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De Sitter to Poincaré contraction and relativistic coherent states

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ABSTRACT. — We continue the analysis of relativistic phase space, identified with the quotient of the Poincaré group in $1+1$ dimensions by the time translation subgroup. Proceeding by contraction from the corresponding (Anti) de Sitter group $SO_0(2, 1)$, we obtain a realisation of the Poincaré representation $\mathscr{P}(m)$ of mass m in a space of functions defined on phase space. The contraction singles out a privileged section in the Poincaré group, with a unique left and right invariant measure. Using that section, we show that the representation $\mathscr{P}(m)$ is square integrable over the coset space, *i.e.* phase space. From this we build a new set of Poincaré coherent states, and more generally weighted coherent states, which have all the usual properties, resolution of the identity,

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overcompleteness, reproducing kernel, orthogonality relations. Finally we derive the corresponding Wigner transform.

RÉSUMÉ. — On poursuit l'analyse de l'espace de phase relativiste, identifié au quotient du groupe de Poincaré en dimension $1+1$ par le sous-groupe des translations temporelles. Par contraction à partir du groupe Anti-de Sitter correspondant $SO_0(2, 1)$, on obtient une réalisation de la représentation $\mathscr{P}(m)$ de masse m du groupe de Poincaré dans un espace de fonctions définies sur l'espace de phase. La contraction sélectionne une section privilégiée dans le groupe de Poincaré, possédant une unique mesure invariante à la fois à gauche et à droite. Avec ce choix de section, on montre que la représentation $\mathscr{P}(m)$ est de carré sommable sur l'espace quotient, c'est-à-dire l'espace de phase. A partir de là, on construit une nouvelle famille d'états cohérents de Poincaré et, plus généralement, d'états cohérents avec poids, qui ont toutes les propriétés usuelles : résolution de l'identité, surcomplétude, noyau reproduisant, relations d'orthogonalité. Enfin, on dérive la transformation de Wigner correspondante.

1. INTRODUCTION

This paper is a sequel to a previous work [1], henceforth referred to as I, in which some questions concerning the square integrability and the existence of coherent states, for a certain representation of the Poincaré group, $\mathscr{P}_+^\uparrow(1, 1)$, in one space and one time dimensions, were considered. The coherent states were constructed by considering a certain section mapping the physical phase space – realized as a homogeneous space of $\mathscr{P}_+^\uparrow(1, 1)$ – into the group. While the particular choice of a section in I was physically motivated, and enabled coherent states to be constructed, the mathematical, as well as physical, question of what would happen to the ensuing construction if a different section were chosen, was left unasked. In the present paper we adopt the point of view that studying higher space-time symmetries, which include the Poincaré group in a local approximation, would shed further light on what other possibilities might exist for the identification of the physical phase space with sections of the group. The homogeneous space of $\mathscr{P}_+^\uparrow(1, 1)$ in question here is $\mathscr{P}_+^\uparrow(1, 1)/T$ (T = time translation subgroup), which can be parametrized by points $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$. One then looks for sections $\beta : \mathbb{R}^2 \rightarrow \mathscr{P}_+^\uparrow(1, 1)$.

The search for higher space-time symmetry groups, in our case, is guided by the fact that for two-dimensional space-times there are the following six different relativities possible [2], along with their respective kinematical groups:

(1) the two de Sitterian relativities, with kinematical groups $SO_0(2, 1)_{\pm}$;

(2) the two Newtonian relativities, with groups $\mathcal{N}_{1\pm}$;

(3) the Einstein-Poincaré relativity, having the kinematical group $\mathscr{P}_+^{\uparrow}(1, 1)$;

(4) the Galilean relativity, having the group $\mathscr{G}_1$.

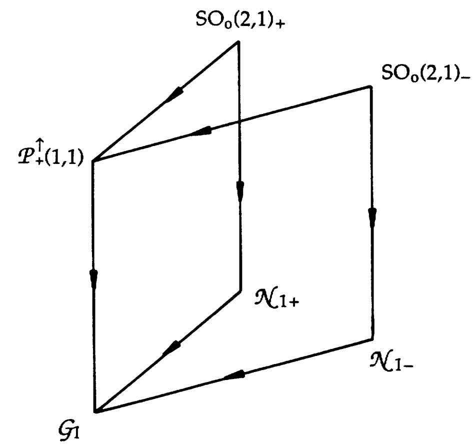

FIG. 1. — Contraction-deformation relationships between two-dimensional space-time relativities. The horizontal lines denote a $\kappa \to 0$ contraction, whereas the vertical ones correspond to the limit $c \to \infty$.

As schematized in Figure 1, these six groups can be related by contraction-deformation procedures, corresponding to letting either the curvature of space-time κ^2 tend to 0 or the velocity of light c to ∞.

The difference between the two maximal symmetry relativities — the anti-de Sitterian, with group $SO_0(2, 1)_+$, and the de Sitterian, with group $SO_0(2, 1)_-$ — is merely one of identification of the subgroup appropriate to space or time translations. In both cases the group is $SO_0(2, 1)$. It is denoted $SO_0(2, 1)_+$ when the maximal compact subgroup $SO(2)$ is identified with time translations (compactified time), and $SO_0(2, 1)_-$ when this subgroup is identified with spatial translations (compactified space). The four dimensional analogue of $SO_0(2, 1)_+$ is $SO_0(2, 3)$ while that of $SO_0(2, 1)_-$ is $SO_0(4, 1)$ [3].

In a quantum theory, elementary systems are associated to (projective) unitary, irreducible representations (UIR) of the (possibly extended) kinematical group (or its universal covering group) [4]. If we denote by $\mathscr{P}(m)$ the UIR of the Poincaré group, $\mathscr{P}_+^{\uparrow}(1, 1)$, which describes a system of mass m, the corresponding representations of $SO_0(2, 1)_+$ or $SO_0(2, 1)_-$, of which $\mathscr{P}(m)$ is the contracted version, differ radically. Indeed, $\mathscr{P}(m)$ arises from the contraction of a representation $D_+(E_0)$ belonging to the discrete series ([5]-[7]) of $SO_0(2, 1)_+$, while it is a contraction of a principal series representation $D_-(E_0)$ of $SO_0(2, 1)_-$ which leads [8] to the direct

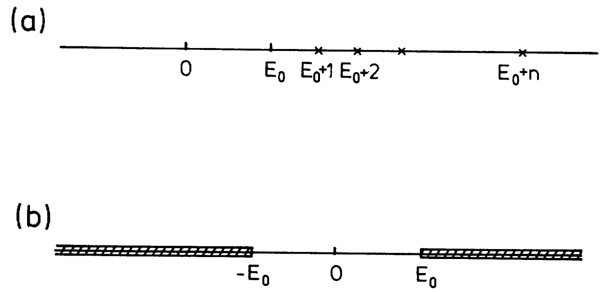

FIG. 2. — (*a*) Spectrum of the compact generator K_0 of $SO_0(2, 1)$ in a discrete series representation $D_+(E_0)$. (*b*) Spectrum of the noncompact generator T_0 of $SO_0(2, 1)$ in a principal series representation $D_-(E_0)$.

sum $\mathscr{P}(m) \oplus \mathscr{P}(-m)$. Here E_0 is the positive lower bound (see *Fig.* 2 *a*) of the discrete spectrum of the compact generator K_0, for the representation $D_+(E_0)$, and as such, is reminiscent of a "minimal energy". In the case of $D_-(E_0)$, the parameter E_0 is associated (see *Fig.* 2 *b*) to the continuous spectrum of the non-compact time translation operator T_0.

In both cases a contraction of the representation is carried out by letting $E_0 \to +\infty$ and $\kappa \to 0$, while holding the product κE_0 constant and equal to m, the mass of the system:

$$D_+(E_0) \xrightarrow[\kappa E_0 = m]{\substack{E_0 \to +\infty \\ \kappa \to 0}} \mathscr{P}(m) \tag{1.1}$$

$$D_-(E_0) \xrightarrow[\kappa E_0 = m]{\substack{E_0 \to +\infty \\ \kappa \to 0}} \mathscr{P}(m) \oplus \mathscr{P}(-m) \tag{1.2}$$

The uniqueness of the limit in the first case, together with the nice properties of a discrete series representation, has motivated the present work. The group $SO_0(2, 1)$, or its $SU(1, 1)$ or $SL(2, \mathbb{R})$ versions, has relatively well-known properties and the contraction procedure should keep trace of some of them. In particular coherent states associated to the discrete series have become classical since the work of Perelomov [7] and some aspects of their existence should subsist in a coherent state theory for the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$, such as the one developed in I.

The original square integrability of the discrete series representations on the entire group manifold has to be replaced by the notion of square integrability modulo some subgroup, namely the time-translation subgroup which, before contraction, was SO(2). Hence a phase space for $\mathscr{P}_{+}^{\uparrow}(1, 1)$ is used in I, and on it the square integrability of the unitary representation $\mathscr{P}(m)$ is studied. This notion was subsequently cast into a more general form by De Bièvre [9], using the geometric language of the Kirillov-Kostant-Souriau quantization procedure [10].

What is really new in our present approach is that contraction from $SO_0(2, 1)_+$ brings out a natural section in the Poincaré phase space besides revealing other striking features, such as an unusual realisation of $\mathscr{P}(m)$ on functions defined on the phase space itself (4). The familiar Wigner realisation of $\mathscr{P}(m)$ is next recovered through a constraint imposed on such functions, reminiscent of the choice of a polarization in the geometric quantization method [10]. The choice of polarization which has to be made here follows from the contraction itself, if we wish to protect the latter against infinities.

The organisation of this paper is as follows. In Section 2 we fix the notation and parameters for $SO_0(2, 1)$, its SU(1, 1) version, their respective homogeneous spaces, and the discrete series representations $D_+(E_0)$. Next we give a brief review of the main features of $D_+(E_0)$.

In Section 3 the contraction procedure $\kappa \to 0$ is described in the $SO_0(2, 1)$ language and in the SU(1, 1) language at the level of the generators of the representations. A privileged section in the Poincaré phase space and a representation of $\mathscr{P}_{+}^{\uparrow}(1, 1)$ on functions $f(\mathbf{p}, \mathbf{q})$ appear in a rather unexpected way. A brief account of the ensuing properties is then given.

In Section 4 the link with I is made. We resolve certain questions raised in the conclusion to that paper and finally present some ideas for possible extensions or applications of our work.

Remark. — Throughout the paper, as in I, we use boldface letters **a**, **q**, **p**, ... to denote the (1-dimensional) space component of vectors. The point is to remind the reader that a substantial part of the formulas are actually valid in the usual $(1+3)$-dimensional Minkowski space-time.

