

ANNALES DE L'I. H. P., SECTION A

S. TWAREQUE ALI

J.-P. ANTOINE

Coherent states of the 1+1 dimensional Poincaré group : square integrability and a relativistic Weyl transform

Annales de l'I. H. P., section A, tome 51, n° 1 (1989), p. 23-44

http://www.numdam.org/item?id=AIHPA_1989__51_1_23_0

© Gauthier-Villars, 1989, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

**Coherent states of the $1+1$ dimensional Poincaré
group: square integrability
and a relativistic Weyl transform**

by

S. Twareque ALI (*)

Department of Mathematics, Concordia University,
Montréal, Canada H4B 1R6

and

J.-P. ANTOINE

Institut de Physique Théorique,
Université Catholique de Louvain,
B-1348 Louvain-la-Neuve, Belgium

ABSTRACT. — We study the notion of square integrability of a group representation over a coset space, as a generalization of the usual notion of square integrability for representations belonging to the discrete series. We work out, explicitly, a parallel theory for the Poincaré group in 1-space and 1-time dimensions, which displays a greater richness of structure than the usually studied square integrable representations. As an application, we derive a relativistic Weyl transform.

(*) Work supported in part by the Natural Science and Engineering Research Council (N.S.E.R.C.) of Canada.

RÉSUMÉ. — On étudie la notion de représentation de groupe de carré intégrable sur un espace homogène, généralisant ainsi la notion usuelle de carré-intégrabilité pour les représentations de la série discrète. La théorie est développée explicitement dans le cas du groupe de Poincaré en deux dimensions (une dimension spatiale et une dimension temporelle); il en résulte une structure plus riche que celle obtenue à partir des représentations de carré intégrable au sens usuel. A titre d'application, on dérive une transformation de Weyl relativiste.

I. INTRODUCTION

The objective of this paper is twofold. First to display, for a specific representation, the coherent states ([1], [2]) of the Poincaré group (in 1-space and 1-time dimensions) built over the points of a certain coset space of this group. The construction illustrates the fact that certain group representations, which are not square integrable ([3]-[7]) in the usual sense (over the entire group), satisfy a weaker condition of square integrability over a coset space (possibly depending on a choice of section). Actually this generalized square integrability has been studied already in the mathematical literature, but only for *semisimple* Lie groups ([8]-[9]) (see also Ref. 2 for a review). Once, however, the definition of a square integrable representation is extended, there immediately arises the question as to whether the characteristic properties of such representations, such as the existence of a resolution of the identity, a formal dimension operator and orthogonality relations, still survive. The second objective of this paper is to study these questions for the particular representation of the Poincaré group that is introduced. One discovers in this way a remarkable richness in the structure underlying (generalized) square integrable representations.

The physical relevance of this paper stems from the fact that such coherent states can be used to extend to the relativistic regime ([10], [11]) a wide variety of coherent state computations that have hitherto been possible only nonrelativistically. Moreover, using the notion of coherent states for general group representations—to the extent that these do in fact exist—, one can set up a quantization procedure ([10], [12]) for classical algebras which resembles in many ways [12] the technique of geometric quantization.

Finally, as a byproduct of our computations, we obtain a relativistic analogue of the Weyl transform ([13]-[15]) used so frequently in non-relativistic statistical mechanics. The relativistic Weyl transform turns out

to be the natural extension of a unitary isomorphism that has long been known to exist [16] between the Hilbert-Schmidt operators on $L^2(\mathbb{R})$ and the Hilbert space $L^2(\mathbb{R}^2)$, and which is constructed using a certain group related transformation for the Weyl group G_w of the canonical commutation relations.

It ought to be pointed out that we restrict our discussion of the Poincaré group in this paper to 1-space and 1-time dimensions for reasons of computational neatness alone. Similar coherent states have in fact been obtained elsewhere ([17]-[19]) for the usual Poincaré group in 3-space and 1-time dimensions, as well as for the Galilei group, although all the other features pertaining to square integrability had not been studied there. Further generalizations of the method do not seem obvious, however. We will come back to this point at the end.

