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The initial value problem of the Poincaré gauge theory in vacuum. II. First order formalism

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First order formalism

by

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ABSTRACT. — The exterior initial value problem of the Poincaré gauge theory is studied in the first order formalism. The Cauchy-Kowalevski conditions on the ten coupling constants of the theory found in paper I are verified. The equations of Poincaré gauge theory take the form of a symmetric hyperbolic system if, and only if, the hyperbolicity conditions found in paper I are satisfied.

RÉSUMÉ. — Nous considérons le problème des données initiales pour la jauge de Poincaré dans le formalisme du premier ordre. Nous vérifions les conditions de Cauchy-Kowalevski sur les dix constantes de couplages introduites dans l'article I. Les équations de la jauge de Poincaré prennent la forme d'un système hyperbolique symétrique si et seulement si les conditions d'hyperbolicité trouvées dans l'article I sont satisfaites.

1. INTRODUCTION

In a previous paper [1], hereafter denoted by I, we studied the exterior initial value problem of Poincaré gauge theory (PGT) [2] in the second

order formalism. Our prototype was the work of Y. Bruhat [3] on the initial value problem of general relativity. We found there sufficient conditions (called *Cauchy-Kowalevski conditions*) on the ten coupling constants of PGT, under which the Cauchy-Kowalevski theorem can be applied in the field equations. We also obtained sufficient conditions on the same constants (called *hyperbolicity conditions*), so that gauge conditions, similar to the harmonic gauge in general relativity, can be found in which the field equations are hyperbolic. We failed to show necessity for both sets of conditions. As mentioned already in I, to prove necessity for the C-K conditions one has to study separately all possible theories arising from their violation. This is hard to do not only because of the big number of possibilities (we found 10 C-K conditions), but also because no standard methods for such an undertaking exist. It is surprising that although the C-K conditions are weak conditions of the form (*linear combination of the coupling constants*) $\neq 0$, they are violated by most of the theories already proposed in the literature (*see* [4] for references).

The hyperbolicity conditions are obtained from the requirement, that the second order terms of the field equations take the form $A \Box u + B \partial F$, $F = C \partial u$, where u denotes the field variables: orthonormal tetrad and connection and A, B, C are matrices depending on the tetrad. $F = 0$ is the generalization of the harmonic gauge. It seems restrictive to require the fild equations of PGT to take this special form motivated from general relativity. Unfortunately general criteria for hyperbolicity of second order equations existing in the mathematic literature ([5], [6]) do not apply directly on our object. Another possibility is to apply K. O. Friedrichs theory of symmetric hyperbolic systems. This theory already applied in general relativity and other gauge theories by A. E. Fischer and J. E. Marsden [7] and H. Friedrich [8] uses first order formalism.

In the present paper we formulate PGT as a system of first order equations in tetrad, connection, torsion and curvature and find conditions under which this system becomes symmetric hyperbolic. The system of equations we study consists of the structure equations, the Bianchi identities and the actual field equations. Using the "time gauge" [9] with respect to the hypersurface S of initial data, we eliminate the gauge degrees of freedom and decompose the field variables and equations in components orthogonal and tangent to S. Since the resulting objects are three dimensional we express the $(3 + 1)$-decomposed equations in the calculus of 3-vectors. We finally obtain a system of first order equations for 84 independent variables. It decomposes into 84 evolution equations and 46 constraints. Using the integrability conditions of the system, we show that the 46 constraints are preserved in time. To apply the Cauchy-Kowalevski theorem on the system of evolution equations the coefficient matrix of the time derivatives of the variables must be invertible. We express the

system in terms of traces, the antisymmetric parts and the trace free, symmetric parts of the tensors appearing in the equations to make this matrix diagonal. The ten C-K condition found in I follow now immediately.

A first order system of differential equations is symmetric hyperbolic in the sence of K. O. Friedrichs, if the coefficient matrices of the derivatives of the variables are symmetric and the coefficient matrix of the time derivatives is additionally positive definite. Our system does not satisfy these conditions. We can modify it by taking combinations of its equations and the constraints. We find that it can be made symmetric hyperbolic if, and only if the hyperbolicity conditions obtained in I are satisfied. Thus we succeed to prove that these conditions are necessary.

The method used here required lengthy and cumbersome calculations. One can recognise that from the unavoidably lengthy lists of equations that follow. We tried to give only these equations, which we thing are absolutely necessary to follow the reasoning without much effort.

2. FIELD EQUATIONS OF PGT IN VACUUM

The Lagrangian of the Poincaré Gauge Theory (PGT) is composed of ten terms, which are at most quadratic in the field strengths: torsion Θ^i and curvature $\Omega^i{}_j$. We write it in the form

$$\mathcal{L} = c_0\,\varepsilon + c_1\,\Omega_{ij} \wedge * (\vartheta^i \wedge \vartheta^j) + \frac{1}{2} a_{ij\,|\,kl}(\vartheta^i \wedge \Theta^j) \wedge * (\vartheta^k \wedge \Theta^l)$$

$$+ \frac{1}{2} b_{ijkl\,|\,mnrs}(\vartheta^i \wedge \vartheta^j \wedge \Omega^{kl}) \wedge * (\vartheta^m \wedge \vartheta^n \wedge \Omega^{rs}). \quad (1)$$

ϑ^i denotes the orthonormal tetrad field and

$$\varepsilon := \frac{1}{4!}\varepsilon_{ijkl}\,\vartheta^i \wedge \vartheta^j \wedge \vartheta^k \wedge \vartheta^l$$

is the volume four form. In terms of the tetrad and the connection one form $\omega^i{}_j$ torsion and curvature are given by the structure equations

$$\Theta^i := d\,\vartheta^j + \omega^i{}_j \wedge \vartheta^j \quad\quad\quad\quad (2)$$

and

$$\Omega^i{}_j := d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \quad\quad\quad\quad (3)$$

c_0, c_1 are coupling constants and the two invariant tensors $a_{ij\,|\,kl}$, $b_{ijkl\,|\,mnrs}$ are given in terms of eight more coupling constants $\bar{p}, \bar{q}, \bar{r}$ and $p, q_1, q_2,$

r_1, r_2 in the form

$$a_{ij\,|\,kl} := \frac{\bar{p}+\bar{q}+\bar{r}}{4}\,\eta_{ik}\,\eta_{jl} - \bar{q}\,\eta_{ij}\,\eta_{kl} - \bar{r}\,\eta_{il}\,\eta_{jk} \qquad (4\,a)$$

and

$$b_{ijkl\,|\,mnrs} := \frac{1}{8}\,\delta^{ab}_{ij}\,\delta^{cd}_{kl}\,\delta^{ef}_{mn}\,\delta^{gh}_{rs}\left(\frac{p+q_1+r_1}{4}\,\eta_{ae}\,\eta_{fb}\,\eta_{cg}\,\eta_{hd}\right.$$

$$+ \frac{q_2-q_1+r_1-r_2}{4}\,\eta_{ag}\,\eta_{hb}\,\eta_{ce}\,\eta_{fd} - (q_1+r_2)\,\eta_{ac}\,\eta_{de}\,\eta_{fg}\,\eta_{hb}$$

$$\left.+\, q_1\,\eta_{ae}\,\eta_{fg}\,\eta_{hc}\,\eta_{db} + r_1\,\eta_{ag}\,\eta_{hc}\,\eta_{de}\,\eta_{fb}\right). \qquad (4\,b)$$

where

$$\delta^{ab}_{ij} := \delta^a_{[i}\,\delta^b_{j]} := \delta^a_i\,\delta^b_j - \delta^a_j\,\delta^b_i.$$

The Lagrangian as given in (1) contains the least possible number of star operators. We made this choice because in the variation process it is $*$ that gives the most complicated expressions.

An *orthonormal* tetrad field ϑ^i and a *metric compatible* connection $\omega^i_{\ j}$ are the dynamical variables of PGT ([1]). In the first step of the variation we set

$$\delta\mathcal{L} = \delta\,\vartheta^a\big|_{\Theta,\,\Omega}\wedge q_a + \delta\,\Theta^a\big|_{\vartheta,\,\Omega}\wedge T_a + \delta\Omega^{ab}\big|_{\vartheta,\,\Theta}\wedge W_{ab}. \qquad (5)$$

Using the structure equations we find

$$\delta\Theta^a = D\,\delta\,\vartheta^a + \delta\omega^{ab}\wedge\vartheta_b, \qquad (6\,a)$$

and

$$\delta\Omega^{ab} = D\,\delta\omega^{ab}, \qquad (6\,b)$$

where D denotes the exterior covariant derivative associated to $\omega^i_{\ j}$. Substituting these expressions in (5) we obtain

$$\delta\mathcal{L} = \delta\,\vartheta^a\wedge p_a + \delta\omega^{ab}\wedge c_{ab} + (\text{exact form}), \qquad (7)$$

with

$$p_a = DT_a + q_a, \qquad (8\,a)$$

$$c_{ab} = DW_{ab} - \frac{1}{2}\vartheta_{[a}\wedge T_{b]}. \qquad (8\,b)$$

The vacuum field equations of PGT are

$$p_a = 0, \qquad c_{ab} = 0. \qquad (9)$$

([1]) The Minkowski metric has here the signature $(-1, +1, +1, +1)$.

Since $\mathscr{L}$ is a scalar four form application of Noether's theorem gives two differential identities

$$Dp_a = (e_a \lrcorner \Theta^b) \wedge p_b + (e_a \lrcorner W^{bc}) \wedge c_{bc}, \qquad (10\,a)$$

$$Dc_{ab} = \frac{1}{2} \vartheta_{[a} \wedge p_{b]} \qquad (10\,b)$$

and an algebraic one

$$q_a = e_a \lrcorner \mathscr{L} - (e_a \lrcorner \Theta^b) \wedge T_b - (e_a \lrcorner \Omega^{bc}) \wedge W_{bc}. \qquad (10\,c)$$

From equations (1) and (5) we obtain

$$T_a = a_{ai\,|\,jk} \, \vartheta^i \wedge *(\vartheta^k \wedge \Theta^j), \qquad (11\,a)$$

$$W_{ab} = c_1 * (\vartheta_a \wedge \vartheta_b) + b_{abij\,|\,klmn} \, \vartheta^i \wedge \vartheta^j \wedge *(\vartheta^m \wedge \vartheta^n \wedge \Omega^{kl}). \qquad (11\,b)$$

We call these *constitutive equations* of PGT.

3. (3 + 1)-DECOMPOSITION AND GAUGE CONDITIONS

Let S denote a spacelike hypersurface given by $t(x) = 0$. Using Gaussian coordinates with respect to S the metric takes the form

$$g = -dt^2 + g_{A'B'} \, dx^{A'} dx^{B'}. \qquad (12)$$

This form of the metric implies that we can take the tetrad field to be

$$\vartheta^0 = dt, \qquad \vartheta^A = \vartheta^A_{A'} \, dx^{A'} \qquad (13)$$

with

$$g_{A'B'} = \delta_{AB} \, \vartheta^A_{A'} \, \vartheta^B_{B'}. \qquad (14)$$

A, B, ... = 1, 2, 3 denote anholonomic and A', B', ... = 1, 2, 3 denote holonomic indices. The dual tetrad is given by

$$e_0 = \partial_t, \qquad e_A = e_A^{A'} \partial_{A'}, \qquad (15)$$

where $e_A^{A'}$ is the inverse of $\vartheta^A_{A'}$.

The choices made above fix some of the gauge degrees of freedom of the theory. The use of Gaussian coordinates fixes obviously the diffeomorphisms and the choice (13) of the tetrad fixes additionally the boost part of the local Lorentz transformations. It is used to call this gauge fixing the *time gauge* [9]. The remaining local Lorentz gauge freedom will be fixed after the 3 + 1 decomposition of the connection is defined.

