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The initial value problem of the Poincaré gauge theory in vacuum I. Second order formalism

by

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ABSTRACT. — The exterior initial value problem of the ten parameter Poincaré gauge theory is studied in the second order formalism. Sufficient conditions on the ten parameters are found under which (i) the Cauchy-Kowalevski theorem can be applied and (ii) the field equations become hyperbolic.

RÉSUMÉ. — Nous considérons le problème des données initiales pour la jauge de Poincaré à dix paramètres dans le cadre du formalisme du second ordre. Nous trouvons des conditions sur les dix paramètres telles que (i) le théorème du Cauchy-Kowalevski peut s'appliquer et (ii) les équations de champ deviennent hyperboliques.

1. INTRODUCTION

The problem studied in the present article can be expressed in the form of two questions: first can one apply the Cauchy-Kowalevski theorem to the field equations of Poincaré gauge theory (PGT) and secondly are these equations hyperbolic? To both questions we give only partially positive answers. We obtain ten conditions of the form (*linear combination of the coupling constants*)$\neq 0$, which can guarantee the applicability of the Cauchy-Kowalevski theorem, and four equations such that if they are satisfied by the coupling constants, gauge fixing conditions can be found, under which the field equations take an obvious hyperbolic form. Although the above conditions are obtained in a very natural and unforced way, they are only sufficient conditions. The problems connected with proving necessity will be exposed in the main part of the article.

In case these conditions hold we achieve a standard situation, which as in general relativity ([1], [2]) enables application of the theory of hyperbolic differential equations.

Poincaré gauge theory ([3], [4]) is a field theory formulated for a spacetime obeying a Riemann-Cartan geometry. Because of the presence of torsion, the connection attains here the status of an independent dynamical variable. Expressed in terms of tetrad and connection the theory is invariant under diffeomorphisms and local Lorentz transformations. In its gauge theoretic version tetrad ϑ and connection ω are interpreted as gauge potentials of the Poincaré group with torsion Q and curvature R as corresponding gauge fields ([5], [6]). Contrary to the situation in Riemannian geometry any Lagrangian of the form $L(\vartheta, Q, R)$ gives second order differential equations in the field variables. Poincaré gauge theory has Yang-Mills theory as a prototype and thus restricts the Lagrangian to be parity non-violating and at most quadratic in the gauge fields. The most general theory underlying these conditions contains ten coupling constants.

The elaboration of the initial value problem of PGT follows in its basic characteristics that of general relativity [7]. PGT leads to a system of forty second order, quasilinear partial differential equations for forty field variables. Since the theory is invariant under the groups of diffeomorphisms and local Lorentz transformations ten of the variables are unessential. That is, by means of transformations of the above groups one can ascribe to the ten unessential variables any functional dependence wished, without affecting the physics described by the initial data. Associated to that, ten of the field equations contain no evolution terms and thus impose constraints on the initial data. The ten differential Noether identities following from the invariances of PGT guarantee, that these constraints hold in every neighbourhood of the "initial time". In case the hypersurface representing initial time is spacelike, the remaining thirty field equations determine locally the evolution of the essential variables, if the determinants of some coefficient matrices do not vanish. This imposes ten conditions on the coupling constants of PGT. We call them *Cauchy-Kowalevski conditions*.

The characteristics of PGT are lightlike hypersurfaces. The number of second order outgoing derivatives of the field variables, which remain undetermined on a characteristic hypersurface depends on the coupling constants. If these satisfy four equations this number becomes a maximum. The same holds for the number of free second order discontinuities of the variables on characteristics.

It is amazing that in this case one can find gauge conditions, similar to the Hilbert-de Donder gauge conditions in general relativity, such that the second order terms of the field equations take the form: (*constant matrix*) $\square$ (*field variables*). The coefficient matrices are invertible because of the Cauchy-Kowalevski conditions and thus the system is obviously hyperbolic. These four equations reduce the number of free coupling constants from ten to six. We call them *hyperbolicity conditions*.

The ten Cauchy-Kowalevski conditions have already been found in the study of the Hamiltonian dynamics of PGT from M. Blagojević and I. A. Nikolić [8], Table I. Of the four hyperbolicity conditions the first is mentioned in another context from R. Kuhfuss and J. Nitsch [9], equation (4.15*b*).

Our conventions are as follows: Greek indices denote holonomic, latin indices anholonomic components. Both run over 0, 1, 2, 3. (Anti-)Symmetrization symbols are used without factors.

2. VACUUM FIELD EQUATIONS OF POINCARÉ GAUGE THEORY

The geometric background of PGT is a four dimensional spacetime with (i) a tetrad field $\vartheta^i{}_\mu$, which together with the Minkowski metric

$$(\eta_{ij}) = \operatorname{diag}(1, -1, -1, -1) \tag{1}$$

determines through

$$\vartheta^i{}_\mu \vartheta^j{}_\nu \eta_{ij} = g_{\mu\nu} \tag{2}$$

a metric structure and (ii) a linear connection $\omega^i{}_{j\mu}$, which is compatible with the metric. In terms of the covariant derivative ∇_μ associated to $\omega^i{}_{j\mu}$ metric compatibility means $\nabla_\mu \eta_{ij} = 0$ and implies

$$\omega_{ij\mu} + \omega_{ji\,\mu} = 0. \tag{3}$$

Tetrad and connection transform like one forms under diffeomorphisms. The metric $g_{\mu\nu}$ is invariant under local Lorentz transformations of the tetrad and $\omega^i{}_{j\mu}$ is a Lorentz connection.

Torsion and curvature are given by the structure equations

$$Q^i{}_{\mu\nu} := \nabla_{[\mu} \vartheta^i{}_{\nu]} = \partial_{[\mu} \vartheta^i{}_{\nu]} + \omega^i{}_{j\,[\mu} \vartheta^j{}_{\nu]}, \tag{4a}$$

$$R^{ij}{}_{\mu\nu} := \partial_{[\mu}\,\omega^{ij}{}_{\nu]} + \omega^{i}{}_{k\,[\mu}\,\omega^{kj}{}_{\nu]}, \tag{4 b}$$

and satisfy the Bianchi identities

$$\nabla_{[\mu}\,Q^{i}{}_{\nu\rho]} = R^{i}{}_{[\mu\nu\rho]}, \tag{5 a}$$

$$\nabla_{[\mu}\,R^{ij}{}_{\nu\rho]} = 0, \tag{5 b}$$

where anholonomic indices transform into holonomic ones by means of the tetrad $\vartheta^{i}{}_{\mu}$ and its dual $e_{i}{}^{\mu}$.

A Lagrangian giving field equations for $\vartheta^{i}{}_{\mu}$, $\omega^{i}{}_{j\mu}$ must transform like a density under diffeomorphisms and must be invariant under local Lorentz transformations. If the field equations have to be at most second order differential equations in $\vartheta^{i}{}_{\mu}$, $\omega^{i}{}_{j\mu}$ we must set

$$\mathscr{L} = (\det\vartheta)\,L(\vartheta, Q, R). \tag{6}$$

The field equations obtained from $\mathscr{L}$ are quasilinear and take a compact form expressed in terms of

$$q_{ij} := \frac{1}{\det\vartheta}\,\vartheta_{j\mu}\,\frac{\partial\mathscr{L}}{\partial\vartheta^{i}{}_{\mu}}\bigg|_{Q,\,R}, \tag{7 a}$$

$$T_{ijk} := \vartheta_{j\mu}\,\vartheta_{k\nu}\,\frac{\partial L}{\partial Q^{i}{}_{\mu\nu}}, \tag{7 b}$$

$$W_{ijkl} := \vartheta_{k\mu}\,\vartheta_{l\nu}\,\frac{\partial L}{\partial R^{ij}{}_{\mu\nu}}. \tag{7 c}$$

