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Schrödinger operators with form-bounded potentials in L^p-spaces

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by

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ABSTRACT. — Let

$$V = V_+ - V_-, \qquad 0 \leq V_\pm \in L^p_{comp}(\mathbb{R}^l),$$
$$\beta < 1, \qquad V_- \leq -\beta \Delta + C,$$

$$V_- \in L^{k,\infty}_{comp}(\mathbb{R}^l), \qquad k = (1 + \sqrt{1-\beta}) \bigg/ \left(\frac{2}{p} - 1 + \sqrt{1-\beta}\right)$$

($L^{p,\infty}$ is the weak $- L^p$-space). If $p \in \left(\dfrac{2}{1 + \sqrt{1-\beta}}, \dfrac{2}{1 - \sqrt{1-\beta}}\right)$, then the closure of the operator $(-\Delta + V + C) \restriction C_0^\infty(\mathbb{R}^l)$ is generator of the contraction C_0-semigroup in $L^p(\mathbb{R}^l)$. For $p = 2$ we obtain a new theorem concerning the essential self-adjointness of the Schrödinger operator. The sharpness of the results is illustrated by the examples.

RÉSUMÉ. — $V = V_+ - V_-, \qquad 0 \leq V_\pm \in L^p_{comp}(\mathbb{R}^l),$
$$\beta < 1, \qquad V_- \leq -\beta \Delta + C,$$

$$V_- \in L^{k,\infty}_{comp}(\mathbb{R}^l), \qquad k = (1 + \sqrt{1-\beta}) \bigg/ \left(\frac{2}{p} - 1 + \sqrt{1-\beta}\right)$$

($L^{p,\infty}$ est l'espace L^p faible). Si $p \in \left(\dfrac{2}{1 + \sqrt{1-\beta}}, \dfrac{2}{1 - \sqrt{1-\beta}}\right)$, la fermeture de l'opérateur $(-\Delta + V + C) \restriction C_0^\infty(\mathbb{R}^l)$ est un générateur de semigroupe C_0 de contraction dans $L^p(\mathbb{R}^l)$. Pour $p = 2$ nous obtenons un

nouveau théorème concernant l'auto-adjonction essentielle de l'opérateur de Schrödinger. Ces résultats sont illustrés sur des exemples.

INTRODUCTION

This article deals with the Schrödinger operators $-\Delta + V$ acting in $L^p \equiv L^p(\mathbb{R}^l, d^l x)$. We want to investigate the first spectral problem, *i. e.* the problem of m-accretivity. Most of the results concerning this problem contain the condition of $-\Delta$-boundedness of the operator $V_- \equiv \max\{-V, 0\}$. Our goal is to investigate the conditions of the following type:

$$V_- \leqq -\beta\Delta + C(\beta), \qquad \beta < 1, \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} \tag{1}$$
$$V_- \in F,$$

where F is some function space.

The first result in this direction is due to Povzner [1] [$p=2$, $\beta=1$, $F = C(\mathbb{R}^l)$]. Recently some results were also obtained. In particular the essential self-adjointness was proved provided that $F = L^k$, $k > \dfrac{2l}{l - \beta(l-2)}$ [2]; and $F = L^k$, $k = (1 + \sqrt{1 - \beta})/\sqrt{1 - \beta}$ [3].

My purpose here is to extend the abovementioned results to the case of L^p-spaces and to choose the broader space F. Even for $p=2$ we obtain a new result which is very close to optimal.

It should be noted that our method is an extension of the Semenov's one [3].

Note that L^p-Δ-boundedness implies the condition (1) (*see* [4]) and in my opinion the condition (1) corresponds to the point of our problem. We want also to emphasize that the exact eigenfunction estimates for the operator $-\Delta + V$ and the exact results concerning the accretivity of the Schrödinger operators have been proved under the same condition (1) (*see* [5], [6]).

