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P. LOCHAK
A. PORZIO

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A realistic exponential estimate for a paradigm hamiltonian

by

P. Lochak

C.M.A. École Normale Supérieure
45, rue d'Ulm, 75230 Paris Cedex, 05 France

and

A. Porzio

C.P.T.H. École Polytechnique, 91128 Palaiseau, France

ABSTRACT. — We prove a Nekhoroshev-type theorem for a simple but important one-dimensional nonautonomous hamiltonian system. By adapting the classical perturbation scheme to our model, we are able to find a realistic threshold of validity of our result. Some generalizations are outlined expecially in the aim of discussing the true dependence on the degrees of freedom of the exponential estimates for the stability times.

RÉSUMÉ. — On démontre un théorème de type Nekhoroshev pour un système hamiltonien unidimensionnel dépendant du temps, simple mais important. La théorie classique des perturbations est adaptée au système étudié et permet d'obtenir des estimations réalistes. On discute des modèles plus généraux, en particulier pour la bonne dépendance par rapport aux degrés de liberté des estimations exponentielles pour les temps de stabilité.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Although less revolutionary — with respect to classical perturbation theory — than the KAM type results, Nekhoroshev's theorem appeared in fact much later [Nekh]. We shall first recall informally some of its main features, so as to put the present work in perspective. As is usual in hamiltonian perturbation theory, one starts with a near integrable system governed by the hamiltonian

$$H(A, \varphi) = h(A) + \varepsilon f(A, \varphi), \qquad A \in D \subset \mathbb{R}^d, \quad \varphi \in \mathbb{T}^d \tag{1}$$

where (A, φ) are the action-angle variables of the unperturbed hamiltonian h, and εf is a perturbation, as emphasized by the explicit appearance of the small parameter ε. One works on a domain $D \times \mathbb{T}^d$ where D is a "nice" — say convex — subset of $\mathbb{R}^d$ and $\mathbb{T}^d$ is the d dimensional torus. In fact, Nekhoroshev's theorem is an essentially *analytic* result in that it requires that the hamiltonian H can be extended to a complex neighborhood of the above domain (this is made precise below in the particular case we shall deal with), or equivalently that the Fourier coefficients decrease at an exponential rate.

The analytical technique then consists in performing the classical algorithm of step by step elimination of the angles to an order N which however depends on ε. Typically it is shown that one may set $N = N(\varepsilon) = \varepsilon^{-a}$, $(0 < a \leqq 1)$, a choice which makes remainders on the order of ε^N exponentially small in ε. In conjunction with some non trivial geometric ideas, this allows to bound the deviation of the action variable(s) *globally* in phase space (*i. e.* without the occurrence of arithmetic conditions as is the case in the KAM theorem) over exponentially long intervals of time, ascertaining in particular that Arnold's diffusion can take place only on this extremely slow time-scale. Complete proofs of the theorem can be found in the original papers (with a generic assumption of steepness on h) and in the physically relevant case of quasi-convex hamiltonians (energy surfaces are convex) in [BGG] and [BG].

As is the case for the KAM theorem, it is however extremely difficult to obtain analytic *realistic* estimates, in particular for what concerns the threshold of validity of the results. Already in the most favorable situation $(d = 2, h$ convex) the above cited papers provide estimates which are valid only for $\varepsilon \leqq 10^{-50}$, a number which is not only physically irrelevant but also numerically inaccessible. It is in fact very likely that no realistic *general* estimate can be obtained and that one has to resort to a case by case method, exploiting the peculiarities of the situation at hand. Very few problems have as yet been examined along these lines; notable among them is the stability of the Lagrange (L4) point of the three-body problem, discussed in [GDFGS] who show how the general estimates can be enormously improved to the point of furnishing a physically satisfactory result.

Here we examine the important—albeit simple—hamiltonian with one and a half degree of freedom

$$H(A, \varphi, t)=\frac{A^2}{2}+\varepsilon(\cos\varphi+\cos(\varphi-t)), \qquad A\in D\subset\mathbb{R}, \quad \varphi\in\mathbb{T} \tag{2}$$

This is certainly one of the simplest non trivial (*i. e.* non integrable) hamiltonians and we postpone to the end of the paper the reasons which make it interesting to apply Nekhoroshev's technique to a system with so few degrees of freedom. The reader will also find there outlined some possible generalizations of the result stated below. If E is taken as the variable conjugate to t, (2) may be written under the form (1) with $h=\frac{1}{2}A^2+E$ which is indeed quasi convex. Systems which may be described by (2) arise (more or less naturally...) in various physical situations ([Es], [ED]); besides, it has been used to build an approximate renormalization scheme ([Es], [ED]) and also to find rigorous—computer assisted—estimates of the break-up threshold of KAM tori [Cel-Ch]. In fact, until now, numerical and realistic theoretical estimates connected with the KAM theorem have almost all been performed on the standard map and the hamiltonian (2); they also almost all deal with the breaking of the "golden torus", with rotation number $\rho=\frac{\sqrt{5}-1}{2}\simeq 0.618$ close to the unperturbed one given by $A=\rho$ [Cel—Ch].

Here we prove the following estimate. We consider the hamiltonian (2) as defined on $D_{\rho,\xi}=I^r_\rho\times\mathbb{T}^2_\xi$, $1\geqq\xi>0$, $\rho>0$. $D_{\rho,\xi}$ is itself defined as follows. We pick $A_0\in(0,1)$ and $(\gamma, 1)$-diophantine, in the sense that

$$|k_1A_0+k_2|\geqq\gamma(|k_1|+|k_2|)^{-1}, \qquad (k_1, k_2)\in\mathbb{Z}^2\diagdown(0,0), \quad \gamma>0 \tag{3}$$

$$\left(\text{For } A_0=\frac{\sqrt{5}-1}{2} \text{ it is } \gamma^{-1}=\frac{\sqrt{5}+3}{2}\right).$$

We set

$$I^r=[A_0-r, A_0+r]\subset\mathbb{R},\ I^r_\rho=\{A\in\mathbb{C},\ \mathrm{dist}(A, I^r)\leqq\rho\}\subset\mathbb{C}$$
$$T^2_\xi=\{(\varphi, t)\in\mathbb{C}^2, |\mathrm{Im}(\varphi, t)|\leqq\xi\}$$

Then we have the following

THEOREM. — *For* $A(0)\in[A_0-50\varepsilon, A_0+50\varepsilon]$, $A_0\in\left(\frac{1}{4}, \frac{3}{4}\right)$, *and* $(\gamma, 1)$-*diophantine and for any t such that*

$$|t|\leqq T(\varepsilon, \gamma)=400\gamma^{-1}(\varepsilon/\gamma)^{3/2}\exp[(5\sqrt{\varepsilon/\gamma})^{-1}]$$

one has $|A(t)-A(0)|\leqq 20\varepsilon$ *if* $|\varepsilon|\leqq\varepsilon^*=2.10^{-4}\gamma$.

