varphi_n - \varphi_m) \upharpoonright \mathfrak{A}(\mathcal{O})\| \geq \delta$  if  $|\ln \ln n - \ln \ln m| \geq 1$ . Hence the sequence  $\varphi_n$  is not precompact in the norm topology induced by the local algebras.

The reason for the failure of Condition  $C_*$  in the theory under consideration is its peculiar infrared-structure. As can be seen from relation (2.12) there exist states in this theory with an infinite “mean field” which are obtained as limits of Fock states of uniformly bounded total energy. (For a more detailed discussion of this pathological feature *cf.* [6].) Such singular states cannot appear in theories satisfying Condition  $C_*$ , where all weak\* limit points of bounded subsets of  $\mathcal{T}_\beta$  are locally normal relative to the underlying vacuum representation. Condition  $C_*$  may thus be regarded as a criterion characterizing theories with decent phase space properties in the infrared domain.

$C_* \rightarrow N$ : It remains to show that there exist quantum field theories satisfying Condition  $C_*$ , but not Condition N. As was already pointed out, Condition N is sensitive to the phase space properties of a theory in the ultraviolet region. In contrast, it is apparent from relation (2.3) that Condition  $C_*$  is insensitive in this respect.

Accordingly we consider the theory of an infinite number of free scalar particles with masses  $m_i > 0$ ,  $i \in \mathbb{N}$ , where we assume that the sequence  $m_i$  tends monotonically (but sufficiently slowly) to infinity. For the proof that this theory satisfies Condition  $C_*$  in any number of space-time dimensions we rely on the result established in Sect. 3 that in the theory of a single scalar non-interacting particle of arbitrary mass  $m_i > 0$  the maps  $\Pi_{\beta, \emptyset}^{(i)}$  are nuclear for sufficiently large  $\beta$ . The present theory involving an infinite number of species of particles is obtained by a standard tensor product construction from the latter one [7]. We recall that the Hilbert space  $\mathcal{H}$  is the “incomplete tensor product”  $\otimes_i \mathcal{H}^{(i)}$  of the underlying Fock spaces  $\mathcal{H}^{(i)}$  with reference vector  $\Omega = \otimes_i \Omega^{(i)}$ , where  $\Omega^{(i)} \in \mathcal{H}^{(i)}$  are the respective Fock vacua. The local algebras  $\mathfrak{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$  are defined as the (closure of the) algebraic tensor product  $\otimes_i \mathfrak{A}^{(i)}(\mathcal{O})$  on  $\mathcal{H}$  of the underlying local algebras  $\mathfrak{A}^{(i)}(\mathcal{O})$ .

Let  $P_E^{(i)}$  be the spectral projections of the underlying Hamiltonians  $H^{(i)}$  on  $\mathcal{H}^{(i)}$  associated with the spectrum in  $[0, E]$ . According to our assumption on the mass spectrum we have  $m_i > E$  for any given  $E$  and almost all  $i \in \mathbb{N}$ . Hence the projection  $P_E^{(i)}$  projects onto the Fock vacuum  $\Omega^{(i)}$  if  $i$  is sufficiently large. We can thus define on  $\mathcal{H}$  the projection  $P_E^\otimes = \otimes_i P_E^{(i)}$ . In

analogy to  $\Lambda_E$ , *cp.* (2.2), we define a map  $\Lambda_E^\otimes : \mathcal{T}_\beta \rightarrow \mathcal{T}_\beta$  by

$$\Lambda_E^\otimes(\varphi) = P_E^\otimes \varphi P_E^\otimes, \quad \varphi \in \mathcal{T}_\beta. \tag{2.13}$$

Let us consider now the maps  $\Pi_{\beta, \emptyset}$ . Representing  $\mathcal{T}_\beta$  as inductive limit of the net

$$\left( \otimes_{j=1}^i \mathcal{T}_\beta^{(j)} \right) \otimes \left( \otimes_{k=i+1}^\infty \omega_0^{(k)} \right), \quad i \in \mathbb{N}, \tag{2.14}$$



where  $\omega_0^{(k)}$  is the vacuum state induced by  $\Omega^{(k)}$ , we have, in an obvious notation,

$$\Pi_{\beta, \varrho} \circ \Lambda_E^{\otimes} = \bigotimes_i \Pi_{\beta, \varrho}^{(i)} \circ \Lambda_E^{(i)}. \tag{2.15}$$

As was mentioned before, each map  $\Pi_{\beta, \varrho}^{(i)}$  is nuclear for sufficiently large  $\beta$ , and consequently the maps  $\Pi_{\beta, \varrho}^{(i)} \circ \Lambda_E^{(i)}$  are nuclear for all  $E$  and  $\beta > 0$ . Moreover, according to the preceding remarks on the projections  $P_E^{(i)}$  almost all maps  $\Pi_{\beta, \varrho}^{(i)} \circ \Lambda_E^{(i)}$  project onto  $\omega_0^{(i)} \mathfrak{A}^{(i)}(\mathcal{O})$ , *i. e.* they are of rank 1. Since a finite tensor product of nuclear maps is nuclear and since the tensor product of a nuclear map with an arbitrary number of maps of rank 1 is nuclear too, we conclude that  $\Pi_{\beta, \varrho} \circ \Lambda_E^{\otimes}$  is nuclear for any  $E$  and any  $\beta > 0$ . On the other hand we have in analogy to relation (2.3)

$$\| \Pi_{\beta, \varrho} - \Pi_{\beta, \varrho} \circ \Lambda_E^{\otimes} \| \leq 2 \cdot e^{-\beta E}. \tag{2.16}$$

This proves that  $\Pi_{\beta, \varrho}$  is compact.