The rest of this paper is organized as follows. In Section II, we first briefly recall the important features of square integrable representations, as they are usually defined in the literature. Then we set up the various structures associated to these representations (Wigner map, coherent state decomposition), which we intend eventually to generalize along the directions indicated in Section III. We work out the proposals explicitly for the case of the Poincaré group in 1-space and 1-time dimensions in Section IV. The representations of this group, which we study, are square integrable over a certain homogeneous space of the group. In Section V we discuss the relativistic Weyl transform, and finally we end in Section VI with a few related comments. Some explicit calculations and proofs are relegated to the Appendix.

II. SOME PRELIMINARIES ON SQUARE INTEGRABLE REPRESENTATIONS

Let G be a locally compact group, dg the left invariant Haar measure on G and $g \mapsto U(g)$ a continuous, unitary irreducible representation of G on a Hilbert space \mathcal{H} .

DEFINITION 2.1. — A vector $\zeta \in \mathcal{H}$ is said to be *admissible* if

$$c(\zeta) = \int_G |\langle U(g)\zeta | \zeta \rangle|^2 dg < \infty. \quad (2.1)$$

Let \mathcal{A} be the set of all admissible vectors in \mathcal{H} . If $\mathcal{A} \neq \emptyset$, then it is dense in \mathcal{H} [5]. In particular, if G is a unimodular group (*i. e.* if the left and right invariant measures are the same), then $\mathcal{A} = \mathcal{H}$. Conversely, if $\mathcal{A} = \mathcal{H}$, then G is unimodular. If $\mathcal{A} \neq \emptyset$, then the representation $g \mapsto U(g)$ is said to be *square integrable*, since for each $\zeta \in \mathcal{A}$, the map

$W_\zeta: \mathcal{H} \rightarrow L^2(G, dg)$, defined for any $\varphi \in \mathcal{H}$ by:

$$(W_\zeta \varphi)(g) = [c(\zeta)]^{-1/2} \langle U(g)\zeta | \varphi \rangle, \quad (2.2)$$

is isometric. Moreover, if U_l is the left regular representation of G on $L^2(G, dg)$, i. e.,

$$(U_l(g)f)(g') = f(g^{-1}g') \quad (2.3)$$

for all $f \in L^2(G, dg)$ and almost all $g' \in G$, then

$$W_\zeta U(g) = U_l(g) W_\zeta. \quad (2.4)$$

Thus W_ζ intertwines U with the left regular representation U_l of G . This also means that every square integrable representation is unitarily equivalent to a subrepresentation of the left regular representation of G . In fact, every representation of G which belongs to the discrete series is square integrable (this is often taken as definition of the discrete series [4]).

Square integrable representations have a number of rather appealing properties. Let us choose an admissible vector $\zeta \in \mathcal{A}$ (in the terminology of Ref. 7, ζ is the *analyzing wavelet* and the map W_ζ is the *wavelet transform*). We denote by \mathcal{H}_ζ the subspace $W_\zeta \mathcal{H}$ of $L^2(G, dg)$ and by \mathbb{P}_ζ the corresponding projection operator,

$$\mathcal{H}_\zeta = \mathbb{P}_\zeta L^2(G, dg). \quad (2.5)$$

Then,

$$W_\zeta W_\zeta^* = \mathbb{P}_\zeta, \quad W_\zeta^* W_\zeta = I_{\mathcal{H}}, \quad (2.6)$$

$I_{\mathcal{H}}$ denoting the identity operator on \mathcal{H} . It can then be proved that

$$[c(\zeta)]^{-1} \int_G |\zeta_g\rangle \langle \zeta_g| dg = I_{\mathcal{H}}, \quad (2.7)$$

where $\zeta_g = U(g)\zeta$. Thus, the orbit \mathcal{O}_ζ of ζ under G forms an overcomplete family of vectors in \mathcal{H} — also known as *coherent states* ([1], [2]). The vectors $\varphi_\zeta \in \mathcal{H}_\zeta$ are actually *continuous functions* on G , which implies that every square integrable representation can be realized on a Hilbert space of continuous functions on the group [20]. As a consequence, there exists a *reproducing kernel* $K: G \times G \rightarrow \mathbb{C}$ which has the following properties:

$$(i) \quad K(g, g') = [c(\zeta)]^{-1} \langle \zeta_g | \zeta_{g'} \rangle \quad (2.8)$$