Observe that the time T may be given a natural interpretation as a number of revolutions around the cylinder, which is very close to $\frac{A(0)T}{2\pi}$.

The core of the paper is devoted to the proof of the above Nekhoroshev-type estimate. In the last section we further explain its meaning and discuss some possible natural extensions.

II. PROOF OF THE THEOREM

1. Formal perturbation theory and recursive relations

As is usual, the estimate in the theorem is proved by building a normal form which locally conjugates the system to an other one which is integrable up to an exponentially small remainder. This conjugacy is performed via a canonical transform S : $S(A, \varphi)=(A', \varphi')$ generated by a generating function $A'\varphi+F$

$$\begin{aligned} A &= A' + \frac{\partial F}{\partial \varphi}(A', \varphi, t) \\ \varphi' &= \varphi + \frac{\partial F}{\partial A'}(A', \varphi, t) \end{aligned} \tag{4}$$

$$F(A', \varphi, t)=\sum_{p=1}^{N} \varepsilon^p \sum_{l,\, m \in \mathbb{Z}^2} F_{lm}^p(A')\, e^{i((l+m)\varphi - mt)} \equiv \sum_{p=1}^{N} \varepsilon^p F^p. \tag{5}$$

The new hamiltonian reads

$$H'(A', \varphi, t)=\frac{A'^2}{2}+A'\frac{\partial F}{\partial \varphi}+\frac{1}{2}\left(\frac{\partial F}{\partial \varphi}\right)^2+\varepsilon \cos \varphi+\varepsilon \cos(\varphi - t)+\frac{\partial F}{\partial t} \tag{6}$$

and the new action variable A' evolves according to

$$\frac{dA'}{dt}=-\frac{\partial H'}{\partial \varphi}\frac{\partial \varphi'}{\partial \varphi}=-\frac{\partial H'}{\partial \varphi}\left(1+\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right)^{-1} \tag{7}$$

so that

$$\left|\frac{dA'}{dt}\right| \leq \left|\frac{\partial H'}{\partial \varphi}\right| \left|\left(1+\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right)^{-1}\right| \tag{8}$$

The generating function F is built directly from (6), requiring that H' be "as independent as possible" of the angles φ, t. We emphasize that no inversion is needed at this level as is seen from (7) and (8), provided $\frac{\partial^2 F}{\partial \varphi\, \partial A'}$ remains sufficiently small.

The recurrence for the p-th order term F^p of the generating function is

$$F^p_{lm} = -\frac{1}{2}[(l+m)A' - m]^{-1} \sum_{\substack{jkq \\ j'k'q'}}^{*} (j+k)(j'+k') F^q_{jk} F^{q'}_{j'k'} \tag{9}$$

where $\sum^*$ indicates the constraints $q+q'=p$, $j+j'=l$, $k+k'=m$. F^1 explicitly reads

$$\begin{gathered} F^1 = -\frac{\sin\varphi}{A'} - \frac{\sin(\varphi - t)}{(A'-1)} \; i.e.\; F^1_{\pm 10} = \mp \frac{1}{2iA'}, \\ F^1_{0\pm 1} = \mp \frac{1}{2i(A'-1)}. \end{gathered} \tag{10}$$

It is important to notice that (as is checked by induction) $F^p_{lm}=0$ if $|l|+|m|>p$.

We now define a norm $\|.\|$ which allows to derive more easily upper bounds on F_p

$$\|G\|_{\rho\xi} \equiv \sum_{lm} |G_{lm}|_\rho e^{2(|m|+|l|)\xi} \quad \text{with} \quad |G_{lm}|_\rho \equiv \sup_{I^r_\rho} |G_{lm}(A)| \tag{11}$$

so that $\|G\|_{\rho,\xi} \geqq |G|_{\rho\xi} \equiv \sup_{D_{\rho\xi}} |G|$.

One obviously has

$$F = \sum_1^N \varepsilon^p F^p \quad \text{with} \quad \|F\|_{\rho\xi} \leq \sum_1^N \varepsilon^p \|F^p\|_{\rho\xi} \equiv \sum_1^N \varepsilon^p f_p \tag{12}$$

One has

$$\begin{aligned} |F^p|_{\rho\xi} \leqq f_p \leqq \sum_{lm} \frac{1}{2} &|[(l+m)A' - m]^{-1}|_\rho \\ &\times \Big\{ \sum_{\substack{jkq \\ j'k'q'}}^{*} |j+k| |j'+k'| |F^q_{jk}|_\rho |F^{q'}_{j'k'}|_\rho e^{2(|j|+|k|)\xi} e^{2(|j'|+|k'|)\xi} \Big\} \end{aligned}$$

using that $e^{2(|m|+|l|)\xi} \leqq e^{2(|j|+|k|)\xi} e^{2(|j'|+|k'|)\xi}$.
We rewrite this as

$$\begin{aligned} f_p \leqq \sum_{lm} 2p\gamma^{-1} \sum_{\substack{jkq \\ j'k'q'}}^{*} &|j+k| |F^q_{jk}|_\rho e^{2(|j|+|k|)\xi} \\ &|j'+k'| |F^{q'}_{j'k'}|_\rho e^{2(|j'|+|k'|)\xi} \leqq 2\rho\gamma^{-1} \sum_{q+q'=p} q f_q q' f_{q'} \end{aligned} \tag{13}$$

We shall later impose that $\rho \leqq r \leqq \gamma/8\,N^2$ so that this last condition will be satisfied. We thus find

$$f_p \leqq 2p\gamma^{-1} \sum_{q+q'=p} q f_q q' f_{q'} \tag{15}$$

Setting $\alpha_p = p f_p$, $c = 2\gamma^{-1}$ this may be rewritten as

$$\alpha_p \leqq c p^r \sum_{q=1}^{p-1} \alpha_q \alpha_{p-q}. \tag{16}$$

We have taken $r \in \mathbb{Z}^+$ arbitrary for future reference; the above case uses $r=2$. Now for any sequence satisfying (16) we have the following

PROPOSITION:

$$\alpha_p \leqq c^{p-1} (p!)^r \frac{2^{2(p-1)}}{p} \alpha_1^p \tag{17}$$

(see the appendix for a proof).