For an arbitrary form ψ we set

$$\overset{\perp}{\psi} := e_0 \lrcorner \psi = \partial_t \lrcorner \psi \qquad (16)$$

and

$$\underline{\psi} := \psi - \vartheta^0 \wedge \overset{\perp}{\psi} = \psi - dt \wedge \overset{\perp}{\psi}. \qquad (17)$$

Both ψ and $\overset{\perp}{\psi}$ are differential forms defined on S. Using these operations we find from (13)

$$\underline{\vartheta}^i = \delta^i{}_0, \qquad \underline{\vartheta}^i = \delta^i{}_A \vartheta^A. \tag{18}$$

For the $3+1$ decomposition of the connection we set following [10]

$$\overset{\perp}{\omega}{}^0{}_A = a_A, \qquad \underline{\omega}{}^0{}_A = -K_A. \tag{19 a}$$

We use now the remaining local Lorentz gauge freedom to set

$$\overset{\perp}{\omega}{}^A{}_B = 0. \tag{19 b}$$

To decompose the field equations of PGT as in (16), (17) we need to know how the exterior product, the star operator and the exterior derivative behave under these operations. We find for all forms ψ and r-forms φ_r

$$\underline{\varphi_r \wedge \psi} = \underline{\varphi}_r \wedge \underline{\psi}, \tag{20 a}$$

$$(\varphi_r \wedge \psi)^\perp = \underline{\varphi}_r \wedge \overset{\perp}{\psi} + (-1)^r \overset{\perp}{\underline{\varphi}}_r \wedge \underline{\psi}, \tag{20 b}$$

and

$$\underline{d\psi} = \underline{d}\,\underline{\psi}, \tag{21 a}$$

$$(d\psi)^\perp = \dot{\underline{\psi}} - \underline{d}\overset{\perp}{\psi} \tag{21 b}$$

and

$$\underline{\ast\psi} = -\underline{\ast}\overset{\perp}{\psi}, \tag{22 a}$$

$$(\ast\varphi_r)^\perp = (-1)^r \underline{\ast}\,\underline{\varphi}_r. \tag{22 b}$$

The symbols used above have the following meaning: $\underline{d}$ denotes the exterior derivative on S, a dot denotes the Lie derivative along e_0 and $\underline{\ast}$ is the star operator on S defined with respect to $g_{A'B'}$. Two further operations are necessary in order to write the decomposed field equations in a compact form. We set

$$\overset{\ast}{\underline{\psi}} := \underline{\ast}\, [(\underline{\ast}\,\underline{\psi})]^{\cdot} \tag{23}$$

and

$$D\,\underline{\psi}^A := \underline{d}\underline{\psi}^A + \underline{\omega}{}^A{}_B \wedge \underline{\psi}^B \tag{24}$$

Since the metric $g_{A'B'}$ depends in general on time, it is obvious that $\overset{\ast}{\psi}' \overset{\cdot}{\psi}$. We have

$$\overset{\ast}{\underline{\psi}} = \underline{\dot{\psi}} + \underline{\ast}\,[\dot{\vartheta}^A \wedge \underline{\ast}(\underline{\psi} \wedge \vartheta_A)] - \dot{\vartheta}^A \wedge \underline{\ast}(\vartheta_A \wedge \underline{\psi}). \tag{25}$$

In the three dimensional case we deal with, the calculus of differential forms is equivalent to the usual vector calculus. Since the later is more familiar we express all one and two forms in terms of vectors. Every one form $\underline{a}$ is equivalent to a vector $\mathbf{a}$ and from a two form φ we can construct a one form with the aid of the star operator. It is easy to prove the following identities: for any two 1-forms $\underline{a}$, $\underline{b}$

$$\underline{*}(\underline{a} \wedge \underline{b}) = \mathbf{a} \times \mathbf{b}, \qquad (26\,a)$$

$$\underline{*}(\underline{a} \wedge \underline{*}\underline{b}) = \mathbf{a} \cdot \mathbf{b}, \qquad (26\,b)$$

where $\times$ (.) denotes the usual cross (dot) product.

For a 0-form φ we have

$$\underline{d}\varphi = \nabla\varphi \qquad (27\,a)$$

and for a 1-form $\underline{a}$

$$\underline{*}\,\underline{d}\underline{a} = \nabla \times \mathbf{a}, \qquad (27\,b)$$

and

$$\underline{*}\underline{d}\,\underline{*}\,\underline{a} = \nabla \cdot \mathbf{a}. \qquad (27\,c)$$

We set also

$$\omega^A := \frac{1}{2}\varepsilon^A{}_{BC}\,\underline{\omega}^{BC} \qquad (28)$$

and we write the covariant nabla operator

$$\overset{\omega}{\nabla}\varphi^A := \underline{D}\,\varphi^A = \nabla\varphi^A - \varepsilon^A{}_{BC}\,\omega^B\varphi^C, \qquad (29\,a)$$

$$\overset{\omega}{\nabla}\cdot\mathbf{a}^A := \underline{*}\,\underline{D}\,\underline{*}\,\underline{a}^A = \nabla\cdot\mathbf{a}^A - \varepsilon^A{}_{BC}\,\omega^B\cdot\mathbf{a}^C, \qquad (29\,b)$$

and

$$\overset{\omega}{\nabla}\times\mathbf{a}^A := \underline{*}\,\underline{D}\,\underline{a}^A = \nabla\times\mathbf{a}^A - \varepsilon^A{}_{BC}\,\omega^B\times\mathbf{a}^C. \qquad (29\,c)$$

The curvature tensor on S is given by

$$\Omega^A := \frac{1}{2}\varepsilon^A{}_{BC}\,\underline{\underline{\Omega}}^{BC} = \nabla\times\omega^A - \frac{1}{2}\varepsilon^A{}_{BC}\,\omega^B\times\omega^C, \qquad (30)$$

where

$$\underline{\underline{\Omega}}^{AB} := \underline{d}\underline{\omega}^{AB} + \underline{\omega}^A{}_C \wedge \underline{\omega}^{CB} \qquad (31)$$

is different from $\underline{\Omega}^{AB}$. Expressed in this calculus equation (25) takes the form

$$\dot{\psi} = \overset{*}{\psi} + \dot{\vartheta}^A \times (\psi \times \vartheta_A) - \dot{\vartheta}^A(\vartheta_A \cdot \psi). \qquad (32)$$

4. (3+1)-DECOMPOSITION OF THE FIELD EQUATIONS OF PGT

To write the equations of PGT as a first order system we use the tetrad field ϑ^i, the connection $\omega^i{}_j$, the torsion Θ^j and the curvature $\Omega^i{}_j$ as independent variables. These give a total number of 100 variables. The equations of the system are:

(i) the structure equations

$$0 = S^i = d\vartheta^i + \omega^i{}_j \wedge \vartheta^j - \Theta^i, \tag{33 a}$$

$$0 = S^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j - \Omega^i{}_j, \tag{33 b}$$

(ii) the Bianchi identities

$$0 = B^i = d\Theta^i + \omega^i{}_j \wedge \Theta^j - \Omega^i{}_j \wedge \vartheta^j, \tag{34 a}$$

$$0 = B^i{}_j = d\Omega^i{}_j + \omega^i{}_k \wedge \Omega^k{}_j - \omega^k{}_j \wedge \Omega^i{}_k, \tag{34 b}$$

and (iii) the field equations

$$0 = p^i = d\mathrm{T}^i + \omega^i{}_j \wedge \mathrm{T}^j - q^i, \tag{35 a}$$

$$0 = c^{ij} = d\mathrm{W}^{ij} + \omega^i{}_k \wedge \mathrm{W}^{kj} + \omega^j{}_k \wedge \mathrm{W}^{ik} - \frac{1}{2}\vartheta^{[i} \wedge \mathrm{T}^{j]}. \tag{35 b}$$

The constitutive equations $(11\,a, b)$ and equation $(10\,c)$ must be used in order to express the last two equations in terms of the independent variables. The above equations are accompanied by their integrability conditions

$$0 = J^i = DS^i - S^i{}_j \wedge \vartheta^j + B^i, \tag{36 a}$$

$$0 = J^i{}_j = DS^i{}_j + B^i{}_j, \quad \cdot \tag{36 b}$$

$$0 = DB^i - S^i{}_j \wedge \Theta^j + \Omega^i{}_j \wedge S^j + B^i{}_j \wedge \vartheta^j, \tag{37 a}$$

$$0 = DB^i{}_j + \Omega^i{}_k \wedge S^k{}_j - \Omega^k{}_j \wedge S^i{}_k, \tag{37 b}$$

and the Noether identities $(10\,a, b)$. Equations $(36\,a, b)$ are 3-forms and their integrability conditions are

$$0 = DJ^i + J^i{}_j \wedge \vartheta^j, \tag{38 a}$$

$$0 = DJ^i{}_j. \tag{38 b}$$

No further integrability conditions arise since equations $(37\,a, b)$ and $(38\,a, b)$ are 4-forms. To express the $3+1$ decomposition of all these equations in the calculus of 3-vectors it is necessary to introduce a big number of new symbols. Since the equations are in general complicated we use different letters. We give therefore tables of the symbols and their meaning.

For the two forms $\mathscr{S}^i = \Theta^i$, T^i, S^i and $\mathscr{S}^{ij} = \Omega^{ij}$, W^{ij}, S^{ij} we set

$\mathscr{S}^i$	$\underline{*}\,\mathscr{S}^0$	$\overset{\perp}{\mathscr{S}}{}^0$	$\underline{*}\,\mathscr{S}^A$	$\overset{\perp}{\mathscr{S}}{}^A$
Θ^i	ζ	η	τ^A	ρ^A
T^i	s	r	m^A	n^A
S^i	A	B	C^A	E^A

$\mathscr{S}^{ij}$	$\underline{*}\,\mathscr{S}^{OA}$	$\overset{\perp}{\mathscr{S}}{}^{OA}$	$\frac{1}{2}\varepsilon^A{}_{BC}\,\underline{*}\,\mathscr{S}^{BC}$	$\frac{1}{2}\varepsilon^A{}_{BC}\,\overset{\perp}{\mathscr{S}}{}^{BC}$
Ω^{ij}	e^A	f^A	b^A	h^A
c^{ij}	u^A	y^A	v^A	z^A
S^{ij}	F^A	G^A	P^A	Z^A

For the three forms $\mathscr{S}^i = q^i$, p^i, B^i, J^i and $\mathscr{S}^{ij} = c^{ij}$, B^{ij}, J^{ij} we set

$\mathscr{S}^i$	$\underline{*}\,\mathscr{S}^0$	$\underline{*}\,\overset{\perp}{\mathscr{S}}{}^0$	$\underline{*}\,\mathscr{S}^A$	$\underline{*}\,\overset{\perp}{\mathscr{S}}{}^A$
q^i	α	$\varkappa$	β^A	λ^A
p^i	π	μ	φ^A	ν^A
B^i	H	L	M^A	N^A
J^i	Γ	Δ	Σ^A	Φ^A

and

$\mathscr{S}^{ij}$	$\underline{*}\,\mathscr{S}^{OA}$	$\underline{*}\,\overset{\perp}{\mathscr{S}}{}^{OA}$	$\frac{1}{2}\varepsilon^A{}_{BC}\,\underline{*}\,\mathscr{S}^{BC}$	$\frac{1}{2}\varepsilon^A{}_{BC}\,\underline{*}\,\overset{\perp}{\mathscr{S}}{}^{BC}$
c^{ij}	γ^A	ψ^A	δ^A	χ^A
B^{ij}	U^A	V^A	X^A	Y^A
J^{ij}	Λ^A	Ψ^A	Ξ^A	Π^A

Using the rules of section 2 we calculate the $3+1$ decomposition of the sections of the system. We find:

from the first structure equation (33 a)

$$0 = A = \vartheta^A \times K_A - \zeta, \tag{39 a}$$

$$0 = B = a_A \vartheta^A - \eta, \tag{39 b}$$

$$0 = C^A = \overset{\omega}{\nabla} \times \vartheta^A - \tau^A, \tag{39 c}$$

$$0 = E^A = \dot{\vartheta}^A + K^A - \rho^A. \tag{39 d}$$

from the second structure equation (33 b)

$$0 = F^A = -\overset{\omega}{\nabla} \times K^A - e^A, \tag{40 a}$$

$$0 = G^A = \dot{K}^A + \overset{\omega}{\nabla} a_A - f^A, \tag{40 b}$$

$$0 = \mathbf{P}^A = \mathbf{\Omega}^A + \frac{1}{2}\varepsilon^A{}_{BC}\mathbf{K}^B \times \mathbf{K}^C - \mathbf{b}^A, \tag{40 c}$$

$$0 = \mathbf{Z}^A = \dot{\boldsymbol{\omega}}^A + \varepsilon^A{}_{BC}\mathbf{K}^B a^C - \mathbf{h}^A, \tag{40 d}$$

from the first Bianchi identity (34 a)

$$0 = \mathbf{H} = \mathbf{\nabla}.\boldsymbol{\zeta} - \mathbf{K}_A.\boldsymbol{\tau}^A - \boldsymbol{\vartheta}^A.\mathbf{e}_A, \tag{41 a}$$

$$0 = \mathbf{L} = \dot{\boldsymbol{\zeta}} - \overset{\omega}{\mathbf{\nabla}} \times \boldsymbol{\eta} + a_A \boldsymbol{\tau}^A + \mathbf{K}_A \times \boldsymbol{\rho}^A - \mathbf{f}_A \times \boldsymbol{\vartheta}^A, \tag{41 b}$$