We obtain two sets of field equations

$$0 = p_{ij} := (\nabla^{k} - Q^{k})\,T_{ijk} + \frac{1}{2}\,T_{ikl}\,Q_{j}{}^{kl} + q_{ij}, \tag{8 a}$$

$$0 = c_{ijk} := (\nabla^{l} - Q^{l})\,W_{ijkl} + \frac{1}{2}\,W_{ijlm}\,Q_{k}{}^{lm} - \frac{1}{2}\,T_{[ij]\,k}, \tag{8 b}$$

where

$$Q_{i} := Q^{j}{}_{ji}. \tag{9}$$

Applying Noether's theorem on $\mathscr{L}$ we obtain an algebraic identity

$$q_{ij} = \eta_{ij}\,L - Q^{kl}{}_{i}\,T_{klj} - R^{klm}{}_{i}\,W_{klmj}, \tag{10}$$

and two differential identities

$$(\nabla^{k} - Q^{k})\,c_{ijk} = -\frac{1}{2}\,p_{[ij]}, \tag{11 a}$$

$$(\nabla^{j} - Q^{j})\,p_{ij} = -Q^{jk}{}_{i}\,p_{jk} - R^{jkl}{}_{i}\,c_{jkl}. \tag{11 b}$$

PGT is the special case of the theory described above, where the Lagrangian depends at most quadratically on $Q^{i}{}_{\mu\nu}$ and $R^{ij}{}_{\mu\nu}$ and does not contain parity voilating terms. This implies that the field equations are quasilinear second order equations, where the coefficients of the second order derivatives depend only on the tetrad. Written with anholonomic indices these equations look semilinear, but this is so only because the tetrad is hidden in the anholonomic indices of the differential operators. The most general Lagrangian satisfying the above conditions contains ten coupling constants. We write it in the form

$$\mathrm{L} = c_0 + c_1 \mathrm{R} + \frac{1}{4} \mathrm{A}_{ijk}\big|^{abc} \mathrm{Q}^{ijk} \mathrm{Q}_{abc} + \frac{1}{4} \mathrm{B}_{ijkl}\big|^{abcd} \mathrm{R}^{ijkl} \mathrm{R}_{abcd}, \tag{12}$$

with

$$\mathrm{A}_{ijk}\big|^{abc} := \frac{1}{2} [\bar{p} \delta_i^a \delta_{jk}^{bc} - \bar{q}\, \delta_{i\,[j}^{bc} \delta_{k]}^a - \bar{r}\, \eta^{a\,[b} \delta_{[j}^{c]} \eta_{k]\,i}], \tag{13 a}$$

$$\begin{aligned}\mathrm{B}_{ijkl}\big|^{abcd} := \frac{1}{4} [(p + q_2 + r_2)\, \delta_{ij}^{ab} \delta_{kl}^{cd} - q_2\, \eta^{mn} \delta_{m\,[i}^{ab} \eta_{j]\,[k} \delta_{l]\,n}^{cd} \\ + (q_1 + r_2)\, \eta^{mn} \delta_{m\,[i}^{cd} \eta_{j]\,[k} \delta_{l]\,n}^{ab} - r_2\, \delta_{[k}^{r} \delta_{l]\,[i}^{ab} \delta_{j]\,r}^{cd} \\ + (q_2 - q_1 + r_1 - r_2)\, \eta_{i\,[k} \eta_{l]\,j} \eta^{a\,[c} \eta^{d]\,b}],\end{aligned} \tag{13 b}$$

where c_0, c_1, $\bar{p}$, $\bar{q}$, $\bar{r}$ and p, q_1, q_2, r_1, r_2 are coupling constants, $\delta_{kl}^{ij} := \delta_{[k}^{i} \delta_{l]}^{i}$ and

$$\mathrm{R}_{ij} := \mathrm{R}^{k}{}_{ikj}, \qquad \mathrm{R} := \eta^{ij} \mathrm{R}_{ij}. \tag{14}$$

In expanded form (12) gives

$$\begin{aligned}\mathrm{L} = c_0 + c_1 \mathrm{R} + \frac{1}{2} [\bar{p}\, \mathrm{Q}^{ijk} \mathrm{Q}_{ijk} - \bar{q}\, \mathrm{Q}^{ijk} \mathrm{Q}_{jki} - \bar{r}\, \mathrm{Q}^{i} \mathrm{Q}_{i}] \\ + \Bigg[\frac{p + q_2 + r_2}{4} \mathrm{R}^{ijkl} \mathrm{R}_{ijkl} - r_2 \mathrm{R}^{ijkl} \mathrm{R}_{lijk} - q_2 \mathrm{R}^{ij} \mathrm{R}_{ij} + (q_1 + r_2) \mathrm{R}^{ij} \mathrm{R}_{ji} \\ + \frac{q_2 - q_1 + r_1 - r_2}{4} \mathrm{R}^2 \Bigg].\end{aligned} \tag{15}$$

The present choice of the coupling constants is almost forced to the author by the calculations in the first order formalism([1]). In terms of them the Cauchy-Kowalevski conditions can be divided into three almost identical subsystems of conditions (*cf.* Section 3). This fact reflects the reduction of the calculations to almost one third of the whole, effected by the use of the present coupling constants. In order for the reader to be able to compare our results with those presented in other formulations of PGT we give the following convertion formulas relating the present coup-

([1]) This is part II of our work on the exterior initial value problem of PGT and will be published as a separate paper.

ling constants to the set used in [10]:

$$\bar{p}=\frac{2}{3}(2a_1+a_3),\qquad \bar{q}=\frac{2}{3}(a_1-a_3),\qquad \bar{r}=-\frac{2}{3}(a_1-a_2),$$

and

$$p=\frac{1}{2}(b_1+b_2+b_4+b_5),$$

$$q_1=\frac{1}{2}(b_1-b_2+b_4-b_5),\qquad q_2=\frac{1}{2}(b_1+b_2-b_4-b_5),$$

$$r_1=-\frac{1}{3}(b_1-b_6),\qquad r_2=-\frac{1}{3}(b_1-b_3),$$

where $a_i:=(1/l^2)\,\mathrm{A}_i$, $i=1,2,3$ and $b_j:=(1/\kappa)\,\mathrm{B}_j$, $j=1,\ldots,6$. Note that because of the Gauss-Bonnet theorem the six curvature square terms used in [10] are not all independent, and one has as a consequence $2b_1-3b_2+b_3=0$ (*cf.* also [6]).

The field equations of PGT are

$$\begin{aligned}0=p_{ij}=\mathrm{A}_{ijk}|^{abc}(\nabla^k-\mathrm{Q}^k)\,\mathrm{Q}_{abc}+\frac{1}{2}\mathrm{A}_{ikl}|^{abc}\,\mathrm{Q}_j{}^{kl}\,\mathrm{Q}_{abc}\\ -\mathrm{A}_{klj}|^{abc}\,\mathrm{Q}^{kl}{}_i\,\mathrm{Q}_{abc}+c_0\,\eta_{ij}-2c_1\left(\mathrm{R}_{ij}-\frac{1}{2}\eta_{ij}\mathrm{R}\right)-\mathrm{B}_{klmj}|^{abcd}\,\mathrm{R}^{klm}{}_i\,\mathrm{R}_{abcd}\\ +\frac{1}{4}\eta_{ij}[\mathrm{A}_{klm}|^{abc}\,\mathrm{Q}^{klm}\,\mathrm{Q}_{abc}+\mathrm{B}_{klmn}|^{abcd}\,\mathrm{R}^{klmn}\,\mathrm{R}_{abcd}]\quad(16\,a)\end{aligned}$$

and

$$\begin{aligned}0=c_{ijk}=\mathrm{B}_{ijkl}|^{abcd}(\nabla^l-\mathrm{Q}^l)\,\mathrm{R}_{abcd}+c_1\,\mathrm{Q}_{kij}\\ +\frac{1}{2}\mathrm{B}_{ijlm}|^{abcd}\,\mathrm{Q}_k{}^{lm}\,\mathrm{R}_{abcd}-\frac{1}{2}\mathrm{A}_{[ij]\,k}|^{abc}\,\mathrm{Q}_{abc}.\quad(16\,b)\end{aligned}$$

For the initial value problem only the first terms on the right side of (16 *a*, *b*) are of interest. We must use of course the structure equations (5 *a*, *b*) to eliminate $\mathrm{Q}^i{}_{jk}$, $\mathrm{R}^i{}_{jkl}$ from (16 *a*, *b*).

