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ALBERTO FRIGERIO
MARCO RUZZIER

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# Relativistic transformation properties of quantum stochastic calculus

by

**Alberto FRIGERIO**

Dipartimento di Matematica e Informatica, Università di Udine
via Zanon 6, 33100 Udine, Italy.

and

**Marco RUZZIER**

Dipartimento di Fisica, sezione di Fisica Teorica, Università di Milano,
via Celoria 16, 20133 Milano, Italy

ABSTRACT. — We propose a relativistic version of quantum stochastic calculus based on the relativistic properties of test functions; these properties are determined on the basis of the interpretation of test functions as interaction rate amplitudes, and lead to the relativistic transformations of the quantum stochastic differential equations and of the reduced dynamics. We obtain for the latter the relativistic time dilatation.

RÉSUMÉ. — Nous proposons une version relativiste du calcul stochastique quantique basée sur les propriétés relativistes des fonctions test. Ces propriétés sont déterminées par l'interprétation des fonctions test comme des amplitudes de taux d'interaction, ce qui conduit aux transformations relativistes des équations différentielles stochastiques quantiques et de la dynamique réduite. Pour cette dernière nous obtenons la dilatation temporelle relativiste.

## 1. INTRODUCTION

Quantum stochastic calculus (QSC) ([1], [2]) may be regarded as the mathematical theory of a kind of quantum noise which bears close relationship with the classical Wiener process. It has been applied (among other things) to the description of irreversible time evolution of open systems ([3], [4], [5]); in this connection, the noise field of QSC is supposed to be a suitable idealization of the physical field whose quanta are emitted and/or absorbed by the system of interest. The noise field may indeed be regarded as an approximation of some physical field in suitable limiting situations (weak coupling limit) ([6], [7]). In several applications, notably quantum optics ([4], [5], [8]), the physical field is the electromagnetic field, which is obviously relativistic: this motivates an investigation of the relativistic transformation properties of the noise field.

In a given reference frame, the noise field of QSC is described by creation and annihilation operators $A_j^+(t)$, $A_j(t) : j=1, 2, \ldots, n$; $t \in \mathbf{R}^+$ acting in the symmetric Fock space over $L^2(\mathbf{R}) \otimes \mathbf{C}^n$ and satisfying the commutation relations $[A_i(t), A_j^+(s)] = \delta_{ij}.\min(t, s)$; their time evolution is given by $V_t[A_j(s)] = A_j(t+s) - A_j(t)$.

Our aim in the present paper is to study the transformation laws of the noise field under transformations of the Poincaré group. Obviously, the noise field cannot satisfy all the axioms of the usual relativistic field theory, since its frequency spectrum extends by necessity to the whole real line; however, there remains the possibility that a consistent relativistic theory of it can be constructed, which bears some resemblances with the model described in [9].

The underlying physical idea is as follows. Consider a physical system which can be assigned a trajectory in space and whose internal states are described by quantum theory (say, an atom with infinitely massive nucleus); the internal states of the system interact with a physical field, which is initially in the vacuum state, with an interaction V of the form

$$V = -i \sum_{j=1}^{n} [B_j \otimes a_j^+(g) - B_j^+ \otimes a_j(g)]$$

where, if $H_s$ denotes the free Hamiltonian of the system, $[H_s, B_j] = -\omega_j B_j$, $\omega_j > 0$ (rotating wave approximation). Then, in some limiting situation ([10], [11], [12]), the reduced dynamics of the system is described by a *quantum dynamical semigroup* $\mathcal{T}_t = \exp[G t]$ ([13], [14]), where the generator G is given by

$$G(X) = K^+ X + XK + \sum_{j=1}^{n} c(\omega_j) \cdot B_j^+ X B_j,$$

where $c(\omega)$ is a positive function and where

$$K = iH - \frac{1}{2} \sum_{j=1}^{n} c(\omega_j) . B_j^+ B_j,$$

H being self-adjoint.

The same dynamical semigroup can be obtained as

$$\mathcal{T}_t = \mathbf{E}^0 [U(t) . X \otimes \mathbf{1} . U^+(t)],$$

where $U(t)$ is the solution of the quantum stochastic differential equation [1]

$$dU(t) = U(t) \left[ \sum_{j=1}^{n} c(\omega_j)^{1/2} [B_j \otimes dA_j^+(t) - B_j^+ \otimes dA_j(t)] + K \otimes \mathbf{1}\, dt \right],$$

where $A_j(t)$, $A_j^+(t)$ are mutually independent (Fock) quantum Brownian motions and $\mathbf{E}^0$ is the vacuum conditional expectation. In analogy with Bohr's atomic model, the field created by $A_j^+(t)$ will be assumed to have "energy $\omega_j$", although its wave function is a square wave, containing all frequencies in its spectrum. Thus we are naturally led to the idea of a *quasi-monochromatic field* [15], meaning that the creation and annihilation operators $A_j^+(t)$, $A_j(t)$ will be ascribed a carrier frequency $\omega_j$, although the (physically small) spread of their frequency spectra around the carrier frequency is (matematically) infinite. In any reference frame, quasi-monochromatic fields of all possible carrier frequencies are supposed to exist; then the system of interest will interact with those quasi-monochromatic fields whose carrier frequency is the same as the (Doppler-shifted, when appropriate) frequency of the system operator $B_j$ to which the field $A_j$ is coupled.