$$0 = \mathbf{M}^A = \mathbf{\nabla}.\boldsymbol{\tau}^A - \mathbf{K}^A.\boldsymbol{\zeta} - \varepsilon^A{}_{BC}\boldsymbol{\vartheta}^B.\mathbf{b}^C, \tag{41 c}$$

$$0 = \mathbf{N}^A = \dot{\boldsymbol{\tau}}^A - \overset{\omega}{\mathbf{\nabla}} \times \boldsymbol{\rho}^A + a^A \boldsymbol{\zeta} + \mathbf{K}^A \times \boldsymbol{\eta} - \mathbf{e}^A + \varepsilon^A{}_{BC}\boldsymbol{\vartheta}^B \times \mathbf{h}^C, \tag{41 d}$$

from the second Bianchi identity (34 b)

$$0 = \mathbf{U}^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{e}^A + \varepsilon^A{}_{BC}\mathbf{K}^B.\mathbf{b}^C, \tag{42 a}$$

$$0 = \mathbf{V}^A = \dot{\mathbf{e}}^A - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{f}^A - \varepsilon^A{}_{BC}a^B\mathbf{b}^C - \varepsilon^A{}_{BC}\mathbf{K}^B \times \mathbf{h}^C, \tag{42 b}$$

$$0 = \mathbf{X}^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{b}^A - \varepsilon^A{}_{BC}\mathbf{K}^B.\mathbf{e}^C, \tag{42 c}$$

$$0 = \mathbf{Y}^A = \dot{\mathbf{b}}^A - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{h}^A + \varepsilon^A{}_{BC}a^B\mathbf{e}^C + \varepsilon^A{}_{BC}\mathbf{K}^B \times \mathbf{f}^C, \tag{42 d}$$

from the first field equation (35 a)

$$0 = \pi = \mathbf{\nabla}.\mathbf{s} - \mathbf{K}_A.\mathbf{m}^A + \alpha, \tag{43 a}$$

$$0 = \varphi^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{m}^A - \mathbf{K}^A.\mathbf{s} + \beta^A, \tag{43 b}$$

$$0 = \mu = \dot{\mathbf{s}} - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{r} + a_A\mathbf{m}^A + \mathbf{K}_A \times \mathbf{n}^A + \varkappa, \tag{43 c}$$

$$0 = v^A = \dot{\mathbf{m}}^A - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{n}^A + a^A\mathbf{s} + \mathbf{K}^A \times \mathbf{r} + \lambda^A, \tag{43 d}$$

from the second field equation (35 b)

$$0 = \gamma^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{u}^A + \varepsilon^A{}_{BC}\mathbf{K}^B.\mathbf{v}^C + \frac{1}{2}\boldsymbol{\vartheta}^A.\mathbf{s}, \tag{44 a}$$

$$0 = \delta^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{v}^A - \varepsilon^A{}_{BC}\mathbf{K}^B.\mathbf{u}^C - \frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\vartheta}^B.\mathbf{m}^C, \tag{44 b}$$

$$0 = \psi^A = \dot{\mathbf{u}}^A - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{y}^A - \varepsilon^A{}_{BC}a^B\mathbf{v}^C - \varepsilon^A{}_{BC}\mathbf{K}^B \times \mathbf{z}^C - \frac{1}{2}\mathbf{m}^A - \frac{1}{2}\boldsymbol{\vartheta}^A \times \mathbf{r}. \tag{44 c}$$

$$0 = \chi^A = \dot{\mathbf{v}}^A - \overset{\omega}{\mathbf{\nabla}} \times \mathbf{z}^A + \varepsilon^A{}_{BC}a^B\mathbf{u}^C + \varepsilon^A{}_{BC}\mathbf{K}^B \times \mathbf{y}^C + \frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\vartheta}^B \times \mathbf{n}^C. \tag{44 d}$$

From the integrability conditions (36) to (37) we obtain

$$0 = \Gamma = \mathbf{\nabla}.\mathbf{A} - \mathbf{K}_A.\mathbf{C}^A - \boldsymbol{\vartheta}^A.\mathbf{F}_A + \mathbf{H}, \tag{45 a}$$

$$0 = \Sigma^A = \overset{\omega}{\mathbf{\nabla}}.\mathbf{C}^A - \mathbf{K}^A.\mathbf{A} - \varepsilon^A{}_{BC}\boldsymbol{\vartheta}^B.\mathbf{P}^C + \mathbf{M}^A, \tag{45 b}$$

$$0 = \Delta = \overset{\omega}{\acute{A}} - \nabla \times \mathbf{B} + a_A\,\mathbf{C}^A + \mathbf{K}_A \times \mathbf{E}^A - \vartheta_A \times \mathbf{G}^A + \mathbf{L}, \qquad (45\,c)$$

$$0 = \Phi^A = \overset{\omega}{\acute{C}}^A - \nabla \times \mathbf{E}^A + a^A\,\mathbf{A} + \mathbf{K}^A \times \mathbf{B} - \mathbf{F}^A + \varepsilon^A{}_{BC}\,\vartheta^B \times \mathbf{Z}^C + \mathbf{N}^A, \quad (45\,d)$$

$$0 = \Lambda^A = \overset{\omega}{\nabla}. \mathbf{F}^A + \varepsilon^A{}_{BC}\,\mathbf{K}^B . \mathbf{P}^C + \mathbf{U}^A, \qquad (46\,a)$$

$$0 = \Xi^A = \overset{\omega}{\nabla}. \mathbf{P}^A - \varepsilon^A{}_{BC}\,\mathbf{K}^B . \mathbf{F}^C + \mathbf{X}^A, \qquad (46\,b)$$

$$0 = \Psi^A = \overset{\omega}{\acute{F}}^A - \nabla \times \mathbf{G}^A - \varepsilon^A{}_{BC}\,a^B\,\mathbf{P}^C - \varepsilon^A{}_{BC}\,\mathbf{K}^B \times \mathbf{Z}^C + \mathbf{V}^A, \qquad (46\,c)$$

$$0 = \Pi^A = \overset{\omega}{\acute{P}}^A - \nabla \times \mathbf{Z}^A + \varepsilon^A{}_{BC}\,a^B\,\mathbf{F}^C + \varepsilon^A{}_{BC}\,\mathbf{K}^B \times \mathbf{G}^C + \mathbf{Y}^A, \qquad (46\,d)$$

$$\overset{\omega}{\acute{H}} - \nabla . \mathbf{L} + a_A\,M^A + \mathbf{K}_A . \mathbf{N}^A + \vartheta_A . \mathbf{V}^A \\ = \tau_A . \mathbf{G}^A + \rho_A . \mathbf{F}^A - \mathbf{f}_A . \mathbf{C}^A - \mathbf{e}_A . \mathbf{E}^A, \qquad (47\,a)$$

$$\overset{\omega}{\acute{M}}^A - \nabla . \mathbf{N}^A + a^A\,H + \mathbf{K}^A . \mathbf{L} - U^A + \varepsilon^A{}_{BC}\,\vartheta^B . \mathbf{Y}^C = \zeta . \mathbf{G}^A + \boldsymbol{\eta} . \mathbf{F}^A \\ - \mathbf{f}^A . \mathbf{A} - \mathbf{e}^A . \mathbf{B} + \varepsilon^A{}_{BC}(\tau^B . \mathbf{Z}^C + \rho^B . \mathbf{P}^C + \mathbf{b}^B . \mathbf{E}^C + \mathbf{h}^B . \mathbf{C}^C). \quad (47\,b)$$

$$\overset{\omega}{\acute{U}}^A - \nabla . \mathbf{V}^A - \varepsilon^A{}_{BC}\,a^B\,X^C - \varepsilon^A{}_{BC}\,\mathbf{K}^B . \mathbf{Y}^C \\ = \varepsilon^A{}_{BC}(\mathbf{f}^B . \mathbf{P}^C + \mathbf{e}^B . \mathbf{Z}^C + \mathbf{h}^B . \mathbf{F}^C + \mathbf{b}^B . \mathbf{G}^C), \quad (48\,a)$$

$$\overset{\omega}{\acute{X}}^A - \nabla . \mathbf{Y}^A + \varepsilon^A{}_{BC}\,a^B\,U^C + \varepsilon^A{}_{BC}\,\mathbf{K}^B . \mathbf{V}^C \\ = -\varepsilon^A{}_{BC}(\mathbf{f}^B . \mathbf{F}^C + \mathbf{e}^B . \mathbf{G}^C - \mathbf{h}^B . \mathbf{P}^C - \mathbf{b}^B . \mathbf{Z}^C), \quad (48\,b)$$

$$\overset{\omega}{\acute{\gamma}}^A - \nabla . \psi^A - \varepsilon^A{}_{BC}\,a^B\,\delta^C - \varepsilon^A{}_{BC}\,\mathbf{K}^B . \chi^C = \frac{1}{2}\,\varphi^A, \qquad (49\,a)$$

$$\overset{\omega}{\acute{\delta}}^A - \nabla . \chi^A + \varepsilon^A{}_{BC}\,a^B\,\gamma^C + \varepsilon^A{}_{BC}\,\mathbf{K}^B . \psi^C = -\frac{1}{2}\,\varepsilon^A{}_{BC}\,\vartheta^B . \mathbf{v}^C, \qquad (49\,b)$$

$$\overset{\omega}{\acute{\pi}} - \nabla . \boldsymbol{\mu} + a_A\,\varphi^A + \mathbf{K}_A . \mathbf{v}^A = -\boldsymbol{\eta} . \boldsymbol{\mu} + \rho_A . \mathbf{v}^A - 2\,\mathbf{f}_A . \psi^A + 2\,\mathbf{h}_A . \chi^A, \quad (50\,a)$$

$$\overset{\omega}{\acute{\varphi}}^A - \nabla . \mathbf{v}^A + a^A\,\pi + \mathbf{K}^A . \boldsymbol{\mu} \\ = (\vartheta^A . \boldsymbol{\eta})\,\pi - (\vartheta^A . \rho_B)\,\varphi^B + \vartheta^A . (\zeta \times \boldsymbol{\mu}) - \vartheta^A . (\tau_B \times \mathbf{v}^B) \\ + 2\,[(\vartheta^A . \mathbf{f}_B)\,\gamma^B - (\vartheta^A . \mathbf{h}_B)\,\delta^B + \vartheta^A . (\mathbf{e}_B \times \psi^B) - \vartheta^A . (\mathbf{b}_B \times \chi^B)], \quad (50\,b)$$

and from equations (38 a, b) we obtain

$$\overset{\omega}{\acute{\Gamma}} - \nabla . \Delta + a_A\,\Sigma^A + \mathbf{K}_A . \Phi^A = \vartheta_A . \Psi^A, \qquad (51\,a)$$

$$\overset{\omega}{\acute{\Sigma}}^A - \nabla . \Phi^A + a^A\,\Gamma + \mathbf{K}^A . \Delta = \Xi^A - \varepsilon^A{}_{BC}\,\vartheta^B . \Pi^C, \qquad (51\,b)$$

$$\overset{\omega}{\acute{\Lambda}}^A - \nabla . \Psi^A - \varepsilon^A{}_{BC}\,a^B\,\Xi^C - \varepsilon^A{}_{BC}\,\mathbf{K}^B . \Pi^C = 0, \qquad (52\,a)$$

$$\overset{\omega}{\acute{\Xi}}^A - \nabla . \Pi^A + \varepsilon^A{}_{BC}\,a^B\,\Lambda^C + \varepsilon^A{}_{BC}\,\mathbf{K}^B . \psi^C = 0, \qquad (52\,b)$$

It is plausible that equations of the main system (39) to (44) which contain a time derivative $(\cdot)$ or $(\acute{})$ are evolution equations, while the remaining equations are constraints on the initial data. The constraints (39 a, b) are algebraic and allow us to eliminate ζ and $\boldsymbol{\eta}$ from the other equations. Equations (41 a, b) reduce to (40 a, b) after the elimination. There remain

84 independent variables ϑ^A, ω^A, a^A, K^A, τ^A, ρ^A, e^A, b^A, f^A, h^A, since 10 are already eliminated by the gauge fixing. The independent evolution equations (39 d), (40 b, d), (41 d), (42 b, d), (43 c, d), (44 c, d) are also 84. There remain 46 constraint equations (39 c), (40 a, c), (41 c), (42 a, c), (43 a, b), (44 a, b).