3. CAUCHY-KOWALEVSKI THEOREM FOR PGT

The Cauchy-Kowalevski theorem proves the existence of analytic solutions for systems of partial differential equations, if analytic initial data are given on some hypersurface of the space of independent variables. The theorem presupposes that the differential equations are themselves analytic expressions of their arguments and can be brought in a standard form with respect to the hypersurface of initial data. This is attained if the system can be solved for the highest hypersurface-outgoing derivatives of all its dependent variables ([11], [12]).

In this section our goal is to bring the field equations of PGT (16 *a*, *b*) in the standard form necessary to apply the Cauchy-Kowalevski theorem. They are obviously analytic in all their arguments. Since they are second order in $\vartheta^i{}_\mu$ and $\omega^i{}_{j\mu}$ the initial data needed on a hypersurface S, given by $t(x)=0$, must cosist of the field variables $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ and their first derivatives. Clearly these data cannot be independent, because all derivatives of $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ interior in S can be obtained by the values of $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ on S.

To define outgoing and interior derivatives, we need the one form

$$n_\rho := \frac{\partial t}{\partial x^\rho} \tag{17}$$

and a timelike unit vector t^ρ on S, such that

$$n_\rho t^\rho = \mathrm{N} \quad \text{with}\, \mathrm{N} > 0. \tag{18}$$

The metric properties of t^ρ "unit", "timelike" are defined with respect to the metric structure induced by the initial data for $\vartheta^i{}_\mu$ on S. Independent of the character of n_ρ, t^ρ points always outside of the hypersurface S. For any field $u(x)$ we define the S-*outgoing* and S-*interior* derivatives of $u(x)$ by

$$\partial_t u := t^\rho \partial_\rho u \qquad \text{and} \qquad \underline{\partial}_\rho u := \partial_\rho u - \mathrm{N}_\rho \partial_t u, \tag{19}$$

with $\mathrm{N}_\rho := n_\rho/\mathrm{N}$. Using this decomposition of partial derivatives in the structure equations we obtain

$$\mathrm{Q}_{abc} = e_{[b}{}^\rho e_{c]}{}^\sigma \mathrm{N}_\rho \partial_t \vartheta_{a\sigma} + \ldots, \tag{20 a}$$
$$\mathrm{R}_{abcd} = e_{[c^\rho} e_{d]^\sigma} \mathrm{N}_\rho \partial_t \omega_{ab\sigma} + \ldots, \tag{20 b}$$

where we are interested only on terms containing the highest outgoing derivatives. Substituting into the field equations we find

$$0 = p_{ij} = -2\,\mathrm{A}_{ijk}\big|^{abc} \mathrm{N}^k \mathrm{N}_c e_b{}^\mu (\partial_t^2 \vartheta_{a\mu}) - \rho_{ij}, \tag{21 a}$$
$$0 = c_{ijk} = -2\,\mathrm{B}_{ijkl}\big|^{abcd} \mathrm{N}^l \mathrm{N}_d e_c{}^\mu (\partial_t^2 \omega_{ab\mu}) - \gamma_{ijk}, \tag{21 b}$$

where ρ_{ij}, γ_{ijk} denote terms of at most first order in ∂_t. We set now

$$\tau_{ij} := e_j{}^\mu \partial_t^2 \vartheta_{i\mu}, \qquad \mathrm{A}_{ij}\big|^{kl} := -2\,\mathrm{A}_{ijm}\big|^{kln} \mathrm{N}^m \mathrm{N}_n, \tag{22 a}$$
$$\chi_{ijk} := e_k{}^\mu \partial_t^2 \omega_{ij\mu}, \qquad \mathrm{B}_{ijk}\big|^{lmn} := -2\,\mathrm{B}_{ijkr}\big|^{lmns} \mathrm{N}^r \mathrm{N}_s, \tag{22 b}$$

and write the field equations in the compact form

$$\mathrm{A}_{ij}\big|^{kl} \tau_{kl} = \rho_{ij}, \tag{23 a}$$
$$\mathrm{B}_{ijk}\big|^{lmn} \chi_{lmn} = \gamma_{ijk}. \tag{23 b}$$

To bring this system in the standard form we have to solve it for τ_{ij}, χ_{ijk}. From the symmetries of $\mathrm{A}_{ijk}\big|^{abc}$ and $\mathrm{B}_{ijkl}\big|^{abcd}$ and (22 *a*, *b*) it is obvious that

$$\mathrm{A}_{ij}\Big|^{kl}\mathrm{N}^j=0, \qquad \mathrm{B}_{ijk}\Big|^{lmn}\mathrm{N}^k=0, \tag{24 a}$$
$$\mathrm{A}_{ij}\Big|^{kl}\mathrm{N}_l=0, \qquad \mathrm{B}_{ijk}\Big|^{lmn}\mathrm{N}_n=0. \tag{24 b}$$

From (23 *a*, *b*) and (24 *a*) we have immediately

$$\rho_{ij}\mathrm{N}^j=0, \qquad \gamma_{ijk}\mathrm{N}^k=0, \tag{25}$$

which means that the components $p_{ij}\mathrm{N}^j=0$, $c_{ijk}\mathrm{N}^k=0$ of the field equations do not contain second order derivatives of the field variables and hence they are *constraints* on the initial data.

Equations (24 *b*) imply that components of the form $\tau_k\mathrm{N}_t$ and $\chi_{lm}\mathrm{N}_n$ do not appear in the field equations. This is related to the fact, that the field equations are invariant under diffeomorphisms and local Lorentz transformations

$$\bar{\vartheta}^i{}_\mu=\Lambda^i{}_j\vartheta^j{}_\nu\left(\frac{\partial\bar{x}^\nu}{\partial x^\mu}\right), \tag{26 a}$$

$$\bar{\omega}^i{}_{j\mu}=\left[\Lambda^i{}_k\Lambda_j{}^l\omega^k{}_{l\nu}-\left(\frac{\partial}{\partial\bar{x}^\nu}\Lambda^i{}_k\right)\Lambda_j{}^k\right]\left(\frac{\partial\bar{x}^\nu}{\partial x^\mu}\right), \tag{26 b}$$

with

$$\det\left(\frac{\partial\bar{x}^\nu}{\partial x^\mu}\right)\neq 0, \qquad \Lambda^i{}_k(x)\,\eta^{kl}\Lambda^j{}_l(x)=\eta^{ij}. \tag{27}$$

Similar to what one does in general relativity ([7], [13]) we choose $\bar{x}^\mu(x)$ and $\Lambda^i{}_j(x)$ satisfying

$$\frac{\partial\bar{x}^\mu}{\partial x^\nu}=\delta^\mu_\nu, \qquad \partial_\rho\left(\frac{\partial\bar{x}^\mu}{\partial x^\nu}\right)=0, \qquad \Lambda^i{}_j=\delta^i_j, \qquad \partial_\mu\Lambda^i{}_j=0, \qquad \partial_\mu\partial_\nu\Lambda^i{}_j=0$$

on S, and such that the only non vanishing third order derivatives of them on S are

$$\partial_t^3\bar{x}^\mu=\tau^\mu, \qquad \partial_t^3\Lambda^i{}_j=\chi^i{}_j+\omega^i{}_{j\mu}\tau^\mu,$$

where $\chi_{ij}+\chi_{ji}=0$ because of the Lorentz relation (27). Then (26 *a*, *b*) imply $\bar{\vartheta}^i{}_\mu=\vartheta^i{}_\mu$, $\bar{\omega}^i{}_{j\mu}=\omega^i{}_{j\mu}$, $\partial_t\bar{\vartheta}^i{}_\mu=\partial_t\vartheta^i{}_\mu$, $\partial_t\bar{\omega}^i{}_{j\mu}=\partial_t\omega^i{}_{j\mu}$ and

$$\bar{\tau}_{ij}=\tau_{ij}+\tau_i\mathrm{N}_j, \tag{28 a}$$
$$\bar{\chi}_{ijk}=\chi_{ijk}-\chi_{ij}\mathrm{N}_k \tag{28 b}$$

on S. Thus $\tau_{ij}t^j$ and $\chi_{ijk}t^k$ can take arbitrary values, because of the invariance of PGT under diffeomorphisms and local Lorentz transformations. We conclude that the components $\vartheta^i{}_\mu\,t^\mu$ and $\omega^i{}_{j\mu}t^\mu$ are *unessential* for the physics of the theory and we need not bother about their evolution.