The paper is organized as follows: in § 2 we describe QSC [1] in its latest version [2]; on the basis of the interpretation of test functions as interaction rate amplitudes, in § 3 we construct a class of $(1+1)$-dimensional models ("counting" models, *cf.* [9]) describing the relativistic properties of test functions. We furthermore present in § 4 a semiclassical model for matter-radiation interaction, which is based on a zero-mass quasi-monochromatic scalar field [15] interacting with a two-level system S; in this model it is possible to construct quantities which have the same meaning as test functions, and which transform according to the rules of counting models. In § 5 we show that these transformations can be expressed in terms of unitary representations of the proper orthochronous Poincaré group on $L^2(\mathbf{R})$; this observation allows their application to QSC by using the functor $\Gamma$ and the transformations induced on the unitary operator group $\mathcal{U}(L^2(\mathbf{R}))$ (§ 6). Thus, we obtain the transformation properties for QSDEs and for the reduced dynamics, finding for the latter the relativistic time dilatation.

## 2. QSC: Multidimensional formalism

We describe multidimensional QSC following [1] and [2]. Let $h_0$ and $h_1$ be two separable Hilbert spaces; we write

$$h = L^2(\mathbf{R}) \otimes h_1 \simeq L^2(\mathbf{R}, h_1); \qquad \mathcal{H} = \Gamma(h); \qquad \tilde{\mathcal{H}} = h_0 \otimes \mathcal{H},$$

where $\Gamma(h)$ denotes the symmetric Fock space over $h$. Let $\{e_i\}$ be an orthonormal basis in $h_1$, and define $P_{ij} = |e_i\rangle\langle e_j|$. Test functions of the form $l \otimes e_i$ can be interpreted as follows:

the quantity $|l(t)|^2 dt$ represents the number of interactions occuring in the infinitesimal time interval $(t, t+dt)$ between an open system S and the $i-$th noise component.

The Fock space $\mathcal{H}$ contains a dense linear manifold $\mathcal{E}$ which is generated by the *exponential vectors* $\{\psi(f); f \in h_1\}$ satisfying $\langle \psi(f) | \psi(g) \rangle = \exp \langle f | g \rangle$. The *annihilation*, *creation* and *conservation operators* are defined on $\mathcal{E}$ as follows:

$$a(f)\psi(g) = \langle f | g \rangle \psi(g), \qquad a^+(f)\psi(g) = \frac{d}{d\varepsilon}\psi(g+\varepsilon f)\Big|_{\varepsilon=0},$$

$$\lambda(T)\psi(g) = \frac{d}{d\varepsilon}\psi(e^{\varepsilon T} g)\Big|_{\varepsilon=0}.$$

The functor $\Gamma$ is an application defined on the space $\mathcal{C}(h)$ of contractions on $h$ with values in the space $\mathcal{C}(\mathcal{H})$ of contractions on $\mathcal{H}$ and is given by

$$\Gamma(S)\psi(f) = \psi(Sf); \qquad S \in \mathcal{C}(h), \quad f \in h. \tag{2.1}$$

The following relations hold:

$$\Gamma(ST^+) = \Gamma(S)\Gamma(T)^+, \qquad \Gamma(\mathbf{1}_h) = \mathbf{1}_{\mathcal{H}},$$
$$\Gamma(U)\, a(f)\, \Gamma(U^+) = a(Uf), \qquad \Gamma(U)\lambda(T)\Gamma(U^+) = \lambda(UTU^+); \tag{2.2}$$

where U is unitary on $h$. The *annihilation*, *creation* and *conservation processes* with initial time $t_0$ are the following families of operators:

$$A_i(t) = \mathbf{1}_0 \otimes a(\chi_{[t_0, t]} \otimes e_i), \qquad A_i^+(t) = \mathbf{1}_0 \otimes a^+(\chi_{[t_0, t]} \otimes e_i),$$
$$\Lambda_{ij}(t) = \mathbf{1}_0 \otimes \lambda(M_{\chi_{[t_0, t]}} \otimes P_{ij}) \tag{2.3}$$

where $\mathbf{1}_0$ is the identity in $h_0$, and $M_l f = l \cdot f$. We furthermore consider an "age" process:

$$T(t) = (t - t_0) \cdot \mathbf{1}_{\tilde{\mathcal{H}}}. \tag{2.4}$$

The *stochastic differentials* are the differential counterparts of the fundamental processes (2.3), (2.4) and may be written as

$$dA_i = \mathbf{1}_0 \otimes a(\chi_{[t, t+dt]} \otimes e_i) \quad \text{etc.}; \tag{2.5}$$

they satisfy the quantum Itô's table

$$d\Lambda_{ij}\cdot d\Lambda_{kl}=\delta_{jk}\,d\Lambda_{il},\qquad d\Lambda_{ij}\,.\,dA_k^+=\delta_{jk}\,dA_i^+,$$
$$dA_i\cdot d\Lambda_{kl}=\delta_{ik}\,dA_l,\qquad dA_i\cdot dA_k^+=\delta_{ik}\,dt;$$

the other products vanish.

Let now $h_1=\mathbf{C}^2$, so that $h=L^2(\mathbf{R},\mathbf{C}^2)\simeq L^2(\mathbf{R})\oplus L^2(\mathbf{R})$; the orthonormal basis $\{e_i\}$ then reduces to the pair $\{e_{(+1)}, e_{(-1)}\}$. We will occasionally write $(+)$, $(-)$ instead of $(+1)$, $(-1)$.

Unitary processes are solutions of QSDEs of the form

$$dU=U\Bigg[(W_{jk}-\delta_{jk})\otimes d\Lambda_{jk}-W_{jk}M_k\otimes dA_j^+ + M_k^+\otimes dA_k+\left(iK-\frac{1}{2}M_k^+M_k\right)\otimes \mathbf{1}_{\mathscr{H}}\,dt\Bigg]$$
$$U(t_0)=\mathbf{1}_{\tilde{\mathscr{H}}};\qquad j,\,k=\pm 1 \tag{2.6}$$

where $W_{jk}$, $M_k$, K are possibly time-dependent, bounded linear operators in $h_0$ such that $W_{kj}^+W_{kl}=W_{jk}W_{lk}^+=\delta_{jl}\mathbf{1}_0$ and K is self-adjoint; summation over repeated indices is understood.

The solution of (2.6) may be interpreted as the evolution of a system composed of a particle S whose Hilbert space is $h_0$, interacting with a two-component noise living in $\mathscr{H}=\Gamma(L^2(\mathbf{R}))\otimes\Gamma(L^2(\mathbf{R}))$. The reduced dynamics of S is given by

$$\mathscr{T}_t(D)=\mathbf{E}^0[U(t)\cdot D\otimes\mathbf{1}_{\mathscr{H}}\cdot U^+(t)];\qquad D\in\mathscr{L}(h_0) \tag{2.7}$$

where

$$\langle u|\mathbf{E}^0(V)|v\rangle=\langle u\otimes\psi(0)|V|v\otimes\psi(0)\rangle;\qquad u,\,v\in h_0,\quad V\in\mathscr{L}(\tilde{\mathscr{H}})$$

and satisfies the equation

$$\frac{d}{dt}\mathscr{T}_t(D)=\mathscr{T}_t\left(i[K,D]-\frac{1}{2}(M_k^+M_kD-2M_k^+DM_k+DM_k^+M_k)\right)$$
$$\mathscr{T}_{t_0}(D)=D. \tag{2.8}$$

## 3. "Counting" models

In order to determine the relativistic transformations of the QSDEs, it is necessary to find the transformation properties of the interaction, represented by the test functions $f\in h$: we shall do this for a $(1+1)$-dimensional model which is somewhat similar to a $(1+1)$-dimensional version of the model in [9].

A space-time event is specified, with respect to an inertial system (I.S.) K, by a column vector $(t, r)^T$. The metric is chosen so that $[g^{\mu\nu}]=\text{diag}(1,$

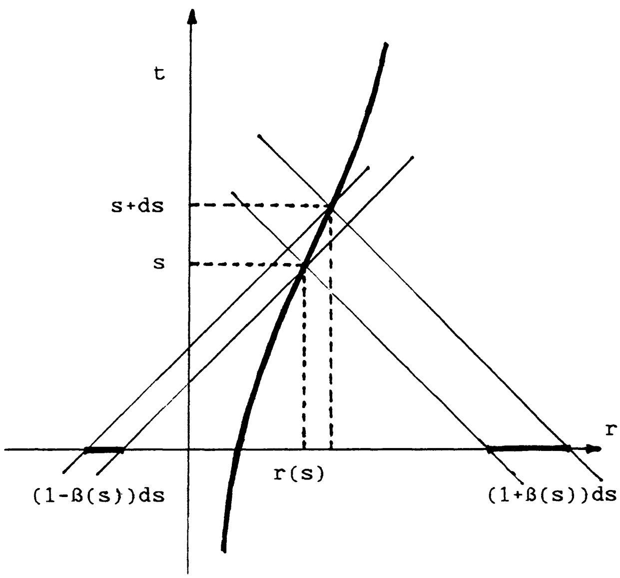
$-1$); the proper orthochronous (P.O.) Poincaré group $P_+^\uparrow$ is written as $P_+^\uparrow = \mathbf{R}^2 \times L_+^\uparrow$, where $L_+^\uparrow$ is the P. O. Lorentz group; every point $p \in P_+^\uparrow$ is written as $p=(a, \Lambda(\alpha))$, where $a=(a^0, a^1)^T \in \mathbf{R}^2$ and

$$\Lambda(\alpha)=\begin{bmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{bmatrix} \in L_+^\uparrow, \qquad \alpha\in\mathbf{R}.$$