(4) The identification of phase space with a suitable homogeneous space of the relativity group (Galilei or Poincaré) has a long history; the idea can be traced back, at least, to H. Bacry and A. Kihlberg [11] and R. Arens [12]. It led ultimately to the methods of geometric quantization [10]. For a review of those questions we refer to the recent monograph of H. Bacry [13].

2. TWO DIMENSIONAL DE SITTER SPACE AND ITS KINEMATICAL GROUP $SO_0(2, 1)$

The de Sitter space with curvature κ^2 can be described by the one-sheeted hyperboloid in $\mathbb{R}^3$ (see *Fig.* 3):

$$u_1^2+u_2^2-u_3^2=\kappa^{-2}. \tag{2.1}$$

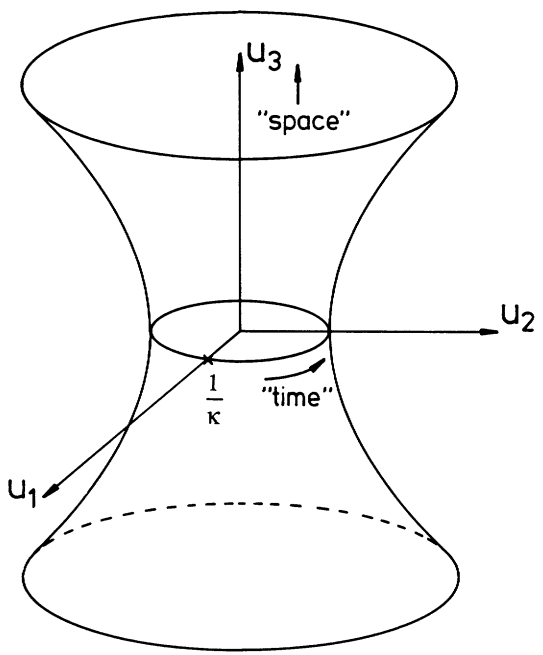

FIG. 3. — (Anti) de Sitter two dimensional space time visualised as the one-sheeted hyperboloid $u_1^2+u_2^2-u_3^2=\kappa^{-2}$, in $\mathbb{R}^3$.

Global coordinates $(x_0, \mathbf{x})$ exist for such a manifold, namely:

$$\begin{aligned} u_1&=(\kappa^{-2}+\mathbf{x}^2)^{1/2}\cos\kappa x_0,\\ u_2&=(\kappa^{-2}+\mathbf{x}^2)^{1/2}\sin\kappa x_0,\\ u_3&=\mathbf{x}, \end{aligned} \tag{2.2}$$

with $-\infty<\mathbf{x}<+\infty$, $-\pi\leqq\kappa x_0<\pi$. $SO_0(2, 1)$ acts transitively on the hyperboloid:

$$(a_{ij})\in SO_0(2, 1), \qquad u_i'=\sum_{j=1}^{3} a_{ij}u_j \tag{2.3}$$

Infinitesimal generators for (pseudo-) rotations in the (i, j)-plane are denoted by L_{ij}: L_{12} for true rotations, L_{23} and L_{31} for hyperbolic rotations.

$$L_{12}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad L_{23}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\quad L_{31}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}. \tag{2.4}$$

They satisfy the commutation rules:

$$\begin{aligned}[L_{12}, L_{23}]&=-L_{31},\\ [L_{31}, L_{12}]&=-L_{23},\\ [L_{23}, L_{31}]&=L_{12}.\end{aligned} \tag{2.5}$$

The homomorphism between $SO_0(2, 1)$ and the group $SU(1, 1)$ of the 2×2 complex unimodular matrices

$$g=\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix},\qquad |\alpha|^2-|\beta|^2=1, \tag{2.6}$$

is easily displayed through the action of the latter on hermitian matrices $\mathcal{U}$ associated to the triplets (u_1, u_2, u_3)

$$\mathcal{U}\equiv\begin{pmatrix}u_3 & u_1+iu_2\\ u_1-iu_2 & u_3\end{pmatrix}\mapsto\mathcal{U}'=g\,\mathcal{U}\,g^+. \tag{2.7}$$

Here g^+ is the hermitian adjoint of g. The 3×3 matrix (a_{ij}) which corresponds to (2.6) through (2.7) is given by

$$(a_{ij})=\begin{pmatrix}\mathrm{Re}(\alpha^2+\beta^2) & -\mathrm{Im}(\alpha^2-\beta^2) & 2\,\mathrm{Re}\,\alpha\beta\\ \mathrm{Im}(\alpha^2+\beta^2) & \mathrm{Re}(\alpha^2-\beta^2) & 2\,\mathrm{Im}\,\alpha\beta\\ 2\,\mathrm{Re}\,\alpha\bar\beta & -2\,\mathrm{Im}\,\alpha\bar\beta & (|\alpha|^2+|\beta|^2)\end{pmatrix}. \tag{2.8}$$

The three basic one-parameter subgroups of $SO_0(2, 1)$

$$\exp\theta\,L_{12},\quad \exp\varphi\,L_{23},\quad \exp\psi\,L_{31}, \tag{2.9}$$

correspond, respectively, to the $SU(1, 1)$ matrices (modulo a factor of -1):

$$\begin{aligned}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}&\equiv\exp\theta\,N_{12},\\ \begin{pmatrix}\cosh\varphi/2 & i\sinh\varphi/2\\ -i\sinh\varphi/2 & \cosh\varphi/2\end{pmatrix}&\equiv\exp\varphi\,N_{23},\\ \begin{pmatrix}\cosh\psi/2 & \sinh\psi/2\\ \sinh\psi/2 & \cosh\psi/2\end{pmatrix}&\equiv\exp\psi\,N_{31}.\end{aligned} \tag{2.10}$$

In other words, in this parametrization, the generators of $SU(1, 1)$ read:

$$N_{12}=\frac{i}{2}\sigma_3,\qquad N_{23}=-\frac{\sigma_2}{2},\qquad N_{31}=\frac{\sigma_1}{2}. \tag{2.11}$$

The Cartan decomposition G=PK of SU(1, 1) is easily performed:

$$g=\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}=\begin{pmatrix}|\alpha| & \beta\alpha/|\alpha|\\ \bar\beta\bar\alpha/|\alpha| & |\alpha|\end{pmatrix}\times\begin{pmatrix}\alpha/|\alpha| & 0\\ 0 & \bar\alpha/|\alpha|\end{pmatrix}\equiv pk \quad (p\in \mathrm{P},\ k\in \mathrm{K}). \tag{2.12}$$

K is the maximal compact subgroup and P can be put in one-to-one correspondence with the symmetric homogeneous space G/K. The latter is homeomorphic to the open unit disk

$$\mathscr{D}=\{\zeta\in\mathbb{C},\ |\zeta|<1\}. \tag{2.13}$$

The above identification is achieved through the choice of the section:

$$g\equiv\begin{pmatrix}\delta & \delta\zeta\\ \delta\bar\zeta & \delta\end{pmatrix}(\mathrm{mod}\,k),\qquad \delta=(1-|\zeta|^2)^{-1/2},\qquad \zeta=\beta\bar\alpha^{-1}. \tag{2.14}$$

Let us introduce the coordinates τ and ω for $\mathscr{D}$:

$$\zeta=\tanh\tau/2\,e^{i\omega},\qquad -\infty<\tau<+\infty,\quad 0\leqq\omega<2\pi. \tag{2.15}$$

These are also the (pseudo-) angular coordinates for the upper sheet $\mathscr{L}_+$ of the unit hyperboloid in $\mathbb{R}^3$:

$$\begin{aligned}\mathscr{L}_+\equiv\{\mathbf{n}=(n_0, n_1, n_2)\in\mathbb{R}^3;\ n_0^2-n_1^2-n_2^2=1,\ n_0\geqq 1\}\\ =\{\mathbf{n}=(\cosh\tau,\ \sinh\tau\cos\omega,\ \sinh\tau\sin\omega)\}.\end{aligned} \tag{2.16}$$

This simply illustrates the well-known correspondence between $\mathscr{L}_+$ and $\mathscr{D}$, the latter being the stereographic projection of the former (*Fig.* 4).

We note the formulas:

$$\zeta=\zeta_1+i\zeta_2=\frac{n_1+in_2}{n_0+1},\qquad |\zeta|^2=\frac{n_0-1}{n_0+1}. \tag{2.17}$$

At the level of the relationship between the groups SU(1, 1) and $\mathrm{SO}_0(2, 1)$ given by equation (2.8), the following $\mathrm{SO}_0(2, 1)$ hermitian matrix corresponds to the section (2.14):

$$\begin{pmatrix}1+\dfrac{n_1^2}{n_0+1} & \dfrac{n_1n_2}{n_0+1} & n_1\\ \dfrac{n_1n_2}{n_0+1} & 1+\dfrac{n_2^2}{n_0+1} & n_2\\ n_1 & n_2 & n_0\end{pmatrix} \tag{2.18}$$

This achieves the description of the homogeneous space $\mathrm{SO}_0(2, 1)/\mathrm{SO}(2)$ in terms of points $\mathbf{n}=(n_0, n_1, n_2)$ lying on $\mathscr{L}_+$.

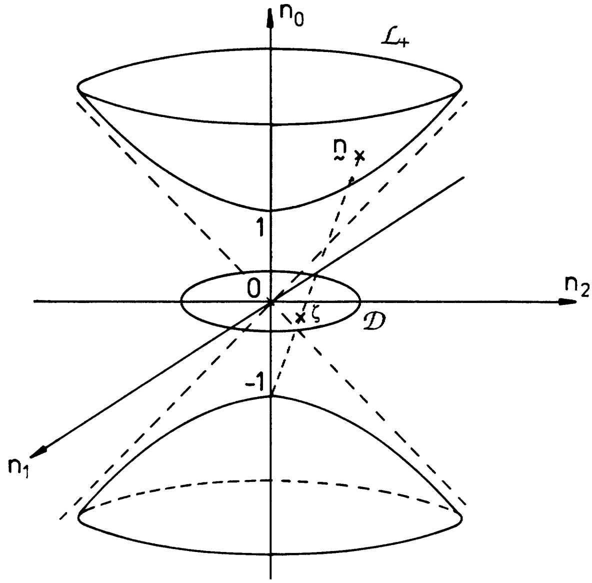

FIG. 4. — $\zeta \in \mathscr{D}$ as the stereographic projection of a point $\mathbf{n}=(n_0, n_1, n_2)$ lying on the upper sheet $\mathscr{L}_+$ of the unit hyperboloid $n_0^2-n_1^2-n_2^2=1$, in $\mathbb{R}^3$.