The proof that the theory under consideration does not satisfy Condition N if the mass spectrum  $m_i$  tends sufficiently slowly to infinity (for example like  $m_0 \cdot \ln \ln i$ ) can be extracted from the literature: it follows from [8], Theorem 4.3, that such a theory does not have the “distal split property”. On the other hand, this property is known to be a consequence of Condition N [3].

For completeness we give here a more direct argument: first we note that if  $A_i \in \mathfrak{A}(\mathcal{O})$ ,  $\|A_i\| \leq 1$  is a sequence of operators such that the vectors  $\Theta_{\beta, \varrho}(A_i)$  are orthogonal, and if  $\Theta_{\beta, \varrho}$  is nuclear, *i. e.*  $\Theta_{\beta, \varrho}(\cdot) = \sum_k \varphi_k(\cdot) \cdot \Phi_k$  where  $\sum_k \|\varphi_k\| \cdot \|\Phi_k\| < \infty$ , then

$$\begin{aligned} \sum_i \|\Theta_{\beta, \varrho}(A_i)\|^4 &= \sum_i \left| \sum_k \varphi_k(A_i) \cdot (\Theta_{\beta, \varrho}(A_i), \Phi_k) \right|^2 \\ &\leq \|\Theta_{\beta, \varrho}\|^2 \cdot \left( \sum_k \|\varphi_k\| \cdot \|\Phi_k\| \right)^2 < \infty. \end{aligned} \tag{2.17}$$

In order to show that this condition is *not* satisfied in the present theory we pick from each  $\mathfrak{A}^{(i)}(\mathcal{O})$  a Weyl operator  $W(\omega_i^{-1/2} g)$ , where  $g$  is a fixed test function with compact support and  $\omega_i$  is the single-particle Hamiltonian which acts in momentum space by multiplication with  $(|\mathbf{p}|^2 + m_i^2)^{1/2}$ . The images of these operators under the natural embedding of  $\mathfrak{A}^{(i)}(\mathcal{O})$  into  $\uparrow \mathfrak{A}(\mathcal{O})$  are denoted by  $W^{(i)}$ . We set  $A_i = (1/2)(W^{(i)} - \omega_0(W^{(i)} \cdot 1))$ , where  $\omega_0$  is the vacuum state induced by  $\Omega$ . The vectors  $e^{-\beta H} A_i \Omega$  are orthogonal and

$$\begin{aligned} \|e^{-\beta H} A_i \Omega\|^2 &= \frac{1}{4} \exp(-\|\omega_i^{-1/2} g\|^2) \cdot \{ \exp(\|\omega_i^{-1/2} e^{-\beta \omega_i} g\|^2) - 1 \}. \end{aligned} \tag{2.18}$$

From the latter equation we see that for large  $i$  (such that the masses  $m_i$  are sufficiently large)

$$\|\Theta_{\beta, \varrho}(A_i)\|^4 \geq C \cdot \frac{e^{-4 \beta m_i}}{m_i^2}, \tag{2.19}$$

where  $C$  is some positive constant. Combining this estimate with relation (2.17) it is clear that the maps  $\Theta_{\beta, \varrho}$  can only be nuclear for some  $\beta > 0$  if the mass values  $m_i$  tend to infinity sufficiently rapidly. Hence the present theory violates Condition N.

### 3. NUCLEARITY IN FREE FIELD THEORY

As is clear from the preceding general discussion, Condition  $N_*$  is a quite stringent requirement on a quantum field theory. Therefore we want to show now that this condition is satisfied in models of physical interest. For simplicity we treat here only the case of free fields. But we think that with the more advanced methods of constructive quantum field theory expounded in [9] Condition  $N_*$  can also be established in less trivial cases.

We present here a generalized and somewhat simplified version of an argument in [10]. Similar methods, designed for the proof of Condition  $C_*$  in the massive case, have been used in [4]. In order to mark the various steps involved in our analysis we split the text into several subsections. We begin with

#### 3.1. Basic facts and general strategy

Let  $\mathcal{H}$  be a separable Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$  bearing an antiunitary involution  $J$ . The symmetric Fock space over  $\mathcal{H}$  will be denoted by  $\mathcal{F}$ , and the scalar product in  $\mathcal{F}$  by  $(\cdot, \cdot)$ . The vector  $\Omega \in \mathcal{F}$  represents the Fock vacuum, and the subspace  $\mathcal{H} \subset \mathcal{F}$  the single-particle states.

The single-particle Hamiltonian acting on  $\mathcal{H}$  is denoted by  $\omega$ . It is assumed to be positive and to commute with  $J$ . Via the method of "second quantization" it determines the Fock space Hamiltonian  $H \geq 0$  on  $\mathcal{F}$ .

Let  $a^*(f)$  and  $a(f)$  be the familiar creation and annihilation operators on  $\mathcal{H}$  which are linear and antilinear in  $f \in \mathcal{H}$ , respectively. With the help of these operators we define the unitary Weyl operators  $W(f) = \exp \{ i(a^*(f) + a(f))^- \}$ ,  $f \in \mathcal{H}$ , for which there hold the Weyl relations

$$W(f) W(g) = e^{(1/2) \langle g, f \rangle - \langle f, g \rangle} \cdot W(f+g). \tag{3.1}$$

Within this framework the local algebras are obtained by the following standard procedure: for each space-time region there exists a certain specific pair of closed subspaces  $\mathcal{L}_\varphi$  and  $\mathcal{L}_\pi$  of  $\mathcal{H}$  which are invariant under  $J$ . These spaces are linked to the Cauchy data  $\varphi$  and  $\pi$  of the underlying local fields. Given  $\mathcal{L}_\varphi$  and  $\mathcal{L}_\pi$  one forms the real-linear subspace  $\mathcal{L} = (1 + J) \cdot \mathcal{L}_\varphi + (1 - J) \cdot \mathcal{L}_\pi$  and takes the von Neumann algebra  $\mathfrak{W}(\mathcal{L}) = \{W(f) : f \in \mathcal{L}\}''$  as the local algebra corresponding to the given region. The explicit definition of the spaces  $\mathcal{L}_\varphi$  and  $\mathcal{L}_\pi$  in the theory of a single scalar particle will be given in the last part of this section.