This implies in this case

$$f_p \leqq ((p-1)!)^2 (8\gamma^{-1})^{(p-1)} f_1^p \tag{18}$$

with

$$f_1 = e^{2\xi} \left| \frac{1}{A'} + \frac{1}{(1-A')} \right|_\rho \tag{19}$$

Requiring $A_0 \in \left(\frac{1}{4}, \frac{3}{4}\right)$ and $\rho \leqq r \leqq \frac{1}{40}$ we obtain

$$f_1 \leqq 6.25\, e^{2\xi}. \tag{20}$$

2. Effective conjugacy and final estimates

Now that we have formally constructed the canonical transform S and found a first estimate on the generating function F, we shall list below the various conditions that we need impose to carry out the construction. Firstly, we should find a complex domain $D_1 \supset [A_0 - r, A_0 + r] \times \mathbb{T}^2$ on which S exists and is invertible, $S^{\pm 1}(D_1) \subset D_{\rho\xi}$. The evolution of the system is then described as usual, using the chain

$$P(0) \equiv (A(0), \varphi(0)) \xrightarrow{S} P'(0) \equiv (A'(0), \varphi'(0))$$

$$\rightarrow P'(T) \equiv (A'(T), \varphi'(T)) \xrightarrow{S^{-1}} P(T) \equiv (A(T), \varphi(T))$$

where the second step refers to the evolution governed by the transformed hamiltonian H'. One has to determine a *real* domain $D_2 = [A_0 - \alpha r,$

$A_0+\alpha r]\times \mathbb{T}^2 (0<\alpha<1)$ such that the above chain makes sense when $P(0)\in D_2$.

The needed estimates write as follows

1. To control the *small divisors* until step N we need that inequality (14) be satisfied; imposing $\rho \leqq r$ this yields the condition

$$r \leqq \frac{\gamma}{8 N^2} \tag{21}$$

which we supplement with $r \leqq 1/40$ [*see* condition above eq. (20)] in order to control the divisors at the first step.

2. The *dynamical condition* requests that $(A'(T), \varphi'(T)) \in D_1$ if $(A(0), \varphi(0)) \in D_2$ and in fact we require the more stringent (since $I^r \times T^2 \subset D_1$)

$$|A(0)-A_0| + \left|\frac{\partial F}{\partial \varphi}\right|_{\rho\xi} + \left(1 - \left|\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right|_{\rho\xi}\right)^{-1} T \left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq r$$

If $P(0) \subset D_2$ that is

$$\alpha r + \left|\frac{\partial F}{\partial \varphi}\right|_{\rho\xi} + \left(1 - \left|\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right|_{\rho\xi}\right)^{-1} T \left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq r \tag{22}$$

The deviation $|A(t)-A(0)|$, $|t| \leqq T$ may be majorized as

$$|A(T)-A(0)| \leqq 2\left|\frac{\partial F}{\partial \varphi}\right|_{\rho\xi} + \left(1 - \left|\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right|_{\rho\xi}\right)^{-1} T \left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq r \tag{23}$$

3. The last ingredient we need is an *inversion theorem*, of which we briefly recall the version we shall use (see [G]). Consider the transformation formulas

$$\begin{aligned} A &= A' + \frac{\partial F}{\partial \varphi}(A', \varphi, t) \\ \varphi' &= \varphi + \frac{\partial F}{\partial A'}(A', \varphi, t) \end{aligned} \tag{24}$$

the inversion in the action variable is made possible by the implicit function theorem, provided the condition

$$\Gamma_1 \left|\frac{\partial F}{\partial \varphi}\right|_{\rho,\, \xi-\delta} \rho^{-1} < 1 \tag{25}$$

is satisfied. Here Γ_1 is a positive constant and δ is the analyticity loss in the periodic variables. Analogously, the inversion of the second equation is possible under the condition

$$\Gamma_2 \left|\frac{\partial F}{\partial A}\right|_{\rho,\, \xi-\delta} \frac{3}{2} \delta^{-1} < 1 \tag{26}$$

where Γ_2 is another positive constant (see below).

This yields well defined transformations

$$\mathrm{S}\begin{cases} \mathrm{A}=\mathrm{A}'+\Xi(\mathrm{A}', \varphi', t') \\ \varphi=\varphi'+\Delta(\mathrm{A}', \varphi', t') \end{cases}$$

and

$$\tilde{\mathrm{S}}\begin{cases} \mathrm{A}'=\mathrm{A}+\tilde{\Xi}(\mathrm{A}, \varphi, t) \\ \varphi'=\varphi+\tilde{\Delta}(A, \varphi, t) \end{cases}$$

which are mutual inverses ($\mathrm{S}\tilde{\mathrm{S}}=\tilde{\mathrm{S}}\mathrm{S}=\mathrm{I}$) on the common domain $\mathrm{D}_{\rho e^{-\tau},\ \xi-(4/3)\delta}\equiv\mathrm{D}_1$. It is possible to choose

$$\Gamma_1=(1-e^{-2\tau})^{-1}, \qquad \Gamma_2=\frac{\pi}{2}e^{\delta/\pi}(e^{2\xi}+e^{2\delta}).$$

We set $\xi=\frac{1}{2}$, $\tau=\frac{3}{2}\log 2$ and $\delta=\frac{\xi}{2}=\frac{1}{4}$, which yields the conditions

$$\begin{aligned} \left|\frac{\partial \mathrm{F}}{\partial \mathrm{A}'}\right|_{\rho\xi} &\leqq \frac{\delta}{10}=\frac{1}{40} \\ \left|\frac{\partial \mathrm{F}}{\partial \varphi}\right|_{\rho\xi} &\leqq \rho(1-e^{-2\tau})=\frac{7}{8}\rho. \end{aligned} \tag{27}$$