To find out if the remaining components of τ_{ij} χ_{ijk} can be determined from the field equations we need to specify the character of n_ρ. We assume therefore here, that S is a spacelike hypersurface, that is n_ρ satisfies

$$n^2 := n^\rho n_\rho > 0. \tag{29}$$

The case $n^2 < 0$ makes physically no sense, nevertheless the results obtained in this section apply equally well to this case. The remaining case $n^2 \doteq 0$ will be discussed in the next section.

Because of condition (29) it is convenient to set $t^\rho = \mathrm{N}^\rho$; then $n = \mathrm{N}$ and $n^\rho = \mathrm{N}\mathrm{N}^\rho$. Using this we obtain from (23 *a, b*), (22 *a, b*) and (13 *a, b*)

$$\begin{aligned}\rho_{ij} = -\mathrm{N}^2 [\bar{p}\,\tau_{ij} + \bar{q}\,\tau_{ji} + \bar{r}\,\eta_{ij}\,\tau^k{}_k - (\bar{p}\,\tau_{i\mathrm{N}}\,\mathrm{N}_j + \bar{q}\,\tau_{\mathrm{N}i}\,\mathrm{N}_j + \bar{q}\,\tau_{j\mathrm{N}}\,\mathrm{N}_i \\ - (\bar{q} + \bar{r})\,\tau_{\mathrm{N}j}\,\mathrm{N}_i + \bar{r}\,\mathrm{N}_i\,\mathrm{N}_j\,\tau^k{}_k + \bar{r}\,\eta_{ij}\,\tau_{\mathrm{NN}})],\end{aligned} \tag{30 a}$$

$$\begin{aligned}\gamma_{ijk} = -\mathrm{N}^2 [(p + q_2 + r_2)\,\chi_{ijk} + r_2\,\chi_{k\,[ij]} + q_2\,\eta_{k\,[i}\,\chi_{j]\,l}{}^l - ((p + q_2 + r_2)\,\chi_{ij\,\mathrm{N}}\,\mathrm{N}_k \\ + r_2\,\chi_{\mathrm{N}\,[ij]}\,\mathrm{N}_k + q_2\,\mathrm{N}_{[i}\,\chi_{j]\,l}{}^l\,\mathrm{N}_k + q_2\,\eta_{k\,[i}\,\chi_{j]\,\mathrm{NN}} + r_2\,\mathrm{N}_{[i}\,\chi_{j]\,k\,\mathrm{N}} \\ - (q_1 + r_2)\,\chi_{k\,\mathrm{N}\,[i}\,\mathrm{N}_{j]} + (q_2 + r_1)\,\eta_{k\,[i}\,\mathrm{N}_{j]}\,\chi_{\mathrm{N}\,l}{}^l - (q_2 + r_2)\,\mathrm{N}_{[i}\,\chi_{j]\,\mathrm{N}\,k})],\end{aligned} \tag{30 b}$$

where the abbreviation $u_\mathrm{N} := u_i\,\mathrm{N}^i$ was used. To analyse further these equations we decompose τ_{kl}, χ_{lmn} in components orthogonal and parallel to N_i. Using the projection tensor

$$\underline{\delta}^i_j := \delta^i_j - \mathrm{N}^i\,\mathrm{N}_j,$$

implicitely defined also in (19) for general N^ρ, we have

$$\tau_{ij} = \underline{\tau}_{ij} + \underline{\tau}_{i\mathrm{N}}\,\mathrm{N}_j + \mathrm{N}_i\,\underline{\tau}_{\mathrm{N}j} + \mathrm{N}_i\,\mathrm{N}_j\,\tau_{\mathrm{NN}}, \tag{31 a}$$

$$\chi_{ijk} = \underline{\varepsilon}_{ij}{}^l\,\underline{\chi}_{lk} + \underline{\varepsilon}_{ij}{}^l\,\underline{\chi}_{l\mathrm{N}}\,\mathrm{N}_k - \mathrm{N}_{[i}\,\underline{\chi}_{j]\,\mathrm{N}k} - \mathrm{N}_{[i}\underline{\chi}_{j]\,\mathrm{NN}}\,\mathrm{N}_k, \tag{31 b}$$

where

$$\underline{\tau}_{ij} := \underline{\delta}^k_i\,\delta^l_j\,\tau_{kl}, \qquad \underline{\tau}_{i\mathrm{N}} := \underline{\delta}^k_i\,\tau_{kl}\,\mathrm{N}^l, \qquad \ldots$$

and

$$\underline{\varepsilon}_{ijk} := \varepsilon_{ijk\mathrm{N}}, \qquad \underline{\chi}_{ij} := -\frac{1}{2}\,\varepsilon_i{}^{kl}\,\underline{\chi}_{klj}, \qquad \underline{\chi}_{i\mathrm{N}} := \frac{1}{2}\,\underline{\varepsilon}_i{}^{jk}\,\underline{\chi}_{jk\,\mathrm{N}}, \qquad \ldots$$

In the same way we decompose also the field equations (30 *a, b*) and express the results in terms of the projected components of τ_{ij}, χ_{ijk}. We find

$$\underline{\rho}_{ij} = -\mathrm{N}^2 (\bar{p}\,\underline{\tau}_{ij} + \bar{q}\,\underline{\tau}_{ji} + \bar{r}\,\underline{\eta}_{ij}\,\underline{\tau}^k{}_k), \tag{32 a}$$

$$\underline{\rho}_{\mathrm{N}i} = -\mathrm{N}^2 (\bar{p} + \bar{q} + \bar{r})\,\underline{\tau}_{\mathrm{N}i}, \tag{32 b}$$

$$\underline{\rho}_{i\mathrm{N}} = 0, \qquad \rho_{\mathrm{NN}} = 0 \tag{32 c}$$

and

$$\underline{\gamma}_{ij} = -\mathrm{N}^2 (p\,\underline{\chi}_{ij} + q_2\,\underline{\chi}_{ji} + r_2\,\underline{\eta}_{ij}\,\underline{\chi}^k{}_k), \tag{33 a}$$

$$\underline{\gamma}_{\mathrm{N}ij} = -\mathrm{N}^2 (p\,\underline{\chi}_{\mathrm{N}ij} + q_1\,\underline{\chi}_{\mathrm{N}ji} + r_1\,\underline{\eta}_{ij}\,\underline{\chi}_{\mathrm{N}k}{}^k), \tag{33 b}$$

$$\underline{\gamma}_{i\mathrm{N}} = 0, \qquad \underline{\gamma}_{i\mathrm{NN}} = 0. \tag{33 c}$$