It is our aim to establish a condition, formulated in terms of the spaces  $\mathcal{L}_\varphi$ ,  $\mathcal{L}_\pi$ , and the single-particle Hamiltonian  $\omega$ , which implies that the map  $\Xi_{\beta, \mathcal{L}} : \mathfrak{W}(\mathcal{L}) \rightarrow \mathcal{B}(\mathcal{H})$  given by

$$\Xi_{\beta, \mathcal{L}}(W) = e^{-\beta H} W e^{-\beta H}, \quad W \in \mathfrak{W}(\mathcal{L}) \tag{3.2}$$

is  $p$ -nuclear. Thus we have to find operators  $T_i \in \mathcal{B}(\mathcal{H})$  and functionals  $\varphi_i \in \mathfrak{W}(\mathcal{L})^*$  such that

$$\Xi_{\beta, \mathcal{L}}(\cdot) = \sum_i \varphi_i(\cdot) \cdot T_i \tag{3.3}$$

and  $\sum_i \|\varphi_i\|^p \|T_i\|^p < \infty$ . To this end we first expand all operators  $W \in \mathfrak{W}(\mathcal{L})$  into a normal-ordered power series of creation and annihilation operators. We adopt the methods in [11], Sect. 6, and introduce for each bounded operator  $V_{mn} : \mathcal{H}_n \rightarrow \mathcal{H}_m$ , where  $\mathcal{H}_k$ ,  $k \in \mathbb{N}_0$  denotes the  $k$ -particle subspace of  $\mathcal{H}$ , a generalized creation and annihilation operator ( $a^{*m} V_{mn} a^n$ ). These operators act on the dense subspace  $\mathcal{D}_0 \subset \mathcal{H}$  of vectors with finite particle number in an obvious manner. We recall from [11] the

LEMMA 3.1. — *To each  $W \in \mathcal{B}(\mathcal{H})$  there exists a unique family of bounded operators  $[W]_{mn} : \mathcal{H}_n \rightarrow \mathcal{H}_m$  such that*

$$W = \sum_{m, n} (a^{*m} [W]_{mn} a^n) \tag{3.4}$$

*in the sense of quadratic forms on  $\mathcal{D}_0 \times \mathcal{D}_0$ .*

Let us briefly recall how the kernels of the operators  $[W]_{mn}$  can be determined: let  $\{e_i\}$  be a fixed orthonormal basis in  $\mathcal{H}$ , and let  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_i, \dots)$  be any multi-index of non-negative integers with  $|\boldsymbol{\mu}| = \sum_i \mu_i < \infty$ . For any such sequence  $\boldsymbol{\mu}$  of ‘‘occupation numbers’’ we

define a vector  $\Phi_{\boldsymbol{\mu}} \in \mathcal{H}$  by forming the  $|\boldsymbol{\mu}|$ -fold symmetrized and normalized tensor product of the basis elements in  $\mathcal{H}$ , where each  $e_i$  appears with multiplicity  $\mu_i$ . The vector assigned to  $\mathbf{0} = (0, 0, \dots)$  is the Fock vacuum  $\Omega$ . By construction, the vectors  $\Phi_{\boldsymbol{\mu}}$  form an orthonormal basis of  $\mathcal{H}$ .

Taking the vacuum expectation value of the  $(m + n)$ -fold commutator of  $W$  with  $m$  annihilation operators and  $n$  creation operators, we obtain

from (3.4)

$$(\Phi_{\mu}, [W]_{mn} \Phi_{\nu}) = \frac{(-1)^n}{m! n!} \cdot (\Omega, [\dots [a(e_i), \dots [a^*(e_k), \dots [., W] \dots ] \Omega). \quad (3.5)$$

Here  $|\mu| = m$ ,  $|\nu| = n$ , and the multi-indices  $\mu$  and  $\nu$  specify the families of annihilation operators  $a(e_i)$  and creation operators  $a^*(e_k)$  appearing on the right-hand side of (3.5). Because of the multiple commutator in this expression it is apparent that the kernels of the operators  $[W]_{mn}$  have good localization properties in configuration space if  $W$  is a local operator.

Localization in momentum space is achieved by multiplying equation (3.4) from the left and right by  $e^{-\beta H}$ . With the help of the relation

$$a(f) e^{-\beta H} = e^{-\beta H} a(e^{-\beta \omega} f) \quad (3.6)$$

this energy cutoff can be transported to the operators  $[W]_{mn}$ . The resulting operators  $\otimes_1^m e^{-\beta \omega} [W]_{mn} \otimes_1^n e^{-\beta \omega}$  have good localization properties, both in configuration and momentum space, and are therefore  $p$ -nuclear for any  $p > 0$ . Moreover, they can be expanded with respect to a fixed system of matrix units which does not depend on  $W$ .