Some common symbols are:

$$\| f \|_p = \left(\int_{\mathbb{R}^l} | f(x) |^p \, d^l x \right)^{1/p} ;$$

$$\langle f, g \rangle = \int_{\mathbb{R}^l} f(x) \overline{g(x)} \, d^l x ;$$

$$(f * g)(x) = \int_{\mathbb{R}^l} f(x-y) g(y) \, d^l y ;$$

$L_{\text{loc}}^p \equiv L_{\text{loc}}^p(\mathbb{R}^l, d^l x) =$ the complex functions f on $\mathbb{R}^l$ such that $f \varphi \in L^p$, $\forall \varphi \in C_0^\infty$; meas $(B) =$ Lebesgue measure of $B \subset \mathbb{R}^l$; $L^{p, \infty} \equiv L^{p, \infty}(\mathbb{R}^l, d^l x) =$ the complex functions f on $\mathbb{R}^l$ such that

$$\sup_{t > 0} t^p \, \text{meas} \{ x: | f(x) | > t \} < \infty$$

(note that $L^p \subset L^{p, \infty}$ and $|x|^{-l/p} \in L^{p, \infty}$). $L_{\text{comp}}^{p, \infty} =$ the set of functions in $L^{p, \infty}$ with compact support. $\Delta = \sum_{j=1}^{l} \dfrac{\partial^2}{\partial x_j^2}$ the Laplacian acting on $\mathscr{D}'(\mathbb{R}^l)$.

MAIN RESULTS

LEMMA 1. — *If* $V \in L_{\text{comp}}^{p, \infty}$, *then* $\| V \|_{p-\varepsilon} = O(\varepsilon^{-1/p})$, $\varepsilon \downarrow 0$.

Proof. — Let $\mu(t) = \text{meas} \{ x: | V(x) | > t \}$. The function $\mu(t)$ is nonincreasing and $\mu(0) = \text{meas} \{ \text{supp } V \} < \infty$. The definition of the space $L^{p, \infty}$ implies that $\mu(t) \leq C t^{-p}$, $\forall t > 0$, so

$$\| V \|_{p-\varepsilon}^{p-\varepsilon} \equiv (p-\varepsilon) \int_0^\infty t^{p-\varepsilon-1} \mu(t) \, dt$$

$$= (p-\varepsilon) \left(\int_0^1 t^{p-\varepsilon-1} \mu(t) \, dt + \int_1^\infty t^{p-\varepsilon-1} \mu(t) \, dt \right)$$

$$\leq (p-\varepsilon) \left(\mu(0) \int_0^1 t^{p-\varepsilon-1} \, dt + C \int_1^\infty t^{-1-\varepsilon} \, dt \right)$$

$$= \mu(0) - C + \frac{C p}{\varepsilon}. \quad \blacksquare$$

DEFINITION. — Let $0 \leq V$, $W \in L_{\text{loc}}^1$. Then $V \in PK_\beta(-\Delta + W)$ if and only if

$$\langle V \varphi, \varphi \rangle \leq \beta (\| \vec{\nabla} \varphi \|_2^2 + \| W^{1/2} \varphi \|_2^2) + C(\beta) \| \varphi \|_2^2,$$
$$\forall \varphi \in C_0^\infty.$$

THEOREM 2. – *Let V be a real valued measurable function such that*

$$V = V_+ - V_-, \qquad 0 \leq V_+ \in L^p_{comp},$$
$$0 \leq V_- \in PK_\beta(-\Delta + V_+) \cap L^{k,\infty},$$
$$\beta < 1, \qquad p \in \left(1, \frac{2}{1 - \sqrt{1-\beta}}\right),$$
$$k = (1 + \sqrt{1-\beta}) \bigg/ \left(\frac{2}{p} - 1 + \sqrt{1-\beta}\right).$$

Then the range of the operator $(-\Delta + V + \lambda) \upharpoonright C_0^\infty$ *is dense in* L^p *for any* $\lambda > C(\beta)$.

Proof. – If $p \in \left(1, \dfrac{2}{1 - \sqrt{1-\beta}}\right)$, then $1 < p < k < \infty$.

Suppose that the range is not dense, then by the Hahn-Banach theorem, there exists a function $0 \neq u \in L^{p'}$ such that

$$\langle (-\Delta + V + \lambda)\varphi, u \rangle = 0, \qquad \forall \varphi \in C_0^\infty. \tag{3}$$

Whithout loss we can assume $u = \mathrm{Re}\, u$. Let $1/t = 1/p' + 1/k'$, $q = p'/k'$. Let $\varepsilon > 0$ be a small parameter and let us set $1/s = 1/t + \varepsilon/p'$, $1/s' = (q - 1 - \varepsilon)/p'$. Since $V_- \in L^{k,\infty}_{comp}$ and $u \in L^{p'}$. It follows that $V_- u \in L^1 \cap L^s$.