Let $\beta(t)$ be the velocity of a particle in the I.S. K, $\gamma(t)=(1-\beta(t)^2)^{-1/2}$ and let $\tanh\xi(t)=\beta(t)$; let K′ be another I.S. such that $K'=p\cdot K$; then, as is well known, in the I.S. K′ we have

$$\xi'(t')=\xi(t)+\alpha.$$

The model can be described as follows (see *fig.* 1):




— S is a positive-mass particle with internal states of different energies, moving along a noise-independent trajectory $x(s)=(s, r(s))^T$.

— noise is represented by a reservoir R composed of noninteracting zero-mass particles (therefore moving at the speed of light $c=1$) which interact only with the internal degrees of freedom of S. R is partitioned into two sub-assemblies described by two functions $F^{(+)}$, $F^{(-)} \in L^2(\mathbf{R})$ which have the following meaning:

$|F^{(\pm)}(r)|^2 dr$ is the number of particles moving with velocity $v=\pm 1$ which are contained in the space interval $(r, r+dr)$ at time $t=0$.
Their time evolution is given by $V_t F^{(\pm)}(r)=F^{(\pm)}(r\mp t)$.

– the interaction between S and a noise particle takes place when their space-time positions coincide; it is fully described by the functions

$$f^{(\pm)}_{[F^{(\pm)},\, x]}(s)=(1\mp\beta(s))^{1/2}\,F^{(\pm)}(r(s)\mp s), \tag{3.2}$$

which depend on the free-noise amplitudes $F^{(\pm)}$ and on the trajectory $x(s)$. The quantities $|f^{(\pm)}_{[F^{(\pm)},\, x]}(s)|^2\, ds$ may be interpreted as the number of interactions taking place between S and the $(+)$, $(-)$-noise component in the time interval $(s,\ s+ds)$; then they correspond to the test functions in the space $h$ over which the Fock space of noise is constructed.

In the I.S. $K'=(a,\ \Lambda(\alpha))\cdot K$ the trajectory takes the form

$$x'(s')=\begin{bmatrix} s' \\ r'(s') \end{bmatrix}=\begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}\begin{bmatrix} s \\ r(s) \end{bmatrix}+\begin{bmatrix} a^0 \\ a^1 \end{bmatrix};$$

if $r(s)=\beta s+r_0=\tanh\xi\cdot s+r_0$ then we have

$$r'(s')=\beta' s'+r'_0=\tanh\xi'\cdot s'+r'_0,$$

where $\xi'=\xi+\alpha$ and $r'_0=\cosh\alpha\cdot r_0+a^1-\tanh(\xi+\alpha)\cdot(\sinh\alpha\cdot r_0+a^0)$. The transformed free noise and interaction amplitudes $F^{(\pm)}(r')$, $f^{(\pm)}_{[F^{(\pm)\prime},\, x']}(s')$ are determined by imposing the condition below:

for every trajectory $x(s)$ and for every time interval $(s,\ s+ds)$ measured by K along $x$, whose counterparts for $K'$ are $x'(s')$, $(s',\ s'+ds')$, the following equations must hold:

$$|f^{(\pm)\prime}_{[F^{(\pm)\prime},\, x']}(s')|^2\, ds'=|f^{(\pm)}_{[F^{(\pm)},\, x]}(s)|^2\, ds.$$

This condition is equivalent to the physical requirement that the number of interactions taking place in any given proper time interval must be reference independent. Up to a phase factor, this condition leads to

$$f^{(\pm)\prime}_{[F^{(\pm)\prime},\, x']}(s')=(\cosh\alpha+\beta(s)\sinh\alpha)^{-1/2}\, f^{(\pm)}_{[F^{(\pm)},\, x]}(s) \tag{3.3}$$

$$F^{(\pm)\prime}(y)=e^{\pm\alpha/2}\,F^{(\pm)}(e^{\pm\alpha}y-e^{\pm\alpha}(a^1\mp a^0)). \tag{3.4}$$