The system of integrability conditions (45) to (50) can be divided similarly into 46 evolution equations (45 d), (46 c, d), (47 b), (48 a, b), (49 a, b), (50 a, b) and 9 constraint equations (45 b), (46 a, b) for the 46 constraints of the main system C^A, F^A, P^A, M^A, U^A, X^A, π, φ^A, γ^A, δ^A. Since A and B vanish identically (45 a, c) reduce to algebraic equations, which can be used to eliminate H, L and show that (47 a) is a consequence of (45 d). After the evolution equations of the main system are solved for some initial data satisfying the constraints of the main system, the system of integrability conditions becomes a linear homogeneous and hyperbolic system in these constraints. Thus the constraints vanish for all times, if they are taken to vanish on the initial surface S. The time conservation of the 9 constraint equations of the system of integrability conditions can be proved in a similar way from the 9 equations (51 b), (52 a, b). No further constraints arise.

5. CAUCHY-KOWALEVSKI CONDITIONS

We apply the $3+1$ decomposition rules on the constitutive equations (11 a, b) and eliminate with there aid s, r, m_A, n_A and u_A, z_A, v_A, y_A from equations (43), (44), consideration of the principle parts of the differential equations shows that the system decomposes into three decoupled — on the level of their first derivatives — subsystems. We will express the vectors in terms of their anholonomic components with respect to ϑ^A.

The first subsystem for the 18 components of ϑ^A and ω^A is given by the *evolution equations:*

$$\hat{\vartheta}_{AB} = \dot{\vartheta}_A \cdot \vartheta_B, \tag{53 a}$$

$$\hat{\omega}_{AB} = \dot{\omega}_{AB}, \tag{53 b}$$

and the *constraints:*

$$\mathfrak{J}_{AB} = \varepsilon_B{}^{CE} (\partial_C \vartheta_A) \cdot \vartheta_E, \tag{54 a}$$

$$\widetilde{\omega}_{AB} = \varepsilon_B{}^{CE} \partial_C \omega_{AE}. \tag{54 b}$$

For any field variable $\mathscr{S}$, $\hat{\mathscr{S}}$ resp. $\tilde{\mathscr{S}}$ will be used alternatively to denote either the terms of zeroth order or as an abbreviation for the principle part of the evolution equation resp. constraint for $\mathscr{S}$. It is obvious that the above subsystem has a well posed initial value problem and is hyperbolic. We will not discuss it any longer.

In the other two subsystems we decompose the tensors $\mathscr{S}_{AB}$ into a trace part $\mathscr{S} := \mathscr{S}^C{}_C$, an antisymmetric part $\mathscr{S}_A := \frac{1}{2}\varepsilon_A{}^{BC}\mathscr{S}_{BC}$ and a trace free symmetric part $\mathscr{S}_{AB} := \frac{1}{2}\mathscr{S}_{(AB)} - \frac{1}{3}\delta_{AB}\mathscr{S}$. In the same way we decompose also the equations for $\mathscr{S}_{AB}$.

The second subsystem is for the 30 components of torsion and second fundamental form.

Evolution equations:

$$\hat{K} = \dot{K} + \partial^A a_A, \tag{55a}$$

$$\hat{\rho} = (\bar{p} + \bar{q} + 3\bar{r})\dot{\rho} - 2(\bar{p} + \bar{q} + 2\bar{r})\partial^A \tau_A + 2\bar{r}\partial^A a_A, \tag{55b}$$

$$\hat{\tau} = \dot{\tau} + 2\partial^A \rho_A, \tag{55c}$$

$$\hat{K}_A = \dot{K}_A - \frac{1}{2}\varepsilon_A{}^{BC}\partial_B a_C, \tag{56a}$$

$$\hat{\rho}_A = (\bar{p} - \bar{q})\dot{\rho}_A + \frac{1}{2}(\bar{p} - 2\bar{q})\partial_A \tau - \frac{1}{2}(\bar{p} + \bar{q})\partial^B \underline{\tau}_{AB}$$
$$+ \frac{1}{2}(\bar{p} + \bar{q} + 2\bar{r})\varepsilon_A{}^{BC}\partial_B \tau_C - \frac{1}{2}(2\bar{q} + \bar{r})\varepsilon_A{}^{BC}\partial_B a_C, \tag{56b}$$

$$\hat{\tau}_A = \dot{\tau}_A - \frac{1}{3}\partial_A \rho - \frac{1}{2}\varepsilon_A{}^{BC}\partial_B \rho_C + \frac{1}{2}\partial^B \underline{\rho}_{AB}, \tag{56c}$$

$$\hat{a}_A = (\bar{p} + \bar{q} + \bar{r})\dot{a}_A + 2\bar{p}\varepsilon_A{}^{BC}\partial_B K_C - \frac{2}{3}\bar{r}\partial_A \rho$$
$$- (2\bar{q} + \bar{r})\varepsilon_A{}^{BC}\partial_B \rho_C + \bar{r}\partial^B \underline{\rho}_{AB}, \tag{56d}$$

$$\underline{\hat{K}}_{AB} = \underline{\dot{K}}_{AB} + \partial_A a_B, \tag{57a}$$

$$\hat{\rho}_{AB} = (\bar{p} + \bar{q})\underline{\dot{\rho}}_{AB} + \frac{1}{2}(\bar{p} + \bar{q})\varepsilon_C{}^E{}_{(A}\partial^C \underline{\tau}_{B)E} + (\bar{p} + \bar{q} + 2\bar{r})\partial_A \tau_B - \bar{r}\partial_A a_B, \tag{57b}$$

$$\hat{\underline{\tau}}_{AB} = \underline{\dot{\tau}}_{AB} - \frac{1}{2}\varepsilon_C{}^E{}_{(A}\partial^C \underline{\rho}_{B)E} - \partial_A \rho_B. \tag{57c}$$

Constraints:

$$\tilde{K} = \partial^A K_A, \tag{58a}$$

$$\tilde{a} = (\bar{p} + \bar{q} + \bar{r})\partial^A a_A - 2\bar{r}\partial^A \tau_A, \tag{58b}$$

$$\tilde{K}_A = \frac{1}{2}\partial^B \underline{K}_{AB} - \frac{1}{2}\varepsilon_A{}^{BC}\partial_B K_C - \frac{1}{3}\partial_A K, \tag{59a}$$

$$\tilde{\rho}_A = (\bar{p} + \bar{q})\partial^B \underline{\rho}_{AB} + (\bar{p} - \bar{q})\varepsilon_A{}^{BC}\partial_B \rho_C$$
$$+ \frac{1}{3}(\bar{p} + \bar{q} + 3\bar{r})\partial_A \rho - 2\bar{r}\varepsilon_A{}^{BC}\partial_B K_C. \tag{59b}$$

$$\tilde{\tau}_A = \partial^B \underline{\tau}_{AB} + \varepsilon_A{}^{BC}\partial_B \tau_C + \frac{1}{3}\partial_A \tau, \tag{59c}$$

$$\underline{\tilde{K}}_{AB} = \frac{1}{2} \varepsilon_C{}^E{}_{(A} \partial^C \underline{K}_{B) E} + \underline{\partial_A K_B},\tag{60}$$

where for all vectors $\mathscr{S}_A$ we write $\underline{\partial_A \mathscr{S}_B}$ to denote the trace free symmetric part of $\partial_A \mathscr{S}_B$.

The third subsystem is for the 36 components of the curvature.

Evolution equations:

$$\hat{e} = \dot{e} + 2 \, \partial^A f_A,\tag{61 a}$$

$$\hat{h} = (p + q_2 + 3 r_2) \, \dot{h} - 2 (p - q_1) \, \partial^A b_A + 2 (q_1 + q_2 + 3 r_2) \, \partial^A f_A,\tag{61 b}$$

$$\hat{b} = \dot{b} + 2 \, \partial^A h_A,\tag{61 c}$$

$$\hat{f} = (p + q_1 + 3 r_1) \, \dot{f} - 2 (p - q_2) \, \partial^A e_A - 2 (q_1 + q_2 + 3 r_1) \, \partial^A h_A,\tag{61 d}$$

$$\hat{e}_A = \dot{e}_A - \frac{1}{3} \partial_A f - \frac{1}{2} \varepsilon_A{}^{BC} \partial_B f_C + \frac{1}{2} \partial^B \underline{f}_{AB},\tag{62 a}$$

$$\hat{h}_A = (p - q_2) \, \dot{h}_A + \frac{1}{3} (p + q_1 + 3 r_1) \, \partial_A b + \frac{1}{2} (p - q_1) \, \varepsilon_A{}^{BC} \partial_B b_C$$
$$- \frac{1}{2} (p + q_1) \, \partial^B \underline{b}_{AB} + \frac{1}{3} (q_1 + q_2 + 3 r_1) \, \partial_A f + \frac{1}{2} (q_1 - q_2) \, \varepsilon_A{}^{BC} \partial_B f_C$$
$$- \frac{1}{2} (q_1 + q_2) \, \partial^B \underline{f}_{AB},\tag{62 b}$$

$$\hat{b}_A = \dot{b}_A - \frac{1}{3} \partial_A h - \frac{1}{2} \varepsilon_A{}^{BC} \partial_B h_C + \frac{1}{2} \partial^B \underline{h}_{AB},\tag{62 c}$$

$$\hat{f}_A = (p - q_1) \, \dot{f}_A + \frac{1}{3} (p + q_2 + 3 r_2) \, \partial_A e + \frac{1}{2} (p - q_2) \, \varepsilon_A{}^{BC} \partial_B e_C$$
$$- \frac{1}{2} (p + q_2) \, \partial^B \underline{e}_{AB} - \frac{1}{3} (q_1 + q_2 + 3 r_2) \, \partial_A h + \frac{1}{2} (q_1 - q_2) \, \varepsilon_A{}^{BC} \partial_B h_C$$
$$+ \frac{1}{2} (q_1 + q_2) \, \partial^B \underline{h}_{AB},\tag{62 d}$$

$$\underline{\hat{e}}_{AB} = \underline{\dot{e}}_{AB} - \frac{1}{2} \varepsilon_C{}^E{}_{(A} \partial \underline{f}_{B) E} - \underline{\partial_A f_B},\tag{63 a}$$

$$\underline{\hat{h}}_{AB} = (p + q_2) \, \underline{\dot{h}}_{AB} + \frac{1}{2} (p + q_1) \, \varepsilon_C{}^E{}_{(A} \partial^C \underline{b}_{B) E} + (p - q_1) \, \underline{\partial_A b_B}$$
$$- \frac{1}{2} (q_1 - q_2) \, \varepsilon_C{}^E{}_{(A} \partial^C \underline{f}_{B) E} - (q_1 + q_2) \, \underline{\partial_A f_B},\tag{63 b}$$

$$\underline{\hat{b}}_{AB} = \underline{\dot{b}}_{AB} - \frac{1}{2} \varepsilon_C{}^E{}_{(A} \partial^C \underline{h}_{B) E} - \underline{\partial_A h_B},\tag{63 c}$$

$$\underline{\hat{f}}_{AB} = (p + q_1) \, \underline{\dot{f}}_{AB} + \frac{1}{2} (p + q_2) \, \varepsilon_C{}^E{}_{(A} \partial^C \underline{e}_{B) E} - (p - q_2) \, \underline{\partial_A e_B}$$
$$- \frac{1}{2} (q_1 - q_2) \, \varepsilon_C{}^E{}_{(A} \partial^C \underline{f}_{B) E} - (q_1 + q_2) \, \underline{\partial_A h_B},\tag{63 d}$$

Constraints:

$$\tilde{e}_A = \frac{1}{2} \partial_A e + \varepsilon_A{}^{BC} \partial_B e_C + \partial^B \underline{e}_{AB}, \qquad (64\,a)$$

$$\tilde{h}_A = (p + q_2 + 3\,r_2)\,\partial_A h + (p - q_2)\,\varepsilon_A{}^{BC}\,\partial_B h_C + (p + q_2)\,\partial^B \underline{h}_{AB}$$
$$- \frac{2}{3}\,r_2\,\partial_A e + (2\,q_1 + r_2)\,\varepsilon_A{}^{BC}\,\partial_B e_C + r_2\,\partial^B \underline{e}_{AB}, \quad (64\,b)$$

$$\tilde{b}_A = \frac{1}{3}\,\partial_A b + \varepsilon_A{}^{BC}\,\partial_B b_C + \partial^B \underline{b}_{AB}, \qquad (64\,c)$$

$$\tilde{f}_A = \frac{1}{3}\,(p + q_1 + 3\,r_1)\,\partial_A f + (p - q_1)\,\varepsilon_A{}^{BC}\,\partial_B f_C + (p + q_1)\,\partial^B \underline{f}_{AB}$$
$$- \frac{1}{3}\,(q_1 - q_2 - 3\,r_1 + r_2)\,\partial_A b - (q_1 + q_2 + r_2)\,\varepsilon_A{}^{BC}\,\partial_B b_C$$
$$- (q_1 - q_2 + r_2)\,\partial^B \underline{b}_{AB}. \quad (64\,d)$$

Since every evolution equation above contains only one time derivative of some quantity, it is obvious that in order to apply the Cauchy-Kowalevski theorem on PGT the coefficients of these terms may not vanish. Looking for these terms we obtain a table of conditions for the coupling constants.