From now on, we shall denote the sections (2.14) and (2.18) by $p(\zeta)$ and $p(\mathbf{n})$ respectively. The action of SU(1, 1) on $\mathscr{D}$ can be found from the usual multiplication of $p(\zeta)$ from the left:

$$gp(\zeta) \equiv p(g.\zeta) (\operatorname{mod} k), \tag{2.19}$$

where

$$g.\zeta \equiv (\alpha\zeta+\beta)(\bar{\beta}\zeta+\bar{\alpha})^{-1} \quad \text{if} \quad g=\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \tag{2.20}$$

In a similar manner,

$$ap(\mathbf{n})=p(a\,\mathbf{n}) \tag{2.21}$$

if $a=(a_{ij}) \in SO_0(2, 1)$.

The spaces $\mathscr{D}$ and $\mathscr{L}_+$ are endowed with beautiful analytic properties, which make them Kählerian ([7], [14]). They have G-invariant metrics and G-invariant 2-forms [G=SU(1, 1) or $SO_0(2, 1)$], both arising from the Kählerian potential $-\ln(1-|\zeta|^2)=\ln\left(\dfrac{1+n_0}{2}\right)$:

$$ds^2=d\mathbf{n}.d\mathbf{n}=\frac{4\,d\zeta\,d\bar{\zeta}}{(1-|\zeta|^2)^2}=[d\tau^2+\sinh^2\tau\,d\omega^2], \tag{2.22}$$

$$\omega = 2i \frac{d\zeta \wedge d\bar{\zeta}}{(1-|\zeta|^2)^2} = \sinh \tau \, d\tau \wedge d\omega. \tag{2.23}$$

In view of their symplectic structure, $\mathscr{L}_+$ and $\mathscr{D}$ may be called phase spaces for the kinematical group $SO_0(2, 1)$ and its double covering $SU(1, 1)$, respectively [15]. Finally we recall that these manifolds are the simplest examples (besides S^2) of Lobachevsky spaces, endowed with a distance ρ between any two points $\mathbf{n}$ and $\mathbf{n}'$ (resp. ζ and ζ'):

$$\cosh \rho/2 = \delta\delta' \, |1 - \bar{\zeta}\zeta'|, \tag{2.24}$$

where

$$\delta = (1 - |\zeta|^2)^{-1/2}.$$

We come now to the description of the discrete series representation of $SU(1, 1)$ [or its universal covering $\overline{SU(1, 1)}$]. We denote by $\mathscr{F}^{E_0} = \{f(\zeta)\}$ the space of functions analytic inside the unit circle, satisfying the condition $\|f\|_{E_0} < +\infty$, where

$$\|f\|^2_{E_0} \equiv \frac{2E_0 - 1}{\pi} \int_{\mathscr{D}} |f(\zeta)|^2 (1 - |\zeta|^2)^{2E_0 - 2} \, d^2\zeta, \tag{2.25}$$

with $E_0 = 1, 3/2, 2, 5/2, \ldots$ (or $E_0 > 1/2$, for the universal covering).

The positive number E_0 will be considered as a minimal weight or minimal energy for reasons which will soon appear. Let us define the representation operators $T^{E_0}(g)$ by

$$T^{E_0}(g) f(\zeta) = (\beta\zeta + \bar{\alpha})^{-2E_0} f({}^t g . \zeta) \tag{2.26}$$

for $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ and ${}^t g \equiv$ transpose of g.

The representatives of the Lie algebra elements (2.11) are given (after adjoining the usual factor of i) by:

$$N_{12} \mapsto K_0 = \zeta \frac{d}{d\zeta} + E_0 \tag{2.27}$$

$$N_{31} \mapsto K_1 = -i \left(\frac{\zeta^2 - 1}{2} \frac{d}{d\zeta} + E_0 \zeta \right) \tag{2.28}$$

$$N_{23} \mapsto K_2 = -\left(\frac{\zeta^2 + 1}{2} \frac{d}{d\zeta} + E_0 \zeta \right) \tag{2.29}$$

with the commutation rules,

$$[K_1, K_2] = -iK_0, \qquad [K_2, K_0] = iK_1, \qquad [K_0, K_1] = iK_2. \tag{2.30}$$

It can directly be checked that the Casimir operator

$$C_2 = K_0^2 - K_1^2 - K_2^2 \tag{2.31}$$

takes the value

$$C_2 = E_0 (E_0 - 1) I \tag{2.32}$$

identically on $\mathscr{F}_0$: no second-order wave equation here!

K_0 is the ladder or "energy" operator whose eigenvalues in the representation space $\mathscr{F}^{E_0}$ are E_0, $E_0 + 1$, $E_0 + 2$, ..., $E_0 + n$, ... The corresponding eigenvectors are the normalized monomials:

$$u_n(\zeta) = \left[\frac{\Gamma(2E_0 + n)}{n!\,\Gamma(2E_0)}\right]^{1/2} \zeta^n. \tag{2.33}$$

These are orthogonal with respect to the form

$$(f_1, f_2)_{E_0} = \frac{2E_0 - 1}{\pi} \int_{\mathscr{D}} \overline{f_1(\zeta)} f_2(\zeta) (1 - |\zeta|^2)^{2E_0 - 2} d^2\zeta. \tag{2.34}$$

Sometimes the (bra) ket notation is used:

$$u_n(\zeta) = |E_0, E_0 + n\rangle.$$

Let us introduce the energy raising and lowering operators

$$K_{\pm} = \mp K_2 \pm i K_1, \tag{2.35}$$

$$K_+ = \zeta^2 \frac{d}{d\zeta} + 2E_0 \zeta, \tag{2.35'}$$

$$K_- = \frac{d}{d\zeta}. \tag{2.35''}$$

They allow us to build up the entire space $\mathscr{F}^{E_0}$, starting from the "fundamental state" $|E_0, E_0\rangle = u_0(\zeta) = 1$:

$$K_+ u_n(\zeta) = [(n+1)(2E_0 + n)]^{1/2} u_{n+1}(\zeta) \tag{2.36}$$

$$K_- u_n(\zeta) = [n(2E_0 + n - 1)]^{1/2} u_{n-1}(\zeta). \tag{2.37}$$

All the above is well-known, and can be traced back to the works of Fock, Bargmann ($\mathscr{F}^{E_0}$ is a Fock-Bargmann space), Gelfand, Vilenkin ([5], [6], [7] *et al.*). It provides us with the ingredients for a theory of coherent states on the Lobachevskian plane, as developed by Perelomov. We shall re-examine this important point in the last part of this paper. Let us end the present review by listing some global features of the representation T^{E_0} [equation (2.26)]. Its matrix elements with respect to the orthonormal basis (2.33) are given by:

$$\begin{aligned} T^{E_0}_{nn'}(g) &= (u_n, T^{E_0}(g)\, u_{n'})_{E_0} \\ &= \left[\frac{n_>!\,\Gamma(2E_0 + n_>)}{n_<!\,\Gamma(2E_0 + n_<)}\right]^{1/2} \frac{h^{n_> - n_<}}{(n_> - n_<)!} \alpha^{n_<} (\bar{\alpha})^{-2E_0 - n_>} \\ &\quad \times {}_2F_1(-n_<, 2E_0 + n_>; n_> - n_< + 1; |\beta/\alpha|^2) \end{aligned} \tag{2.38}$$

where $n_{\gtrless} = \begin{matrix}\sup\\ \inf\end{matrix}(n, n')$ and $h = \begin{cases} \bar{\beta} \text{ if } n_{>} = n' \\ -\beta \text{ if } n_{>} = n \end{cases}$.

In the range $1/2 < E_0 < +\infty$, they are square integrable with respect to the Haar measure of SU(1, 1). The latter reads

$$dg = \frac{1}{2\pi^2}(1 - |\zeta|^2)^{-2}\, d\theta\, d^2\zeta, \qquad \zeta \in \mathcal{D}, \quad 0 \leqq \theta \leqq 2\pi, \tag{2.39}$$

in the Cartan parametrization (2.12):

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = p(\zeta) \exp i\theta\sigma_3, \qquad \zeta = \beta\bar{\alpha}^{-1}, \qquad \theta = \operatorname{Arg}\alpha.$$

More generally, they satisfy the orthogonality relations:

$$\int \overline{T^{E_0}_{n_1 n_2}(g)}\, T^{E_0}_{m_1 m_2}(g)\, dg = \frac{\delta_{n_1 m_1}\, \delta_{n_2 m_2}}{2E_0 - 1}, \tag{2.40}$$

which occur as a particular case of orthogonality relations holding for any representation of the so called discrete series [16]. It is precisely the discrete series which occurs here, for any real $E_0 > 1/2$, with the holomorphic discrete series occurring for the particular values: $E_0 = 1, 3/2, 2, 5/2, \ldots$

3. CONTRACTION PROCEDURE TOWARD POINCARÉ

Doing physics in de Sitter space means that we have at our disposal a universal length κ^{-1}. Any contraction process toward a flat space-time implies a rescaling of what we consider as lengths before contraction *because* they tend to lengths in our own "flatland". This applies to the global coordinates $\mathbf{x}$ and x_0 introduced in equation (2.2) and this will be the case for the $SO_0(2, 1)$ parameters $\mathbf{a}$ and a_0, that we are going to use instead of the ψ and θ appearing in equations (2.9):

$$\left.\begin{aligned} \psi &\equiv \kappa\mathbf{a}, \\ \theta &\equiv \kappa a_0. \end{aligned}\right\} \tag{3.1}$$

Now, following Inönü [17], we take the limit $\kappa \to 0$ of the product of matrices:

$$\Delta(\kappa) . \exp \kappa\, \mathbf{a}\, L_{31} . \exp \kappa\, a_0\, L_{12} . \exp \varphi\, L_{23} . \Delta^{-1}(\kappa), \tag{3.2}$$

$\Delta(\kappa)$ being the similitude (or "rescaling") matrix:

$$\Delta(\kappa) = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

the form of which is imposed by the asymptotic behavior of the coordinates u_i [equation (2.2)]: $u_1 \approx 1/\kappa$, $u_2 \approx x_0$, $u_3 = \mathbf{x}$. The results is the three-dimensional matrix representation of the Poincaré group $\mathscr{P}_+^\uparrow(1, 1)$:

$$(a, \Lambda) \equiv \begin{pmatrix} 1 & 0 & 0 \\ a_0 & \cosh\varphi & \sinh\varphi \\ \mathbf{a} & \sinh\varphi & \cosh\varphi \end{pmatrix}, \tag{3.4}$$

which acts on the column vectors $x \equiv {}^t(1, x_0, \mathbf{x})$. The contraction is said to be performed with respect to the Lorentz subgroup of $SO_0(2, 1)$ whose parameter is the unmodified de Sitterian "rapidity" φ.