## 4. Noise as a scalar field

A concrete counting model is provided by a scalar, zero-mass field with a carrier frequency (quasi-monochromatic field, *cf.* [15]). A scalar zero-mass field $\varphi$ in $(1+1)$ dimensions is a solution of

$$\Box\varphi=0; \qquad \Box=g^{\mu\nu}\partial_\mu\partial_\nu=\partial_0^2-\partial_1^2. \tag{4.1}$$

The energy-momentum tensor is given by

$$T^{\mu\nu}=\partial_\mu\varphi\cdot\partial^\nu\varphi-(1/2)\,g^{\mu\nu}\partial_\rho\varphi\,.\,\partial^\rho\varphi,$$

or

$$T^{00}=T^{11}=\frac{1}{2}[(\partial^0 \varphi)^2+(\partial^1 \varphi)^2]; \qquad T^{01}=T^{10}=\partial^1 \varphi . \partial^0 \varphi. \tag{4.2}$$

The general solution of the wave equation (4.1) can be decomposed into its progressive and regressive components $\varphi^{(+)}$ and $\varphi^{(-)}$ as $\varphi=\varphi^{(+)}+\varphi^{(-)}$; these components satisfy the relations

$$\begin{aligned} &\varphi^{(+)}(t, r)=\varphi^{(+)}(0, r-t); \qquad \partial^0 \varphi^{(+)}=\partial^1 \varphi^{(+)}; \\ &\varphi^{(-)}(t, r)=\varphi^{(-)}(0, r+t); \qquad \partial^0 \varphi^{(-)}=-\partial^1 \varphi^{(-)}. \end{aligned} \tag{4.3}$$

It may be shown that an analogous decomposition holds for the energy-momentum tensor: $T^{\mu\nu}=T^{\mu\nu}_{(+)}+T^{\mu\nu}_{(-)}$, where

$$\begin{aligned} &T^{00}_{(\pm)}=T^{11}_{(\pm)}=\frac{1}{2}[(\partial^0 \varphi^{(\pm)})^2+(\partial^1 \varphi^{(\pm)})^2]=[\partial^0 \varphi^{(\pm)}]^2; \\ &T^{01}_{(\pm)}=T^{10}_{(\pm)}=\pm T^{00}_{(\pm)}=\partial^0 \varphi^{(\pm)} . \partial^1 \varphi^{(\pm)}=\pm[\partial^0 \varphi^{(\pm)}]^2. \end{aligned} \tag{4.4}$$

These decompositions are Lorentz-invariant because of the zero field mass; $T^{\mu\nu}_{(\pm)}$ transform as

$$T^{\mu\nu}_{(\pm)}(t', r')=e^{\pm 2\alpha} T^{\mu\nu}_{(\pm)}(t, r) \tag{4.5}$$

which, in view of (4.3), may be written as

$$\begin{aligned} &T^{\mu\nu'}_{(+)}(0, y)=e^{2\alpha} T^{\mu\nu}_{(+)}(0, e^{\alpha} y-e^{\alpha}(a^1-a^0)) \\ &T^{\mu\nu'}_{(-)}(0, y)=e^{-2\alpha} T^{\mu\nu}_{(-)}(0, e^{-\alpha} y-e^{-\alpha}(a^1+a^0)). \end{aligned} \tag{4.6}$$

We suppose that $\varphi^{(+)}$ and $\varphi^{(-)}$ each have a carrier energy-momentum vector (or frequency-wavenumber vector, $\hbar=1$) respectively of the form $k_{(+)}=(k^0_{(+)}, k^0_{(+)})^T$ and $k_{(-)}=(k^0_{(-)}, -k^0_{(-)})^T$, where $k^0_{(\pm)}>0$. The carrier frequencies $k^0_{(\pm)}$ transform as

$$k^{0'}_{(+)}=e^{\alpha} k^0_{(+)}; \qquad k^{0'}_{(-)}=e^{-\alpha} k^0_{(-)}. \tag{4.7}$$

We define

$$N^{(\pm)}(t, r)=\frac{T^{00}_{(\pm)}(t, r)}{k^0_{(\pm)}}=\frac{[\partial^0 \varphi^{(\pm)}(t, r)]^2}{k^0_{(\pm)}}.$$

The quantities $N^{(\pm)}(t, r)\,dr$ may be interpreted as the number of field particles moving with velocity $v=\pm 1$, which are contained in the space interval $(r, r+dr)$ at time $t$. This interpretation follows directly from the definitions, for $N^{(\pm)}dr$ is the ratio between a total energy $T^{00}_{(\pm)}dr$ and a one-particle energy $k^0_{(\pm)}$. Up to a phase factor, we may define the *free noise amplitudes* as

$$F^{(\pm)}(y)=\sqrt{N^{(\pm)}(0, y)}=\sqrt{\frac{T^{00}_{(\pm)}(0, y)}{k^0_{(\pm)}}}=\frac{\partial^0 \varphi^{(\pm)}(0, y)}{\sqrt{k^0_{(\pm)}}}. \tag{4.8}$$

Their transformation properties can be derived from (4.6), (4.7) and coincide with (3.4).