$$(ii) \quad (\mathbb{P}_\zeta \Phi)(g) = \int_G K(g, g') \Phi(g') dg', \quad \forall \Phi \in L^2(G, dg) \quad (2.9)$$

$$(iii) \quad \int_G K(g, g'') K(g'', g') dg'' = K(g, g'). \quad (2.10)$$

In particular, since the elements in \mathcal{H}_ζ are continuous functions, it follows from (2.9) that, for every $\varphi_\zeta \in \mathcal{H}_\zeta$,

$$\varphi_\zeta(g) = \int_G K(g, g') \varphi_\zeta(g') dg' \tag{2.11}$$

for all $g \in G$. This is the reproducing property of the kernel K . The Hilbert space \mathcal{H}_ζ is then called a reproducing kernel Hilbert space.

Finally, for square integrable representations one can prove the following *orthogonality relations* ([5]-[7]):

There exists a unique positive invertible operator C on \mathcal{H} with domain equal to \mathcal{A} (the set of all admissible vectors) such that,

$$\int_G \langle \varphi_1 | U(g)\zeta_1 \rangle \langle U(g)\zeta_2 | \varphi_2 \rangle dg = \langle C\zeta_2 | C\zeta_1 \rangle \langle \varphi_1 | \varphi_2 \rangle, \tag{2.12}$$

$\forall \zeta_1, \zeta_2 \in \mathcal{A}$ and $\forall \varphi_1, \varphi_2 \in \mathcal{H}$. In the special case where G is a unimodular group, $C = I_{\mathcal{H}}$.

Let us cast the orthogonality relation (2.12) into a somewhat different form, which will be useful for us in the next section. Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert space of all Hilbert-Schmidt operators on \mathcal{H} , with scalar product,

$$\langle \rho_1 | \rho_2 \rangle_{\mathcal{B}_2(\mathcal{H})} = \text{tr}[\rho_1^* \rho_2]. \tag{2.13}$$

Then, denoting by $\rho_{\varphi, \zeta}$ the rank one (thus Hilbert-Schmidt) operator,

$$\rho_{\varphi, \zeta} = |\varphi\rangle \langle \zeta|, \tag{2.14}$$

the relations (2.12) can be rewritten as:

$$\int_G \overline{\text{tr}[U(g)^* \rho_{\varphi_1, \zeta_1} C^{-1}]} \text{tr}[U(g)^* \rho_{\varphi_2, \zeta_2} C^{-1}] dg = \langle \rho_{\varphi_1, \zeta_1} | \rho_{\varphi_2, \zeta_2} \rangle_{\mathcal{B}_2(\mathcal{H})} \tag{2.15}$$

for all ζ_1, ζ_2 in the *range* of C and all $\varphi_1, \varphi_2 \in \mathcal{H}$. Let $\mathcal{D} \subset \mathcal{B}_2(\mathcal{H})$ be the linear space of all Hilbert-Schmidt operators of the type (2.14) with ζ in the range of C (i. e., in the domain of C^{-1}). Since $\text{Range}(C)$ is dense in \mathcal{H} , \mathcal{D} is dense in $\mathcal{B}_2(\mathcal{H})$. Define a map $\mathcal{W} : \mathcal{D} \rightarrow L^2(G, dg)$ by the relation:

$$(\mathcal{W} \rho)(g) = \text{tr}[U(g)^* \rho C^{-1}]. \tag{2.16}$$

Then, in view of (2.15), \mathcal{W} is a linear isometry, which can be extended by continuity to the whole of $\mathcal{B}_2(\mathcal{H})$. The extended map (denoted by the same symbol) $\mathcal{W} : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(G, dg)$, which on the dense set \mathcal{D} is defined by (2.16), will be called the *Wigner transform map*, and for any $\rho \in \mathcal{B}_2(\mathcal{H})$, $\mathcal{W} \rho \in L^2(G, dg)$ is its *Wigner transform*. In general \mathcal{W} is only a partial

isometry, but in many important cases, the range of \mathcal{W} is in fact the whole of $L^2(G, dg)$.