Having derived conditions (21), (22) and (27) we now turn to the necessary estimates on F and its derivatives. The one for F will be easily derived from (18) and (19), in fact $\frac{\partial \mathrm{F}}{\partial \varphi}$ writes

$$\frac{\partial \mathrm{F}}{\partial \varphi}=\sum_{p>0}\varepsilon^p\frac{\partial \mathrm{F}^p}{\partial \varphi}=\sum_{lm} i(l+m)\mathrm{F}^p_{lm}e^{i((l+m)\varphi-mt)} \tag{28}$$

from which there follows

$$\left\|\frac{\partial \mathrm{F}^p}{\partial \varphi}\right\|_{\rho\xi}\leqq pf_p \tag{29}$$

In order to estimate $\dfrac{\partial \mathrm{F}}{\partial \mathrm{A}'}$ one needs to turn back to the recursive relation (9)

$$\frac{\partial \mathrm{F}^p_{lm}}{\partial \mathrm{A}'} = -\frac{1}{2}[(l+m)\,\mathrm{A}'-m]^{-2}(l+m) \sum_{\substack{jkq\\ j'k'q'}}^{*} (j+k)(j'+k')\,\mathrm{F}^q_{jk}\,\mathrm{F}^{q'}_{j'k'}$$

$$= -\frac{1}{2}[(l+m)\,\mathrm{A}'-m]^{-1}$$

$$\left\{ \sum_{\substack{jkq\\ j'k'q'}}^{*} (j+k)(j'+k')\left[\mathrm{F}^q_{jk}(\mathrm{A}')\frac{\partial \mathrm{F}^{q'}_{j'k'}}{\partial \mathrm{A}'}(\mathrm{A}') + \mathrm{F}^{q'}_{j'k'}(A')\frac{\partial \mathrm{F}^q_{jk}}{\partial \mathrm{A}'}(\mathrm{A}')\right]\right\} \quad (30)$$

where $\sum^*$ refers to the constraints written out under eq. (9). We set $\left|\dfrac{\partial \mathrm{F}^p}{\partial \mathrm{A}'}\right|_{\rho\xi} \equiv g_p$ so that (30) entails

$$g_p \leqq 8p^3\gamma^{-2}\sum_{q=1}^{p-1} qf_q q'\, f_{q'} + 4p\gamma^{-1}\sum_{q=1}^{p-1} qf_q q'\, g_{q'}$$

Changing to $\beta_p = pg_p$ so that $\left|\dfrac{\partial \mathrm{F}}{\partial \mathrm{A}'}\right|_{\rho\xi} \leqq \sum_{p=1}^{p-1} \varepsilon^p \dfrac{\beta_p}{p}$ we have that

$$\beta_p \leqq 4p^2\gamma^{-1}\left[\alpha_p + \sum_{q=1}^{p-1}\alpha_q\beta_{p-q}\right] \quad (31)$$

where α_p verifies

$$\alpha_p = 2p^2\gamma^{-1}\sum_{q=1}^{p-1}\alpha_p\alpha_{p-}q. \quad (32)$$

We again obtain a relation of type (16) and as it is shown in the appendix this implies that

$$g_p \leqq (p\,!)^2\frac{(8\gamma^{-1}f_1)^p}{p}. \quad (33)$$

We are now ready to estimate F and its derivatives, fixing the hitherto arbitrary parameter $\mathrm{N}=\mathrm{N}(\varepsilon)$, *i. e.* the number of steps. We use the norm $|\,.\,|_{\rho\xi}$, the other norm $\|\,.\,\|_{\rho\xi}$ having been introduced for technical convenience only (so as to entail the set of recursive inequalities (15)). We start with :

$$|F|_{\rho\xi} \leqq \sum_{p\geqq 1}^{N} f_p \varepsilon^p \leqq \frac{\gamma}{8} \sum_{p\geqq 1}^{N} (8\gamma^{-1} f_1 \varepsilon)^p ((p-1)!)^2$$
$$\leqq \frac{\gamma}{8} \sum_{p\geqq 1}^{N} \frac{(50\gamma^{-1} \varepsilon e^{2\xi})^p}{p} (p!)^2 \leqq 0.93\gamma \sum_{p\geqq 1}^{N} \eta^p p^{2p-1} \quad (34)$$

having defined $\eta \equiv 50\gamma^{-1} e^{2(\xi-1)} \varepsilon$. This definition and the formula above stress that (ε/γ) is in fact the natural, dimensionless small parameter. We have used (20) to estimate f_1 and Stirling's formula in the next to last step in the rather precise form

$$p! \leqq e^{1/12} \sqrt{2\pi p}\, p^p e^{-p}$$

for what concerns the derivatives of the generating function we get in a similar way

$$\left|\frac{\partial F}{\partial \varphi}\right|_{\rho\xi} \leqq \sum_{p=1}^{N} p f_p \varepsilon^p \leqq 0.93\gamma \sum_{p=1}^{N} \eta^p p^{2p} \quad (35)$$

Furthermore

$$\left|\frac{\partial F}{\partial A'}\right|_{\rho\xi} \leqq \sum_{p=1}^{N} \varepsilon^p g_p \leqq \sum_{p=1}^{N} (\varepsilon f_1 8\gamma^{-1})^p p((p-1)!)^2 \leqq 7.44 \sum_{p=1}^{N} \eta^p p^{2p} \quad (36)$$

and lastly

$$\left|\frac{\partial^2 F}{\partial \varphi\, \partial A'}\right|_{\rho\xi} \leqq 7.44 \sum_{p=1}^{N} \eta^p p^{2p+1}. \quad (37)$$

Finally to get an estimate for $\dfrac{\partial H'}{\partial \varphi}$ observe that

$$\begin{aligned} H'(A', \varphi, t) &= h(A', \varepsilon) + f'(A', \varphi, t; \varepsilon) \\ f'(A', \varphi, t; \varepsilon) &= \sum_{p=N+1}^{2N} \varepsilon^p \left(\sum_{\substack{r+s=p \\ r,s\geqq 1}} \frac{\partial F^s}{\partial \varphi} \frac{\partial F^r}{\partial \varphi} \right) \end{aligned} \quad (38)$$

so that

$$\left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq 2 \sum_{p=N+1}^{2N} \varepsilon^p \left(\sum_{\substack{r+s=p \\ r,s\geqq 1}} \left|\frac{\partial F^r}{\partial \varphi}\right|_{\rho\xi} \left|\frac{\partial^2 F^s}{\partial \varphi^2}\right|_{\rho\xi} \right). \quad (39)$$