The interaction with S can be described in a semiclassical way as follows. Let $r(t)=\beta\cdot t+r_0$ be the time evolution of the mean value of the position operator relative to S; if the accuracy in the position measures is low with respect to its de Broglie wavelength, we may say that S moves along the trajectory $x(t)=(t, r(t))$. The only interaction allowed to S is by emission and absorption of "photons" (zero-mass scalar particles) of two kinds:

– particles with velocity $v=+1$ and frequency $p^0_{(+)}$

– particles with velocity $v=-1$ and frequency $p^0_{(-)}$.

The distinction between the frequencies is due to the Doppler effect, by which $p^0_{(+)}=\gamma(1+\beta)\cdot\bar{p}^0$, $p^0_{(-)}=\gamma(1-\beta)\cdot\bar{p}^0$ where $\bar{p}^0$ is the frequency corresponding to the energy transition coupled with noise, measured by an observer co-moving with S. These frequencies transform as $p^0_{(\pm)}=\gamma'(1\pm\beta')\bar{p}^0=e^{\pm\alpha}\gamma(1\pm\beta)\bar{p}^0=e^{\pm\alpha}p^0_{(\pm)}$ [*see* (3.1)].

We describe these noise particles by means of the quasi-monochromatic (q.m.) field $\varphi$, whose carrier frequencies $k^0_{(\pm)}$ must then satisfy $k^0_{(\pm)}=p^0_{(\pm)}$. The energy $k^0_{(\pm)}$ involved in each interaction is supposed to be much smaller than the energy of S, to avoid effects on the motion of S at least for finite time intervals.

## 5. Noise amplitudes and unitary representations of $P^\uparrow_+$

It may be shown that the transformation properties (3.4) relative to the free noise amplitudes $F^{(\pm)}$ are given by two unitary representations of $P^\uparrow_+$, say $\rho^{(+)}$ and $\rho^{(-)}$, which may be written as

$$\begin{aligned} &\rho^{(+)} : P^\uparrow_+ \to \mathcal{U}(L^2(\mathbf{R})) \\ &(a, \Lambda(\alpha)) \mapsto \rho^{(+)}(a, \Lambda(\alpha)) = e^{-i(a^1-a^0)P}\cdot e^{i\alpha D} \\ &\rho^{(-)} : P^\uparrow_+ \to \mathcal{U}(L^2(\mathbf{R})) \\ &(a, \Lambda(\alpha)) \mapsto \rho^{(-)}(a, \Lambda(\alpha)) = e^{-i(a^1+a^0)P}\cdot e^{-i\alpha D} \end{aligned} \tag{5.1}$$

where $\mathcal{U}(L^2(\mathbf{R}))$ is the group of unitary operators on $L^2(\mathbf{R})$, $P=-i\,d/dx$ is the momentum operator, $M=M_{\{x\}}$ is the position operator and $D=(1/2)\{M, P\}=(1/2)(MP+PM)$ is the infinitesimal generator of the dilatations in $L^2(\mathbf{R})$:

$$(e^{i\lambda D} f)(y)=e^{\lambda/2} f(e^\lambda y).$$

Furthermore, the relations (3.2) between free noise and interaction amplitudes may be written in a similar manner as

$$f^{(+)}_{[F^{(+)}, x]}=R\cdot e^{i\cdot\log(1-\beta)\cdot D}\cdot e^{ir_0P}\cdot F^{(+)},$$
$$f^{(1)}_{[F^{(-)}, x]}=e^{i\cdot\log(1+\beta)\cdot D}\cdot e^{ir_0P}\cdot F^{(-)},$$

where R is the reflection operator on $L^2(\mathbf{R})$: $(\mathrm{R}\,f)(x)=f(-x)$.

Finally, the transformations of the interaction amplitudes (3.3) can be written as

$$f^{(\pm)'}_{[\mathrm{F}^{(\pm)'},\, x']}=\mathrm{R}_x(a,\,\Lambda(\alpha))\cdot f^{(\pm)}_{[\mathrm{F}^{(\pm)},\, x]},$$

where

$$\begin{gathered}\mathrm{R}_x : \mathrm{P}^{\uparrow}_{+} \to \mathscr{U}(\mathrm{L}^2(\mathbf{R}))\\ (\alpha,\,\Lambda(\alpha))\mapsto \mathrm{R}_x(a,\,\Lambda(\alpha))=e^{-i\,(a^0+\sinh\alpha\cdot r_0)\,\mathrm{P}}\cdot e^{-i\cdot\log(\cosh\alpha+\beta\sinh\alpha)\cdot\mathrm{D}}. \quad (5.2)\end{gathered}$$

These unitary operators are the basis for the relativistic form of QSC we propose.