Consider now the unitary representation $\tilde{U}(g)$ of G on $\mathcal{B}_2(\mathcal{H})$ defined by multiplication by the operators $U(g)$ (of the representation $g \mapsto U(g)$ on \mathcal{H}):

$$\tilde{U}(g) \rho = \rho' = U(g) \rho. \quad (2.17)$$

Then, it is easy to see that

$$\mathcal{W} \tilde{U}(g) = U_1(g) \mathcal{W}, \quad (2.18)$$

where $U_1(g)$ is the left regular representation given in (2.3). On the other hand, the representation $\tilde{U}(g)$ is highly reducible. In fact, for each $\zeta \in \text{Range}(C)$, the set of vectors in $\mathcal{B}_2(\mathcal{H})$ of the type (2.14), with $\varphi \in \mathcal{H}$, is isomorphic to \mathcal{H} , and by (2.15), if $\zeta_1, \zeta_2 \in \text{Range}(C)$ and $\zeta_1 \perp \zeta_2$, then the corresponding sets of vectors in $\mathcal{B}_2(\mathcal{H})$ live in orthogonal subspaces. Let us choose a complete orthonormal basis $\{\zeta_i\}_{i=1}^\infty$ in \mathcal{H} , with each $\zeta_i \in \text{Range}(C)$. For each ζ_i , denote by $\mathcal{H} \otimes \zeta_i$ the subspace of $\mathcal{B}_2(\mathcal{H})$, defined by (2.14), $\forall \varphi \in \mathcal{H}$. Then each $\mathcal{H} \otimes \zeta_i$ is stable under $\tilde{U}(g)$ and in fact the restriction $\tilde{U}_i(g)$ of $\tilde{U}(g)$ to this subspace gives an irreducible representation of G (namely $g \mapsto U(g)$ on \mathcal{H} , itself). On the other hand, \mathcal{W} restricted to $\mathcal{H} \otimes \zeta_i$ is precisely a map W_{ζ_i} of the type given in (2.2).

In other words, since $\mathcal{B}_2(\mathcal{H})$ decomposes as the direct sum:

$$\mathcal{B}_2(\mathcal{H}) = \bigoplus_{i=1}^{\infty} \mathcal{H} \otimes \zeta_i, \quad (2.19)$$

$$\tilde{U}(g) = \bigoplus_{i=1}^{\infty} \tilde{U}_i(g),$$

we get:

$$\begin{aligned} \mathbb{P}_{\mathcal{W}} L^2(G, dg) &= \bigoplus_{i=1}^{\infty} \mathcal{H}_{\zeta_i}, \\ U_1(g) \mathbb{P}_{\mathcal{W}} &= \bigoplus_{i=1}^{\infty} U_{1,i}(g), \end{aligned} \quad (2.20)$$

where $\mathbb{P}_{\mathcal{W}} = \mathcal{W} \mathcal{W}^*$ is the projection operator, in $L^2(G, dg)$, onto the range of \mathcal{W} , the space \mathcal{H}_{ζ_i} is a Hilbert space of the type \mathcal{H}_{ζ} in (2.5) and $U_{1,i}(g)$ is the restriction of $U_1(g)$ to \mathcal{H}_{ζ_i} . Thus, the image of the Wigner transform decomposes completely into a direct sum of Hilbert spaces \mathcal{H}_{ζ_i} , each of which carries an irreducible representation $U_{1,i}(g)$ of G and is a space of continuous functions on G admitting a reproducing kernel. The decomposition of (2.20) may thus be called a *coherent state decomposition* of the part of the left regular representation which acts in the image of the Wigner transform.

To make contact with familiar situations, we mention the two standard examples (see [1], [2], [7] and references therein):

(i) G = the Weyl-Heisenberg group, which is unimodular. Then $g = (q, p)$, a point in phase space, and one gets the *canonical* coherent states.

(ii) G = the “ $ax + b$ ” group, which is *not* unimodular. This leads to the affine coherent states.