The evolution of the following quantities is determined	if	the following conditions hold
$a_A = \eta_A = Q^0{}_{0A}$		$\bar{p} + \bar{q} + \bar{r} \neq 0,$
$\rho = Q^A{}_{0A}$		$\bar{p} + \bar{q} + 3\,\bar{r} \neq 0,$
$\rho_A = -\frac{1}{2}\,\varepsilon_A{}^{BC}\,Q_{BC0}$		$\bar{p} - \bar{q} \neq 0,$
$\underline{\rho}_{AB} = -\frac{1}{2}\,Q_{(AB)\,0} + \frac{1}{3}\,\delta_{AB}\,Q^C{}_{C0}$		$\bar{p} + \bar{q} \neq 0,$
$f = R^{0A}{}_{0A}$		$p + q_1 + r_1 \neq 0,$
$f_A = -\frac{1}{2}\,\varepsilon_A{}^{BC}\,R^0{}_{BC0}$		$p - q_1 \neq 0,$
$\underline{f}_{AB} = -\frac{1}{2}\,R^0{}_{(AB)\,0} + \frac{1}{3}\,\delta_{AB}\,R^{0C}{}_{C0}$		$p + q_1 \neq 0,$
$h = \frac{1}{2}\,\varepsilon^{ABC}\,R_{BC0A}$		$p + q_2 + 3\,r_2 \neq 0,$
$h_A = \frac{1}{2}\,R_A{}^C{}_{C0}$		$p - q_2 \neq 0,$
$\underline{h}_{AB} = -\frac{1}{2}\,\varepsilon_{CE(A}\,R^{CE}{}_{B)\,0} + \frac{1}{6}\,\delta_{AB}\,\varepsilon^{CEF}\,R_{EFC0}$		$p + q_2 \neq 0,$

where

$$\Theta^i = \frac{1}{2} Q^i{}_{jk} \vartheta^j \wedge \vartheta^k, \qquad \Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} \vartheta^k \wedge \vartheta^l.$$

These ten conditions are the same with those found and discussed in theorem 1 of **I**. We note again that they are sufficient but not necessary conditions. The advantage here is that the components of torsion and curvature are given whose propagation depend on these condition. Thus if some of these conditions are violated we can say how many free functions are to be expected in a generic solution of the theory.

6. HYPERBOLICITY CONDITIONS OF PGT

A system of differential equations

$$\hat{u} = A \dot{u} + A^A \partial_A u, \tag{65}$$

where A, A^A are square matrices and u is the column vector of dependent variables is *symmetric hyperbolic* in the sence of K. O. Friedrichs ([5], [6]), if (i) all matrices A, A^A are symmetric and (ii) A is positive definite. Although the theory of symmetric hyperbolic systems is initially developed for linear systems it extends to the quasi-linear case ([7], [8]).

Let (65) denote the system of evolution equations of PGT (53), (55) to (57) and (61) to (63). This is accompanied by the system of constraint equations (54), (58) to (60) and (64). We denoted it by

$$\tilde{u} = B^A \partial_A u, \tag{66}$$

where B^A need not be square matrices.

We see that for our system A is diagonal but the matrices A^A, $A = 1, 2, 3$ are not in general symmetric. We have a symmetric hyperbolic system immediately if all coupling constants but c_0, c_1, $\bar{p}$, p vanish (*see* [4], p. 46 for references). In order to make this system symmetric in the general case we can apply three operations: (i) change the dependent variables, (ii) add a combination of the constraints and (iii) multiply on the left by an invertible matrix. Applying these operations on (65) we obtain

$$\hat{u} = CAE \dot{u} + (CA^A + FB^A) E \partial_A u, \tag{67}$$

where C, E, F are matrices depending in general on all variables. Since A, A^A are constant matrices (the tetrad $e_A{}^{A'}$ is hidden in ∂_A) it suffices to take C, E, F to be also constant. Thus the problem reduces to find constant matrices C, E and F such that $A' := CAE$ is positive definite and A', $A'^A := (CA^A + FB^A) E$ are symmetric. This is simplified by the observation that the two properties of positive definiteness and symmetry

are invariant under similarity transformations. Thus transforming (A', A'^A) to $(E^{-1})^T (A', A'^A) (E^{-1})$ the two properties needed for hyperbolicity are invariant. Therefore we can set from the beginning E = identity. Another simplification comes from the fact that A is diagonal. This and the fact that A' must be symmetric restricts C to be almost symmetric. Finally it is obvious that it will not help symmetrization or positive definiteness to combine either uncoupled subsystems or even equations in the same subsystem determining the evolution of the trace of some variable with that for the antisymmetric part or the trace free symmetric part of the same or an other variable. Thus it remains to look only for combinations in the "trace parts", the "antisymmetric parts" and the "trace free symmetric parts" of the subsystems. These observations reduce the number of independent entries of the matrices C, F drastically. For the torsion system there remains a total number of 31 and for the curvature system a number of 26 entries to be determined. In the appendices we determine these entries, so that a symmetric hyperbolic system of the PGT can be constructed. Here we give only the results.

In order for the torsion system to be symmetric hyperbolic the coupling constants $\bar{p}, \bar{q}, \bar{r}$ must satisfy condition

$$\bar{p}(\bar{q}+\bar{r})+\bar{q}^2 = 0. \tag{68}$$

Similarly the coupling constants p, q_1, q_2, r_1, r_2 must satisfy conditions

$$q_1 = q_2 =: q, \qquad r_1 = r_2 =: r \tag{69 a}$$

and

$$p(q+r)+q(q+2r)=0 \tag{69 b}$$

in order for the curvature system to be symmetric hyperbolic. These conditions are both necessary and sufficient. They are identical with those found in paper I. There we were able to prove only sufficiency.

Under these condition we obtain two symmetric hyperbolic subsystems.

The symmetric hyperbolic torsion system:

$$\hat{K}_{AB} = 2\dot{K}_{AB} - \dot{K}_{BA} + \alpha\,\delta_{AB}\,\dot{\rho}^C{}_C - \partial_A a_B + 2\,\partial_B a_A - \delta_{AB}\,\partial^C a_C, \tag{70 a}$$

$$\hat{\rho}_{AB} = \alpha\delta_{AB}\dot{K}^C{}_C + \dot{\rho}_{AB} + 3\,\alpha^2\,\delta_{AB}\,\dot{\rho}^C{}_C - 2\alpha\delta_{AB}\,\partial^C a_C - \alpha\partial_B a_A$$
$$+ \alpha\delta_{AB}\,\varepsilon^{CEF}\,\partial_C\,\tau_{EF} - \alpha\varepsilon_{AB}{}^C\,\partial^E\,\tau_{CE} + \alpha\varepsilon_{AB}{}^C\,\partial^E\,\tau_{EC}$$
$$+ 2(1+\alpha)\,\varepsilon_B{}^{CE}\,\partial_C\,\tau_{AE} - \alpha\varepsilon_B{}^{CE}\,\partial_C\,\tau_{EA}, \tag{70 b}$$

$$\hat{\tau}_{AB} = (1+\alpha^2)\,\dot{\tau}_{AB} - \alpha^2\,\dot{\tau}_{BA} - \alpha^2\,\varepsilon_{AB}{}^C\,\dot{a}_C + \alpha\varepsilon_{AB}{}^C\,\partial_C\,\rho^E{}_E$$
$$- \alpha\varepsilon_A{}^{CE}\,\partial_B\,\rho_{CE} + \alpha\varepsilon_B{}^{CE}\,\partial_A\,\rho_{CE} + 2(1+\alpha)\,\varepsilon_A{}^{CE}\,\partial_C\,\rho_{AE} + \alpha\varepsilon_A{}^{CE}\,\partial_C\,\rho_{BE}, \tag{70 c}$$

$$\hat{a}_A = -\alpha^2\,\varepsilon_A{}^{BC}\,\dot{\tau}_{BC} + (2+\alpha^2)\,\dot{a}_A$$
$$- \partial_A K^B{}_B + 2\,\partial^B K_{AB} - \partial^B K_{BA} - 2\,\alpha\partial_A\,\rho^B{}_B - \alpha\partial^B\,\rho_{AB}, \tag{70 d}$$

where

$$\alpha := \frac{\overline{q}}{p}.$$

The symmetric hyperbolic curvature system:

$$\hat{e}_{AB} = (1+\beta^2)\,\dot{e}_{AB} - \beta^2\,\dot{e}_{BA} - \beta^2\,(\dot{h}_{AB} - \dot{h}_{BA})$$
$$- \varepsilon_B{}^{CE}\,\partial_C f_{AE} - \beta\varepsilon_{AB}{}^C\,\partial^E f_{CE}, \quad (71\,a)$$

$$\hat{b}_{AB} = (1+\beta^2)\,\dot{b}_{AB} - \beta^2\,\dot{b}_{BA} + \beta^2\,(\dot{f}_{AB} - \dot{f}_{BA})$$
$$- \varepsilon_B{}^{CE}\,\partial_C h_{AE} - \beta\varepsilon_{AB}{}^C\,\partial^E h_{CE}, \quad (71\,b)$$

$$\hat{f}_{AB} = \beta^2\,(\dot{b}_{AB} - \dot{b}_{BA}) + (1+\beta^2)\,\dot{f}_{AB} - \beta^2\,\dot{f}_{BA} + \varepsilon_B{}^{CE}\,\partial_C e_{AE}$$
$$- \beta\varepsilon_A{}^{CE}\,\partial_B e_{CE} + \beta\varepsilon_A{}^{CE}\,\partial_B h_{CE} - \beta\varepsilon_{AB}{}^C\,\partial^E h_{CE}, \quad (71\,c)$$

$$\hat{h}_{AB} = -\beta^2\,(\dot{e}_{AB} - \dot{e}_{BA}) + (1+\beta^2)\,\dot{h}_{AB} - \beta^2\,\dot{h}_{BA}$$
$$+ \varepsilon_B{}^{CE}\,\partial_C b_{AE} - \beta\varepsilon_A{}^{CE}\,\partial_B b_{CE} + \beta\varepsilon_{AB}{}^C\,\partial^E f_{CE} - \beta\varepsilon_A{}^{CE}\,\partial_B f_{CE}, \quad (71\,d)$$

where

$$\beta := \frac{q}{p+q}.$$

The two constants $\overline{r}$ and r are eliminated with the aid of (68) and (69 b), and $\overline{p} \neq 0 \neq p+q$ because of the Cauchy-Kowalevski conditions.

The characteristics of the two subsystems can be calculated from

$$\det(A\,\xi_0 + A^A\,\xi_A) = 0, \quad (72)$$

in terms of ξ_0, ξ_A. The light cone will belong to them as we show in our first paper. It is interesting to know how the characteristics behave when the hyperbolicity conditions (68), (69) are not satisfied. Since the matrices involved are 30×30 and 36×36 this is a problem for the computer.

We can summarize the results of this section in the form of a theorem.

THEOREM. — *The evolution equations of PGT in vacuum, in first order formalism can be brought in the form (65) with A, A^A symmetric and A semipositive definite if, and only if the hyperbolicity conditions (68), (69) are satisfied. If further the Cauchy-Kowalevski conditions hold, then A becomes positive definite and we obtain the symmetric hyperbolic system of equations (53), (70), (71).*

This is stronger than theorem 2 of paper I, since it proves that the hyperbolicity conditions are also necessary to obtain a hyperbolic system out of the field equations of PGT in vacuum. The only objection one can raise to this result is that it may depend on the time gauge used here. To this we note that the hyperbolicity conditions are conditions on the coupling constants alone and thus *cannot depend* on any gauge condition. Further we have complete agreement with the results of paper I, where no gauge conditions are used. The contrary assumption, *i. e.* that the necessary and sufficient hyperbolicity conditions derived depend on the

gauge would lead to the strange possibility of fixing some gauge by gauge independent means.