If the creation and annihilation operators were bounded (as is the case in a theory of fermions), we would have arrived at an expansion of  $e^{-\beta H} W e^{-\beta H}$  of the desired form (3.3). In the present case we have to make use of the fact that there hold so-called energy bounds for the operators  $a(f)$  and  $a^*(f)$  of the form

$$\|a(f) e^{-\beta H}\| = \|e^{-\beta H} a^*(f)\| \leq (2e\beta)^{-1/2} \|\omega^{-1/2} f\|. \quad (3.7)$$

Note that on the right-hand side of (3.7) there appears the operator  $\omega^{-1/2}$ . This operator is bounded in theories of massive particles and does not cause any problems there. But in theories of massless particles, where the spectrum of  $\omega$  extends to 0, the operator is unbounded, and it is at this point where the dimension of space matters. It will be convenient to absorb the operator  $\omega^{-1/2}$  into the above  $p$ -nuclear operators, leading to the expression

$$\Gamma_{mn}(W) = \left( \otimes_1^m \omega^{-1/2} e^{-\beta \omega/2} \right) [W]_{mn} \left( \otimes_1^n \omega^{-1/2} e^{-\beta \omega/2} \right). \quad (3.8)$$

Here we have made use of relation (3.6) for  $e^{-\beta H/2}$  in order to leave over a factor  $e^{-\beta H/2}$  as mollifier for the annihilation and creation operators. We will see that the operators  $\Gamma_{mn}(W)$  are still  $p$ -nuclear and can be expanded with respect to a  $W$ -independent basis of matrix units if the dimension of space is appropriate. It then follows that the map  $W \rightarrow e^{-\beta H} (a^{*m} [W]_{mn} a^n) e^{-\beta H}$  is  $p$ -nuclear. Making  $\beta$  sufficiently large we

can also control the sums over  $m$  and  $n$  and thereby establish the  $p$ -nuclearity of  $\Xi_{\beta, \mathcal{L}}$ . We now proceed to the details.

### 3.2. Analysis of $\Gamma_{mn}(W)$

In the first step we will establish a connection between the operators  $\Gamma_{mn}(W)$  introduced in (3.8) and the single-particle operators  $E_\phi$ ,  $E_\pi$ , and  $\omega$ , where  $E_\phi$  and  $E_\pi$  are the orthogonal projections onto  $\mathcal{L}_\phi$  and  $\mathcal{L}_\pi$ , respectively.

There appears a little problem at the very beginning: the forms  $\Gamma_{mn}(W)$ ,  $W \in \mathfrak{B}(\mathcal{L})$  need not be well-defined operators for arbitrary  $\mathcal{L} \subset \mathcal{H}$ . A simple necessary condition is the requirement that  $\mathcal{L}$  is contained in the domain  $D(\omega^{-1/2})$ , which we will assume henceforth. By an elementary calculation one then finds that  $\Gamma_{mn}(W)$  is bounded for any finite linear combination  $W = \sum c_i \cdot W(f_i)$  of Weyl operators  $W(f_i)$  with  $f_i \in \mathcal{L}$ . The  $*$ -algebra generated by these operators is denoted by  $\mathfrak{B}_0(\mathcal{L})$ . This algebra is weakly dense in  $\mathfrak{B}(\mathcal{L})$ , but it is *a priori* not clear whether  $\Gamma_{mn}(W)$  is well-defined for all  $W \in \mathfrak{B}(\mathcal{L})$ . This problem will be solved at the very end of our argument. In the intermediate steps we deal only with  $\mathfrak{B}_0(\mathcal{L})$ .

In order to handle the combinatorial problems involved in the analysis of the operators  $\Gamma_{mn}(W)$  we will make use of the generating functional techniques expounded in the Appendix of [2]. We define for  $W \in \mathfrak{B}(\mathcal{H})$  and  $f, g \in \mathcal{H}$  the functional

$$\mathcal{G}(f, g | W) = (e^{a^*(f)} \Omega, W e^{a^*(g)} \Omega), \tag{3.9}$$

and the corresponding ‘‘truncated’’ functional

$$\mathcal{G}_T(f, g | W) = \exp(-\langle f, g \rangle) \cdot \mathcal{G}(f, g | W). \tag{3.10}$$

A useful relation is

$$\mathcal{G}_T(f, g | W) = \sum_{m, n} (m! n!)^{1/2} \mathcal{G}(f, g | [W]_{mn}). \tag{3.11}$$

The sum appearing here is absolutely convergent, and the operators  $[W]_{mn}$ , introduced in Lemma 3.1, are naturally extended to  $\mathcal{H}$  by putting them equal to 0 on the orthogonal complement of  $\mathcal{H}_n$ . Equation (3.11) can be solved for  $\mathcal{G}(f, g | [W]_{mn})$  by Fourier analysis,

$$\begin{aligned} & (m! n!)^{1/2} \mathcal{G}(f, g | [W]_{mn}) \\ &= (2\pi)^{-2} \cdot \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi e^{i(m\phi - n\psi)} \mathcal{G}_T(e^{i\phi} f, e^{i\psi} g | W). \end{aligned} \tag{3.12}$$

It is an advantage of the truncated functionals that the consequences of ‘‘locality’’ can be studied quite easily. Making use of the definition of Weyl operators in terms of creation and annihilation operators one can

represent these functionals in the form

$$\mathcal{G}_T(f, g | W) = (e^{a^*(f-g)} \Omega, W(ig) \cdot W \cdot W(ig)^{-1} \Omega). \quad (3.13)$$

It is crucial here that the unitary Weyl operators  $W(ig)$ ,  $g \in \mathcal{H}$  induce automorphisms of  $\mathfrak{B}_0(\mathcal{L})$ , as is seen from the Weyl relations (3.1). Hence we obtain from (3.13) the estimate for  $W \in \mathfrak{B}_0(\mathcal{L})$

$$|\mathcal{G}_T(f, g | W)| \leq \sup \{ |(e^{a^*(f-g)} \Omega, V \Omega)| : V \in \mathfrak{B}_0(\mathcal{L}), \|V\| \leq \|W\| \}. \quad (3.14)$$

The right-hand side of this inequality has been analyzed in [2]. Making use of these results we get

$$|\mathcal{G}_T(f, g | W)| \leq \exp \left( \frac{1}{8} \|E_\phi(1-J)(f-g)\|^2 + \frac{1}{8} \|E_\pi(1+J)(f-g)\|^2 \right) \cdot \|W\|. \quad (3.15)$$