Thus

$$\left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq 2 \sum_{p=N+1}^{2N} \varepsilon^p \Big(\sum_{r+s=p} r f_r s^2 f_s \Big)$$
$$\leqq 2 \sum_{p=N+1}^{2N} \varepsilon^p f_1^p (8\gamma^{-1})^{p-2} \left[\sum_{q=1}^{p-1} q((q-1)!)^2 (p-q)^2 ((p-q-1)!)^2 \right] \quad (40)$$

Observing that the largest term is obtained for $q=1$ and $q=p-1$ and using Stirling's formula we get (assuming $N \geqq 2$)

$$\left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\xi} \leqq 0.21\,\gamma^2 \sum_{p=N+1}^{2N} \eta^p p^{2p} \quad (41)$$

We have now to choose N so as to minimize the values of the sums [see (35), (36), (41)]

$$\sum_{p=1}^{N} \eta^p p^{2p}, \qquad \sum_{p=N+1}^{2N} \eta^p p^{2p}$$

Before we do that, it may be useful to notice that these are typical of the behavior of the perturbation series. Observe that the important feature is to get a factor p^{2p+c}, where c a constant. This can be traced back to the diophantine inequality. It will yield (see below) $N(\varepsilon) \sim \varepsilon^{-1/2}$ and eventually a time of the order $\exp(c\,\varepsilon^{-1/2})$. Ideally, 2 should be replaced by d in dimension d, (see section III).

We now work with the natural variable η and translate in fine the estimates in terms of ε. We simply compute

$$\frac{d}{dp}(\eta^p p^{2p}) = (\eta + 2\log p + 2)\,\eta^p p^{2p}$$

The minimum is for $p = \dfrac{1}{e\sqrt{\eta}}$. We impose $N = \dfrac{1}{2e\sqrt{\eta}}$ so that $\eta^p p^{2p}$ decreases for $p \leqq 2N$. If we also require $N \geqq 3$ and write the first two terms explicitly, we get

$$\sum_{p=1}^{N} \eta^p p^{2p} \leqq \eta + 2^4 \eta^2 + \eta^3 3^6 \frac{1}{2e\sqrt{\eta}} \leqq 1.16\,\eta \quad (42)$$

Also

$$\sum_{p=N+1}^{2N} \eta^p p^{2p} \leqq N\,\eta^N (N)^{2N} \leqq \frac{1}{2e\sqrt{\eta}} e^{(-c/\sqrt{\eta})} \quad (43)$$

where $c=\frac{1}{e}(1+\log 2)\simeq 0.623$. The estimates for the sums $\sum_{p=1}^{N}\eta^p p^{2p\pm 1}$ pertaining to the evaluation of F and $\frac{\partial^2 F}{\partial\varphi\,\partial A'}$ are obtained similarly (with of course the *same* value of N). One gets

$$\sum_{p=1}^{N}\eta^p p^{2p-1}\leqq 1.12\,\eta,\qquad \sum_{p=1}^{N}\eta^p p^{2p+1}\leqq 1.41\,\eta. \tag{44}$$

Collecting the estimates we get

$$\begin{aligned}
&|F|_{\rho\xi}\leqq 1.03\,\gamma\eta\\
&\left|\frac{\partial H'}{\partial\varphi}\right|_{\rho\xi}\leqq\frac{0.21}{2e}\gamma^2\frac{1}{\sqrt{\eta}}e^{-c/\sqrt{\eta}},\qquad c=\frac{1}{e}(1+\log 2)\\
&\left|\frac{\partial F}{\partial\varphi}\right|_{\rho\xi}\leqq 1.12\,\gamma\eta\\
&\left|\frac{\partial F}{\partial A'}\right|_{\rho\xi}\leqq 9.23\,\eta\\
&\left|\frac{\partial^2 F}{\partial\varphi\,\partial A'}\right|_{\rho\xi}\leqq 10.6\,\eta
\end{aligned}\tag{45}$$

The theorem is now within reach. In fact, we need only verify conditions (21), (22) and (27) keeping in mind that one must also have $\rho\leqq r\leqq 1/40$ and $A_0\in\left(\frac{1}{4},\frac{3}{4}\right)$. As we already noticed, we shall use $\xi=1/2$ and it turns out that we can set $\rho=r$. We then have $\eta=\frac{50}{e}\gamma^{-1}\varepsilon\leqq 18.4\,\varepsilon/\gamma$. This value we use in order to check the invertibility conditions (27); however, in the dynamical condition (22), we notice that everything takes place on a *real* domain. Since $\eta=50\gamma^{-1}e^{2(\xi-1)}\varepsilon$, we may set there $\xi=0$, that is $\eta\to\eta/e=\frac{50}{e^2}\gamma^{-1}\varepsilon\leqq 6.77\,\gamma^{-1}\varepsilon$, i. e., η has been divided by a factor e. On the other hand the number of steps remains unchanged. The reader will easily convince himself that the estimates on $\frac{\partial F}{\partial\varphi}$ and $\frac{\partial^2 F}{\partial\varphi\,\partial A'}$ may be divided by a factor e, whereas $\frac{\partial H'}{\partial\varphi}$ is now estimated as

$$\left|\frac{\partial H'}{\partial\varphi}\right|_{\rho 0}\leqq 0.21\,\gamma^2\sum_{p=N+1}^{2N}\left(\frac{\eta}{e}\right)^p p^{2p}$$

so that we find the *same* upper bound as in (45) with a constant $c' = \frac{1}{e}\left(\log 2 + \frac{3}{2}\right) = 0.807$ and with the *same* value of $\eta = \frac{50}{e}\gamma^{-1}\varepsilon$.

We now proceed to check the various conditions. (21) reads

$$r \leqq \frac{\gamma}{8\,N^2} = \frac{e^2}{2}\gamma\eta \simeq 68\,\varepsilon \tag{46}$$

(22) can be written as

$$\left|\frac{\partial F}{\partial \varphi}\right|_{\rho\,0} + \left(1 - \left|\frac{\partial^2 F}{\partial \varphi\,\partial A'}\right|_{\rho\,0}\right)^{-1} T \left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\,0} \leqq (1-\alpha)\,r. \tag{47}$$

At the same time the second condition in (27) requires that $\left|\frac{\partial F}{\partial \varphi}\right|_{\rho\,0} \leqq \frac{7}{8}\rho = \frac{7}{8}r$ or in view of (45), $1.12\,\gamma\eta \leqq \frac{7}{8}r$. That is $r \geqq 24\,\varepsilon$.