APPENDIX A

The most general reasonable combinations of the equations of the torsion system (55) to (60) are

$$\hat{K}' = \lambda_1 \hat{K} + \lambda_2 \hat{\rho} + \lambda_5 \hat{\tau} + \beta_1 \tilde{a},$$
$$\hat{\rho}' = (\bar{p} + \bar{q} + 3\bar{r}) \lambda_2 \hat{K} + \lambda_3 \hat{\rho} + \beta_2 \tilde{a} + \beta_4 \tilde{K},$$
$$\hat{\tau}' = \lambda_5 \hat{K} + \lambda_4 \hat{\tau} + \beta_3 \tilde{K},$$

$$\hat{K}'_A = \mu_1 \hat{K}_A + \mu_2 \hat{\rho}_A + \nu_4 \hat{\tau}_A + \gamma_1 \tilde{\tau}_A,$$
$$\hat{\rho}'_A = (\bar{p} - \bar{q}) \mu_2 \hat{K}_A + \mu_3 \hat{\rho}_A + \gamma_2 \tilde{\tau}_A + \tilde{\gamma}_7 \tilde{K}_A,$$
$$\hat{\tau}'_A = \nu_4 \hat{K}_A + \nu_1 \hat{\tau}_A + \nu_2 \hat{a}_A + \gamma_3 \tilde{K}_A + \gamma_4 \tilde{\rho}_A,$$
$$\hat{a}'_A = (\bar{p} + \bar{q} + r) \nu_2 \hat{\tau}_A + \nu_3 \hat{a}_A + \gamma_5 \tilde{K}_A + \gamma_6 \tilde{\rho}_A + \gamma_8 \tilde{\tau}_A.$$

$$\underline{\hat{K}}'_{AB} = \eta_1 \underline{\hat{K}}_{AB} + \eta_2 \underline{\hat{\rho}}_{AB} + \eta_5 \underline{\hat{\tau}}_{AB},$$
$$\underline{\hat{\rho}}'_{AB} = (\bar{p} + \bar{q}) \eta_2 \underline{\hat{K}}_{AB} + \eta_3 \underline{\hat{\rho}}_{AB} + \vartheta_2 \underline{\tilde{K}}_{AB},$$
$$\underline{\hat{\tau}}_{AB} = \eta_5 \underline{\hat{K}}_{AB} + \eta_4 \underline{\hat{\tau}}_{AB} + \vartheta_1 \underline{\tilde{K}}_{AB}.$$

We demand from this system to be symmetric hyperbolic. That is written in the form

$$A \dot{u} + A^A \partial_A u = \hat{u}$$

the matrices A, A^A must be symmetric and A must be positive definite. It is cumbersome to write down the matrices explicitly, instead we can make use the of following fact: variation of the Lagrangian

$$L = u^T A \dot{u} + u^T A^A \partial_A u$$

with respect to u leads to

$$(A^T - A) \dot{u} + (A^{AT} - A^A) \partial_A u = 0.$$

Thus if the matrices A, A^A are symmetric these equations vanish identically in $\dot{u}$ and $\partial_A u$. In our case L is given by

$$
\begin{aligned}
L = \lambda_1 K\hat{K} &+ \lambda_2 [K \hat{\rho} + (\bar{p} + \bar{q} + 3\bar{r}) \rho \hat{K}] + \lambda_3 \rho\hat{\rho} + \lambda_4 \tau\hat{\tau} + \lambda_5 (K \hat{\tau} + \tau \hat{K}) \\
&+ \mu_1 K^A \hat{K}_A + \mu_2 [K^A \hat{\rho}_A + (\bar{p} - \bar{q}) \rho^A \hat{K}_A] + \mu_3 \hat{\rho}_A \rho^A \\
&+ \nu_1 \tau^A \hat{\tau}_A + \nu_2 [\tau^A \hat{a}_A + (\bar{p} + \bar{q} + r) a^A \hat{\tau}_A] + \nu_4 (K^A \hat{\tau}_A + \tau^A \hat{K}_A) \\
&+ \eta_1 \underline{K}^{AB} \underline{\hat{K}}_{AB} + \eta_2 [\underline{K}^{AB} \underline{\hat{\rho}}_{AB} + (\bar{p} + \bar{q}) \rho^{AB} \underline{\hat{K}}_{AB}] \\
&+ \eta_3 \rho^{AB} \underline{\hat{\rho}}_{AB} + \eta_4 \tau_{AB} \hat{\tau}^{AB} + \eta_5 [\underline{K}^{AB} \underline{\hat{\tau}}_{AB} + \underline{\tau}^{AB} \underline{\hat{K}}_{AB}) \\
+ \beta_1 K \tilde{a} + \beta_2 \rho \tilde{a} &+ \beta_3 \tau \tilde{K} + \beta_4 \rho \tilde{K} + \gamma_1 K^A \tilde{\tau}_A + \gamma_2 \rho^A \tilde{\tau}_A + \gamma_3 \tau^A \tilde{K}_A + \gamma_4 \tau^A \tilde{\rho}_A \\
&+ \gamma_5 a^A \tilde{K}_A + \gamma_6 a^A \tilde{\rho}_A + \gamma_7 \rho^A \tilde{K}_A + \gamma_8 a^A \tilde{\tau}_A + \vartheta_1 \underline{\tau}^{AB} \underline{\tilde{K}}_{AB} + \vartheta_2 \rho^{AB} \underline{\tilde{K}}_{AB}.
\end{aligned}
$$

We vary now L with respect to all variables and demand the equations to vanish identically. This leads to a linear homogeneous system for the 31 unknowns $\lambda_1, \ldots, \vartheta_2$. The system decomposes into two independent subsystems. The first part is for 7 unknowns and can be solved immediately in terms of a parameter y:

$$\lambda_5 = y, \qquad \nu_4 = 6y, \qquad \eta_5 = 3y, \qquad \beta_4 = -2y,$$
$$\gamma_7 = -6y, \qquad \gamma_8 = 3y, \qquad \vartheta_2 = 3y.$$

Four of the remaining unknowns can be eliminated with the aid of the following equations

$$\vartheta_1 = -(\bar{p} + \bar{q}) \eta_2, \qquad \eta_4 = (\bar{p} + \bar{q}) \eta_3,$$
$$\gamma_5 = 2 \eta_1 - 2\bar{r} \eta_2, \qquad \gamma_3 = 2(\bar{p} + \bar{q} + 2\bar{r}) \eta_2.$$

There remains a system of 14 equations for 20 unknowns:

$$(\bar{p} + \bar{q} + 2\bar{r}) \lambda_2 + \bar{r} \beta_1 - \frac{1}{3}(\bar{p} + \bar{q} + 2\bar{r}) \eta_2 = 0, \qquad \text{(A 1)}$$

$$\lambda_1 + 2\bar{r} \lambda_2 + (\bar{p} + \bar{q} + \bar{r}) \beta_1 + \frac{2}{3} \eta_1 - \frac{2}{3}\bar{r} \eta_2 = 0, \qquad \text{(A 2)}$$

$$(\bar{p} + \bar{q} + 2\bar{r}) \lambda_3 - \frac{1}{6} \nu_1 - \frac{1}{3}\bar{r} \nu_2 + \bar{r} \beta_2 + \frac{1}{6}(\bar{p} + \bar{q} + 3\bar{r}) \gamma_4 = 0, \qquad \text{(A 3)}$$

$$(\bar{p} + \bar{q} + 3\bar{r}) \lambda_2 + 2\bar{r} \lambda_3 + \frac{1}{3}(\bar{p} + \bar{q} + \bar{r}) \nu_2$$
$$+ \frac{2}{3}\bar{r} \nu_3 + (\bar{p} + \bar{q} + \bar{r}) \beta_2 - \frac{1}{3}(\bar{p} + \bar{q} + 3\bar{r}) \gamma_6 = 0, \qquad \text{(A 4)}$$

$$(\bar{p} - 2\bar{q}) \mu_2 - 3\beta_3 + \gamma_1 = 0, \qquad \text{(A 5)}$$

$$6\lambda_4 - (\bar{p} - 2\bar{q}) \mu_3 - \gamma_2 = 0, \qquad \text{(A 6)}$$

$$(\bar{p} + \bar{q} + 2\bar{r}) \mu_2 + 4\bar{p} \nu_2 + 2\gamma_1 - 2(\bar{p} + \bar{q} + 2\bar{r}) \eta_2 - 4\bar{q} \gamma_4 = 0, \qquad \text{(A 7)}$$

$$\mu_1 + (2\bar{q} + \bar{r}) \mu_2 - 4\bar{p} \nu_3 + 2\eta_1 - 2\bar{r} \eta_2 + 4\bar{q} \gamma_6 = 0, \qquad \text{(A 8)}$$

$$(\bar{p} + \bar{q}) \mu_2 - 2\gamma_1 - 2(\bar{p} + \bar{q}) \eta_2 = 0, \qquad \text{(A 9)}$$

$$(\bar{p} + \bar{q} + 2\bar{r}) \mu_3 - \nu_1 - 2(2\bar{q} + \bar{r}) \nu_2 + 2\gamma_2 + 2(\bar{p} - \bar{q}) \gamma_4 = 0, \qquad \text{(A 10)}$$

$$(\bar{p} - \bar{q}) \mu_2 + (2\bar{q} + \bar{r}) \mu_3 + (\bar{p} + \bar{q} + \bar{r}) \nu_2$$
$$+ 2(2\bar{q} + \bar{r}) \nu_3 - 2(\bar{p} - \bar{q}) \gamma_6 = 0, \qquad \text{(A 11)}$$

$$(\bar{p} + \bar{q}) \mu_3 - 2(\bar{p} + \bar{q}) \eta_3 - 2\gamma_2 = 0, \qquad \text{(A 12)}$$

$$\nu_1 + 2\bar{r} \nu_2 - 2(\bar{p} + \bar{q} + 2\bar{r}) \eta_3 + 2(\bar{p} + \bar{q}) \gamma_4 = 0, \qquad \text{(A 13)}$$

$$(\bar{p} + \bar{q} + \bar{r}) \nu_2 + 2\bar{r} \nu_3 - 2(\bar{p} + \bar{q}) \eta_2 + 2\bar{r} \eta_3 + 2(\bar{p} + \bar{q}) \gamma_6 = 0. \qquad \text{(A 14)}$$

Matrix A of the system is block diagonal consisting of

$$\begin{bmatrix} \lambda_1 & (\bar{p} + \bar{q} + 3\bar{r}) \lambda_2 & y \\ (\bar{p} + \bar{q} + 3\bar{r}) \lambda_2 & (\bar{p} + \bar{q} + 3\bar{r}) \lambda_3 & 0 \\ y & 0 & \lambda_4 \end{bmatrix},$$

three times

$$\begin{bmatrix} \mu_1 & (\bar{p}-\bar{q})\mu_2 & 6y & 0 \\ (\bar{p}-\bar{q})\mu_2 & (\bar{p}-\bar{q})\mu_3 & 0 & 0 \\ 6y & 0 & \nu_1 & (\bar{p}+\bar{q}+\bar{r})\nu_2 \\ 0 & 0 & (\bar{p}+\bar{q}+\bar{r})\nu_2 & (\bar{p}+\bar{q}+\bar{r})\nu_3 \end{bmatrix}$$

and five times

$$\begin{bmatrix} \eta_1 & (\bar{p}+\bar{q})\eta_2 & 3y \\ (\bar{p}+\bar{q})\eta_2 & (\bar{p}+\bar{q})\eta_3 & 0 \\ 3y & 0 & (\bar{p}+\bar{q})\eta_3 \end{bmatrix}.$$

To prove positive definiteness of these three matrices we use a well known theorem of matrix theory [11], § 10.4: a square matrix is positive definite if and only if all its main diagonal determinants are positive. Applying the theorem to the above matrices we obtain among other conditions the following:

$$\lambda_4 > 0, \tag{A 15}$$

$$(\bar{p}-\bar{q})[\mu_1\mu_3 - (\bar{p}-\bar{q})\mu_2{}^2] > 0, \tag{A 16}$$

$$\eta_1 > 0, \tag{A 17}$$

$$(\bar{p}+\bar{q})[\eta_1\eta_3 - (\bar{p}+\bar{q})\eta_2{}^2] > 0, \tag{A 18}$$

and

$$(\bar{p}+\bar{q})\eta_3 > 0, \tag{A 19}$$

which follows from (A 17) and (A 18).