The equality (3) gives

$$(-\Delta + V_+ + \lambda)u = V_- u \in L^1 \cap L^s \tag{4}$$

in the distributional sense; hence

$$u \in L^1 \cap L^s \cap L^{p'}; \tag{5}$$

(*cf.* [7]).

Moreover, $u \in L^{p'}$ implies

$$|u|^{q-1} \in L^{s'}. \tag{6}$$

Let us consider the operator H_z^+ with the domain

$$\mathscr{D}(H_z^+) = \left\{ u \in L^z : V_+ u \in L^1_{loc}, (-\Delta + V_+)u \in L^z \right\},$$

where $(-\Delta + V_+)$ is taken in the distributional sense and $H_z^+ u \equiv (-\Delta + V_+)u$, $\forall u \in \mathscr{D}(H_z^+)$.

H_z^+ is the generator of the strongly continuous contraction semigroup $T_z(t) \equiv \exp(-t H_z^+)$, $\forall t > 0$ in L^z which has properties such as follows below

$$\left. \begin{array}{l} \| T_z(t)\varphi \|_w \leq C(t, w, y, l) \| \varphi \|_{y'}, \\ \forall \varphi \in C_0^\infty, \quad \forall t > 0, \quad \forall w \geq y \geq 1 \end{array} \right\} \tag{7}$$

$$T_z(t) \upharpoonright [L^z \cap L^w] = T_w(t) \upharpoonright [L^z \cap L^w] \tag{8}$$

(*cf.* [7]).

Let $\varphi_r = T_s\left(\dfrac{1}{r}\right) u$. It follows from (5), (8), (4) and C_0-property that

(1) $\varphi_r \underset{L^y}{\to} u, \forall y \in [1, p']$;

(2) $(-\Delta + V_+) \varphi_r \underset{L^s}{\to} (-\Delta + V_+) u$.

According to (1) we can choose subsequence (which we shall denote by the same symbol $\{\varphi_r\}$) such that

$$\varphi_r(x) \to u(x) \quad \text{for a. e. } x \in \mathbb{R}^l.$$

Then

(3) $\varphi_r |\varphi_r|^{q-\varepsilon-2} \underset{L^{s'}}{\to} u |u|^{q-\varepsilon-2}$.

Indeed,

$$\varphi_r |\varphi_r|^{q-\varepsilon-2} \to u |u|^{q-\varepsilon-2} \quad \text{a. e. on } \mathbb{R}^l,$$

because $\varphi_r(x) \to u(x)$ a. e. Hence in order to conclude the proof of (3) it is sufficient to prove the convergence of the norms (a nice proof of this fact may be found in [8]) but

$$\| \varphi_r |\varphi_r|^{q-\varepsilon-2} \|_{s'} = \| \varphi_r \|_{p'}^{q-\varepsilon-1}$$

and the convergence $\| \varphi_r \|_{p'} \to \| u \|_{p'}$ is the consequence of (1).
In the same way we obtain

(4) $|\varphi_r|^{q-\varepsilon} \underset{L^y}{\to} |u|^{q-\varepsilon}, \forall y < k'$.

Let us set $j_m(x) = j(mx) m^{-l}$, where $0 \leq j \in C_0^\infty$, $\displaystyle\int_{\mathbb{R}^l} j(x) d x = 1$.