III. GENERALIZATION: SQUARE INTEGRABILITY OVER A COSET SPACE

The notion of square integrability introduced in the last section—and this is the one usually found in the literature ([3]-[7])—hinges on the integral (2.1) existing over the entire group G . However, it often happens that one has an analogous situation over a transitive homogeneous space X of the group. In other words, there exists a closed subgroup H of G , for which

$$X \cong G/H \tag{3.1}$$

and for which there exists a Borel section

$$\beta: X \rightarrow G \tag{3.2}$$

such that the following integral converges, for some $\zeta \in \mathcal{H}$:

$$c_\beta(\zeta) = \int_X |\langle U(\beta(x))\zeta | \zeta \rangle|^2 dv(x) < \infty. \tag{3.3}$$

The measure dv on X is assumed to be invariant under the action of the group: $(g, x) \mapsto gx \in X$, $\forall (g, x) \in G \times X$. On the other hand, for the same representation U , there may not exist any vectors ζ which are admissible in the sense of (2.1). Nevertheless it turns out that the existence of vectors ζ which satisfy (3.3) (and possibly some additional conditions) is often enough to give rise to properties of the representation which parallel those described in the previous section ([17]-[19]). Besides, as mentioned there, square integrable representations are representations of the discrete series of G , so that studying a generalized notion of square integrability might enable one to catch some of the other representations of the group as well. We will work out these ideas explicitly in a specific case in Section IV below. Before that, we make some additional comments on the general situation.

Square integrability of a representation over a coset space has in fact been discussed previously. In the simplest case, G is locally compact and H is the center $Z(G)$ of G . Indeed, strict square integrability, in the sense

of Section II, implies that $Z(G)$ be compact [21]. But this is an artificial limitation, hence the natural notion is that of *square integrability modulo the center*, that is, precisely the situation described above, with $H \equiv Z(G)$. This is the case commonly treated in the mathematical literature ([2], [21]) for a locally compact group. Another situation which has been extensively studied is that where G is a *semisimple Lie group* and $X \cong G/H$ is a symmetric space ([2], [8], [9]). In that case, H is usually the maximal compact subgroup of G or the stability subgroup of a maximal weight vector. A familiar example is $G = \text{SU}(2)$, for which one gets the spin coherent states ([1], [2]). Other examples are the Lorentz group $\text{SO}(3,1)$ or the group $\text{SU}(1,1)$ [2].

For non-semisimple groups, the emphasis has been put on the existence of systems of (generalized) coherent states based on a coset space $X \cong G/H$ (see Ref. 2, § 10.5 for a review). For G a *nilpotent* Lie group, rather complete results have been obtained by Moscovici [22], by an extensive use of Kirillov's method of orbits [23]. Some of those results remain valid when G is a *solvable* or a *reductive* Lie group [24]. In any case, for an arbitrary Lie group G , if the representation U of G admits a *covariant* system (in the sense of Scutaru [25]) of coherent states based on $X \cong G/H$, then U is a subrepresentation of the representation of G induced by the restriction of U to H . This clearly generalizes the notion of square integrable representation.

It is interesting to notice that, for coherent states in the usual sense of Perelomov [2,26], the covariance condition implies that the restriction of U to H is a unitary *character*. It follows that the integral in (3.3) is independent of the choice of the section β , two different sections being related by an element of H : $\beta'(x) = \beta(x)h(x)$, with $h(x) \in H$ an element of the fibre over x . In the case discussed below, with G the Poincaré group in 1+1 dimensions, $U \upharpoonright H$ is *not* a character, yet the discussion goes through and coherent states may be defined with all the expected properties. To be sure, the whole setup depends now on the choice of the section $\beta: X \rightarrow G$ (which, in the case to be studied, has physical meaning). Of course, we don't expect the construction to work for a general pair G, U , and it is an interesting problem to find precise conditions under which it does. For instance, it is probably too restrictive to require U to be irreducible, the whole machinery should apply to *cyclic* representations as well. Beyond that, the question is open.

IV. RELATIVISTIC COHERENT STATES

We denote by $\mathcal{P}_+^\uparrow(1,1)$ the Poincaré group in 1+1 dimensions. Its elements are written as (a, Λ) , where $a = (a_0, \mathbf{a})$ is a space-time translation

and Λ is a Lorentz boost. The matrix Λ may be parametrized by a vector $p=(p_0, \mathbf{p})$:

$$\Lambda = \Lambda_p = m^{-1} \begin{pmatrix} p_0 & \mathbf{p} \\ \mathbf{p} & p_0 \end{pmatrix}, \quad p \in \mathcal{V}_m^+, \quad (4.1)$$

where \mathcal{V}_m^+ denotes the forward mass hyperbola:

$$\mathcal{V}_m^+ = \{(p_0, \mathbf{p}) \in \mathbb{R}^2 \mid p_0 > 0, p_0^2 - \mathbf{p}^2 = m^2\}. \quad (4.2)$$

The elements Λ_p of the Lorentz group act on \mathcal{V}_m^+ in the natural manner,

$$k \rightarrow k' = \Lambda_p k, \quad k \in \mathcal{V}_m^+. \quad (4.3)$$

This action is transitive and the corresponding invariant measure on \mathcal{V}_m^+ is easily seen to be dk/k_0 . Further details are given in the Appendix.