In (47) the most relevant part is $\left|\frac{\partial F}{\partial \varphi}\right|_{\rho\,0}$, the size of the action part of the canonical transform, which is bounded by $1.12\,\gamma\frac{\eta}{e} \leqq 7.6\,\varepsilon$.

At this point there are obviously arbitrary choices to be made, and we simply present below a reasonable set of parameters. Namely we set $r = 60\,\varepsilon$, $\alpha = 5/6$. On the other hand it is easy to see that the first of the invertibility conditions (27) implies that also $\left|\frac{\partial^2 F}{\partial \varphi\,\partial A'}\right|_{\rho\,0} \leqq 1/40$, so that with these choices (47) reads

$$T\left|\frac{\partial H'}{\partial \varphi}\right|_{\rho\,0} \leqq 3.4\,\varepsilon$$

This yields a time of validity

$$T \leqq 400\,\gamma^{-1}(\varepsilon/\gamma)^{3/2}\,e^{(1/5\,\sqrt{\varepsilon/\gamma})}.$$

Finally the deviation is controlled by

$$|A(T) - A(0)| \leqq (1-\alpha)\,r + \left|\frac{\partial F}{\partial \varphi}\right|_{\rho\,0} \leqq 20\,\varepsilon.$$

The value of the threshold is obtained by requiring that the number of steps $N = \frac{1}{2e\sqrt{\eta}}$ be at least 3; we have used this to get estimates (45) and anyway the whole theory supposes that N is "large" enough, otherwise everything can be done by hand.

We obtain $\varepsilon/\gamma \leqq 2.10^{-4}$, which also guarantees that the invertibility conditions (27) are satisfied. This completes the proof of the theorem.

III. COMMENTS AND EXTENSIONS

We now come to some comments about the above result.

1. We did not prove a *global* estimate, *i. e.* one which applies to any initial condition with $A(0)$ inside the interval—say—$(0, 1)$. One rather trivial reason is that the width of the primary resonances near $A=0$ and $A=1$ is on the order of $\sqrt{\varepsilon}$ and is thus much larger than the order of the deviation we allowed. One has to take $A(0)$ far enough from 0 or 1 such that the denominators appearing at the first step are not too small. A much more serious inconvenience is the arithmetic property we had to impose on A_0 (but *not* $A(0)$).

In fact the basic dimensionless perturbation parameter appears to be the combination ε/γ which depends very sensitively on the value of the action. What we did basically consisted in working as if the unperturbed hamiltonian could locally be considered as a harmonic oscillator and we find a result which is valid as far as the local frequency does not differ too much from this value. This local reduction to the linear case is in fact a common feature of the classical perturbation theories and the geometric part of the proof of Nekhoroshev's theorem consists essentially in pasting these local estimates together (including those around the resonances). We refer to [BG] for a general treatment of the purely linear case ($h(A)=\omega \cdot A$, $A \in R^d$).

In any case our result provides at least partial barriers in the sense that the zones described by the allowed initial conditions cannot be traversed in less than $T=T(\varepsilon, \gamma)$.

To get a global estimate, one should be able to use *resonant* normal forms inside the resonant subdomains and then paste together the various estimates. This is done in the general proof of Nekhoroshev theorem but at the expense of a great loss in precision. This is not only a matter of worsening the value of the constants, but also reflects a real theoretical problem, in particular for what concerns the exponent of ε appearing in the value of $T(\varepsilon, \gamma)$ (see below).

Here we confine ourselves to stating a proposition which uses (almost optimally) the above obtained estimates in order to formulate a global statement.

PROPOSITION. — *Let* $M \in Z^+$ *and set* $\gamma=\dfrac{1}{M+2} \leqq \dfrac{1}{3}$. *Pick any* $\alpha=A(0) \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ *and expand it as a continuous fraction* $\alpha=[a_1, \ldots, a_n, \ldots]$, $a_i \in Z^+$. *Suppose* α *is not* $(\gamma, 1)$-*diophantine* (*otherwise the theorem applies*). *Then there exists* $n \in Z^+$ *such that* $a_i \leqq M$, $i=0, 1, \ldots, n-1$ *and* $a_n \geqq M$ (*if* $\alpha \in Q$, *we write* $\alpha=[a_1, \ldots, a_k, \infty]$, *so that* *this condition holds). Under these circumstances, one has*

$$|A(t)-A(0)| \leqq \delta(\gamma, n)+50\,\varepsilon$$

when $|t| \leqq T$ *(the same as in the statement of the theorem), with*

$$\delta(\gamma, n)=\frac{2\gamma}{1-2\gamma}\,\frac{1}{q_{n+1}^2} \leqq 6\,\gamma\sigma^{2(n-1)}$$

where $\sigma=\dfrac{\sqrt{5}-1}{2}$ *and* q_n *is the* n^{th} *Fibonacci number* $(q_0=0,\ q_1=1,\ q_{n+1}=q_n+q_{n-1})$.

One sees that $\delta(\gamma, n)$ is independent of ε so that we loose much of the power of the usual Nekhoroshev type results. The above proposition depends simply on the distribution of the diophantine numbers (see e. g. [Khi]).

2. It is often said that for two dimensional hamiltonians KAM tori partition the phase space and provide barriers which trap any trajectory inside the gap between two tori. This would of course make such an estimate as the one presented above rather useless. This assertion however assumes that one works with a value of ε which lies *below* the threshold of destruction of "many" tori. But this in turn requires ε to be much too small to be of any physical or even numerical significance. In fact the validity threshold for our estimate lies below the known numerical breack-up parameter for the golden torus ($\varepsilon=0.027$, see [Gre]) and even below the best rigorous computer assisted estimate ($\varepsilon=0.015$, see [Cel-Chi]). It lies however most likely much *above* the break-up threshold of most other tori (including the noble ones) for which no numerical or realistic analytical estimates are available. Predictions can however be made using an approximate renormalization scheme [Es]. In a less dramatic way, the argument of the perpetual trapping between tori would also necessitate an upper bound on the width of the gap between two successive surviving tori. As a matter of fact, the distortion of a given torus (distance of the perturbed to the unperturbed torus) is itself on the order of ε/γ and thus strongly γ-dependent.