Let $\varphi_{mr} = j_m * \varphi_r$. By well known properties of the operation $j_m *$ (see, e. g., [9]), we have (1), (3), (4) for the subsequence of $\{\varphi_{mr}\}$. (7) implies $\varphi_r \in L^\infty$ ($r = 1, 2, \ldots$) so that $V_+ \varphi_{mr} \underset{L^s}{\to} V_+ \varphi_r$. This fact and the equality

$$\Delta(j_m * \varphi_r) = j_m * (\Delta \varphi_r)$$ show that

$$(-\Delta + V_+) \varphi_{mr} \underset{L^s}{\to} (-\Delta + V_+) \varphi_r.$$

Let us choose any $\omega \in C_0^\infty$ having $\omega(0) = 1$ and let $\omega_d(x) = \omega\left(\dfrac{x}{d}\right)$ and $\varphi_{dmr} = \omega_d \varphi_{mr}$. It is easy to see that properties (1), (3), (4) hold for $\{\varphi_{dmr}\}$.

Using the standard diagonal process we can choose the subsequence $u_r = \varphi_{d_r m_r r}$ such that

$$u_r \underset{L^y}{\to} u, \qquad \forall\, y \in [1, p'], \tag{9}$$

$$(-\Delta + V_+)\, u_r \underset{L^s}{\to} (-\Delta + V_+)\, u, \tag{10}$$

$$u_r \,|\, u_r\,|^{q-\varepsilon-2} \underset{L^{s'}}{\to} u\,|\,u\,|^{q-\varepsilon-2}, \tag{11}$$

$$|\,u_r\,|^{q-\varepsilon} \underset{L^y}{\to} |\,u\,|^{q-\varepsilon}, \qquad \forall\, y \leqq k'. \tag{12}$$

Observe that $u_r \in C_0^\infty$, so that $u_r \,|\, u_r\,|^{q-\varepsilon-2}$, $|\,u_r\,|^{(q-\varepsilon)/2} \in \mathcal{D}\,(\vec{V})$ (see, e. g., [10], Chapter 2), and the direct calculation yields the identity

$$\langle\, u_r \,|\, u_r\,|^{q-\varepsilon-2}, \; -\Delta u_r \,\rangle = \frac{4\,(q-\varepsilon-1)}{(q-\varepsilon)^2} \|\, \vec{V}\,|\, u_r\,|^{(q-\varepsilon)/2} \,\|_2^2. \tag{13}$$

(4)-(6) imply the finiteness of the quantities

$$\langle\, u\,|\,u\,|^{q-\varepsilon-2}, (-\Delta+V_+)\, u\,\rangle; \qquad \langle\, u\,|\,u\,|^{q-\varepsilon-2}, V_-\, u\,\rangle;$$
$$\langle\, u\,|\,u\,|^{q-\varepsilon-2}, \lambda\, u\,\rangle,$$

so that the simple approximation arguments show that (3) is valid provided that $\varphi = u_r \,|\, u_r\,|^{q-\varepsilon-2}$ and (9)-(12) imply

$$\lim_{r \to \infty} \langle\, u_r \,|\, u_r\,|^{q-\varepsilon-2}, (-\Delta+V+\lambda)\, u_r \,\rangle$$
$$= \langle\, u\,|\,u\,|^{q-\varepsilon-2}, (-\Delta+V+\lambda)\, u\,\rangle = 0. \tag{14}$$

Using (13), (14) and the condition $V_- \in PK_\beta\,(-\Delta+V_+)$ we have

$$\lambda\,\|\,u\,\|_{q-\varepsilon}^{q-\varepsilon} + \langle\, u\,|\,u\,|^{q-\varepsilon-2}, (-\Delta+V_+)\, u\,\rangle$$
$$= \langle\, u\,|\,u\,|^{q-\varepsilon-2}, V_-\, u\,\rangle = \lim_{r \to \infty} \langle\, u_r\,|\,u_r\,|^{q-\varepsilon-2}, V_-\, u_r\,\rangle$$
$$= \lim_{r \to \infty} \langle\, V_-\,|\,u_r\,|^{(q-\varepsilon)/2}, \;|\,u_r\,|^{(q-\varepsilon)/2}\,\rangle$$
$$\leqq \lim_{r \to \infty}\, [\beta\,\|\,\vec{V}\,|\,u_r\,|^{(q-\varepsilon)/2}\,\|_2^2 + \beta$$
$$\times \|\,V_+^{1/2}\,|\,u_r\,|^{(q-\varepsilon)/2}\,\|_2^2 + C\,(\beta)\,\|\,|\,u_r\,|^{(q-\varepsilon)/2}\,\|_2^2]$$
$$= \lim_{r \to \infty} \left[\frac{\beta\,(q-\varepsilon)^2}{4\,(q-\varepsilon-1)} \langle\, u_r\,|\,u_r\,|^{q-\varepsilon-2}, \; -\Delta u_r\,\rangle \right.$$
$$\left. + \beta\,\langle\, u_r\,|\,u_r\,|^{q-\varepsilon-2}, V_+\, u_r\,\rangle + C\,(\beta)\,\|\,u_r\,\|_{q-\varepsilon}^{q-\varepsilon} \right] \equiv \lim_{r \to \infty} I_r. \tag{15}$$