Consider next the following unitary irreducible representation of $\mathcal{P}_+^\uparrow(1,1)$. The Hilbert space is $\mathcal{H}_w = L^2(\mathcal{V}_m^+, dk/k_0)$, whose elements are really functions of the single variable $\mathbf{k} \in \mathbb{R}$, square integrable with respect to dk/k_0 . The unitary operators constituting the representation will be denoted by $U_w(a, \Lambda)$, $(a, \Lambda) \in \mathcal{P}_+^\uparrow(1,1)$ and their action is:

$$(U_w(a, \Lambda) \varphi_w)(k) = e^{ik \cdot a} \varphi_w(\Lambda_p^{-1} k), \quad \varphi_w \in \mathcal{H}_w, \quad (4.4)$$

where $k \cdot a = k_0 a_0 - \mathbf{k} \cdot \mathbf{a}$. We shall call U_w the *Wigner representation* of $\mathcal{P}_+^\uparrow(1,1)$ for mass m .

It is easy to see that the Wigner representation is not square integrable in the sense of Definition 2.1. Indeed, for any $\varphi_w \in \mathcal{H}_w$,

$$\int_{\mathcal{P}_+^\uparrow(1,1)} |\langle U_w(a, \Lambda) \varphi_w \mid \varphi_w \rangle|^2 da_0 d\mathbf{a} d\mathbf{p}/p_0 = \infty. \quad (4.5)$$

However, we shall now show that in a certain sense U_w is square integrable with respect to a particular homogeneous space. Consider for this purpose the subgroup T of time translations of $\mathcal{P}_+^\uparrow(1,1)$ and denote by Γ_l and Γ_r the corresponding left and right coset spaces,

$$\Gamma_l = \mathcal{P}_+^\uparrow(1,1)/T, \quad \Gamma_r = T \backslash \mathcal{P}_+^\uparrow(1,1). \quad (4.6)$$

It is easy to see (*see* the Appendix) that points in both Γ_l and Γ_r can be parametrized by $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$, and that the map $\beta: \Gamma_{l,r} \rightarrow \mathcal{P}_+^\uparrow(1,1)$ defined by

$$\beta(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), \Lambda_p), \quad p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}) \quad (4.7)$$

is a Borel section for both Γ_l and Γ_r . Since $\mathcal{P}_+^\uparrow(1,1)$ and T are unimodular, both coset spaces Γ_l, Γ_r have unique left, resp. right, invariant measures [4]. A straightforward computation then shows that the measure

$$d\mu_l(\mathbf{q}, \mathbf{p}) = d\mathbf{q} d\mathbf{p} \quad (4.8)$$

is left-invariant on Γ_l , while the measure

$$d\mu_r(\mathbf{q}, \mathbf{p}) = d\mathbf{q} d\mathbf{p}/p_0 \quad (4.9)$$

is right-invariant on Γ_r . It is important to notice here that, whereas we may take the two coset spaces $\mathcal{P}_+^\dagger(1,1)/T$ and $T \setminus \mathcal{P}_+^\dagger(1,1)$ to be equal, *i. e.* $\Gamma_l = \Gamma_r = \Gamma$, the two invariant measures $d\mu_l$ and $d\mu_r$ are indeed different.