3. The above estimate may be extended in various directions. Let us first consider the "multiwave" perturbation

$$H(A, \varphi, t)=\frac{1}{2}A^2+\varepsilon \sum_{|k| \leqq K} f_k \cos(\varphi-kt).$$

Almost everything remains unchanged except that here $F_{lm}^p=0$ for $|l|+|m|>Kp$. This induces a change in the recursive inequalities (15) which in fact amounts to introducing a factor K^2 in front of the r.h.s.

Also, the value of the initial term f_1 is altered. In fact, one has

$$F^1 = - \sum_{|k| \leqq K} f_k \frac{\sin(\varphi - kt)}{A' - k}.$$

Hence (compare (19))

$$f_1 = e^{2\xi} \Big(\sum_{|k| \leqq K} |f_k| |A' - 1|^{-1} \Big)_\rho.$$

Since $A' \in (0, 1)$, $|A' - k|^{-1}$ decreases as $|k|$ increases and, if the amplitudes f_k are not too large, f_1 remains close to its value for the two-wave hamiltonian.

To summarize, we must change $c \to c K^2$ in (16) with a new $\alpha_1 = f_1$; accordingly, the value of η is multiplied by K^2 with the new value of f_1. N in turn is given in terms of ε as $N \simeq \dfrac{1}{K\sqrt{\varepsilon}}$, which leaves the exponent of ε in the double exponential in $T(\varepsilon)$ unmodified.

We observe that we have not taken *any* trigonometric polynomial as a perturbation (see 4), nor have we considered the case of out-of-phase waves, which would correspond to changing $\varphi - kt$ into $\varphi - kt + \varphi_k^0$ in the hamiltonian, where $\varphi_k^0 \in (0, 2\pi)$ is an arbitrary initial phase. In some cases (in particular when $\varphi_k^0 = 0$ for k even, and $\varphi_k^0 = \pi$ for k odd), on physical grounds one expects better estimates, but the method we use is too rough to make this clear.

4. What we have done above can also be extended almost word for word to a class of simple d-dimensional systems (see also [GG]). We sketch below the qualitative scheme, which emphasizes the d-dependence of the exponent appearing in $N(\varepsilon)$ and consequently in $T(\varepsilon)$.

Let us consider the d-degrees of freedom hamiltonian

$$H(A, \varphi) = \frac{A^2}{2} + \varepsilon \sum_{|k| \leqq K} f_k e^{ik \cdot \varphi}$$

where $A \in R^d$, $\varphi \in T^d$, $k \in Z^d$, $K \in Z^+$, $|k| \equiv \sum_i |k_i|$ if $k = (k_1, \ldots k_d)$, $k_i \in Z$.

The perturbation f is thus a trigonometric polynomial independent of the action variables A ($f_k \in R$). The formal perturbation theory provides, as above, the following recursive expression for the generating function F.

$$F_k^1 = f_k/(ik \, . \, A)$$
$$F_k^p = (2iA \cdot k)^{-1} \sum_{q, q'; j, j'}^{*} (j \, . \, j') F_j^q F_{j'}^{q'}$$

where $(j \, . \, j')$ indicates the scalar product and * indicates the constraints $j + j' = k$, $q + q' = p$. By induction one checks that $F_k^p = 0$ if $|k| > pK$. One uses then the diophantine inequality $|A' \, . \, k|^{-1} \leqq \gamma^{-1} |k|^{d-1}$ and introduces as above the norm (see (11))

$$\| G \|_{\rho\xi} = \sum_{l} | G_l |_{\rho} e^{\xi |l|}$$

so that $\| F \|_{\rho\xi} \leqq \sum_{1}^{N} \varepsilon^p \| F^p \|_{\rho\xi} \equiv \sum_{1}^{N} \varepsilon^p f_p$. Setting as above $\alpha_p = p f_p$ we see that α_p verifies

$$\alpha_p \leqq c \gamma^{-1} p^d \sum_{k=1}^{p-1} \alpha_k \alpha_{p-k}$$

which yields (see (17))

$$f_p \leqq \left(\frac{8c}{\gamma} \right)^{p-1} (p!)^d \frac{f_1^p}{p}.$$

The same analysis as in the two dimensional case for the sums (see (34), (35), (36))

$$\sum_{p} \eta^p (p!)^d p^{-s} \eta = 8 \varepsilon \gamma^{-1} c f_1$$

(where $s = 0, 1, 2$ according to the various quantities involved), allows to choose $N \simeq (1/\varepsilon)^{1/d}$. In this way one obtains that if A_0 is $(\gamma, d-1)$-diophantine and $A(0)$ is "near" A_0 then the deviation in action remains "small" (of order ε) for an interval of time of the order of $\exp(c/\varepsilon^{1/d})$.

5. Very roughly speaking, Nekhoroshev's theorem says that for a—say—quasi convex unperturbed hamiltonian $h(A)$, the deviation of the action variable(s) for the nearby hamiltonian $H(A, \varphi)$ (see (1)) can be controlled (and remains small in the sense that it vanishes together with ε) over an interval of time on the order of $\exp(\varepsilon^{-\alpha})$, $(0 < \alpha \leqq 1)$.

The exponent α may well be the only intrinsically defined parameter in these exponential estimates. For instance in the quasi convex case one may give a lower bound for this quantity which depends only on the dimension d (see [Nekh] or [BG]). In fact, $\alpha = \alpha(d)$ may be defined as the upper bound of the α' such that $\forall \delta > 0 \, \exists \varepsilon_* = \varepsilon_*(h, \delta)$ such that if $\varepsilon \leqq \varepsilon_*$ and $(A(t), \varphi(t))$ is a trajectory of $H(A, \varphi) = h(A) + \varepsilon f(A, \varphi)$, then one has

$$\sup_{|t| < T(\varepsilon)} |A(t) - A(0)| < \delta \qquad \text{for} \quad A(0) \in D, \operatorname{dist}(A(0), \partial D) \geqq \delta$$

and

$$\liminf_{\varepsilon \to 0} - \frac{\log \log T(\varepsilon)}{\log \varepsilon} \geqq \alpha'.$$

This is of course nothing but a formal way of expressing that the deviation in action remains $o(1)$ (independently of f) for

$|t|<T(\varepsilon)\sim\exp\varepsilon^{-\alpha}$. Here we require only a $o(1)$ deviation because it is likely than any other requirement (such as $\delta=o(\varepsilon^{\beta})$, $0<\beta\leqq 1$) would define the same α (taking β small enough). In terms of small denominators, this is linked with the rate of divergence of the relevant perturbation series, which should behave at worst as $\sum_p (c\,\varepsilon)^p (p!)^{1/\alpha}$. The growth of the terms in these series is in turn connected with the diophantine exponent.