What does the de Sitter phase space $SO_0(2, 1)/SO(2)$, or the set of sections (2.18), become under the contraction? Because we have in mind a Poincaré phase-space parametrized by $(\mathbf{q}, \mathbf{p})$, let us adopt a different notation in equation (3.4). First, replace $a = (a_0, \mathbf{a})$ by $q = (q_0, \mathbf{q})$. Secondly reparametrize $\Lambda \equiv \Lambda_p$ by a vector $p = (p_0, \mathbf{p})$ belonging to the forward mass hyperbola $\mathscr{V}_m^+ = \{(p_0, \mathbf{p}) \in \mathbb{R}^2 \mid p_0 > 0, p_0^2 - \mathbf{p}^2 = m^2\}$ (as in I):

$$\sinh\varphi = \frac{\mathbf{p}}{m}, \qquad \cosh\varphi = \frac{p_0}{m}. \tag{3.5}$$

Now the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ q_0 & \dfrac{p_0}{m} & \dfrac{\mathbf{p}}{m} \\ \mathbf{q} & \dfrac{\mathbf{p}}{m} & \dfrac{p_0}{m} \end{pmatrix} \tag{3.6}$$

should be the contracted version of a certain rescaled $SO_0(2, 1)$ matrix (a_{ij}). Hence, for small κ, its matrix elements behave as follows (*see* also the Appendix):

$$\begin{aligned} &a_{11} \sim 1, && a_{12}\,\text{finite}, && a_{13}\,\text{finite}, \\ &a_{21} \sim \kappa q_0, && a_{22} \sim \frac{p_0}{m}, && a_{23} \sim \frac{\mathbf{p}}{m} \\ &a_{31} \sim \kappa\mathbf{q}, && a_{32} \sim \frac{\mathbf{p}}{m}, && a_{33} \sim \frac{p_0}{m}. \end{aligned} \tag{3.7}$$

For the matrix $p(\mathbf{n})$, given by (2.18), it follows that

$$n_1 \sim \kappa\mathbf{q}, \qquad n_2 \sim \frac{\mathbf{p}}{m}, \qquad n_0 \sim \frac{p_0}{m} \tag{3.8}$$

and

$$\frac{n_1 n_2}{n_0 + 1} \sim \kappa q_0. \tag{3.8'}$$

The relation (3.8′) imposes a very specific value on the time-translation parameter q_0:

$$q_0 = \frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}. \tag{3.9}$$

Therefore, the contraction procedure has provided the left coset space $\Gamma_l \equiv \mathscr{P}_+^\uparrow(1, 1)/\mathrm{T}$, where T is the subgroup of time translations (notation borrowed from I), with the following Borel section $\beta : \Gamma_l \to \mathscr{P}_+^\uparrow(1, 1)$:

$$\beta(\mathbf{q}, \mathbf{p}) = \left(\left(\frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{q}\right), \Lambda_p\right) \tag{3.10}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{\mathbf{q}\cdot\mathbf{p}}{p_0+m} & \dfrac{p_0}{m} & \dfrac{\mathbf{p}}{m} \\ \mathbf{q} & \dfrac{\mathbf{p}}{m} & \dfrac{p_0}{m} \end{pmatrix}.$$

The space-like vector

$$q_s = \left(\frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{q}\right), \qquad q_s \cdot q_s = -\frac{2m}{p_0+m}\mathbf{q}^2, \tag{3.11}$$

enjoys remarkable properties (see *Fig.* 5). First,

$$\begin{aligned} \Lambda_p^{-1} q_s &= \left(-\frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{q}\right) \\ &\equiv \Theta\, q_s, \end{aligned} \tag{3.12}$$

where Θ denotes time inversion. Property (3.12) is characteristic of the section β and can actually be taken to be its definition. Thus an equivalent formulation would be to assert that β be the (unique) section which obeys the equation

$$[\beta(\mathbf{q}, \mathbf{p})]^{-1} = \beta(-\mathbf{q}, -\mathbf{p}). \tag{3.13}$$

It follows that $\beta(\mathbf{q}, \mathbf{p})$ admits an almost symmetrical factorization, on the left and on the right:

$$\begin{aligned} \beta(\mathbf{q}, \mathbf{p}) &= (q_s, \Lambda_p) = (q_s, \mathrm{I})(0, \Lambda_p) \\ &= (0, \Lambda_p)(\Theta\, q_s, \mathrm{I}). \end{aligned} \tag{3.14}$$

Next an arbitrary element $(q, \Lambda_p) \in \mathscr{P}_+^\uparrow(1, 1)$ may be factorized either as

$$(q, \Lambda_p) = (q_s', \Lambda_p)\left(\left(q_0 - \frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{0}\right), \mathrm{I}\right) \tag{3.15}$$

according to the left coset $\Gamma_l \equiv \mathscr{P}_+^\uparrow(1, 1)/\mathrm{T}$, or as

$$(q, \Lambda_p) = \left(\left(q_0 - \frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{0}\right), \mathrm{I}\right)(q_s, \Lambda_p) \tag{3.15'}$$

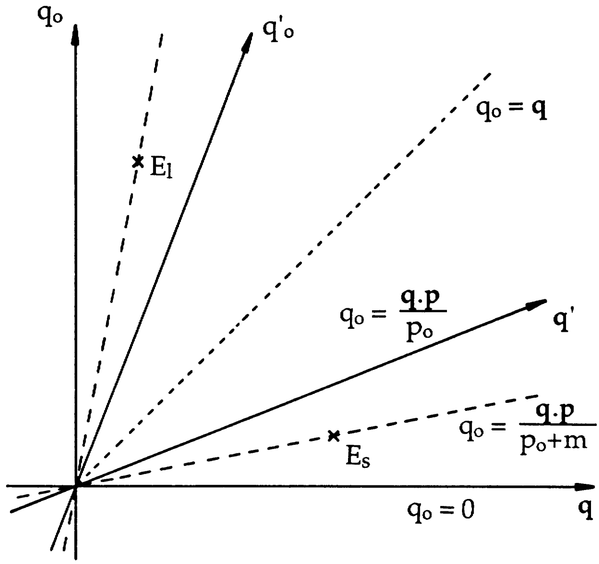

FIG. 5. — Space-time diagram for the section $\beta(\mathbf{q}, \mathbf{p}) = (q_s, \Lambda_p) \in \mathscr{P}_+^\uparrow(1, 1)$. Two Lorentz frames K and K′ are shown, K′ being the transform of K under the Lorentz transformation Λ_p^{-1}. For any such pair of frames, there exists one and only one "bisector" frame K_s which moves with opposite velocities with respect to K and K′: $u \equiv u_{K_s/K} = -u_{K_s/K'} = \dfrac{\mathbf{p}}{p_0 + m}$. The coordinates, with respect to K and K′, of any event E_1 located on the time-axis of K_s are related by $q_0' = q_0$, $\mathbf{q}' = -\mathbf{q}$, *i. e.* E_1 occurs simultaneously, but is oppositely located in K and K′. On the other hand, for any event E_s located on the space-axis of K_s, we have $q_0' = -q_0$ whereas $\mathbf{q}' = \mathbf{q}$. Since the space-axis of K_s is defined by the relation $q_0 = u\mathbf{q} = \dfrac{\mathbf{q}\cdot\mathbf{p}}{p_0 + m}$, we conclude that it is the geometrical locus of the space-like vector q_s defining the section $\beta(\mathbf{q}, \mathbf{p})$.

according to the right coset $\Gamma_r \equiv T \diagdown \mathscr{P}_+^\uparrow(1, 1)$. In the former case, $q_s' = \left(\dfrac{\mathbf{q}'\cdot\mathbf{p}}{p_0 + m}, \mathbf{q}'\right)$ and q' is built from q through the Lorentz boost Λ_p^{-1}:

$$\mathbf{q}' = \frac{1}{m}[p_0 \mathbf{q} - \mathbf{p}\, q_0]. \tag{3.16}$$

Therefore, the section (3.10) is valid for both Γ_l and Γ_r and points of both of them can be parametrized by $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$ according to β. The left and right actions of $\mathscr{P}_+^\uparrow(1, 1)$ on Γ_l and Γ_r, respectively, are then perfectly symmetrical unlike what was encountered in I. On Γ_l, the action is

$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}') = (a, \Lambda_k)(\mathbf{q}, \mathbf{p}) \tag{3.17}$$

where

$$\mathbf{q}' = \mathbf{q} + \frac{1}{m}[p_0\,\mathbf{a}' - \mathbf{p}\,a'_0], \tag{3.18}$$

with $a' = \Lambda_k^{-1} a$, and

$$\mathbf{p}' = \frac{1}{m}[k_0\,\mathbf{p} + \mathbf{k}\,p_0]. \tag{3.19}$$

Similarly, on Γ_r:

$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}'', \mathbf{p}'') = (\mathbf{q}, \mathbf{p})\,(a, \Lambda_k), \tag{3.20}$$

where now

$$\mathbf{q}'' = \mathbf{q} + \frac{1}{m}[p_0\,\mathbf{a} + \mathbf{p}\,a_0], \tag{3.21}$$

$$\mathbf{p}'' = \frac{1}{m}[k_0\,\mathbf{p} + \mathbf{k}\,p_0] = \mathbf{p}'. \tag{3.22}$$

It is then obvious that both Γ_l and Γ_r have the *same* invariant measure

$$d\mu(\mathbf{q}, \mathbf{p}) = d\mathbf{q} \wedge d\mathbf{p}/p_0 \tag{3.23}$$

with respect to (3.17) or (3.20). This fact could have been easily inferred through contraction from the (unique!) invariant measure (2.23) on the Lobachevskian plane in its version "$\mathscr{D}$" or "$\mathscr{L}_+$". A curvature-dependent $(\mathbf{q}, \mathbf{p})$ parametrization of these phase spaces is possible through equation (3.8) and equation (2.17).

$$n_0 = \frac{\pi_0}{m}, \qquad n_1 = \kappa\,\mathbf{q}, \qquad n_2 = \frac{\mathbf{p}}{m}, \tag{3.24}$$

$$\zeta = \frac{m\kappa\,\mathbf{q} + i\,\mathbf{p}}{\pi_0 + m}, \qquad |\zeta|^2 = \frac{\pi_0 - m}{\pi_0 + m}, \tag{3.25}$$

where π_0 depends on $(\mathbf{q}, \mathbf{p})$:

$$\begin{aligned} \pi_0 &= [m^2(1 + \kappa^2\,\mathbf{q}^2) + \mathbf{p}^2]^{1/2} \\ &= [p_0^2 + m^2\,\kappa^2\,\mathbf{q}^2]^{1/2}. \end{aligned} \tag{3.26}$$

The 2-form ω of equation (2.23) then reads

$$\omega = \kappa\, d\mathbf{q} \wedge d\mathbf{p}/\pi_0 \tag{3.27}$$

and gives by rescaling and contraction

$$\frac{1}{\kappa}\omega \underset{\kappa \to 0}{\to} d\mu(\mathbf{q}, \mathbf{p}). \tag{3.28}$$