Note that $\beta < \beta (q-\varepsilon)^2/4(q-\varepsilon-1)$. Therefore,

$$\lim_{r\to\infty} I_r \leq \frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} \lim_{r\to\infty} \langle u_r | u_r |^{q-\varepsilon-2}, (-\Delta + V_+) u_r \rangle$$

$$+ C(\beta) \|u\|_{q-\varepsilon}^{q-\varepsilon} = \frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}$$

$$\times \langle u | u |^{q-\varepsilon-2}, (-\Delta + V_+) u \rangle + C(\beta) \|u\|_{q-\varepsilon}^{q-\varepsilon}. \quad (16)$$

According to (15), (16),

$$(\lambda - C(\beta)) \|u\|_{q-\varepsilon}^{q-\varepsilon} \leq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 \right] \langle u | u |^{q-\varepsilon-2}, (-\Delta + V_+) u \rangle.$$

From (14) we have that the right side of the last inequality equals

$$\left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 \right] \langle u | u |^{q-\varepsilon-2}. (V_- - \lambda) u \rangle$$

$$\leq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 \right] \langle u | u |^{q-\varepsilon-2}. V_- u \rangle.$$

So, using Hölder inequality we obtain

$$(\lambda - C(\beta)) \|u\|_{q-\varepsilon}^{q-\varepsilon} \leq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 \right] \langle | u |^{q-\varepsilon}, V_- \rangle$$

$$\leq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 \right] \|u\|_{p'}^{q-\varepsilon} \| V_- \|_{z(\varepsilon)}, \quad (17)$$

where $z(\varepsilon) = k/(1 + \varepsilon k/p')$.

According to the lemma 1 $\| V_- \|_{z(\varepsilon)} = O(\varepsilon^{-1/k})$. On the other hand

$$\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} - 1 = O(\varepsilon).$$

Now assuming that $\varepsilon \downarrow 0$ in (17) and using $k > 1$ we obtain

$$(\lambda - C(\beta)) \|u\|_q^q \leq 0.$$

Then $u \equiv 0$ as $\lambda > C(\beta)$. ∎

Remarks. – 1. The condition $V_- \in PK_\beta(-\Delta + V_+) \cap L_{loc}^p$ is necessary in the theorem 2 but it is not sufficient as it may be seen in the example $-\Delta - \dfrac{\kappa}{|x|^2}$ (*see* [11], Chapter X). The same example shows the sharpness of the theorem 2. Inded, let $2p < m \leq l$,

$$V(x) = \kappa\chi(x) \left(\sum_{j=1}^m x_j^2 \right)^{-1},$$

where $\chi(.)$ is the indicator of the unit ball in $\mathbb{R}^l$. Then the range of the operator $(-\Delta - V - \lambda) C_0^\infty$ is dense in L^p if and only if

$\kappa \leq m(m-2p)(p-1)/p^2$ [4]. On the other hand, $V \in L_{comp}^{m/2, \infty}$, *i. e.*, $\kappa = m(m-2p)(p-1)/p^2$ implies $k = (1 + \sqrt{1-\beta}) / \left(\dfrac{2}{p} - 1 + \sqrt{1-\beta} \right)$.

2. The condition $V_- \in PK_\beta(-\Delta + V_+)$ may be replaced by $V_- \in PK_\beta\left(-\Delta + \dfrac{p^2}{4(p-1)}V_+\right)$.

3. The analysis of the proof shows that the restriction $V_- \in L_{comp}^{k, \infty}$ may be replaced by the weaker condition $\|V_-\|_{k-\varepsilon} = o(\varepsilon)$.