Equations (32 *c*), (33 *c*) are the constraint equations (25) and the unessential components $\underline{\tau}_{iN}$, τ_{NN}, χ_{ijN}, χ_{iNN} do not appear in both systems (32), (33). To solve the remaining equations for the essential components conditions must be imposed on the coupling constants. From (32 *a*, *b*) we find the conditions:

$$\bar{p}+\bar{q}\neq 0 \quad \text{for the trace free, symmetric part of } \underline{\tau}_{ij}, \tag{34 a}$$
$$\bar{p}-\bar{q}\neq 0 \quad \text{for the antisymmetric part of } \underline{\tau}_{ij}, \tag{34 b}$$
$$\bar{p}+\bar{q}+3\bar{r}\neq 0 \quad \text{for the trace part of } \underline{\tau}_{ij}, \tag{34 c}$$
$$\bar{p}+\bar{q}+\bar{r}\neq 0 \quad \text{for } \underline{\tau}_{Ni}, \tag{34 d}$$

and similarly from (33 *a*, *b*) the conditions:

$$p+q_1\neq 0 \quad \text{for the trace free, symmetric part of } \underline{\chi}_{Nij}, \tag{35 a}$$
$$p-q_1\neq 0 \quad \text{for the antisymmetric part of } \underline{\chi}_{Nij}, \tag{35 b}$$
$$p+q_1+3r_1\neq 0 \quad \text{for the trace of } \underline{\chi}_{Nij}, \tag{35 c}$$
$$p+q_2\neq 0 \quad \text{for the trace free, symmetric part of } \underline{\chi}_{ij}, \tag{35 d}$$
$$p-q_2\neq 0 \quad \text{for the antisymmetric part of } \underline{\chi}_{ij}, \tag{35 e}$$
$$p+q_2+3r_2\neq 0 \quad \text{for the trace of } \underline{\chi}_{ij}. \tag{35 f}$$

We can express these results in the following form.

THEOREM 1. — *The Poincaré gauge theory in vacuum has a well posed analytic, initial value problem, if the Cauchy-Kowalevski conditions* (34) *and* (35) *are satisfied. In this case, equations* $0=\underline{p}_{ij}=\underline{p}_{Ni}$, $0=\underline{c}_{ijk}=\underline{c}_{Nij}$ *are evolution equations for the essential components of* $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ *and the Cauchy-Kowalevski theorem applies.*

Going back to the Noether identities (11 *a*, *b*) and decomposing the partial derivatives according to (19) we obtain

$$\left.\begin{array}{l}\partial_t(p_{ij}\mathrm{N}^j)+\underline{\partial}^j p_{ij}= \\ \partial_t(c_{ijk}\mathrm{N}^k)+\underline{\partial}^k c_{ijk}=\end{array}\right\} \text{linear homogeneous in } p_{ij},\, c_{ijk}. \tag{36}$$

This system reduces to a first order linear system for the constraints only, if we suppose, that the evolution equations are already solved for some initial data satisfying $p_{ij}\mathrm{N}^j=0$ and $c_{ijk}\mathrm{N}^k=0$ on S. The initial value problem for (36) possesses a unique solution, which for the initial data $p_{ij}\,\mathrm{N}^j=0=c_{ijk}\mathrm{N}^k$ for $t=0$, is obviously $p_{ij}\mathrm{N}^j=0=c_{ijk}\mathrm{N}^k$ for all t. Thus we have shown that under the Cauchy-Kowalevski conditions the exterior initial value problem of PGT can be formulated as in general relativity.

Conditions (34), (35) are sufficient but not necessary. The cases where some or all of them are violated must be studied separately. For example, if $\bar{p}=\bar{q}=\bar{r}=p=q_1=q_2=r_1=r_2=0$, PGT reduces to Einstein-Cartan theory with cosmological constant, which is first order in $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ and possesses a well posed initial value problem [14]. To study how the present results go over into the initial value problem of the Einstein-Cartan theory one has to include in (21) also terms of first order. The transition from the initial value problem of the full PGT to that of Einstein-Cartan theory is a difficult problem and will not be undertaken here. The teleparallelism theory can be obtained if one replaces equation $c_{ijk}=0$ by $\mathrm{R}_{ijkl}=0$. Here conditions (35) are no more relevant, but (34) still hold. More complicated becomes if only few of (34), (35) are violated. An example is the von der Heyde-Hehl theory ([15], [16]), where $\bar{p}+\bar{q}+\bar{r}=0$. In this theory equation (32 *b*) becomes

$$\underline{\rho}_{\mathrm{N}i}=0. \tag{37 a}$$

This contains only interior derivatives and hence is a constraint of the initial data. Additionally the term $\underline{e}_i{}^{\mu}\,\mathrm{N}^j\,\partial_t^2\,\vartheta_{j\mu}$ does not appear now in the field equations, and thus the evolution of the corresponding components of the tetrad is not clear. One way out of this problem is given, if $\underline{\rho}_{\mathrm{N}i}$ contains the term $\underline{e}_i{}^{\mu}\,\mathrm{N}^j\,\partial_t\,\vartheta_{j\mu}$ and (37 *a*) can be solved for it. In this case (37 *a*) takes the form

$$\underline{e}_i{}^{\mu}\,\mathrm{N}^j\,\partial_t\,\vartheta_{j\mu}=\underline{\sigma}_{\mathrm{N}i} \tag{37 b}$$

and can be interpreted again as an evolution equation. The conditions under which this can be done will in general restrict the initial data further. It will be therefore necessary to show that these new conditions are preserved in time. Another possibility is that the theory possesses some hidden symmetry, which will make these components of the tetrad field to be unessential. The Noether identity associated to the hidden symmetry will then perhaps guarantee the time conservation of the new constraint. We will not discuss such degenerate cases any further.

4. CHARACTERISTIC HYPERSURFACES OF PGT

The study of the field equations in the last section was based essentially on the assumption $n^2\neq 0$. If

$$n^2=g^{\mu\nu}\frac{\partial t}{\partial x^{\mu}}\,\frac{\partial t}{\partial x^{\nu}}=0, \tag{38}$$

then the situation changes drastically. The field equations do not determine any more the propagation of all essential field variables. S becomes a characteristic hypersurface of PGT. We shall now investigate how many of the outgoing second order derivatives of the essential components of the field variables can be determined on a characteristic hypersurface of PGT.

On a characteristic hypersurface the field equations become

$$\rho_{ij}=\bar{p}\,\tau_{iN}\,N_j+\bar{q}\,\tau_{Ni}\,N_j+\bar{q}\,\tau_{jN}\,N_i -(\bar{q}+\bar{r})\,\tau_{Nj}\,N_i+\bar{r}\,N_i\,N_j\,\tau^k{}_k+\bar{r}\,\eta_{ij}\,\tau_{NN}, \quad (39\,a)$$

$$\gamma_{ijk}=(p+q_2+r_2)\,\chi_{ijN}\,N_k+r_2\,\chi_{N\,[ij]}\,N_k +q_2\,N_{[i}\chi_{j]\,l}{}^l\,N_k+q_2\,\eta_{k\,[i}\,\chi_{j]\,NN}+r_2\,N_{[i}\,\chi_{j]\,k\,N}-(q_1+r_2)\,\chi_{kN\,[i}\,N_{j]} +(q_2+r_1)\,\eta_{k\,[i}\,N_{j]}\,\chi_{Nl}{}^l-(q_2+r_2)\,N_{[i}\,\chi_{j]\,N\,k}. \quad (39\,b)$$