## 6. Relativistic QSC

Let

$$\left.\begin{gathered}\mathscr{R}_x=\mathrm{R}_x\oplus\mathrm{R}_x : \mathrm{P}^{\uparrow}_{+}\to\mathscr{U}(\mathrm{L}^2(\mathbf{R})\oplus\mathrm{L}^2(\mathbf{R}))\\ p\mapsto\mathscr{R}_x(p)=\mathrm{R}_x(p)\oplus\mathrm{R}_x(p);\end{gathered}\right. \quad (6.1)$$

$\mathscr{R}_x(p)$ acts on the test function space $h=\mathrm{L}^2(\mathbf{R})\oplus\mathrm{L}^2(\mathbf{R})$.

Using the functor $\Gamma$ [*cf.* (2.1), (2.2)] we associate to it the operator $\Gamma(\mathscr{R}_x(p)) : \mathscr{H}\to\mathscr{H}$ and then, in a canonical way, the transformation

$$\left.\begin{gathered}\mathscr{G}_x(p) : \mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})\\ \mathrm{A}\mapsto\mathscr{G}_x(p)\cdot\mathrm{A}=\Gamma(\mathscr{R}_x(p))\cdot\mathrm{A}\cdot\Gamma(\mathscr{R}_x(p))^{-1}.\end{gathered}\right\} \quad (6.2)$$

$\mathscr{G}_x(p)$ is taken as the transformation law for the fundamental processes and stochastic differentials: we have

THEOREM 6.1: (i) $\mathscr{G}_x(p).\mathrm{A}_i(t)=(\cosh\alpha+\beta\sinh\alpha)^{-1/2}.\mathrm{A}'_i(t')$

(ii) $\mathscr{G}_x(p)\cdot\mathrm{A}^+_i(t)=(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\cdot\mathrm{A}^{+\prime}_i(t')$

(iii) $\mathscr{G}_x(p)\cdot\Lambda_{ij}(t)=\Lambda'_{ij}(t')$

(iv) $\mathscr{G}_x(p)\cdot\mathrm{T}(t)=(\cosh\alpha+\beta\sinh\alpha)^{-1}\,\mathrm{T}'(t')$

$$=(\cosh\alpha+\beta\sinh\alpha)^{-1}(t'-t'_0)\cdot\mathbf{1}_{\tilde{\mathscr{H}}} \quad (6.3)$$

*Proof:* (i)–(ii): we have

$$\begin{aligned}\mathrm{R}_x(a,\,\Lambda(\alpha))\,\chi_{[t_0,\,t]}(s)&=e^{-i\,(a^0+\sinh\alpha\cdot r_0)\cdot\mathrm{P}}\\ \times e^{-i\cdot\log(\cosh\alpha+\beta\sinh\alpha)\cdot\mathrm{D}}\cdot\chi_{[t_0,\,t]}(s)&=(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\cdot e^{-i\,(a^0+\sinh\alpha\cdot r_0)\cdot\mathrm{P}}\\ &\quad\times\chi_{[(\cosh\alpha+\beta\sinh\alpha)\,t_0,\,(\cosh\alpha+\beta\sinh\alpha)\,t]}(s)\\ &=(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\\ &\quad\times\chi_{[(\cosh\alpha+\beta\sinh\alpha)\,t_0+a^0+\sinh\alpha\cdot r_0,\,(\cosh\alpha+\beta\sinh\alpha)\,t+a^0+\sin^h\alpha\cdot r_0]}(s)\\ &=(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\,\chi_{[t'_0,\,t']}(s).\end{aligned}$$

(6.3) (i)–(ii) follow by (2.2), (2.3), (6.1), (6.2).

(iii) we have

$$R_x(p)\cdot M_{\chi[t_0,t]}\cdot R_x(p)^* = M_{\chi[t_0',t']}$$

and (6.3) (iii) follows by (2.2), (2.3), (6.1), (6.2).

(iv)T$(t)$ is a multiple of the identity, so it is not changed by a unitary transformation and we have

$$\begin{aligned}\mathscr{G}_x(p)\cdot T(t) = T(t) = (t-t_0)\cdot \mathbf{1}_{\tilde{\mathscr{H}}} \\ &= (\cosh\alpha+\beta\sinh\alpha)^{-1}(t'-t_0')\cdot \mathbf{1}_{\tilde{\mathscr{H}}} \\ &= (\cosh\alpha+\beta\sinh\alpha)^{-1}\,T'(t').\end{aligned}$$

Q.E.D.