Next we introduce on  $\mathcal{H}$  the operators

$$S_\phi = E_\phi \cdot \omega^{-1/2} e^{-\beta\omega/2}, \quad S_\pi = E_\pi \cdot \omega^{-1/2} e^{-\beta\omega/2}. \quad (3.16)$$

In order to get on we have to assume that these operators are bounded. (This implies in particular that  $\mathcal{L} \subset D(\omega^{-1/2})$ , cp. the preceding remarks.) Under these premises there exists the “least upper bound”  $S$  of the positive operators  $|S_\phi|$  and  $|S_\pi|$ , respectively, which is given by (cf. [12], p. 316/317)

$$S = w - \lim_n \left( \frac{1}{2} [|S_\phi|^n + |S_\pi|^n] \right)^{1/n}. \quad (3.17)$$

Since the operators  $E_\phi$ ,  $E_\pi$  and  $\omega$  commute with  $J$  it is clear that  $S$  commutes with  $J$ , too. Assuming for a moment that  $f, g \in D(\omega^{-1/2})$  we therefore obtain from (3.15) the estimate

$$|\mathcal{G}_T(\omega^{-1/2} e^{-\beta\omega/2} f, \omega^{-1/2} e^{-\beta\omega/2} g | W)| \leq \exp \left( \frac{1}{2} \|S(f-g)\|^2 \right) \cdot \|W\|. \quad (3.18)$$

After these preparations we can turn to the analysis of the operators  $\Gamma_{mn}(W)$ ,  $W \in \mathfrak{B}_0(\mathcal{L})$ , which are also extended to  $\mathcal{H}$  in the obvious way. With the same restrictions on  $f, g$  as before we have

$$\mathcal{G}(f, g | \Gamma_{mn}(W)) = \mathcal{G}(\omega^{-1/2} e^{-\beta\omega/2} f, \omega^{-1/2} e^{-\beta\omega/2} g | [W]_{mn}). \quad (3.19)$$

Expressing the right-hand side of this equation in terms of the truncated functional  $\mathcal{G}_T$ , cp. (3.12), and applying to  $\mathcal{G}_T$  relation (3.18), we obtain the bound

$$|\mathcal{G}(f, g | \Gamma_{mn}(W))| \leq (m! n!)^{-1/2} e^{\|S f\|^2 + \|S g\|^2} \cdot \|W\|. \quad (3.20)$$

Since  $\Gamma_{mn}(W)$  is bounded, the left-hand side of this inequality is norm continuous in  $f, g$ , and the same is true for the right-hand side, because  $S$  is bounded. Hence, by continuity, relation (3.20) holds for any  $f, g \in \mathcal{H}$ . It is apparent from this result that the properties of  $\Gamma_{mn}(W)$  are governed by spectral properties of the single-particle operator  $S$ .

We restrict now our attention to the cases, where  $S$  is compact. The eigenvalues of  $S$ , counted according to multiplicity, are denoted by  $s_i, i \in \mathbb{N}$  and the corresponding orthonormal eigenvectors in  $\mathcal{H}$  by  $e_i$ . As was discussed before, we can construct from this basis an orthonormal basis  $\Phi_\mu$  in  $\mathcal{H}$ , where  $\mu = (\mu_1, \dots, \mu_i, \dots)$  are the occupation numbers of the respective "modes"  $e_i$ . From the inequality (3.20) we then obtain as in [2], Lemma 1, the bound for any  $\mu, \nu$  with  $|\mu| = m, |\nu| = n$

$$|(\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu)| \leq (m!n!)^{-1/2} 2^{(m+n)/2} \cdot \|W\| \cdot \prod_i (1 + \mu_i) s_i^{\mu_i} \cdot \prod_k (1 + \nu_k) s_k^{\nu_k}, \quad (3.21)$$

where we set  $0^0 = 1$ . For later reference we also introduce the matrix-units  $E_{\mu\nu} : \mathcal{H}_n \rightarrow \mathcal{H}_m$  given by

$$E_{\mu\nu} \Psi = (\Phi_\nu, \Psi) \cdot \Phi_\mu, \quad \Psi \in \mathcal{H}_n. \quad (3.22)$$

We are now in the position to prove

LEMMA 3.2. — *Let  $S_\phi$  and  $S_\pi$  be  $p$ -nuclear <sup>(2)</sup> for some  $0 < p \leq 1$ . Then  $S$  is compact and there holds for any  $W \in \mathfrak{B}_0(\mathcal{L})$  the norm convergent expansion*

$$\Gamma_{mn}(W) = \sum_{\mu, \nu} (\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu) \cdot E_{\mu\nu}.$$

Moreover,

$$\sum_{\mu, \nu} \sup |(\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu)|^p \leq (m!n!)^{-p/2} (8^{p/2} [\|S_\phi\|_p^p + \|S_\pi\|_p^p])^{m+n},$$

where the supremum extends over all  $W \in \mathfrak{B}_0(\mathcal{L})$  with  $\|W\| \leq 1$ .

Remark. — This result says that the map  $\Gamma_{mn}$  from  $\mathfrak{B}_0(\mathcal{L})$  into the bounded operators from  $\mathcal{H}_n$  to  $\mathcal{H}_m$  is  $p$ -nuclear.

Proof. — It was discussed in [12] that if  $|S_\phi|^p$  and  $|S_\pi|^p$  are of trace-class for some  $0 < p \leq 1$ , then the least upper bound  $S$  of  $|S_\phi|$  and  $|S_\pi|$  satisfies

$$\|S^p\|_1 \leq \| |S_\phi|^p \|_1 + \| |S_\pi|^p \|_1.$$

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<sup>(2)</sup> We recall that an operator  $T$  on  $\mathcal{H}$  is  $p$ -nuclear iff  $|T|^p$  is of trace-class. Denoting by  $\|\cdot\|_1$  the trace-norm, one has  $\|T\|_p = \| |T|^p \|_1^{1/p}$ .