It is also interesting to see what happens to the invariant metric ds^2, the other Kählerian attribute of $\mathscr{D}$. In terms of $\mathbf{q}$ and $\mathbf{p}$ it reads

$$ds^2 = \frac{1}{\pi_0^2}[\kappa^2 p_0^2\, d\mathbf{q}^2 + (1+\kappa^2 \mathbf{q}^2)\, d\mathbf{p}^2 - 2\kappa^2 \mathbf{q}.\mathbf{p}\, d\mathbf{q}\, d\mathbf{p}]. \tag{3.29}$$

We obtain, in the limit $\kappa \to 0$,

$$ds^2 = d\mathbf{p}^2/p_0^2. \tag{3.30}$$

We now turn to the task of contracting the infinitesimal generators (2.27)-(2.29) of the representation T^{E_0} introduced in Section 2. Once again, and as well known, a rescaling is necessary [17]:

$$\tilde{K}_0 = \kappa K_0, \qquad \tilde{K}_1 = \kappa K_1, \qquad \tilde{K}_2 = K_2. \tag{3.31}$$

The commutation rules (2.30) become:

$$[\tilde{K}_1, \tilde{K}_2] = -i\tilde{K}_0, \qquad [\tilde{K}_2, \tilde{K}_0] = i\tilde{K}_1, \qquad [\tilde{K}_0, \tilde{K}_1] = i\kappa^2 \tilde{K}_2 \tag{3.32}$$

and should yield the Poincaré commutation rules in the limit $\kappa \to 0$. However, things are not so straightforward, if we examine more carefully the limit of the operators (3.31), after changing the variable ζ into $(\mathbf{q}, \mathbf{p})$, according to (3.25). We then obtain, for small κ (*see* the Appendix for more details):

$$\tilde{K}_0 \approx i\frac{\mathbf{p}}{2m}\frac{\partial}{\partial \mathbf{q}} + m \equiv \tilde{P}_0 \tag{3.33}$$

$$\tilde{K}_1 \approx i\frac{p_0}{2m}\frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0+m} \equiv \tilde{P} \tag{3.34}$$

$$\tilde{K}_2 \approx i\frac{p_0}{2}\frac{\partial}{\partial \mathbf{p}} - \frac{m^2\mathbf{q}}{p_0+m} - \frac{i}{\kappa}\left[-\frac{i}{2}\frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0+m}\right] \equiv \tilde{K} - \frac{i}{\kappa}\tilde{\Pi}. \tag{3.35}$$

Two problems arise here: the first and more puzzling one is the presence of a singularity in the expression of $\tilde{K}_2$. To get rid of it we must impose the following condition on the space of functions $\tilde{f}(\mathbf{q}, \mathbf{p})$ carrying the contracted representation:

$$\tilde{\Pi}\,\tilde{f}(\mathbf{q}, \mathbf{p}) = 0. \tag{3.36}$$

Then we are left, and this is the second problem, with a nonstandard representation of $\mathscr{P}_+^\uparrow(1, 1)$, due to the embarrassing presence of a factor of 1/2 in the commutation rules:

$$[\tilde{K}, \tilde{P}_0] = \frac{i}{2}\tilde{P}, \qquad [\tilde{K}, \tilde{P}] = \frac{i}{2}\tilde{P}_0, \qquad [\tilde{P}_0, \tilde{P}] = 0 \tag{3.37}$$

[compare with the limit $\kappa \to 0$ of the rules (3.32)].

Actually we should not be so worried about this discrepancy. After all, the contraction procedure forces us to a change of representation space, by imposing on the original one a subsidiary condition (or polarisation condition in the Kirillov-Kostant-Souriau terminology), namely the condition (3.36). In a certain sense the latter parallels the disappearance of the real part of ζ when we too cavalierly take the limit $\kappa \to 0$ in equation (3.25). The operators $\tilde{\mathrm{K}}_0$, $\tilde{\mathrm{K}}_1$, $\tilde{\mathrm{K}}_2$ are defined on a space of analytic functions, and if there are some compensating terms on the right-hand side of their commutation rules, they will not survive after contraction. The limit space on which $\tilde{\mathrm{P}}_0$, $\tilde{\mathrm{P}}$, $\tilde{\mathrm{K}}$ and $\tilde{\Pi}$ act is genuinely singular. This point will be further clarified when we eventually identify it as the momentum space carrying the Wigner representation of $\mathscr{P}_+^\uparrow(1, 1)$ for mass $m>0$.

Now, to remove the factor 1/2, we rescale (again!) $\tilde{\mathrm{K}}$ and the variable $\mathbf{q}$:

$$2\mathbf{q} \to \mathbf{q}, \qquad 2\tilde{\mathrm{K}} \to \mathrm{K}. \tag{3.38}$$

Let us adopt the (definitive!) symbols:

$$\mathrm{P}_0 = i\frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{q}} + m \tag{3.39}$$

for the time-translation generator,

$$\mathrm{P} = i\frac{p_0}{m}\frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0+m} \tag{3.40}$$

for the space-translation generator,

$$\mathrm{K} = ip_0\frac{\partial}{\partial \mathbf{p}} - \frac{m^2\mathbf{q}}{\mathrm{p}_0+\mathrm{m}} \tag{3.41}$$

for the Lorentz boost generator,

$$\Pi = -i\frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0+m} \tag{3.42}$$

for what we shall call the "polarization operator". Their commutation rules look familar

$$[\mathrm{K}, \mathrm{P}_0] = i\mathrm{P}, \qquad [\mathrm{K}, \mathrm{P}] = i\mathrm{P}_0, \qquad [\mathrm{P}_0, \mathrm{P}] = 0, \tag{3.43}$$

$$[\Pi, \mathrm{K}] = [\Pi, \mathrm{P}_0] = [\Pi, \mathrm{P}] = 0. \tag{3.44}$$

The polarization condition coming from (3.36),

$$\Pi f(\mathbf{q}, \mathbf{p}) = 0, \tag{3.45}$$

is perfectly consistent with our original aim, namely reaching through a contraction the UIR $\mathscr{P}(m)$ of $\mathscr{P}_+^\uparrow(1, 1)$. The carrier space of the latter should be characterized by the Klein-Gordon condition:

$$(\mathrm{P}_0^2 - \mathrm{P}^2 - m^2) f(\mathbf{q}, \mathbf{p}) = 0. \tag{3.46}$$

Now the representation (3.39)-(3.42) has a remarkable feature:

$$\Pi^2 = -(P_0^2 - P^2 - m^2). \tag{3.47}$$

So the condition (3.45) is really what we need to describe our representation space. More precisely, the latter is made up of solutions to:

$$\left(-i\frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0+m}\right) f(\mathbf{q},\mathbf{p}) = 0 \tag{3.48}$$

namely,

$$\begin{aligned} f(\mathbf{q},\mathbf{p}) &= \exp\left(-im\frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}\right)\varphi(\mathbf{p}) \\ &= e^{iq_s\cdot p}\varphi(\mathbf{p}), \end{aligned} \tag{3.49}$$

where q_s is the section vector given by equation (3.11):

$$q_s = \left(q_0 = \frac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}, \mathbf{q}\right),$$
$$q\cdot p \equiv q_0 p_0 - \mathbf{q}\cdot\mathbf{p},$$

and $\varphi(\mathbf{p})$ is chosen to lie in $L^2(\mathscr{V}_m^+, d\mathbf{p}/p_0)$.

The operators of energy, P_0, and momentum, P, are diagonal in such a representation. Indeed, for a function f of the type (3.49):

$$P_0 f(\mathbf{q},\mathbf{p}) = p_0 f(\mathbf{q},\mathbf{p}), \tag{3.50}$$
$$\mathbf{P} f(\mathbf{q},\mathbf{p}) = \mathbf{p} f(\mathbf{q},\mathbf{p}). \tag{3.51}$$

On the other hand, since the exponential factor $e^{iq_s\cdot p}$ is "transparent" with respect to the action of the boost:

$$\mathbf{K} e^{iq_s\cdot p} = 0, \tag{3.52}$$

we simply have

$$\mathbf{K} f(\mathbf{q},\mathbf{p}) = e^{iq_s\cdot p} i p_0 \frac{d}{d\mathbf{p}}\varphi(\mathbf{p}). \tag{3.53}$$

The Poincaré global actions are then easily deduced from the above.

If we do ignore the phase factor $\exp iq_s.p$, *i. e.* if we just look at the Poincaré action on the functions φ, (3.50), (3.51) and (3.53) are clearly recognized as the original infinitesimal Wigner action on $L^2(\mathscr{V}_m^+, d\mathbf{p}/p_0)$:

$$P_0\varphi(\mathbf{p}) = p_0\varphi(\mathbf{p}), \tag{3.54}$$
$$\mathbf{P}\varphi(\mathbf{p}) = \mathbf{p}\, q(\mathbf{p}), \tag{3.55}$$
$$\mathbf{K}\varphi(\mathbf{p}) = ip_0\frac{d}{d\mathbf{p}}\varphi(\mathbf{p}), \tag{3.56}$$

or equivalently,

$$U_m(a,\Lambda_k)\varphi(\mathbf{p}) = e^{ia\cdot p}\varphi(\Lambda_k^{-1}\underline{p}), \tag{3.57}$$

where $\underline{k} \equiv \mathbf{k}$ is another notation for the space component of a vector k. So the polarization condition (3.45) definitely forbids the use of $L^2(\mathbb{R}^2, d\mathbf{q}\, d\mathbf{p}/p_0)$ as a representation space, reintroduces a wave equation, namely the Klein-Gordon equation (3.46) and, up to a phase factor, leads to the momentum version (3.57) of the Wigner representation $\mathscr{P}(m)$.

4. WEIGHTED POINCARÉ COHERENT STATES

The Perelomov construction of coherent states on the Lobachevskian plane $\mathscr{D} \cong SU(1, 1)/U(1)$ is based on the systematic identification (proper to quantum mechanics):

$$\varphi \equiv e^{i\alpha}\varphi, \qquad \varphi \in \mathscr{F}^{E_0}, \tag{4.1}$$

for any constant phase $\alpha \in \mathbb{R}$. So the objects considered are rays $|\varphi\rangle$ instead of simple elements φ in the Hilbert space. The second ingredient in the Perelomov construction is the existence in $\mathscr{F}^{E_0}$ of specific rays $|\varphi_0\rangle$ left invariant under the action of the representation operators $T^{E_0}(k)$, for any element k of the compact subgroup U(1):

$$T^{E_0}(k)|\varphi_0\rangle = |\varphi_0\rangle, \tag{4.2}$$

i. e. $T^{E_0}(k)\varphi_0 = e^{i\alpha(k)}\varphi_0$, $\forall k \in U(1)$. Examples of such states are provided by the basis elements (2.33). Indeed, we have

$$T^{E_0}(k)\, u_n = e^{i(E_0+n)\theta}\, u_n \tag{4.3}$$

for $k = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$.