We make a (2+2)-decomposition of the field variables using a lightlike vector L_i, which satisfies $L_i N^i=1$ (²). With it we construct a projection tensor

$$\underline{\delta}^i_j:=\delta^i_j-L^i N_j-N^i L_j \quad (40)$$

and decompose τ_{ij} χ_{ijk} to obtain

$$\tau_{ij}=\underline{\tau}_{ij}+L_i\,\underline{\tau}_{Nj}+N_i\,\underline{\tau}_{Lj} +(\underline{\tau}_{iN}+\tau_{NN}\,L_i+\tau_{LN}\,N_i)\,L_j+(\underline{\tau}_{iL}+\tau_{NL}\,L_i+\tau_{LL}\,N_i)\,N_j \quad (41\,a)$$

and

$$\chi_{ijk}=\underline{\varepsilon}_{ij}\,\underline{\chi}_k-L_{[i}\,\underline{\chi}_{j]\,Nk}-N_{[i}\,\underline{\chi}_{j]\,Lk}+\underline{\chi}_{LNk}\,N_{[i}\,L_{j]} +(\underline{\varepsilon}_{ij}\,\chi_N-L_{[i}\,\underline{\chi}_{j]\,NN}-N_{[i}\,\chi_{j]\,LN}+\chi_{LNN}\,N_{[i}\,L_{j]})\,L_k +(\underline{\varepsilon}_{ij}\,\chi_L-L_{[i}\,\underline{\chi}_{j]\,NL}-N_{[i}\,\underline{\chi}_{j]\,LL}+\chi_{LNL}\,N_{[i}\,L_{j]})\,L_k. \quad (41\,b)$$

Here as in (31 *a*, *b*) we have set

$$\underline{\varepsilon}_{ij}:=\varepsilon_{ijNL}, \qquad \underline{\chi}_k:=-\frac{1}{2}\underline{\varepsilon}^{ij}\,\underline{\chi}_{ijk}, \qquad \chi_{L,N}:=-\frac{1}{2}\underline{\varepsilon}^{ij}\,\underline{\chi}_{ij\,L,N}, \qquad \dots$$

Decomposing also the field equations we obtain

$$\underline{\rho}_{ij}=\bar{r}\,\underline{\eta}_{ij}\,\tau_{NN}, \qquad \underline{\rho}_{Ni}=0, \qquad \underline{\rho}_{iN}=0, \qquad \rho_{NN}=0, \quad (42\,a)$$

$$\left.\begin{aligned} \rho_{LN}=0, \qquad \rho_{NL}=(\bar{p}+\bar{q}+\bar{r})\,\tau_{NN}, \\ \rho_{LL}=(\bar{p}+\bar{q}+\bar{r})\,\tau_{LN}+\bar{r}\,\underline{\tau}^k{}_k, \end{aligned}\right\} \quad (42\,b)$$

$$\underline{\rho}_{iL}=\bar{p}\,\underline{\tau}_{iN}+\bar{q}\,\underline{\tau}_{Ni}, \qquad \underline{\rho}_{Li}=\bar{q}\,\underline{\tau}_{iN}-(\bar{q}+\bar{r})\,\underline{\tau}_{Ni}. \quad (42\,c)$$

The last row of (41 *a*) contains the 4 unessential components of τ_{ij}. These remain undetermined by the field equations on characteristic hypersurfaces. Of the other components τ_{NN}, τ_{LN} and $\underline{\tau}_{iN}$ are determined, and give the second row of (41 *a*). From the first row of (41 *a*) only $\underline{\tau}_{Ni}$ is determined or not according to whether

$$\bar{\Delta}=\det\begin{pmatrix}\bar{p} & \bar{q}\\ \bar{q} & -(\bar{q}+\bar{r})\end{pmatrix}=-[\bar{p}(\bar{q}+\bar{r})+\bar{q}^2]$$

(²) One can set for example $L^\rho:=t^\rho-(1/2)\,N^\rho$.

is different from or equal to zero. Thus of the twelve essential components of τ_{ij} we have either six undetermined components $\underline{\tau}_{ij}$, $\underline{\tau}_{Lj}$, if $\bar{\Delta}\neq 0$, or eight $\underline{\tau}_{ij}$, $\underline{\tau}_{Lj}$, $\underline{\tau}_{Nj}$, if $\bar{\Delta}=0$. In the later case

$$\bar{p}(\bar{q}+\bar{r})+\bar{q}^2=0 \tag{43}$$

and we can eliminate $\bar{r}$ from the field equations. Then (39 *a*) can be written in the form

$$\rho_{ij}=\frac{1}{\bar{p}}\alpha_{ij}|^{kl}\mathrm{A}_k\mathrm{N}_l, \qquad \mathrm{A}_i:=\alpha_{ij}|^{kl}\mathrm{N}^j\tau_{kl} \tag{44 a}$$

with

$$\alpha_{ij}|^{kl}:=\bar{p}\,\delta_i^k\,\delta_j^l+\bar{q}\,\delta_j^k\,\delta_i^l-\bar{q}\,\eta_{ij}\,\eta^{kl}, \tag{44 b}$$

$\bar{p}\neq 0$ follows from (34) and (43).

From the second field equation (39 *b*) we find

$$\gamma_{\mathrm{N}}=0, \qquad \underline{\gamma}_{\mathrm{N}ij}=0, \qquad \underline{\gamma}_{i\mathrm{NN}}=0, \qquad \underline{\gamma}_{i\mathrm{LN}}=0, \qquad \gamma_{\mathrm{NLN}}=0, \tag{45 a}$$

$$\underline{\gamma}_i=-q_2\,\underline{\varepsilon}_i{}^j\,\underline{\chi}_{j\mathrm{NN}}, \qquad \underline{\gamma}_{\mathrm{NL}i}=-q_1\,\underline{\chi}_{i\mathrm{NN}}, \qquad \underline{\gamma}_{i\mathrm{NL}}=p\,\underline{\chi}_{i\mathrm{NN}}, \tag{45 b}$$

$$\underline{\gamma}_{i\mathrm{LL}}=p\,\underline{\chi}_{i\mathrm{LN}}-q_2\,\underline{\varepsilon}_i{}^j\,\underline{\chi}_j-q_1\,\underline{\chi}_{\mathrm{LN}i}, \tag{45 c}$$

$$\left.\begin{aligned}\gamma_{\mathrm{L}}&=(p+q_2+r_2)\,\chi_{\mathrm{N}}-r_2\,\underline{\varepsilon}^{ij}\,\underline{\chi}_{\mathrm{N}ij},\\ \gamma_{\mathrm{NLL}}&=-(p+q_1+r_1)\,\chi_{\mathrm{LNN}}+r_1\,\underline{\chi}_{\mathrm{N}k}{}^k,\end{aligned}\right\} \tag{45 d}$$

$$\underline{\gamma}_{\mathrm{L}ij}=r_1\,\underline{\eta}_{ij}\,\chi_{\mathrm{LNN}}+r_2\,\underline{\varepsilon}_{ij}\,\chi_{\mathrm{N}}+(q_2+r_2)\,\underline{\chi}_{\mathrm{N}ij} -(q_1+r_2)\,\underline{\chi}_{\mathrm{N}ji}-(q_2+r_1)\,\underline{\eta}_{ij}\,\underline{\chi}_{\mathrm{N}k}{}^k. \tag{45 e}$$

Decomposing the last equation (45 *e*) into a trace free, symmetric part $\check{\underline{\gamma}}_{\mathrm{L}ij}$, an antisymmetric part $\hat{\gamma}_{\mathrm{L}}:=-\frac{1}{2}\underline{\varepsilon}^{ij}\,\underline{\gamma}_{\mathrm{L}ij}$ and a trace part $\tilde{\gamma}_{\mathrm{L}}:=\underline{\eta}^{ij}\,\underline{\gamma}_{\mathrm{N}ij}$ we obtain from (45 *d*, *e*)

$$\check{\underline{\gamma}}_{\mathrm{L}ij}=(q_2-q_1)\,\check{\underline{\chi}}_{\mathrm{N}ij}, \tag{46 a}$$

$$\left.\begin{aligned}\gamma_{\mathrm{L}}&=(p+q_2+r_2)\,\chi_{\mathrm{N}}+2\,r_2\,\hat{\chi}_{\mathrm{N}}\\ \hat{\gamma}_{\mathrm{L}}&=r_2\,\chi_{\mathrm{N}}+(q_1+q_2+2\,r_2)\,\hat{\chi}_{\mathrm{N}},\\ \Delta_2&:=\det\begin{pmatrix}p+q_2+r_2 & 2\,r_2\\ r_2 & q_1+q_2+2\,r_2\end{pmatrix},\end{aligned}\right\} \tag{46 b}$$

$$\left.\begin{aligned}\gamma_{\mathrm{NLL}}&=-(p+q_1+r_1)\,\chi_{\mathrm{LNN}}+r_1\,\tilde{\chi}_{\mathrm{N}}\\ \tilde{\gamma}_{\mathrm{L}}&=2\,r_1\,\chi_{\mathrm{LNN}}-(q_1+q_2+2\,r_1)\,\tilde{\chi}_{\mathrm{N}},\\ \Delta_1&:=\det\begin{pmatrix}p+q_1+r_1 & -r_1\\ -2\,r_1 & q_1+q_2+2\,r_1\end{pmatrix}.\end{aligned}\right\} \tag{46 c}$$