4. The main device in the proof of the theorem 2 is the inequality

$$\langle -\Delta u, u|u|^{p-2} \rangle \geq 4\frac{p-1}{p^2} \||\vec{\nabla}|u|^{p/2}|\|_2^2.$$

N. Th. Varopoulos [12] has proved the abstract version of this inequality for the generators of submarkovian semigroups. Thus our result may be generalized for these operators.

COROLLARY 3. — *Let*

$$V = V_+ - V_-, \qquad 0 \leq V_\pm \in L_{comp}^2,$$
$$\beta < 1, \qquad V_- \in PK_\beta(-\Delta + V_+) \cap L^{k, \infty}, \qquad k = (1 + \sqrt{1-\beta})/\sqrt{1-\beta}.$$

Then $(-\Delta + V) \upharpoonright C_0^\infty$ *is essentially self-adjoint operator in* L^2.

Corollary 3 is proved under the restriction $V \in L_{comp}^2$, but using the results of C. Simader [13] or H. Brezis [14], we have

THEOREM 4. — *Let*

$$V = V_+ - V_-, \qquad 0 \leq V_\pm \in L_{loc}^2,$$
$$\beta < 1, \qquad V_- \in PK_\beta(-\Delta + V_+) \cap L_{loc}^{k, \infty}, \qquad K = (1 + \sqrt{1-\beta})/\sqrt{1-\beta}.$$

Then $(-\Delta + V) C_0^\infty$ *is essentially self-adjoint.*

Remark. — In the case $l \geq 5$ theorem 4 is a generalization of the Kalf-Walter-Schmincke-Simon theorem (*see* [11], § X.4).

Example. — We consider the N-particle Hamiltonian

$$H = H_0 + V,$$
$$H_0 = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \frac{1}{2N}\left(\sum_{i=1}^{N}\vec{\nabla}_i\right)^2,$$
$$-\alpha\sum_{i<j}|x_i - x_j|^{-2} \leq V \in L_{loc}^2(\mathbb{R}^{l(N-1)}),$$
$$x_j \in \mathbb{R}^l, \qquad l \geq 5.$$

Hardy inequality implies $V_- \in PK_\beta(H_0)$, $\beta = \dfrac{2N\alpha}{(l-2)^2}$. One sees easily that $V_- \in L_{loc}^{l/2, \infty}$. If $\alpha \leq \dfrac{(l-4)}{2N}$ it follows from theorem 4 that $H \upharpoonright C_0^\infty$ is essentially self-adjoint (cf. [15], [16]). Note that for the relativistic Hamiltonian

$$H_R + V_C \equiv \sum_{i=1}^{N} (-\Delta_i + m^2)^{1/2} - \alpha \sum_{i<j} |x_i - x_j|^{-1}, \qquad l = 3.$$

Lieb and Thirring [17] have proved that $V_C \in PK_\beta(H_R)$ with $\beta = O(N\alpha)$ and $H_R + V_C$ is unbounded from below for $\alpha = C/N$, $C < \infty$. This correlation between β and N appears to be true for our nonrelativistic Hamiltonian as well so that the dependence $\alpha = O(1/N)$ is optimal.

To prove m-accretivity we need.

THEOREM 5 [6]. – *Let*

$$V = V_+ - V_-, \qquad 0 \leq V_\pm \in L_{loc}^p,$$
$$V_- \in PK_\beta(-\Delta + V_+), \qquad \beta < 1.$$

If $p \in \left[\dfrac{2}{1 + \sqrt{1-\beta}}, \dfrac{2}{1 - \sqrt{1-\beta}} \right]$, *then the operator* $(-\Delta + V + C(\beta)) \upharpoonright C_0^\infty$ *is accretive operator in* L^p.