A Perelomov coherent state is then defined by

$$\begin{aligned} |\zeta\rangle &= T^{E_0}(p(\zeta))|\varphi_0\rangle \\ &= T^{E_0}(g)|\varphi_0\rangle, \end{aligned} \tag{4.4}$$

where φ_0 obeys (4.2) and $p(\zeta)$ is the left factor in the Cartan decomposition (2.12), $g = p(\zeta)k$ of $g \in SU(1, 1)$. Hence the collection [the SU(1, 1)-orbit through φ_0] of coherent states $|\zeta\rangle$, when g runs throughout SU(1, 1), is labeled by points in $\mathscr{D}$.

A wide set of properties are displayed by such states: nonorthogonality, overcompleteness, existence of a reproducing kernel, minimization of certain inequalities, etc. We refer to reference [7] for a comprehensive inventory of them. Let us just mention the crucial one for our purposes:

$$\int_{\mathscr{D}} |\zeta\rangle\langle\zeta|\, d\mu_{E_0}(\zeta) = I \tag{4.5}$$

(overcompleteness or resolution of the identity), where

$$d\mu_{E_0}(\zeta) = \frac{2E_0 - 1}{\pi} \frac{d^2\zeta}{(1 - |\zeta|^2)^2}. \tag{4.6}$$

The proof of (4.5) stems from Schur's lemma, applied to the irreducible representation T^{E_0}. However, equation (4.5) can be considered as a direct consequence of orthogonality relations holding on the whole group SU(1, 1), for discrete-series representations:

$$\int_{SU(1,1)} \overline{(T^{E_0}(g)\varphi_1, \varphi_1')_{E_0}} (T^{E_0}(g)\varphi_2, \varphi_2')_{E_0} dg = (\varphi_2, \varphi_1)_{E_0} (\varphi_1', \varphi_2')_{E_0}, \tag{4.7}$$

where the form $(\ ,\ )_{E_0}$ and the Haar measure dg are given by (2.34) and (2.39) respectively. Note that equation (2.40) is a particular case of the above relations. Picking $\varphi_1 = \varphi_2 = \varphi_0$ and integrating over the U(1) parameter leads to the resolution of the identity (4.5).

The latter can be cast into a form seemingly more appropriate to contraction. We reintroduce the $(\mathbf{q}, \mathbf{p})$ variables (3.25) for $\mathscr{D}$ and the expression (3.27) for the 2-form ω. The result reads (with $m = \kappa E_0$).

$$\frac{m - \kappa/2}{2\pi} \int_{\mathbb{R}^2} |\mathbf{q}, \mathbf{p}\rangle \langle \mathbf{q}, \mathbf{p}| \frac{d\mathbf{q} \wedge d\mathbf{p}}{\pi_0} = I, \tag{4.8}$$

where $|\mathbf{q}, \mathbf{p}\rangle$ stands for $T^{E_0}(g)\varphi_0$, *i. e.* (*see* (4.4))

$$|\mathbf{q}, \mathbf{p}\rangle = T^{m/\kappa}(p(\mathbf{q}, \mathbf{p}))|\varphi_0\rangle. \tag{4.9}$$

We could now choose an appropriate φ_0 and perform a contraction on (4.8) to arrive at a parallel resolution of the identity for $\mathscr{P}_+^\uparrow(1, 1)$. However, as a cursory examination indicates, this is fraught with too many distracting technical problems.

Let us therefore stick to the original de Sitterian structure, namely the contracted phase-space $\mathscr{D} \to \mathscr{P}_+^\uparrow(1, 1)/T \equiv \Gamma$ and its particular section (3.10):

$$\beta(\mathbf{q}, \mathbf{p}) = (q_s, \Lambda_p).$$

Hence we pick an element φ in the Wigner representation space $\mathscr{H}_m = L^2(\mathscr{V}_m^+, d\mathbf{k}/k_0)$ and following I we study the square-integrability of the function $f_{\varphi, \varphi'} : \Gamma \to \mathbb{C}$ defined by:

$$f_{\varphi, \varphi'}(\mathbf{q}, \mathbf{p}) = \langle U_m(\mathbf{q}, \mathbf{p})\varphi | \varphi' \rangle, \qquad \varphi' \in \mathscr{H}_m. \tag{4.10}$$

Here $\langle . | . \rangle$ is the invariant form on $\mathscr{H}_m$:

$$\langle \varphi | \varphi' \rangle = \int_{\mathbb{R}} \overline{\varphi(\mathbf{k})} \varphi'(\mathbf{k}) d\mathbf{k}/k_0, \tag{4.11}$$

and $U_m(\mathbf{q}, \mathbf{p})$ is a shortened notation for $U_m((q_s, \Lambda_p))$. Note that we consider the elements of $\mathscr{H}_m$ as functions of the single variable $\mathbf{k}$.

The key integral to be investigated for constructing coherent states is the following one:

$$\begin{aligned} \mathrm{I}(\varphi, \varphi') = \int_{\Gamma} | f_{\varphi, \varphi'}(\mathbf{q}, \mathbf{p}) |^2 \, d\mathbf{q} \, d\mathbf{p}/p_0 \\ = \int_{\mathbb{R}^2} | \langle \mathrm{U}_m(\mathbf{q}, \mathbf{p}) \varphi \mid \varphi' \rangle |^2 \, d\mathbf{q} \, d\mathbf{p}/p_0 \\ = \int_{\mathbb{R}^4} d\mathbf{q} \frac{d\mathbf{p}}{p_0} \frac{d\mathbf{k}}{k_0} \frac{d\mathbf{k}'}{k_0'} e^{i\,[(\mathbf{k}-\mathbf{k}') - ((k_0 - k_0')\,\mathbf{p}/p_0 + m)]\,.\,\mathbf{q}} \\ \times \overline{\varphi(\underline{\Lambda}_p^{-1} \underline{k})} \varphi(\underline{\Lambda}_p^{-1} \underline{k}') \varphi'(\mathbf{k}) \overline{\varphi'(\mathbf{k}')}. \end{aligned} \quad (4.12)$$

Integrating over $\mathbf{q}$ leads to the delta function

$$\delta\left(\mathbf{k} - \mathbf{k}' - (k_0 - k_0') \frac{\mathbf{p}}{p_0 + m} \right) \equiv \delta(\mathrm{X}(\mathbf{k})). \quad (4.13)$$

X is a strictly increasing function of $\mathbf{k}$ and vanishes if and only if $\mathbf{k} = \mathbf{k}'$:

$$d\mathrm{X} = \frac{k.p + k_0 m}{p_0 + m} \frac{dk}{k_0}. \quad (4.14)$$

Next, performing the $d\mathbf{k}$ integration, changing $\mathbf{p}$ into $-\mathbf{p}$, using the fact that $\underline{\Lambda}_p \underline{k}' = \underline{\Lambda}_{k'} \underline{p}$ and the invariance of the measures, we obtain:

$$\mathrm{I}(\varphi, \varphi') = \frac{2\pi}{m^2} \iint_{\mathbb{R}^2} \frac{d\mathbf{k}}{k_0} \frac{d\mathbf{k}'}{k_0'} \left(\frac{k.k' + m^2}{k_0 + k_0'} \right) | \varphi(\mathbf{k}) |^2 | \varphi'(\mathbf{k}') |^2. \quad (4.15)$$

This integral is finite if φ and φ' are in the domain of the quadratic form associated to the energy operator P_0, *i.e.* $\varphi, \varphi' \in \mathrm{D}(\mathrm{P}_0^{1/2})$. Recall that

$$\mathrm{P}_0 \varphi(\mathbf{k}) = (\mathbf{k}^2 + m^2)^{1/2} \varphi(\mathbf{k}) = k_0 \varphi(\mathbf{k}). \quad (4.16)$$

More precisely, we have the following estimate

$$\mathrm{I}(\varphi, \varphi') \leqq \frac{\pi}{m^2} [\, \| \mathrm{P}_0^{1/2} \varphi \|^2 \, \| \varphi' \|^2 + \| \varphi \|^2 \, \| \mathrm{P}_0^{1/2} \varphi' \|^2 \,]. \quad (4.17)$$

This inequality should be compared to its SU(1, 1) counterpart

$$\int_{\mathbb{R}^2} | (\mathrm{T}^{m/\kappa}(\mathbf{q}, \mathbf{p}) \varphi, \varphi') |^2 \, d\mathbf{q} \frac{d\mathbf{p}}{\pi_0} = \frac{2\pi}{m - \kappa/2} \| \varphi \|^2 \, \| \varphi' \|^2, \quad (4.18)$$

where the equality is easily deduced from equation (4.8).

Inequality (4.17) clearly indicates that we lose one of the most attractive features of the coherent states, namely the resolution of the identity, if we define the latter in the canonical way [compare to equations (4.4) or (4.9)]:

$$\text{"coherent state"} = c_\varphi \mathrm{U}_m(\mathbf{q}, \mathbf{p}) \varphi \quad (4.19)$$

for a given $\varphi \in D(P_0^{1/2})$, c_φ being some normalisation constant.

Fortunately there exists a way to avoid the above difficulty. The calculation of the integral (4.15) has brought out the bounded positive symmetric kernel

$$\frac{2\pi}{m^2}\left(\frac{k.k'+m^2}{k_0+k_0'}\right). \tag{4.20}$$

Its appearance can be prevented by just "weighting" the action of U_m in Definition (4.19), by the adjunction of a multiplication operator $T(\mathbf{p})$. Thus we introduce the states

$$\varphi(\mathbf{q}, \mathbf{p}) \equiv \left(\frac{m}{2\pi}\right)^{1/2} T(\mathbf{p})\, U_m(\mathbf{q}, \mathbf{p})\,\varphi, \tag{4.21}$$

where $\varphi \in D(P_0^{1/2})$, the operator $T(\mathbf{p})$ is defined by:

$$T(\mathbf{p}) = \frac{1}{\sqrt{m}}\left(P_0 - \frac{\mathbf{p}.\mathbf{P}}{p_0+m}\right)^{1/2}, \tag{4.22}$$

and $\mathbf{P}$ is the momentum operator:

$$\mathbf{P}\,\varphi(\mathbf{k}) = \mathbf{k}\,\varphi(\mathbf{k}). \tag{4.23}$$

This definition makes sense, since the operator $\dfrac{1}{m}\left(P_0 - \dfrac{\mathbf{p}.\mathbf{P}}{p_0+m}\right) = T(\mathbf{p})^2$ is positive definite: its spectrum, for a given $p \in \mathscr{V}_m^+$, is the real interval $\left[\dfrac{2m}{p_0+m}, +\infty\right)$. More explicitly the state $\varphi_{(\mathbf{q}, \mathbf{p})}$ reads:

$$\varphi_{(\mathbf{q}, \mathbf{p})}(\mathbf{k}) = \left(\frac{m}{2\pi}\right)^{1/2}\left(\frac{(\Lambda_p^{-1}k)_0+k_0}{p_0+m}\right)^{1/2} \times \exp\left[-\frac{im}{p_0+m}(\underline{\Lambda}_p^{-1}k+\mathbf{k}).\mathbf{q}\right]\varphi(\underline{\Lambda}_p^{-1}\underline{k}) \tag{4.24}$$

The following statements about square-integrability on $\mathscr{V}_m^+$ and Γ respectively, then follow directly.