The transformation properties of the stochastic differentials can be derived in a similar manner by observing that $f(s)\chi_{[t,t+dt]} = f(t)\chi_{[t,t+dt]}$, and we have

(i) $\mathscr{G}_x(p)\cdot dA_i(t) = (\cosh\alpha+\beta\sinh\alpha)^{-1/2}\cdot dA_i'(t')$
(ii) $\mathscr{G}_x(p)\cdot dA_i^+(t) = (\cosh\alpha+\beta\sinh\alpha)^{-1/2}\cdot dA_i^{+\prime}(t')$
(iii) $\mathscr{G}_x(p)\cdot d\Lambda_{ij}(t) = d\Lambda_{ij}'(t')$
(iv) $\mathscr{G}_x(p)\cdot dt\cdot \mathbf{1}_{\tilde{\mathscr{H}}} = (\cosh\alpha+\beta\sinh\alpha)^2\, dt'\cdot \mathbf{1}_{\tilde{\mathscr{H}}}$.

*If* $h_0$ is the Hilbert space relative to internal degrees of freedom of S, then the operators in $\mathscr{L}(h_0)$ are not changed by the Poincaré transformations (they transform according to the identical representation); thus, if for an I.S. K the QSDE determining the time evolution of a particle interacting with a two-component noise is given by (2.6), then for the I.S. $K' = p\cdot K$ the transformed QSDE takes the form

$$\begin{aligned} dU' = U'[(W_{jk}-\delta_{jk})\otimes d\Lambda_{jk}' \\ &-(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\, W_{jk}M_k\otimes dA_j^{+\prime} \\ &+(\cosh\alpha+\beta\sinh\alpha)^{-1/2}\, M_k^+\otimes dA_k' \\ &+(\cosh\alpha+\beta\sinh\alpha)^{-1}\left(iK-\frac{1}{2}M_k^+M_k\right)\otimes \mathbf{1}_{\tilde{\mathscr{H}}}\,dt'\Big].\end{aligned} \tag{6.4}$$

THEOREM 6.2. — *The reduced dynamics* $\mathscr{T}_{t'}'$ *deduced from* (6.4) *is related to* $\mathscr{T}_t$ *by the equation*

$$\mathscr{T}_{t'}' = \mathscr{T}_{(\cosh\alpha+\beta\sinh\alpha)^{-1}\cdot t'}. \tag{6.5}$$

*Proof:* By (2.8), the reduced dynamics $\mathscr{T}_{t'}'$ satisfies the equation

$$\frac{d}{dt'}\mathscr{T}_{t'}'(D) = \mathscr{T}_{t'}'\Big(i(\cosh\alpha+\beta\sinh\alpha)^{-1}[K, D] + \\ -\frac{1}{2}(\cosh\alpha+\beta\sinh\alpha)^{-1}(M_k^+M_kD - 2M_k^+DM_k + DM_k^+M_k)\Big)$$

$$\mathscr{T}_{t_0}'(D) = D.$$

It can be easily verified that $\mathcal{T}_{(\cosh\alpha+\beta\sinh\alpha)^{-1}\cdot t'}(\mathrm{D})$ satisfies the same equation and initial condition, so that (6.5) is proved.

Q.E.D.

*Remarks.* — 1. Theorem (6.2) represents the relativistic time dilatation. Indeed, consider for example S to be a two-level system at rest in the I.S. K ($\beta=0$); let B be its annihilator, $\mathrm{M}_{(+)}=\mathrm{M}_{(-)}=z\,\mathrm{B}$, $z\in\mathbf{C}$, $\mathrm{K}=\mathrm{B}^+\mathrm{B}$. Then the time evolution of the projection on the upper level is given by

$$\mathcal{T}_t(\mathrm{B}^+\mathrm{B})=e^{-2|z|^2(t-t_0)}\cdot\mathrm{B}^+\mathrm{B},$$

so that the mean life of the upper level is $\tau=1/2|z|^2$. In the I.S. K′, by Theorem (6.2) we have

$$\mathcal{T}'_{t'}(\mathrm{B}^+\mathrm{B})=e^{-2|z|^2\cosh\alpha^{-1}(t'-t'_0)}\cdot\mathrm{B}^+\mathrm{B},$$

so that the transformed mean life $\tau'=\cosh\alpha\cdot 1/2|z|^2=\cosh\alpha\cdot\tau$ is dilatated by the appropriate factor $\cosh\alpha$.

2. We did not modify the structure of QSC, but simply added the transformation rules. The time dilatation we obtain indicates the internal consistency of the framework; more interesting results could perhaps be obtained by a multidimensional extension of the model.

3. In this paper we have dealt with uniform rectilinear motions only; however an extension to arbitrary motions is possible, in which case we have

$$\mathcal{T}'_{t'}=\mathcal{T}_{\sigma(t')},$$

where

$$\sigma(t')=\int_{t'_0}^{t'}ds'(\cosh\alpha+\beta(s)\sinh\alpha)^{-1}+t_0$$

(integrated effect of instantaneous time dilatations).

4. It seems possible to consider interactions between the noise and some external degrees of freedom of S, such as momentum and position, in which case the trajectory would became a dynamical variable.

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