Counting the number of components of χ_{ijk}, which are determined from the field equations, we find (i) the last row of (41 *b*) consists of the unessential components, which are undetermined, (ii) the second row of (41 *b*) is completely determined and (iii) from the first row of (41 *b*) all components are undetermined except of $\underline{\chi}_{Nij}$, which is determined if $q_1 - q_2 \neq 0$, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$. Thus if these conditions hold we have eight undetermined essential components χ_i, $\underline{\chi}_{Lij}$, $\underline{\chi}_{LNi}$ and in the other extrem, if $q_1 = q_2$, $\Delta_1 = 0 = \Delta_2$ we have twelve undetermined essential components. In this case

$$q_1 = q_2 =: q, \qquad r_1 = r_2 =: r, \qquad (p+q)\,q + r\,(p+2\,q) = 0 \tag{47}$$

and equation (39 *b*) takes the compact form

$$\gamma_{ijk} = \frac{1}{p+2\,q}\,\beta_{ijk}\big|^{lmn}\,\mathrm{B}_{lm}\,\mathrm{N}_n, \qquad \mathrm{B}_{ij} := \beta_{ijk}\big|^{lmn}\,\mathrm{N}^k\,\chi_{lmn} \tag{48 a}$$

with

$$\beta_{ijk}\big|^{lmn} := \frac{1}{2}\left[(p+q)\,\delta^{lm}_{ij}\,\delta^n_k - q\,\delta^{lm}_{k\,[i}\,\delta^n_{j]} + q\,\eta_{k\,[i}\,\delta^{[l}_{j]}\,\eta^{m\,]n}\right], \tag{48 b}$$

Again $p+2\,q \neq 0$ because of (35) and (47).

It is obvious that conditions (43) and (47) will be important in the characteristic initial value problem of PGT. Note also that, if we set $\rho_{ij} = 0$, $\gamma_{ijk} = 0$ in (39 *a*, *b*), then the resulting equations are satisfied by the second order discontinuities of the field variables on characteristic hypersurfaces. The situation here is again different from that in general relativity ([17], [18]). Since (43), (47) decide the behaviour of the field equations on characteristics we expect them to play an important role also in the next section.

5. HYPERBOLICITY OF PGT

The restriction on analytic solutions of the field equations is an unnatural restriction on the physics of the theory. We lòok therefore for conditions under which PGT takes the form of a hyperbolic system of partial differential equations. In this undertaking we will be guided by the situation in general relativity, where the second order terms of the field equations take the form

$$\Box\, g_{\mu\nu} - \frac{1}{2}\,\partial_{[\mu}\,\Gamma_{\nu]}, \qquad \Gamma^\mu := g^{\rho\sigma}\left\{ {\mu \atop \rho\sigma} \right\}.$$

Condition $\Gamma^\mu = 0$, known as Hilbert-de Donder gauge condition, reveals then the hyperbolic character of the field equations of general relativity [1]. Having this in mind we try to write the second order terms of the field equations of PGT in the form

$$\mathrm{A}\,\Box\, u + \mathrm{B}\,\partial\,\mathrm{F}, \qquad \mathrm{F} = \mathrm{C}\,\Box\, u,$$

where u represents the field variables $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ and A, B, C are matrices to be determined. Isolating the second order terms of the field equations (16 *a*, *b*) we find

$$0=p_{ij}=-2\,A_{ijm}\big|^{klm}e^{m\mu}e_n{}^\nu(e_l{}^\rho\,\partial_\mu\,\partial_\nu\,\vartheta_{k\rho})+U_{ij} \tag{49 a}$$

and

$$0=c_{ijk}=-2\,B_{ijkr}\big|^{lmns}e^{r\mu}e_s{}^\nu(e_n{}^\rho\,\partial_\mu\,\partial_\nu\,\omega_{lm\rho})+U_{ijk}, \tag{49 b}$$

where U_{ij}, U_{ijk} represent terms of order lower than two. Limited by the fact, that we must construct expressions F_i out of $\partial_\nu\vartheta_{k\rho}$ and F_{ij} out of $\partial_\nu\omega_{lm\rho}$, we find that the most general Ansatz ([3]) for the F's is

$$F_i=a_{ij}\big|^{kl}e^{j\nu}(e_l{}^\rho\,\partial_\nu\,\vartheta_{k\rho}), \tag{50 a}$$

$$F_{ij}=b_{ijk}\big|^{lmn}e^{k\nu}(e_n{}^\rho\,\partial_\nu\,\omega_{lm\rho}) \tag{50 b}$$

with

$$a_{ij}\big|^{kl}=x_{ij}\big|^{kl}(a_1,a_2,a_3):=a_1\,\delta_i^k\,\delta_j^l+a_2\,\delta_j^k\,\delta_i^l+a_3\,\eta_{ij}\,\eta^{kl} \tag{51 a}$$

and

$$b_{ijk}\big|^{lmn}=y_{ijk}\big|^{lmn}(b_1,b_2,b_3)$$

$$:=\frac{1}{2}(b_1\,\delta_{ij}^{lm}\,\delta_k^n+b_2\,\delta_{k[i}^{lm}\,\delta_{j]}^n+b_3\,\eta_{k[i}\,\delta_{j]}^{[l}\,\eta^{m]n}), \tag{51 b}$$

where $a_1,\ldots,b_3$ are constants to be determined. A first restriction on these constants comes from the fact, that for given functions $f_i(x)$, $f_{ij}(x)$ the expressions $F_i=f_i$, $F_{ij}=f_{ij}$ must be gauge conditions. Thus it must be possible, for given coordinate system x'^μ with $\vartheta'^i{}_\mu$, $\omega'^i{}_{j\mu}$, to find in an unique way a coordinate transformation $x^\mu(x')$ and a local Lorentz transformation $\Lambda^i{}_j(x)$, such that the transformed field variables $\vartheta^i{}_\mu$, $\omega^i{}_{j\mu}$ satisfy the gauge conditions $F_i=f_i$, $F_{ij}=f_{ij}$.