First we have the estimate in $\mathscr{H}_m$:

$$\|\varphi_{(\mathbf{q}, \mathbf{p})}\|^2 \leqq \frac{1}{2\pi}\left(1+\frac{|\mathbf{p}|}{p_0+m}\right)\|P_0^{1/2}\varphi\|^2. \tag{4.25}$$

Next it is trivial to derive the following result from the calculation of $I(\varphi, \varphi')$:

$$J(\varphi, \varphi') \equiv \int_\Gamma |\langle \varphi_{(\mathbf{q}, \mathbf{p})} | \varphi' \rangle|^2\, d\mathbf{q}\, d\mathbf{p}/p_0 = \|\varphi\|^2 \|\varphi'\|^2, \tag{4.26}$$

for all $\varphi \in D(P_0^{1/2})$. This equality allows one to establish:

THEOREM 4.1. – *If $\eta \in \mathcal{H}_m$ is in the domain of $P_0^{1/2}$ (or of the quadratic form associated to P_0), then the map W_η^β: $\mathcal{H}_m \to L^2(\Gamma, d\mathbf{q}\, d\mathbf{p}/p_0)$ given, for any $\varphi \in \mathcal{H}_m$, by the relation*

$$(W_\eta^\beta \varphi)(\mathbf{q}, \mathbf{p}) = \| \eta \|^{-1/2} \langle \eta_{(\mathbf{q}, \mathbf{p})} | \varphi \rangle \tag{4.27}$$

is an isometry. □

Borrowing again the terminology of I and [9], it can be said that any $\eta \in D(P_0^{1/2})$ is "admissible" and U_m is "square integrable" mod(T, β). Given an admissible η, we may consider its orbit under U_m:

$$\mathfrak{S}_\beta(\eta) = \{ | \mathbf{q}, \mathbf{p} \rangle_\eta \equiv \| \eta \|^{-1/2} \eta_{(\mathbf{q}, \mathbf{p})} \} \tag{4.28}$$

From the above, $\mathfrak{S}_\beta$ is overcomplete in $\mathcal{H}_m$ and moreover

$$\int_\Gamma | \mathbf{q}, \mathbf{p} >_{\eta\,\eta} < \mathbf{q}, \mathbf{p} | d\mathbf{q}\, d\mathbf{p}/p_0 = I_{\mathcal{H}_m}. \tag{4.29}$$

The family of vectors

$$\mathfrak{S}_\beta(T, \beta) = \bigcup_{\eta \in D(P_0^{1/2})} \mathfrak{S}_\beta(\eta) \tag{4.30}$$

will be called the set of *weighted Poincaré coherent states* on the Poincaré phase space Γ, for the section β.

The usual consequences follow from the resolution of the identity (4.29). For instance, the existence of a reproducing kernel

$$K_\eta: \quad \Gamma \times \Gamma \to \mathbb{C}, \tag{4.31}$$

such that

(i) $$K_\eta(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') = {}_\eta\langle \mathbf{q}, \mathbf{p} | \mathbf{q}', \mathbf{p}' \rangle_\eta; \tag{4.32}$$

(ii) $$\mathbb{P}_\eta \psi(\mathbf{q}, \mathbf{p}) = \int K_\eta(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') \psi(\mathbf{q}', \mathbf{p}')\, d\mathbf{q}'\, d\mathbf{p}'/p_0', \quad \text{for all} \quad \psi \in L^2(\Gamma, d\mathbf{q}, d\mathbf{p}/p_0), \tag{4.33}$$

and $\mathbb{P}_\eta \equiv W_\eta^\beta W_\eta^{\beta *}$ is the projection operator onto the subspace $\mathcal{H}_\eta$ of $L^2(\Gamma, d\mathbf{q}\, d\mathbf{p}/p_0)$ which is the image of $\mathcal{H}_m$ under W_η^β;

(iii) $$\int_\Gamma K_\eta(\mathbf{q}, \mathbf{p}; \mathbf{q}'', \mathbf{p}'') K_\eta(\mathbf{q}'', \mathbf{p}''; \mathbf{q}', \mathbf{p}')\, d\mathbf{q}''/p_0'' = K_\eta(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'). \tag{4.34}$$

Next we recover orthogonality relations similar to those given by (4.7) in the semi-simple case.

THEOREM 4.2. — *The following orthogonality relation holds for all* η_1, $\eta_2 \in D(P_0^{1/2})$ *and all* φ_1, $\varphi_2 \in \mathscr{H}_m$:

$$\int_\Gamma \overline{\langle \eta_{1\,(\mathbf{q},\,\mathbf{p})} | \varphi_1 \rangle} \langle \eta_{2\,(\mathbf{q},\,\mathbf{p})} | \varphi_2 \rangle \, d\mathbf{q}\, d\mathbf{p}/p_0 = \langle \eta_2 | \eta_1 \rangle \langle \varphi_1 | \varphi_2 \rangle \quad \square. \tag{4.35}$$

A Wigner transform may also be introduced. To do this, we denote by $\mathscr{B}_2(\mathscr{H}_m)$ the Hilbert space of all Hilbert-Schmidt operators on $\mathscr{H}_m$, with scalar product

$$\langle \rho_1 | \rho_2 \rangle_{\mathscr{B}_2} = tr(\rho_1^* \rho_2). \tag{4.36}$$

Then if $\rho_{\varphi,\,\eta}$ stands for the rank one (thus Hilbert-Schmidt) operator

$$\rho_{\varphi,\,\eta} = | \varphi \rangle \langle \eta |, \tag{4.37}$$

the orthogonality relations (4.35) can be recast into the form

$$\frac{m}{2\pi} \int_\Gamma \overline{tr[U_m(\mathbf{q},\,\mathbf{p})^* T(\mathbf{p}) \rho_{\varphi_1,\,\eta_1}]} \times tr[U_m(\mathbf{q},\,\mathbf{p})^* T(\mathbf{p}) \rho_{\varphi_2,\,\eta_2}] \, d\mathbf{q}\, d\mathbf{p}/p_0 = tr[(\rho_{\varphi_1,\,\eta_1})^* \rho_{\varphi_2,\,\eta_2}] = \langle \rho_{\varphi_1,\,\eta_1} | \rho_{\varphi_2,\,\eta_2} \rangle_{\mathscr{B}_2} \tag{4.38}$$

for all η_1, η_2 in $D(P_0^{1/2})$.

The domain $D(P_0^{1/2})$ is dense in $\mathscr{H}_m$. It follows that the linear space of all $\rho_{\varphi,\,\eta}$, $\eta \in D(P_0^{1/2})$, is dense in $\mathscr{B}_2(\mathscr{H}_m)$. Next let us define a map:

$$W: \ \{\text{linear span of all } \rho_{\varphi,\,\eta}\} \to L^2(\Gamma;\, m/2\pi\, d\mathbf{q}\, d\mathbf{p}/p_0)$$

by the relation

$$(W\rho)(\mathbf{q},\,\mathbf{p}) = tr[U_m(\mathbf{q},\,\mathbf{p})^* T(\mathbf{p}) \rho]. \tag{4.39}$$

Then W is a linear isometry, which can be extended by continuity to the whole of $\mathscr{B}_2(\mathscr{H}_m)$.

$$W: \ \mathscr{B}_2(\mathscr{H}_m) \to L^2(\Gamma;\, m/2\pi\, d\mathbf{q}\, d\mathbf{p}/p_0) \tag{4.40}$$

will be called the *Wigner transform map* and $W\rho$ is the *Wigner transform* of ρ. The range $\mathscr{R}_w$ of the linear map W is dense in $L^2(\Gamma;\, m/2\pi\, d\mathbf{q}\, d\mathbf{p}/p_0)$ and coincides with the domain of the operator $\hat{H}_0$:

$$(\hat{H}_0 \Phi)(\mathbf{q},\,\mathbf{p}) = (\mathbf{p}^2 + m^2)^{1/2} \Phi(\mathbf{q},\,\mathbf{p}). \tag{4.41}$$

5. CONCLUSION

A representation $D_+(E_0)$ of $SO_0(2, 1)_+$, as displayed in (2.26), belongs to the discrete series and hence admits coherent states in the sense of Perelomov. On the other hand, the representation $\mathscr{P}(m)$ of $\mathscr{P}_+^\uparrow(1, 1)$ which is the contracted version of $D_+(E_0)$ is not a discrete series representation [there are none for $\mathscr{P}_+^\uparrow(1, 1)$]. Hence $\mathscr{P}(m)$ does not admit Perelomov type coherent states. As was pointed out, and explicitly demonstrated in I, an extended notion of square integrability of a group representation enables one to construct physically interesting coherent states for $\mathscr{P}_+^\uparrow(1, 1)$.

Their construction is, however, dependent on the judicious choice of a section. One such choice was made in I. In this paper we looked at a second possible choice of a section, namely the one given in (3.10). While this in no way exhausts all possibilities for obtaining sections, the use of (3.10) demonstrates how square integrability features of the representation $D_+(E_0)$ persist in the transition to $\mathscr{P}(m)$ by the contraction procedure. It is also indicative of the reason behind the existence of a more general notion of square integrability for the representations of $\mathscr{P}_+^\uparrow(1, 1)$.

It ought to be pointed out, however, that the introduction of the notion of a weighted coherent state in (4.21) is new in this context. It enabled us to obtain a resolution of the identity (4.29), without any further conditions on the resolution generator (analyzing vector) η, unlike in I, where coherent states without weighting – but with additional admissibility conditions on η – had been constructed. Weighted coherent states could also have been introduced in I, and results completely analogous to the present ones could have been obtained. As a matter of fact, as we shall demonstrate in a succeeding paper [18], there exists a whole family of sections of $\mathscr{P}_+^\uparrow(1, 1)$ for which similar notions can be introduced. This also seems to indicate that the existence of orthogonality conditions on homogeneous spaces, of the type (4.38), is generic and not just limited to the Poincaré group.