Hence  $S$  is in particular compact, and we can apply the estimate (3.21). Since  $\sum_i \mu_i = m$  there holds  $\prod_i (1 + \mu_i) \leq 2^m$ , and similarly  $\prod_k (1 + \nu_k) \leq 2^n$ .

Consequently we obtain from (3.21) the bound

$$\sum_{\mu, \nu} \sup |(\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu)|^p \leq (m! n!)^{-p/2} (8^{p/2} \cdot \|S^p\|_1)^{m+n}.$$

From the above relation between the trace-norms of  $|S_\phi|^p$ ,  $|S_\pi|^p$  and  $S^p$  the second part of the statement then follows. The first part is an immediate consequence of the fact that the given expansion holds in the sense of sesquilinear forms. Since  $\|E_{\mu\nu}\| = 1$  and since  $p \leq 1$  it then holds also in the sense of norm convergence.

### 3.3. Energy bounds

In the next step we want to establish the energy-bounds for the creation and annihilation operators given in relation (3.7). Similar estimates have been derived in the literature (cf. for example [13]). We give here a proof for completeness.

Let  $f \in D(\omega^{1/2})$  be any non-zero vector. We set  $f_1 = \|f\|^{-1} \cdot f$  and complement this vector to an orthonormal basis  $f_i \in \mathcal{K} \cap D(\omega^{1/2})$ . For  $g, h \in D(\omega)$  we then have

$$\sum_i \langle g, \omega^{1/2} f_i \rangle \langle \omega^{1/2} f_i, h \rangle = \langle g, \omega h \rangle. \tag{3.22}$$

Consequently,

$$\sum_i a^*(\omega^{1/2} f_i) a(\omega^{1/2} f_i) = H \tag{3.23}$$

in the sense of quadratic forms on  $\mathcal{D}_1 \times \mathcal{D}_1$ , where  $\mathcal{D}_1 \subset \mathcal{D}_0$  is the linear span of all finite tensor products of vectors in  $D(\omega)$ . Note that  $\mathcal{D}_1$  is a core for  $a(\omega^{1/2} f)$  as well as  $H^{1/2}$ , and  $e^{-\delta H} \mathcal{D}_1 \subset \mathcal{D}_1$  if  $\delta \geq 0$ . Taking matrix elements of relation (3.23) with respect to any vector  $\Psi \in \mathcal{D}_1$ , we obtain the estimate

$$\|a(\omega^{1/2} f) \Psi\| \leq \|f\| \cdot \|H^{1/2} \Psi\| \tag{3.24}$$

which holds for all  $f \in D(\omega^{1/2})$ . It is now easy to prove

LEMMA 3.3. — Let  $f_1, \dots, f_n \in D(\omega^{1/2})$ . Then

$$\|a(\omega^{1/2} f_1) \cdots a(\omega^{1/2} f_n) e^{-\beta H/2}\| = \|e^{-\beta H/2} a^*(\omega^{1/2} f_n) \cdots a^*(\omega^{1/2} f_1)\| \leq (n/e\beta)^{n/2} \|f_1\| \cdots \|f_n\|.$$

Remark. — This result allows to extend the operators  $a(\omega^{1/2} f_1) \cdots a(\omega^{1/2} f_n) e^{-\beta H/2}$  and their adjoints to arbitrary  $f_1, \dots, f_n \in \mathcal{K}$ . We use the same symbol for these extensions.



*Proof.* — Setting  $\Psi = e^{-\delta H} \Phi$ ,  $\delta > 0$  in relation (3.24) and making use of  $\|H^{1/2} e^{-\delta H}\| \leq (2e\delta)^{-1/2}$  the given bound follows for  $n=1$ . Now according to relation (3.6) (which can easily be established on  $\mathcal{D}_1$ ) we have for  $\delta, \varepsilon > 0$

$$a(\omega^{1/2} f_1) \cdots a(\omega^{1/2} f_n) e^{-(\varepsilon + \delta)H} = a(\omega^{1/2} f_1) \cdots a(\omega^{1/2} f_{n-1}) e^{-\varepsilon H} a(\omega^{1/2} e^{-\varepsilon \omega} f_n) e^{-\delta H}.$$

Taking into account that  $\|e^{-\varepsilon \omega}\| \leq 1$ , the given bound follows for any  $n$  by induction. The stated equality of norms is obvious since the second operator is the adjoint of the first one.

### 3.4. A sufficient condition for nuclearity

The results of the preceding two subsections allow to formulate a simple criterion in terms of the operators  $S_\phi, S_\pi$  which implies that the map  $\Xi_{\beta, \mathcal{L}}$  defined in (3.2) is  $p$ -nuclear for sufficiently large  $\beta$ .

We assume that  $S_\phi$  and  $S_\pi$  are  $p$ -nuclear for some  $\beta$  and some  $0 < p \leq 1$ . It then follows from the specific dependence of these operators on  $\beta$ , cp. definition (3.16), that they are  $p$ -nuclear for all larger values of  $\beta$  and that their respective  $p$ -norms do not increase with  $\beta$ . Choosing  $\beta$  sufficiently large we may therefore assume that

$$(\|S_\phi\|_p^p + \|S_\pi\|_p^p) < (\beta/8)^{p/2}. \tag{3.25}$$

Let  $S$  be the least upper bound of  $|S_\phi|$  and  $|S_\pi|$  and let  $e_i$  be the orthonormal system of eigenvectors of  $S$ . We then define for each pair of multi-indices  $\mu, \nu$  with  $|\mu| = m, |\nu| = n$  the operator

$$T_{\mu\nu} = e^{-\beta H/2} \prod_i a^*(\omega^{1/2} e_i)^{\mu_i} \cdot e^{-\beta H} \cdot \prod_k a(\omega^{1/2} e_k)^{\nu_k} e^{-\beta H/2}. \tag{3.26}$$