Since under a coordinate transformation we have

$$\vartheta^i{}_\mu=\vartheta^i{}_{\mu'}\,\partial_\mu\,x^{\mu'} \tag{52 a}$$

and under a local Lorentz transformation

$$\omega_{ij\mu}=\Lambda_i{}^k\,\Lambda_j{}^l\,\omega'_{kl\mu}-(\partial_\mu\,\Lambda_i{}^k)\,\Lambda_{jk},\qquad \vartheta^i{}_\mu=\Lambda^i{}_j\,\vartheta'^j{}_\mu, \tag{52 b}$$

substitution of these expressions in $F_i=f_i$, $F_{ij}=f_{ij}$ gives for $x^\mu(x')$, $\Lambda^i{}_j(x)$

$$a_1\,\Box'\,x^\mu+(a_2+a_3)(\partial_{\mu'}\,x^\mu)(\partial_\nu\,x^{\nu'})(\partial^{\mu'}\,\partial_{\nu'}\,x^\nu)+\ldots=-f_i\,e^{i\mu} \tag{53 a}$$

and

$$b_1\,\Box\,\Lambda_i{}^j-(b_2+b_3)\,e'^\nu_k\,e'^\rho_l\,(\partial_\nu\,\partial_\rho\,\Delta_{[i}{}^k)\,\Lambda_{m]}{}^l\,\Lambda^{mj}+\ldots=-f_{ik}\,\Lambda^{kj}, \tag{53 b}$$

([3]) This was motivated by unpublished work H. Goenner's on gauge conditions of curvature square theories in Riemannian geometry.

with $\Box' = g^{\mu'\nu'}\partial_{\mu'}\partial_{\nu'}$, $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$. These differential equations become obviously hyperbolic if

$$a_2 + a_3 = 0 = b_2 + b_3, \qquad a_1 \neq 0 \neq b_1$$

and they give unique solutions for $x^\mu(x')$, $\Lambda_i{}^j(x)$. We set therefore

$$a_1 = b_1 = 1, \qquad a_2 = -a_3 =: a, \qquad b_2 = -b_3 =: b \tag{54}$$

and

$$a_{ij}|^{kl} := x_{ij}|^{kl}(1, a, -a), \qquad b_{ijk}|^{lmn} := y_{ijk}|^{lmn}(1, b, -b). \tag{55}$$

Note that for $a=1$ equation $F_i = 0$ is the Hilbert-de Donder gauge condition in tetrad formalism.

To construct the terms containing the gauge conditions in the field equations we have expressions $\partial_\mu F_i$, $\partial_\mu F_{ij}$ from which objects with the index structure of p_{ij}, c_{ijk} must be obtained. Again the most general Ansatz is of the form

$$\bar{F}_{ij} := \bar{a}_{ij}|^{kl}(e_l{}^\mu \partial_\mu F_k), \qquad \bar{F}_{ijk} := \bar{b}_{ijk}|^{lmn}(e_n{}^\mu \partial_\mu F_{lm}) \tag{56}$$

with

$$\bar{a}_{ij}|^{kl} := x_{ij}|^{kl}(a_1, a_2, a_3), \qquad \bar{b}_{ijk}|^{lmn} := y_{ijk}|^{lmn}(b_1, b_2, b_3). \tag{57}$$

Substitution of (50 a, b) in (56) gives up to first order terms

$$\bar{F}_{ij} = \bar{a}_{ij}|^{rn}\, a_{rm}|^{kl}\, e^{m\mu} e_n{}^\nu (e_l{}^\rho \partial_\mu \partial_\nu \vartheta_{k\rho}) + \dots,$$
$$\bar{F}_{ijk} = \bar{b}_{ijk}|^{abs}\, b_{abr}|^{lmn}\, e^{r\mu} e_s{}^\nu (e_n{}^\rho \partial_\mu \partial_\nu \omega_{lm\rho}) + \dots$$

Inserting these expressions in the field equations we obtain

$$0 = p_{ij} = -[2\,A_{ijm}|^{kln} + \bar{a}_{ij}|^{rn}\, a_{rm}|^{kl}]\;\; e^{m\mu} e_n{}^\nu (e_l{}^\rho \partial_\mu \partial_\nu \vartheta_{k\rho}) + \bar{F}_{ij} + U_{ij}, \tag{58 a}$$

$$0 = c_{ijk} = -[2\,B_{ijkr}|^{lmns} + \bar{b}_{ijk}|^{abs}\, b_{abr}|^{lmn}] \times e^{r\mu} e_s{}^\nu (e_n{}^\rho \partial_\mu \partial_\nu \omega_{lm\rho}) + \bar{F}_{ijk} + U_{ijk} \tag{58 b}$$

with U_{ij}, U_{ijk} properly redefined. We demand now that the first terms on the right sides of (58 a, b) can be written in the form

$$-\frac{1}{2}[2\,A_{ijm}|^{kln} + \bar{a}_{ij}|^{rn}\, a_{rm}|^{kl}]\, e^{m(\mu} e_n{}^{\nu)} = -a'_{ij}|^{kl} g^{\mu\nu},$$

$$-\frac{1}{2}[2\,B_{ijkr}|^{lmns} + \bar{b}_{ijk}|^{abs}\, b_{abr}|^{lmn}]\, e^{r(\mu} e_s{}^{\nu)} = -b'_{ijk}|^{lmn} g^{\mu\nu}$$

for some matrices $a'_{ij}|^{kl}$, $b'_{ijk}|^{lmn}$. This gives equations for the constants a, a_1, a_2, a_3 and b, b_1, b_2, b_3 to be solved in terms of the coupling constants. In fact these equations can be solved only, if the coupling constants satisfy conditions (43) and (47) of the last section. In this case the gauge conditions become

$$F_i = \frac{1}{p}\, \alpha_{ij}|^{kl} e^{j\nu} e_l{}^\rho \partial_\nu \vartheta_{k\rho}, \tag{59 a}$$

$$\mathrm{F}_{ij} = \frac{1}{p+q} \beta_{ijk}\big|^{lmn} e^{k\nu} e_n{}^{\rho} \partial_\nu \omega_{lm\rho} \tag{59 b}$$

and the field equations take the form

$$0 = p_{ij} = -\alpha_{ij}\big|^{kl} e_l{}^{\rho} (\Box \vartheta_{k\rho} - \partial_\rho \mathrm{F}_k) + \mathrm{U}_{ij}, \tag{60 a}$$

$$0 = c_{ijk} = -\frac{p+q}{p+2q} \beta_{ijk}\big|^{lmn} e_n{}^{\rho} (\Box \omega_{lm\rho} - \partial_\rho \mathrm{F}_{lm}) + \mathrm{U}_{ijk}, \tag{60 b}$$

where the matrices $\alpha_{ij}|^{kl}$, $\beta_{ijk}|^{lmn}$ are defined in (44 *b*) and (48 *b*). Note that $\bar{p} \neq 0$, $p+q \neq 0$, $p+2q \neq 0$ and the matrices α, β are invertible because of the Cauchy-Kowalevski conditions (34), (35) of PGT. Equations (43) and (47) we call *hyperbolicity conditions* of PGT. We summarize these results as follows:

THEOREM 2. — *The field equations of PGT take the form* $\mathrm{A}\,\Box u + \mathrm{B}\,\partial \mathrm{F} + \ldots = 0$ *with* $\mathrm{F} = \mathrm{C}\,\partial u$, *if the hyperbolicity conditions* (43), (47) *hold. Matrix* A *is invertible, if additionality the Cauchy-Kowalevski conditions* (34), (35) *are satisfied. In this case* PGT *becomes under the gauge fixing conditions* $\mathrm{F}_i = 0$, $\mathrm{F}_{ij} = 0$ *the hyperbolic system of equations* (60 *a*, *b*).

The hyperbolicity conditions restrict the number of coupling constants of PGT from ten to six. Equation (43) is identical to condition (4.15 *b*) of [9] obtained from the requirement that p^{-4} (in their notation) terms cancel in the linearization of $p_{ij} = 0$.

One can object that our method of proving hyperbolicity is not the most general one, since it demands the field equations to be written in the form $\mathrm{A}\,\Box\, u + \ldots$ In case of equations with constant coefficients more general methods are known, as is Gårding's hyperbolicity condition [12]. Thus a test for the necessity of our hyperbolicity conditions (43), (47) can be obtained, if Gårding's method can be applied on linearized PGT. This is not a trivial task since Gårding's method is given for one field variable and is based on the assumption that the initial value problem can be put in a standard form, which in PGT is prevented by the constraint equations.

A method which applies also to quasilinear systems is to use first order formalism and apply K. O. Friedrichs theory of symmetric hyperbolic systems ([11], [12]). We have done this already (*see* [19] and [20] for general relativity) and obtained the same hyperbolicity conditions as here. We report on this in a separate paper.

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