On these grounds, one could expect that $\alpha(d)\geqq\frac{1}{d}$ "or at least $\alpha(d)=o\left(\frac{1}{d}\right)$", locally around the good frequencies, *i.e.* the values of A such that $\omega(A)=\frac{\partial h}{\partial A}$ is γ-diophantine with exponent $d-1$. We recall that these numbers (or rather d-vectors) have Lebesgue measure 0 and Hausdorff measure d in the space of frequencies ω. This local bound reduces to a statement concerning the perturbation of a linear hamiltonian $(h(A)=\omega\cdot A)$; it is however unproved in general, perhaps because the general proof is done recursively, using Cauchy estimates. Above, we have shown that this holds good in the case of a paradigmatic hamiltonian, or in fact in the linear case in any dimension when the perturbation is a trigonometric polynomial of the angles (and is independent of the action variables). A *global* estimate is much harder to obtain, if one wants to go beyond the comparatively rough way of pasting the local behaviours, which is used in the proof of Nekhoroshev's theorem. It is worth noticing finally that the exponent α is also connected, on the geometrical side, with the splitting of the separatrices, as is best seen by analysing Arnold's original example [A], so that this quantity has both an "elliptic" and a "hyperbolic" meaning (see also [N]).

APPENDIX

We first study the set of *equalities*

$$\alpha_p=\sum_{q=1}^{p-1}\alpha_q\,\alpha_{p-q},\quad \alpha_1\text{ given} \tag{A.1}$$

Using the generating function

$$\mathscr{F}(z)=\sum_{k>0}\alpha_k z^k$$

(A.1) ensures that $\mathscr{F}$ solves the equation

$$\mathscr{F}^2(z)=\mathscr{F}(z)-\alpha_1 z,\qquad \mathscr{F}(0)=0 \tag{A.2}$$

Hence :

$$\mathcal{F}=\frac{1}{2}-\frac{1}{2}(1-4\alpha_1 z)^{1/2}=\frac{1}{2}-\frac{1}{2}\sum_{k\geq 0}(4\alpha_1 z)^k\binom{1/2}{k} \tag{A.3}$$

so that :

$$\alpha_k=\frac{(2k-3)!!\,2^{k-1}\,\alpha_1^k}{k!}$$

$$\text{where}\,(2k-3)!!\equiv(2k-3)(2k-5)\dots 3.1\leqq(k-1)!\,2^{k-1}$$

We get :

$$\alpha_k\leqq\frac{2^{2(k-1)}\alpha_1^k}{k} \tag{A.4}$$

The more general recurrence is dealt with by a variation of the constants method :

$$\alpha_p=p^r\sum_{q=1}^{p-1}\alpha_q\alpha_{p-q},\qquad r\in\mathrm{Z}^+ \tag{A.5}$$

Set $\alpha_p=\gamma(p)\beta_p$, if γ satisfies $p^r\gamma(q)\gamma(p-q)\leqq\gamma(p)$ then

$$\gamma(p)\beta_p\leqq p^r\leqq p^r\sum_{q=1}^{p-1}\gamma(q)\beta_q\gamma(p-q)\beta_{p-q}\leqq\gamma(p)\sum_{q=1}^{p-1}\beta_q\beta_{p-q}$$

that is β_p verifies the recursive *inequalities* with $r=0$. A—nearly optimal—choice is seen to be $\gamma(p)=(p!)^r$.

Combining the above inequality (A.4) and a scaling to take the constant c into account one finds inequality (17).

We turn now to the proof of (33).

Starting from (31), (32), we set $\beta_p=\gamma(p)b_p$ and $\alpha_p=\gamma(p)a_p$; again if $p^2\gamma(q)\gamma(p-q)\leqq\gamma(p)$; $\forall q\in[1,p-1]$ one has

$$b_p\leqq 4\gamma^{-1}\left[a_p+\sum_{q=1}^{p-1}a_p b_{p-q}\right] \tag{A.6}$$

where :

$$a_p\leqq 2\gamma^{-1}\sum_{q=1}^{p-1}a_p a_{p-q} \tag{A.7}$$

We choose as above $\gamma(p)=(p!)^2$. We obviously have that $a_p\leqq a'_p$ where a'_p verifies (A.7) with an *equality* so that (see (A.3))

$$\mathcal{F}'(z)\equiv\sum_{p>0}a'_p z^p=\frac{\gamma}{4}[1-(1-8\gamma^{-1}f_1 z)^{1/2}],\qquad a'_1=f_1. \tag{A.8}$$

Setting $a''_p = 4\gamma^{-1} a'_p$ which is the coefficient of the expansion of $1-(1-8\gamma^{-1} f_1 z)^{1/2} \equiv \mathscr{F}''(z)$ one has $b_p \leqq b'_p$ where this last quantity satisfies

$$b'_p \leqq a''_p + \sum_{q=1}^{p-1} a''_p \, b'_{p-q}$$

Finally, letting

$$\mathscr{G}'(z) = \sum_{p>0} b'_p z^p$$

we find $\mathscr{G}' = \mathscr{F}'' + \mathscr{F}'' \mathscr{G}'$ or $\mathscr{G}' = \dfrac{\mathscr{F}''}{1-\mathscr{F}''}$ *i.e.* $\mathscr{G}' = (1-8\gamma^{-1} f_1 z)^{-1/2} - 1$

that is

$$\mathscr{G}'(z) = \sum_{p>0} b'_p z^p = \sum_{p>0} \frac{-\frac{1}{2}}{\rho} (-1)^p (8\gamma^{-1} f_1)^p z^p$$

Tracing back all the scalings we find

$$g_p \leqq (p!)^2 \frac{b'_p}{p} \leqq (p!)^2 \frac{(8\gamma^{-1} f_1)^p}{p}$$

which is (33).

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