This operator is well-defined according to Lemma 3.3 and the remark subsequent to it. (We note that the vectors  $e_i$  corresponding to the eigenvalues  $s_i \neq 0$  of  $S$  are in fact in the domain of  $\omega^{1/2}$ . Only for such vectors the operators  $T_{\mu\nu}$  are needed.) For the norm of  $T_{\mu\nu}$  we obtain the bound

$$\|T_{\mu\nu}\| \leq (m/e\beta)^{m/2} \cdot (n/e\beta)^{n/2}. \tag{3.27}$$

Hence, because of Lemma 3.2, the sum

$$\sum_{\mu, \nu} (\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu) \cdot T_{\mu\nu} \tag{3.28}$$

is norm convergent for all  $W \in \mathfrak{B}_0(\mathcal{L})$ , i. e. it defines a bounded operator. After a straightforward computation one also finds that

$$e^{-\beta H} (a^{*m} [W]_{mn} a^n) e^{-\beta H} = \sum_{\mu, \nu} (\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu) \cdot T_{\mu\nu} \tag{3.29}$$

in the sense of quadratic forms on  $\mathcal{D}_0 \times \mathcal{D}_0$ . But this implies according to Lemma 3.1 that

$$e^{-\beta H} W e^{-\beta H} = \sum_{m, n} \left( \sum_{\mu, \nu} (\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu) \cdot T_{\mu\nu} \right). \tag{3.30}$$

Applying Lemma 3.2 and relation (3.27) a second time we get the bound

$$\sum_{\mu, \nu} \sup |(\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu)|^p \cdot \|T_{\mu\nu}\|^p \leq ((8/\beta)^{p/2} [\|S_\phi\|_p^p + \|S_\pi\|_p^p])^{m+n}, \tag{3.31}$$

where the supremum extends over all  $W \in \mathfrak{B}_0(\mathcal{L})$  with  $\|W\| \leq 1$ . In view of condition (3.25) the right-hand side of this inequality is summable with respect to  $m$  and  $n$ , hence we conclude that the map  $\Xi_{\beta, \mathcal{L}} \upharpoonright \mathfrak{B}_0(\mathcal{L})$  is  $p$ -nuclear.

For the proof that  $\Xi_{\beta, \mathcal{L}}$  itself is  $p$ -nuclear we can now rely on general arguments, cf. [14], Chap. III.2. As we have seen, there exist functionals  $\varphi_i \in \mathfrak{B}_0(\mathcal{L})^*$  and operators  $T_i \in \mathcal{B}(\mathcal{H})$  such that

$$(\Xi_{\beta, \mathcal{L}} \upharpoonright \mathfrak{B}_0(\mathcal{L}))(\cdot) = \sum_i \varphi_i(\cdot) \cdot T_i \tag{3.32}$$

and  $\sum_i \|\varphi_i\|^p \cdot \|T_i\|^p < \infty$ . We extend each functional  $\varphi_i$  (in the  $C^*$ -algebraic sense) to the von Neumann algebra  $\mathfrak{B}(\mathcal{L})$  and decompose it into its normal part  $\nu_i$  and its singular part  $\sigma_i$ . Since  $\|\varphi_i\| = \|\nu_i\| + \|\sigma_i\|$ , the sum in (3.32) then splits into a normal and a singular part. But the map  $\Xi_{\beta, \mathcal{L}}$  (between the von Neumann algebras  $\mathfrak{B}(\mathcal{L})$  and  $\mathcal{B}(\mathcal{H})$ ) is normal, hence the sum over the singular contributions vanishes. We may therefore assume without loss of generality that the functionals  $\varphi_i$  are normal, and since  $\mathfrak{B}_0(\mathcal{L})$  is weakly dense in  $\mathfrak{B}(\mathcal{L})$ , relation (3.32) extends to  $\mathfrak{B}(\mathcal{L})$ . (We note that with some minor effort one can also show directly that the functionals  $W \rightarrow (\Phi_\mu, \Gamma_{mn}(W) \Phi_\nu)$  are normal.) Hence we have proved

LEMMA 3.4. — *Let  $S_\phi$  and  $S_\pi$  be  $p$ -nuclear for some  $0 < p \leq 1$ . Then there exists a  $\beta > 0$  such that the map  $\Xi_{\beta, \mathcal{L}} : \mathfrak{B}(\mathcal{L}) \rightarrow \mathcal{B}(\mathcal{H})$  given by*

$$\Xi_{\beta, \mathcal{L}}(W) = e^{-\beta H} W e^{-\beta H}, \quad W \in \mathfrak{B}(\mathcal{L})$$

*is  $p$ -nuclear.*

### 3.5 The free scalar field

These general results make possible a discussion of models of an arbitrary number of free bosons with any spin. For simplicity we restrict here our attention to a single scalar particle of mass  $m \geq 0$  in  $(s+1)$  space-time dimensions.

In this example the single-particle space is  $\mathcal{K} = L^2(\mathbb{R}^s)$ , the antiunitary involution  $J$  is the operator of complex conjugation in configuration space,  $(Jf)(\mathbf{x}) = \overline{f(\mathbf{x})}$ , and the single-particle Hamiltonian acts in momentum space by  $(\omega f)(\mathbf{p}) = (|\mathbf{p}|^2 + m^2)^{1/2} \cdot \tilde{f}(\mathbf{p})$ , where the tilde denotes Fourier transformation.