APPENDIX

In this appendix, we examine in more details the effect of the contraction $\kappa \to 0$ on the commutation rules (2.30) of the de Sitter Lie algebra. We keep using boldface letters to denote space components of the 2-vectors. Indeed, it turns out [19] that a substantial part of the formulas obtained here admit a straightforward generalization to the $(1+3)$-dimensional case, when one performs a similar contraction from $SO_0(2, 3)$ toward $\mathscr{P}_+^\uparrow(1, 3)$.

First of all, we define the dimensionless parameter $\xi = \kappa/m$. Writing properly the matrix elements a_{ij} of (3.7) to the first order in ξ, we obtain for the section matrix $p(\mathbf{n})$:

$$\begin{aligned} n_1 &= \xi m \mathbf{q} + \xi^2 \mathrm{F}_1(\mathbf{q}, \mathbf{p}), \\ n_2 &= \frac{\mathbf{p}}{m} + \xi \mathrm{F}_2(\mathbf{q}, \mathbf{p}), \\ n_0 &= \frac{p_0}{m} + \xi \mathrm{F}_0(\mathbf{q}, \mathbf{p}), \end{aligned} \tag{A.1}$$

together with the condition $q_0 = \dfrac{\mathbf{q}\cdot\mathbf{p}}{p_0+m}$, where F_i $(i=0, 1, 2)$ are differentiable functions of $\mathbf{q}$ and $\mathbf{p}$, analytic in ξ in a neighborhood of $\xi=0$. Let us write

$$\left.\begin{aligned} \mathrm{F}_i(\mathbf{q}, \mathbf{p}) = f_{i0}(\mathbf{q}, \mathbf{p}) + \xi f_{i1}(\mathbf{q}, \mathbf{p}) + O(\xi^2) \\ (i=0, 1, 2). \end{aligned}\right\} \tag{A.2}$$

Then the relation $n_0^2 - n_1^2 - n_2^2 = 1$ imposes the constraint

$$p_0 f_{00} = \mathbf{p} f_{20}. \tag{A.3}$$

Defining the function $h(\mathbf{q}, \mathbf{p}) = m/p_0 f_{20}(\mathbf{q}, \mathbf{p})$, the complex variable $\zeta \in \mathscr{D}$ reads, according to (2.17):

$$\zeta = \zeta_1 + i\zeta_2,$$

with

$$\begin{aligned} \zeta_1 &= \xi \frac{m^2 \mathbf{q}}{p_0+m}, \\ \zeta_2 &= \frac{\mathbf{p}}{p_0+m} + \xi \frac{m h(\mathbf{q}, \mathbf{p})}{p_0+m}. \end{aligned} \tag{A.4}$$

The arbitrary function h is to be chosen in such a way that the commutation relations of the rescaled $\mathrm{SO}_0(2, 1)$ generators $\tilde{\mathrm{K}}_0$, $\tilde{\mathrm{K}}_1$, $\tilde{\mathrm{K}}_2$ go over, as $\xi \to 0$, to those of the Poincaré Lie algebra. Going back to the $\mathbf{q}$, $\mathbf{p}$ parametrization, we obtain, to the zeroth order in ξ:

$$\frac{\partial}{\partial \zeta} = \xi^{-1} \frac{p_0+m}{2m^2} \frac{\partial}{\partial \mathbf{q}} - \frac{1}{2m^2}\left(\frac{\partial h}{\partial \mathbf{q}} + im\right)\left[\mathbf{p}\mathbf{q}\frac{\partial}{\partial \mathbf{q}} + p_0(p_0+m)\frac{\partial}{\partial \mathbf{p}}\right]. \tag{A.5}$$

Inserting (A.4) and (A.5) into (2.27)-(2.29), we obtain (writing as usual $\kappa \mathrm{E}_0 = m$) to the zeroth order in ξ:

$$\tilde{\mathrm{K}}_0 = m\xi \mathrm{K}_0 \approx i \frac{\mathbf{p}}{2m} \frac{\partial}{\partial \mathbf{q}} + m \equiv \tilde{\mathrm{P}}_0, \tag{A.6.a}$$

$$\tilde{\mathrm{K}}_1 = m\xi \mathrm{K}_1 \approx i \frac{p_0}{2m} \frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0 + m} \equiv \tilde{\mathrm{P}}, \tag{A.6.b}$$

$$\tilde{\mathrm{K}}_2 = \mathrm{K}_2 \approx \tilde{\mathrm{K}} - \frac{i}{m\xi} \tilde{\Pi}, \tag{A.6.c}$$

with

$$\tilde{\mathrm{K}} = \left(\frac{\partial h}{\partial \mathbf{q}} + im \right) \frac{p_0}{2m} \frac{\partial}{\partial \mathbf{p}} - \frac{\mathbf{p}(\mathbf{q}(\partial h/\partial \mathbf{q}) - ih)}{2m(p_0 + m)} - \frac{m(m\mathbf{q} + ih)}{p_0 + m} \tag{A.7.a}$$

$$\tilde{\Pi} = -\frac{i}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{m\mathbf{p}}{p_0 + m}. \tag{A.7.b}$$

To get a finite result from (A.6.c), we are forced to impose on the functions $\tilde{f}(\mathbf{q}, \mathbf{p})$ in the representation space the polarization constaint:

$$\tilde{\Pi} \tilde{f}(\mathbf{q}, \mathbf{p}) = 0. \tag{A.8}$$

This yields:

$$\tilde{\mathrm{P}}_0 = p_0, \qquad \tilde{\mathrm{P}} = \mathbf{p}, \tag{A.9.a}$$

$$\tilde{\mathrm{K}} = i \frac{p_0}{2} \frac{\partial}{\partial \mathbf{p}} - \frac{m^2 \mathbf{q}}{p_0 + m} + \frac{\partial h}{\partial \mathbf{q}} \frac{p_0}{2m} \frac{\partial}{\partial \mathbf{p}} - \frac{p_0 - m}{p_0 + m} \left(h + i\mathbf{q} \frac{\partial h}{\partial \mathbf{q}} \right). \tag{A.9.b}$$

Next we rescale $2\mathbf{q} \to \mathbf{q}$, $2\tilde{\mathrm{K}} \to \mathrm{K}$, but we recover the Poincaré commutation relations *only* if $\partial h/\partial \mathbf{q} = 0$, *i.e.* $h = h(\mathbf{p})$, and then:

$$\mathrm{K} = i p_0 \frac{\partial}{\partial \mathbf{p}} - \frac{m^2 \mathbf{q}}{p_0 + m} - \frac{p_0 - m}{p_0 + m} 2h(\mathbf{p}). \tag{A.10}$$

So, finally, on functions of the form (3.49), $f(\mathbf{q}, \mathbf{p}) = e^{i\, q_s \cdot p} \varphi(\mathbf{p})$, the boost generator K acts as:

$$\mathrm{K} f(\mathbf{q}, \mathbf{p}) = e^{i\, q_s \cdot p} \left[i p_0 \frac{d}{d\mathbf{p}} + \mathrm{F}(\mathbf{p}) \right] \varphi(\mathbf{p}), \tag{A.11}$$

where $\mathrm{F}(\mathbf{p})$ is an arbitrary function of $\mathbf{p}$ — which we may safely put equal to zero.

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REFERENCES

[1] S. T. ALI and J.-P. ANTOINE, *Ann. Inst. H. Poincaré*, Vol. **50**, 1989, n° 4.

[2] H. BACRY and J. M. LÉVY-LEBLOND, *J. Math. Phys.*, Vol. **9**, 1968, p. 1605.

[3] E. P. WIGNER, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. **36**, 1950, p. 184. C. FRONSDAL, *Rev. Mod. Phys.*, Vol. **37**, 1965, p. 221. J.-P. GAZEAU and M. HANS, *J. Math. Phys.*, Vol. **29**, 1988, p. 2533.

[4] E. P. WIGNER, *Ann. Math.*, Vol. **40**, 1939, p. 149. V. BARGMANN, *Ann. Math.*, Vol. **59**, 1954, p. 1.

[5] I. M. GEL'FAND, M. I. GRAEV and N. Ya. VILENKIN, *Generalized Functions*, Vol. **5**, *Integral Geometry and Representation Theory*, Academic Press, N. Y., 1966.

[6] N. Ya. VILENKIN, *Special Functions and the Theory of Group Representations*, (*Transl. RussianMath. Monographs*, Vol. **22**), Amer. Math. Soc., Providence, R.I., 1968.

[7] A. PERELOMOV, *Generalized Coherent States and their Applications*, Springer-Verlag, Berlin, 1986.

[8] T. O. PHILIPS and E. P. WIGNER, in *Group Theory and its Applications*, E. M. LOEBL Ed., Academic Press, New York, 1968.

[9] S. DE BIÈVRE, *J. Math. Phys.*, Vol. **30**, 1989, p. 1401.

[10] A. A. KIRILLOV, *Elements of the Theory of Representations*, Springer-Verlag, Berlin, 1976.

[11] H. BACRY and A. KIHLBERG, *J. Math. Phys.*, Vol. **10**, 1969, p. 2132.

[12] R. ARENS, *Commun. Math. Phys.*, Vol. **21**, 1971, pp. 125, 139, and *J. Math. Phys.*, Vol. **12**, 1971, p. 2415.

[13] H. BACRY, *Localizability and Space in Quantum Physics*, (*Lecture Notes in Physics*, Vol. **308**), Springer-Verlag, Berlin, 1988.

[14] B. DOUBROVINE, S. NOVIKOV and A. FOMENKO, *Géométrie Contemporaine, Méthodes et Applications* (Première partie), Mir, Moscou, 1982.

[15] A. and J. UNTERBERGER, *Ann. Scient. Ec. Norm. Sup.*, Vol. **17**, 1984, p. 83.

[16] R. L. LIPSMAN, *Group Representations*, (*Lecture Notes in Mathematics*, Vol. **388**), Springer-Verlag, Berlin, 1974,

[17] E. INÖNÜ and E, P, WIGNER, *Proc. Nat. Acad. Sci. U.S.A.*, Vol, **39**, 1953, p. 510. E. INÖNÜ, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, F. GÜRSEY Ed,, p, 365, Gordon and Breach, N. Y., 1964.

[18] S. T. ALI, J.-P. ANTOINE and J.-P. GAZEAU, *Square Integrability of Group Representations on Homogeneous Spaces I, II* (in preparation).

[19] R. BALBINOT, A. ELGRADECHI, J.-P. GAZEAU and B. GIORGINI, *Phase Spaces for de Sitterian and Einsteinian Quantum Elementary Systems* (in preparation).

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