We sketch now how the system (A 1) to (A 14) can be solved in terms of λ_1, ν_2, η_1, η_2, η_3, γ_4 and give only the expressions for the quantities in (A 15)-(A 19). We first solve (A 13) for ν_1 and (A 7), (A 9) for γ_1 and

$$\mu_2 = 2\left[\eta_2 - \frac{1}{\bar{p}+\bar{q}+\bar{r}}(\bar{p}\nu_2 - \bar{q}\gamma_4)\right], \tag{A 20}$$

after that (A 10), (A 12) can be solved for γ_2 and

$$\mu_3 = 2\left[\eta_3 + \frac{1}{\bar{p}+\bar{q}+\bar{r}}(\bar{q}\nu_2 - \bar{p}\gamma_4)\right], \tag{A 21}$$

Next we solve (A 6) for

$$\lambda_4 = -\frac{1}{6}(\bar{p}+\bar{q})\eta_3 + \frac{1}{2}(\bar{p}-\bar{q})\left[\eta_3 + \frac{1}{\bar{p}+\bar{q}+\bar{r}}(\bar{q}\nu_2 - \bar{p}\gamma_4)\right], \tag{A 22}$$

and (A 5) for β_3, (A 1), (A 2) for β_1, λ_2. Equations (A 11) and (A 14) can now be brought in the form

$$\bar{q}\left\{\nu_3 + \eta_3 + \frac{1}{2}\frac{1}{\bar{p}+\bar{q}+\bar{r}}[(2\bar{p}+2\bar{q}+\bar{r})\nu_2 - (\bar{p}+\bar{q})\gamma_4]\right\}$$

$$-\bar{p}\left\{\gamma_6 - \eta_2 + \frac{1}{2}\frac{1}{\bar{p}+\bar{q}+\bar{r}}[(\bar{p}+\bar{q})\nu_2 + \bar{r}\gamma_4]\right\} = 0, \tag{A 23}$$

$$(\bar{q}+\bar{r})\left\{v_3+\eta_3+\frac{1}{2}\frac{1}{\bar{p}+\bar{q}+\bar{r}}[(2\bar{p}+2\bar{q}+\bar{r})v_2-(\bar{p}+\bar{q})\gamma_4]\right\}$$
$$+\bar{q}\left\{\gamma_6-\eta_2+\frac{1}{2}\frac{1}{\bar{p}+\bar{q}+\bar{r}}[(\bar{p}+\bar{q})v_2+\bar{r}\gamma_4]\right\}=0. \quad (A\,24)$$

The determinant of this system is

$$\Delta=\bar{p}(\bar{q}+\bar{r})+\bar{q}^2. \tag{A 25}$$

If $\Delta\neq0$ then we solve (A 23), (A 24) for v_3, γ_6 and substitute in (A 8) to obtain

$$\mu_1=-2\,\eta_1-8\,\bar{q}\,\eta_2-4\,\bar{p}\,\eta_3$$
$$-2\frac{2\bar{p}^2-\bar{q}(\bar{p}+\bar{q})}{\bar{p}+\bar{q}+\bar{r}}v_2+2\frac{(\bar{p}-\bar{q})(\bar{p}+2\,\bar{q})}{\bar{p}+\bar{q}+\bar{r}}\gamma_4. \quad (A\,26)$$

We show now that the expressions found for λ_4, μ_1, μ_2, μ_3 lead to contradictions if they are substituted in (A 15), (A 16). In fact from (A 22), (A 15) and (A 19) we obtain

$$(\bar{p}-\bar{q})\left[\eta_3+\frac{1}{\bar{p}+\bar{q}+\bar{r}}(\bar{q}v_2-\bar{p}\gamma_4)\right]>0, \tag{A 27}$$

further from (A 16), (A 20), (A 21) and (A 26) we find

$$(\bar{p}-\bar{q})^2\left[\eta_2-\eta_3-\frac{\bar{p}+\bar{q}}{\bar{p}+\bar{q}+\bar{r}}(v_2-\gamma_4)\right]^2+(\bar{p}-\bar{q})\left[\eta_1+(\bar{p}+\bar{q})(2\,\eta_2+\eta_3)\right]$$
$$\times\left[\eta_3+\frac{1}{\bar{p}+\bar{q}+\bar{r}}(\bar{q}v_2-\bar{p}\gamma_4)\right]<0, \quad (A\,28)$$

which with the aid of (A 17), (A 19) and (A 27) gives

$$0<\eta_1+(\bar{p}+\bar{q})\eta_3<-2(\bar{p}+\bar{q})\eta_2. \tag{A 29}$$

Squaring the last inequality and using (A 18) leads to a contradiction

$$[\eta_1-(\bar{p}+\bar{q})\eta_3]^2<0.$$

We conclude if $\bar{p}(\bar{q}+\bar{r})+\bar{q}^2\neq0$ the torsion system is not symmetric hyperbolic.

If we set

$$\bar{p}(\bar{q}+\bar{r})+\bar{q}^2=0, \tag{A 30}$$

the following values of the unknowns lead to the symmetric hyperbolic system (70).

$$y=0, \qquad \lambda_1=\frac{1}{3}, \qquad \lambda_2=-\frac{1}{p}\frac{\alpha}{(1+\alpha)(1-3\,\alpha)},$$
$$\lambda_3=\frac{1}{\bar{p}}\frac{1}{(1+\alpha)(1-3\,\alpha)}\left(3\,\alpha^2+\frac{1}{3}\right),$$

$$\lambda_4 = \frac{1}{3}, \qquad \mu_1 = 6, \qquad \mu_2 = 0, \qquad \mu_3 = \frac{2}{p} \frac{1}{(1-\alpha)},$$

$$\nu_1 = 2(1+2\alpha^2), \qquad \nu_2 = -\frac{2}{p} \frac{\alpha^2}{1-\alpha^2},$$

$$\nu_3 = \frac{1}{p} \frac{2+\alpha^2}{1-\alpha^2}, \qquad \eta_1 = 1, \qquad \eta_2 = 0, \qquad \eta_3 = \frac{1}{p} \frac{1}{1+\alpha},$$

$$\eta_4 = 1, \qquad \beta_1 = -\frac{1}{p} \frac{1-2\alpha}{(1+\alpha)(1-3\alpha)},$$

$$\beta_2 = \frac{2}{p} \frac{2(2-3\alpha)}{(1+\alpha)(1-3\alpha)}, \qquad \beta_3 = 0, \qquad \gamma_1 = 0,$$

$$\gamma_2 = 2\frac{\alpha}{1-\alpha}, \qquad \gamma_4 = -\frac{2}{p} \frac{\alpha}{1-\alpha^2},$$

$$\gamma_5 = 0, \qquad \gamma_6 = \frac{3}{p} \frac{\alpha}{1-\alpha^2}, \qquad \vartheta_1 = 0, \qquad \gamma_3 = 0,$$

where

$$\alpha := \frac{\dot{q}}{\dot{p}}.$$

In the above calculations we have assumed that the C-K conditions hold. As is obvious from the matrices, if this is not the case A is singular. Allowing also this possibility is now not difficult to handle. This will then verify the necessity of the hyperbolicity conditions for the singular case.

APPENDIX B

The method used in Appendix A will be applied now to the curvature system (61) to (64). The Lagrangian containing the combinations of the equations of the curvature system is

$$
\begin{aligned}
L = {} & \lambda_1\, e\hat{e} + \lambda_2\, [e\hat{h} + (p+q_2+3\,r_2)\, h\hat{e}] + \lambda_3\, h\hat{h} \\
& + \mu_1\, b\hat{b} + \mu_2\, [b\hat{f} + (p+q_1+3\,r_1)\, f\hat{b}] + \mu_3\, f\hat{f} \\
& + \nu_1\, e^A\, \hat{e}_A + \nu_2\, [e^A\, \hat{h}_A + (p-q_2)\, h^A\, \hat{e}_A] + \nu_3\, h^A\, \hat{h}_A \\
& + \xi_1\, b^A\, \hat{b}_A + \xi_2\, [b^A\, \hat{f}_A + (p-q_1)\, f^A\, \hat{b}_A] + \xi_3\, f^A\, \hat{f}_A \\
& + \eta_1\, \underline{e}^{AB}\, \hat{e}_{AB} + \eta_2\, [\underline{e}^{AB}\, \hat{h}_{AB} + (p+q_2)\, \underline{h}^{AB}\, \hat{e}_{AB}] + \eta_3\, \underline{h}^{AB}\, \hat{h}_{AB} \\
& + \vartheta_1\, \underline{b}^{AB}\, \hat{b}_{AB} + \vartheta_2\, [\underline{b}^{AB}\, \hat{f}_{AB} + (p+q_1)\, \underline{f}^{AB}\, \hat{b}_{AB}] + \vartheta_3\, \underline{f}^{AB}\, \hat{f}_{AB} \\
& + \beta_1\, e^A\, \tilde{b}_A + \beta_2\, h^A\, \tilde{f}_A + \beta_3\, e^A\, \tilde{f}_A + \beta_4\, h^A\, \tilde{b}_A \\
& + \gamma_1\, b^A\, \tilde{e}_A + \gamma_2\, f^A\, \tilde{h}_A + \gamma_3\, b^A\, \tilde{h}_A + \gamma_4\, f^A\, \tilde{e}_A.
\end{aligned}
$$

We vary L with respect to all variables and require, that the equations resulting vanish identically. There results a system of 24 linear homoge-

neous equations for the 26 unknowns $\lambda_1, \ldots, \gamma_4$.

$$2\lambda_1 + 2(q_1 + q_2 + 3r_2)\lambda_2 - \frac{1}{3}(p + q_2 + 3r_2)\xi_3 + \frac{2}{3}r_2\gamma_2 - \frac{1}{3}\gamma_4 = 0, \quad \text{(B 1)}$$

$$2(p - q_1)\lambda_2 + \frac{1}{3}(p + q_2 + 3r_2)\xi_2 + \frac{1}{3}\gamma_1 - \frac{2}{3}r_2\gamma_3 = 0, \quad \text{(B 2)}$$

$$2\mu_1 - 2(q_1 + q_2 + 3r_1)\mu_2 - \frac{1}{3}(p + q_1 + 3r_1)\nu_3$$
$$+ \frac{1}{3}(q_1 - q_2 - 3r_1 + r_2)\beta_2 - \frac{1}{3}\beta_4 = 0, \quad \text{(B 3)}$$

$$2(p - q_2)\mu_2 + \frac{1}{3}(p + q_1 + 3r_1)\nu_2 + \frac{1}{3}\beta_1 - \frac{1}{3}(q_1 - q_2 - 3r_1 + r_2)\beta_3 = 0, \quad \text{(B 4)}$$

$$(p + q_2 + 3r_2)\left(2\lambda_2 - \frac{1}{3}\gamma_2\right) + (q_1 + q_2 + 3r_2)$$
$$\times \left(2\lambda_3 + \frac{1}{3}\xi_3\right) + \frac{1}{3}(p - q_1)\xi_2 = 0, \quad \text{(B 5)}$$

$$\text{(B 6)}$$

$$2(p - q_1)\lambda_3 - \frac{1}{3}\xi_1 - \frac{1}{3}(q_1 + q_2 + 3r_2)\xi_2 + \frac{1}{3}(p + q_2 + 3r_2)\gamma_3 = 0,$$

$$(p + q_1 + 3r_1)\left(2\mu_2 - \frac{1}{3}\beta_2\right) - (q_1 + q_2 + 3r_1)$$
$$\times \left(2\mu_3 + \frac{1}{3}\nu_3\right) - \frac{1}{3}(p - q_2)\nu_2 = 0, \quad \text{(B 7)}$$

$$2(p - q_2)\mu_3 - \frac{1}{3}\nu_1 + \frac{1}{3}(q_1 + q_2 + 3r_1)\nu_2 + \frac{1}{3}(p + q_1 + 3r_1)\beta_3 = 0, \quad \text{(B 8)}$$

$$\frac{1}{2}\nu_1 - \frac{1}{2}(q_1 - q_2)\nu_2 - \frac{1}{2}(p - q_2)\xi_3$$
$$- (p - q_1)\beta_3 - (2q_1 + r_2)\gamma_2 - \gamma_4 = 0, \quad \text{(B 9)}$$

$$\frac{1}{2}\nu_1 - \frac{1}{2}(q_1 + q_2)\nu_2 - (p - q_2)\vartheta_3 + (p + q_1)\beta_3 = 0, \quad \text{(B 10)}$$

$$\frac{1}{2}(p - q_1)\nu_2 + \frac{1}{2}(p - q_2)\xi_2 + \beta_1$$
$$- (q_1 + q_2 + r_2)\beta_3 + \gamma_1 + (2q_1 + r_2)\gamma_3 = 0, \quad \text{(B 11)}$$

$$\frac{1}{2}(p + q_1)\nu_2 + (p - q_2)\vartheta_2 - \beta_1 + (q_1 - q_2 + r_2)\beta_3 = 0, \quad \text{(B 12)}$$

$$\frac{1}{2}(p - q_1)\nu_3 - \frac{1}{2}\xi_1 + \frac{1}{2}(q_1 - q_2)\xi_2$$
$$- (q_1 + q_2 + r_2)\beta_2 + \beta_4 + (p - q_2)\gamma_3 = 0, \quad \text{(B 13)}$$