The subspaces  $\mathcal{L}_\phi$  and  $\mathcal{L}_\pi$  of  $\mathcal{K}$ , corresponding to double cones  $\mathcal{O}$  in Minkowski space with base  $\mathbf{O} \subset \mathbb{R}^s$  in the time  $t=0$  plane, are given by (assuming  $s > 1$  if  $m=0$ )

$$\mathcal{L}_\phi = \overline{\omega^{-1/2} \mathcal{D}(\mathbf{O})}, \quad \mathcal{L}_\pi = \overline{\omega^{1/2} \mathcal{D}(\mathbf{O})}.$$

Here  $\mathcal{D}(\mathbf{O})$  is the space of test functions with support in  $\mathbf{O}$ , and the bar denotes closure. The local algebras corresponding to  $\mathcal{O}$  are given by  $\mathfrak{A}(\mathcal{O}) := \mathfrak{B}(\mathcal{L})$ . It suffices to consider these special regions since any bounded space-time region is contained in some such double cone. We need

LEMMA 3.5. — *Let  $\mathbf{O} \subset \mathbb{R}^s$  be any bounded region and let  $\beta > 0$ . If the mass  $m$  is positive, then the operators*

$$S_\phi = E_\phi \omega^{-1/2} e^{-\beta\omega/2}, \quad S_\pi = E_\pi \omega^{-1/2} e^{-\beta\omega/2}$$

*are  $p$ -nuclear for any  $p > 0$ . If  $m=0$ , then this statement holds for  $s > 2$ .*

*Proof.* — Let  $\chi_0$  be a test function on  $\mathbb{R}^s$  which is equal to 1 on  $\mathbf{O}$ . We define on  $\mathcal{K}$  an operator  $\chi$  which acts in configuration space by multiplication with  $\chi_0$ ,  $(\chi f)(\mathbf{x}) = \chi_0(\mathbf{x}) \cdot f(\mathbf{x})$ . We also introduce for each  $i \in \mathbb{N}$  the operators

$$h_i = \omega (1 + \omega^2)^{(i-1)s} \chi (1 + \omega^2)^{-is} \omega^{-1}$$

$$k_i = (1 + \omega^2)^{(i-1)s} \chi (1 + \omega^2)^{-is}.$$

By inspection of the kernels of these operators in momentum space we find that they are in the Hilbert-Schmidt class. (At this point the assumption enters that  $s > 2$  in the massless case.) The same is also true for  $\omega^{-1/2} h_1$ , and  $\omega^{-1/2} k_1$ . It thus follows from the identities

$$S_\phi = E_\phi \omega^{-1/2} e^{-\beta\omega/2} = E_\phi \omega^{-1/2} h_1 \cdots h_n (1 + \omega^2)^{ns} e^{-\beta\omega/2}$$

$$S_\pi = E_\pi \omega^{-1/2} e^{-\beta\omega/2} = E_\pi \omega^{-1/2} k_1 \cdots k_n (1 + \omega^2)^{ns} e^{-\beta\omega/2}$$

that the operators  $S_\phi$  and  $S_\pi$  can be represented as a product of an arbitrary number of Hilbert-Schmidt operators. Hence they are  $p$ -nuclear for any  $p > 0$ .

This result, combined with Lemma 3.4, yields the

PROPOSITION. — *Consider the quantum field theory of a free scalar particle of mass  $m > 0$  in  $(s + 1)$  space-time dimensions. There exists for each bounded region  $\mathcal{O}$  and each  $p > 0$  a number  $\beta > 0$  such that the map*

$\Xi_{\beta, \mathcal{O}}: \mathfrak{A}(\mathcal{O}) \rightarrow \mathcal{B}(\mathcal{H})$  given by

$$\Xi_{\beta, \mathcal{O}}(A) = e^{-\beta H} A e^{-\beta H}, \quad A \in \mathfrak{A}(\mathcal{O})$$

is  $p$ -nuclear. If  $s > 2$ , this statement holds also if  $m = 0$ .

We thus have established Condition  $N^\#$  for the present class of theories. According to the remark in Sect. 2 it is then clear that Condition  $N_\#$  is also satisfied.

#### 4. OUTLOOK

In the present paper we have analyzed the relation between various compactness and nuclearity conditions characterizing the phase space properties of a quantum field theory. We were led to the new and stringent Condition  $N_\#$ , respectively its dual version  $N^\#$ , by combining the desirable features of the hitherto existing conditions into a single criterion.

Nuclearity conditions have proved to be a powerful tool in the general structural analysis of quantum field theory ([2], [3], [15], [16]). We therefore believe that the present sharpened version will be a key to the understanding of some longstanding problems. We have in mind here, on the one hand, the relation between local algebras and quantum fields ([17], [18]). It was surmised by Fredenhagen and Hertel [4] that Condition  $C_\#$  puts certain limitations on the number of independent fields with a fixed ultraviolet behaviour (“dimension”), and that this fact should allow establishing the existence of short distance expansions for these fields. In view of its more stringent restrictions on the ultraviolet behaviour, our new condition seems to be a promising starting point for a fresh look at this problem.

On the other hand, our condition should be relevant for the understanding of the particle aspects in quantum field theory. We conjecture that a quantitative version of Condition  $N_\#$ , amended by the principle of primitive causality [19], implies that the underlying theory has an (infra-) particle interpretation, cp. [20]. The specification “quantitative” here means that the  $p$ -norms of the respective maps have to depend on the region  $\mathcal{O}$  in an appropriate manner, cp. the discussion in [2], Sect. 4.

In this context it is of interest that Conditions  $N_\#$  and  $N^\#$  can be expressed in any superselection sector of a theory, their formulation does not depend on the existence of a vacuum state. Yet one can show by the method of “large translations” [21] that whenever these conditions are satisfied in some superselection sector, then there exists a vacuum state which is locally normal with respect to this sector, and Conditions  $N_\#$  and  $N^\#$  are also satisfied in the corresponding vacuum representation.

## ACKNOWLEDGEMENTS

We are indebted to Klaus Fredenhagen and Joachim Hertel for informing us of their unpublished work on compactness conditions. The second author (M. P.) would also like to thank Peter Junglas for helpful discussions.

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(Manuscript received June 30, 1989.)