Proof. – We will show that

$$[(-\Delta + V + C(\beta))u, u] \geq 0, \qquad \forall u \in C_0^\infty, \tag{18}$$

where $[v, w]$ is semi-inner product (*see* [11], § X.8). In the case of L^p spaces

$$[v, w] = \langle v, w|w|^{p-2} \rangle / \|w\|^{p-2},$$

so that (18) may be written in the following way

$$\langle V_- u, u|u|^{p-2} \rangle \leq \langle (-\Delta + V_+ + C(\beta))u, u|u|^{p-2} \rangle.$$

But $V_- \in PK_\beta(-\Delta + V_+)$ so that

$$\begin{aligned}
\langle V_- u, u|u|^{p-2} \rangle &= \langle V_- |u|^{p/2}, |u|^{p/2} \rangle \\
&\leq \beta \| \vec{\nabla} |u|^{p/2} \|_2^2 + \beta \langle V_+ |u|^{p/2}, |u|^{p/2} \rangle + C(\beta) \| |u|^{p/2} \|_2^2 \\
&= \frac{\beta p^2}{4(p-1)} \langle -\Delta u, u|u|^{p-2} \rangle \\
&\qquad + \beta \langle V_+ u, u|u|^{p-2} \rangle + C(\beta) \langle u, u|u|^{p-2} \rangle.
\end{aligned}$$

Since $\dfrac{\beta p^2}{4(p-1)} \leq 1$ provided that $p \in \left[\dfrac{2}{1 + \sqrt{1-\beta}}, \dfrac{2}{1 - \sqrt{1-\beta}} \right]$ we have proved (18). ∎

Remark. − 1. The sharpness of the result is shown in [6] by the example
$-\Delta - \dfrac{\kappa}{|x|^2}$.

2. T. Kato [7] raised the problem of quasi-accretivity of the operator $-\Delta + V$ in the limiting case $p=1$ for the potentials such that

$$\lim_{\alpha \downarrow 0} \sup_{x} \int_{|x-y|\leq\alpha} |x-y|^{-l+2}\,|V(y)|\,d^l y = 0, \qquad l \geq 3.$$

We will show that the answer is negative, *i. e.*, if $0 \leq V \in L^1_{\text{loc}}$ and if $-\Delta - V$ is quasi-accretive in L^1, then $V \in L^\infty$. Indeed, the semi-inner product in L^1 is $[v, w] = \left\langle v, \dfrac{w}{|w|} \right\rangle \|w\|_1$. The quasi-accretivity implies that

$$\left\langle (-\Delta - V)\,u, \dfrac{u}{|u|} \right\rangle \geq -C\,\|u\|_1,$$
$$C < \infty, \qquad \forall u \in C_0^\infty.$$

Note that $\langle \Delta u, 1 \rangle = 0$, $\forall u \in C_0^\infty$. Therefore, $\langle V, u \rangle \leq C\,\|u\|_1$, *i. e.* $V \in L^\infty$.

3. If $0 \leq V \notin L^\infty$, $V \in L^p$, $p > l/2$, $l \geq 3$, then $-\Delta - V$ is the generator of the positivity preserving C_0-semigroup in L^1 which is not quasi-contractive (other examples may be found in [18], [6]).

Theorems 2, 5 and Lumer-Phillips theorem (*see* [11], § X.8) imply

THEOREM 6. − *Let*

$$\beta < 1, \qquad p \in \left(\dfrac{2}{1+\sqrt{1-\beta}}, \dfrac{2}{1-\sqrt{1-\beta}} \right),$$
$$V = V_+ - V_-, \qquad 0 \leq V_\pm \in L^p_{\text{comp}},$$
$$V_- \in PK_\beta(-\Delta + V_+) \cap L^{k,\,\infty}$$

where $k = (1 + \sqrt{1-\beta}) \Big/ \left(\dfrac{2}{p} - 1 + \sqrt{1-\beta} \right)$. *Then the closure of the operator* $(-\Delta + V + C(\beta)) \restriction C_0^\infty$ *is the generator of C_0-semigroup of contractions in* L^p.

Remarks. − 1. Theorem 6 is generalization (when supp V is compact) of the result stated in [4]. It is of interest to prove the L^p-version of the result of C. Simader [13] and H. Brezis [14].

2. Having completed this paper the author was informed by Yu. A. Semenov about the proof of the theorem 6 provided $0 \leq V_\pm \in L^p_{\text{loc}}$, $p \in \left(\dfrac{2}{1+\sqrt{1-\beta}}, \dfrac{2}{1-\sqrt{1-\beta}} \right)$.

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