$$\frac{1}{2}\xi_1+\frac{1}{2}(q_1+q_2)\xi_2-(p-q_1)\eta_3+(p+q_2)\gamma_3=0, \qquad \text{(B 14)}$$

$$\frac{1}{2}(p+q_2)\xi_2+(p-q_1)\eta_2-\gamma_1-r_2\gamma_3=0, \qquad \text{(B 15)}$$

$$(p-q_2)\left(\frac{1}{2}v_2-\gamma_2\right)-\frac{1}{2}(q_1-q_2)(v_3+\xi_3)$$

$$+(p-q_1)\left(\frac{1}{2}\xi_2-\beta_2\right)=0, \quad \text{(B 16)}$$

$$\frac{1}{2}(p-q_1)\xi_2+(q_1+q_2)\left(\frac{1}{2}\xi_3+\eta_3\right)+(p+q_2)(\eta_2+\gamma_2)=0, \quad \text{(B 17)}$$

$$\frac{1}{2}(p+q_2)\xi_3-\eta_1-(q_1+q_2)\eta_2-r_2\gamma_2-\gamma_4=0, \qquad \text{(B 18)}$$

$$\frac{1}{2}(p-q_2)v_2-(q_1+q_2)\left(\frac{1}{2}v_3+\vartheta_3\right)+(p+q_1)(\vartheta_2+\beta_2)=0, \quad \text{(B 19)}$$

$$\frac{1}{2}(p+q_1)v_3-\vartheta_1+(q_1+q_2)\vartheta_2+(q_1-q_2+r_2)\beta_2-\beta_4=0, \quad \text{(B 20)}$$

$$\eta_1+(q_1-q_2)\eta_2-(p+q_2)\vartheta_3=0, \qquad \text{(B 21)}$$
$$(p+q_1)\eta_2+(p+q_2)\vartheta_3=0, \qquad \text{(B 22)}$$
$$(p+q_1)\eta_3-\vartheta_1-(q_1-q_2)\vartheta_2=0, \qquad \text{(B 23)}$$
$$(p-q_2)\eta_2+(q_1-q_2)(\eta_3+\vartheta_3)+(p+q_1)\vartheta_3=0. \qquad \text{(B 24)}$$

The coefficient matrix of the time derivatives is again block diagonal with components:

$$\begin{bmatrix} \lambda_1 & (p+q_2+3r_2)\lambda_2 \\ (p+q_2+3r_2)\lambda_2 & (p+q_2+3r_2)\lambda_3 \end{bmatrix},$$

$$\begin{bmatrix} \mu_1 & (p+q_1+3r_1)\mu_2 \\ (p+q_1+3r_1)\mu_2 & (p+q_1+3r_1)\mu_3 \end{bmatrix},$$

three times

$$\begin{bmatrix} v_1 & (p-q_2)v_2 \\ (p-q_2)v_2 & (p-q_2)v_3 \end{bmatrix}, \quad \begin{bmatrix} \xi_1 & (p-q_1)\xi_2 \\ (p-q_1)\xi_1 & (p-q_1)\xi_3 \end{bmatrix},$$

and five times

$$\begin{bmatrix} \eta_1 & (p+q_2)\eta_2 \\ (p+q_2)\eta_2 & (p+q_2)\eta_3 \end{bmatrix}, \quad \begin{bmatrix} \vartheta_1 & (p+q_1)\vartheta_2 \\ (p+q_1)\vartheta_2 & (p+q_1)\vartheta_3 \end{bmatrix}.$$

For a 2×2 symmetric matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

the conditions of positive definiteness read

$$a > 0, \qquad ac > b^2.$$

We look for solutions of the system (B 1) to (B 24) which make the above matrices positive definite. From (B 22) we find

$$\vartheta_2 = -\frac{p+q_1}{p+q_2} \eta_2, \tag{B 25}$$

and from (B 22) and (B 24)

$$(q_1 - q_2)(\eta_2 - \vartheta_2 - \eta_3 - \vartheta_3) = 0. \tag{B 26}$$

Thus if $q_1 \neq q_2$ we obtain

$$\vartheta_3 = -\eta_3 + \frac{2p+q_1+q_2}{p+q_2} \eta_2, \tag{B 27}$$

anf from (B 21)

$$\eta_1 = -(p+q_3)\eta_3 + 2(p+q_2)\eta_2. \tag{B 28}$$

Substituting (B 28) in the conditions for positive definiteness of the η-matrix we find a contradiction

$$0 > (p+q_2)^2(\eta_2 - \eta_3)^2. $$

Thus we must set

$$q_1 = q_2 =: q. \tag{B 29}$$

We solve now (B 1), (B 3), (B 14), (B 10), (B 21), (B 23), (B 12), (B 20), (B 15) and (B 18) to obtain

$$\lambda_1 = -(2q+3r_2)\lambda_2 + \frac{1}{4}(p+q+2r_2)\xi_2$$

$$-\frac{1}{3}q\eta_2 - \frac{1}{6}(p+q)\vartheta_3 - \frac{1}{2}r_2\gamma_2, \tag{B 30}$$

$$\mu_1 = (2q+3r_1)\mu_2 + \frac{1}{4}(p+q+2r_1)\nu_3$$

$$+\frac{1}{3}q\vartheta_2 - \frac{1}{6}(p+q)\eta_3 + \frac{1}{2}r_1\beta_2, \tag{B 31}$$

$$\xi_1 = -2q\xi_2 + 2(p-q)\eta_3 - 2(p+q)\gamma_3, \tag{B 32}$$

$$\nu_1 = 2q\nu_2 + 2(p-q)\vartheta_3 - 2(p+q)\beta_3 \tag{B 33}$$

$$\eta_1 = (p+q)\vartheta_3, \tag{B 34}$$

$$\vartheta_1 = (p+q)\eta_3, \tag{B 35}$$

$$\beta_1 = \frac{1}{2}(p+q)\nu_2 + (p-q)\vartheta_2 + r_2\beta_3, \tag{B 36}$$

$$\beta_4 = \frac{1}{2}(p+q)(\nu_3 - 2\eta_3) + 2q\vartheta_2 + r_2\beta_2, \tag{B 37}$$

$$\gamma_1 = \frac{1}{2}(p+q)\,\xi_2 + (p-q)\,\eta_2 - r_2\,\gamma_3, \tag{B 38}$$

$$\gamma_4 = \frac{1}{2}(p+q)\,(\xi_3 - 2\,\vartheta_3) - 2\,q\,\eta_2 - r_2\,\gamma_2. \tag{B 39}$$

We eliminate these quantities from the remaining equations and obtain a simpler system.

$$(p+q+3\,r_2)\left(2\,\lambda_2 - \frac{1}{3}\gamma_2\right)$$
$$+ (2\,q+3\,r_2)\left(2\,\lambda_3 + \frac{1}{3}\xi_3\right) + \frac{1}{3}(p-q)\,\xi_2 = 0, \quad (B 40)$$

$$(p+q+3\,r_1)\left(2\,\mu_2 - \frac{1}{3}\beta_2\right)$$
$$+ (2\,q+3\,r_1)\left(2\,\mu_3 + \frac{1}{3}\nu_3\right) + \frac{1}{3}(p-q)\,\nu_2 = 0, \quad (B 41)$$

$$(p-q)\left(2\,\lambda_2 + \frac{1}{3}\eta_2\right) + \frac{1}{2}(p+q+2\,r_2)\,\xi_2 - r_2\,\gamma_3 = 0, \tag{B 42}$$

$$(p-q)\left(2\,\mu_2 + \frac{1}{3}\vartheta_2\right) + \frac{1}{2}(p+q+2\,r_1)\,\nu_2 + r_1\,\beta_3 = 0, \tag{B 43}$$

$$(p-q)\left(2\,\lambda_3 - \frac{2}{3}\eta_3\right) - r_2\,\xi_2 + (p+q+r_2)\,\gamma_3 = 0, \tag{B 44}$$

$$(p-q)\left(2\,\mu_3 - \frac{2}{3}\vartheta_3\right) + r_1\,\nu_2 + (p+q+r_1)\,\beta_3 = 0, \tag{B 45}$$

$$(p-q)\,\xi_2 + 2\,q\,(\xi_3 + 2\,\eta_3) + 2\,(p+q)\,(\eta_2 + \gamma_2) = 0, \tag{B 46}$$

$$(p-q)\,\nu_2 - 2\,q\,(\nu_3 + 2\,\vartheta_3) + 2\,(p+q)\,(\vartheta_2 + \beta_2) = 0, \tag{B 47}$$

$$p\,(\xi_3 - 2\,\vartheta_3 + 2\,\beta_3) - q\,(\nu_2 + 2\,\eta_2 - 2\,\gamma_2) = 0, \tag{B 48}$$

$$p\,(\nu_3 - 2\,\eta_3 + 2\,\gamma_3) + q\,(\xi_2 + 2\,\vartheta_2 - 2\,\beta_2) = 0, \tag{B 49}$$

$$\eta_2 + \vartheta_2 = 0, \tag{B 50}$$

$$p\,(\nu_2 + \xi_2) - 2\,q\,(\beta_3 - \gamma_3) = 0, \tag{B 51}$$

$$q\,(\xi_3 - \nu_3 + 2\,\eta_3 - 2\,\vartheta_3) + 2\,p\,(\beta_2 + \gamma_2) = 0. \tag{B 52}$$

Now solve (B 46) for ξ_2, (B 42) for λ_2, (B 44) for λ_3 and substitute these quantities in (B 40) to obtain

$$\frac{(p+q)(q+r_2)+qr_2}{(p-q)^2}\,[(p+q)\,(\xi_3 + 2\,\eta_3)$$
$$+ 4\,p\,(\eta_2 + \gamma_2) - 2\,(p-q)\,\gamma_3] = 0. \quad (B 53)$$

From (B 47), (B 43), (B 45) and (B 41) we obtain similarly

$$\frac{(p+q)(q+r_1)+qr_1}{(p-q)^2}\,[(p+q)\,(\nu_3 + 2\,\vartheta_3)$$
$$- 4\,p\,(\vartheta_2 + \beta_2) - 2\,(p-q)\,\beta_3] = 0. \quad (B 54)$$

If $(p+q)(q+r_2)+qr_2 \neq 0$, then we can solve (B 53) for ξ_3 and eliminated it from the expressions for ξ_1 and ξ_2. We observe that

$$\xi_1 + (p+q)\xi_3 = 2(p+q)\xi_2$$

holds, which together with the positive definiteness condition for the ξ-matrix

$$\xi_1(p+q)\xi_3 > (p+q)^2\xi_2^2$$

leads to a contradiction

$$(p+q)^2(\xi_2-\xi_3)^2 < 0.$$

Similarly if $(p+q)(q+r_1)+qr_1 \neq 0$ we obtain a contradiction from the positive definiteness conditions of the v-matrix. Thus we must set

$$(p+q)(q+r_2)+qr_2 = 0 = (p+q)(q+r_1)+qr_1.$$

These conditions are equivalent to

$$r_1 = r_2 =: r, \tag{B 55}$$
$$(p+q)q+(p+2q)r = 0. \tag{B 56}$$

If conditions (B 29), (B 55) and (B 56) holds, then we can eliminate r from all equations with the aid of (B 56). The following values for the unknowns $\lambda_1, \ldots, \gamma_4$ lead to the symmetric hyperbolic system (71).

$$\lambda_1 = \mu_1 = \frac{1}{3}, \qquad \lambda_2 = -\mu_2 = 0, \qquad \lambda_3 = \mu_3 = \frac{1}{3}\frac{1+\beta}{1-2\beta},$$

$$\xi_1 = \nu_1 = 2(1+2\beta^2), \qquad \xi_2 = -\nu_2 = \frac{4\beta^2}{1-2\beta}, \qquad \xi_3 = \nu_3 = 2\frac{1+2\beta^2}{1-2\beta},$$

$$\eta_1 = \vartheta_1 = 1, \qquad \eta_2 = -\vartheta_2 = 0, \qquad \eta_3 = \vartheta_3 = \frac{1}{p+q},$$

$$\gamma_1 = -\beta_1 = \frac{4\beta^3}{(1+\beta)(1-2\beta)}, \qquad \gamma_2 = -\beta_2 = -\frac{1}{p+q}\frac{2\beta(2-\beta)}{1-2\beta},$$

$$\gamma_3 = \beta_3 = -\frac{1}{p+q}\frac{2\beta(1-\beta)}{1-2\beta}, \qquad \gamma_4 = \beta_4 = \frac{2\beta(1+2\beta^2)}{(1+\beta)(1-2\beta)},$$

where

$$\beta = \frac{q}{p+q}.$$

The remark made at the end of appendix A holds here again if the C-K conditions are violated.

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