For the section β in (4.7), let us write,

$$U_w(\mathbf{q}, \mathbf{p}) = U_w(\beta(\mathbf{q}, \mathbf{p})) = U_w((0, \mathbf{q}), \Lambda_p), \quad (4.10)$$

where $U_w((0, \mathbf{q}), \Lambda_p)$ is defined as in (4.4). We now show that there exist vectors $\zeta \in \mathcal{H}_w$ for which the functions $f_{\varphi, \zeta}: \Gamma \rightarrow \mathbb{C}$, defined by

$$f_{\varphi, \zeta}(\mathbf{q}, \mathbf{p}) = \langle U_w(\mathbf{q}, \mathbf{p}) \zeta \mid \varphi \rangle, \quad \varphi \in \mathcal{H}_w, \quad (4.11)$$

are square integrable. Let H_0 be the free Hamiltonian operator on \mathcal{H}_w :

$$(H_0 \varphi)(k) = k_0 \varphi(k), \quad (4.12)$$

which is defined on a dense set of vectors $\varphi \in \mathcal{D}(H_0)$. Clearly, H_0 is a positive operator with spectrum $[m, \infty)$. Hence $H_0^{1/2}$ is defined, also on a dense domain, $\mathcal{D}(H_0^{1/2}) \subset \mathcal{H}_w$, and it has spectrum $[m^{1/2}, \infty)$. The inverse $H_0^{-1/2}$ of $H_0^{1/2}$ is a bounded operator with spectrum $[0, m^{-1/2}]$.

According to the scheme of Section II, we need first an appropriate notion of admissible vectors. The following lemma is straightforward (a proof is given in the Appendix):

LEMMA 4.1. — For arbitrary $\varphi \in \mathcal{H}_w$, the integrals

$$I_l(\varphi, \zeta) = \int_{\Gamma} |f_{\varphi, \zeta}(\mathbf{q}, \mathbf{p})|^2 d\mu_l(\mathbf{q}, \mathbf{p}) \quad (4.13 a)$$

and

$$I_r(\varphi, \zeta) = \int_{\Gamma} |f_{\varphi, \zeta}(\mathbf{q}, \mathbf{p})|^2 d\mu_r(\mathbf{q}, \mathbf{p}) \quad (4.13 b)$$

exist iff $\zeta \in \mathcal{D}(H_0^{1/2})$. ■

Next we introduce the following bounded self-adjoint operator on \mathcal{H}_w :

$$(B \varphi_w)(k) = k/k_0 \varphi_w(k), \quad \forall \varphi_w \in \mathcal{H}_w. \quad (4.14)$$

Then, as a consequence of Lemma 4.1, we obtain the important result:

THEOREM 4.2. — Let $\zeta \in \mathcal{H}_w$ satisfy the conditions:

$$(i) \quad \zeta \in \mathcal{D}(H_0^{1/2}), \quad (4.15)$$

$$(ii) \quad \langle B H_0^{1/2} \zeta \mid H_0^{1/2} \zeta \rangle = 0. \quad (4.16)$$

Then the map $W_\zeta^\beta: \mathcal{H}_w \rightarrow L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ given, for any $\varphi_w \in \mathcal{H}$, by the relation:

$$(W_\zeta^\beta \varphi_w)(\mathbf{q}, \mathbf{p}) = [c_\beta(\zeta)]^{-1/2} \langle U_w(\mathbf{q}, \mathbf{p}) \zeta \mid \varphi_w \rangle, \quad (4.17)$$

is an isometry. The normalization factor in (4.17) is defined as:

$$c_\beta(\zeta) = 2\pi m^{-1} \int_{\mathbb{R}} |\zeta(k)|^2 dk = 2\pi m^{-1} \|H_0^{1/2} \zeta\|^2, \quad \blacksquare \quad (4.18)$$

The proof is again given in the Appendix. A number of consequences can be derived from this theorem. We state some of these without proof, since the proofs are analogous to similar results obtained in References 18 and 19. However, in view of the theorem above and its consequences, we adopt the following definition for the admissibility of a vector $\zeta \in \mathcal{H}_W$ for the representation U_W .

DEFINITION 4.3. — A vector $\zeta \in \mathcal{H}_W$ is said to be *admissible mod* (T, β) if it satisfies the conditions (i) and (ii) of Theorem 4.2 above.

Since the representation U_W admits such vectors, we shall say that it is *square integrable mod* (T, β) .

Note again that the set $\mathcal{A}_{(T, \beta)}$ of all vectors in \mathcal{H}_W which are admissible mod (T, β) is dense in \mathcal{H}_W .

Given an admissible vector $\zeta \in \mathcal{A}_{(T, \beta)}$, we consider its orbit under U_W :

$$\mathfrak{G}_\beta(\zeta) = \{ \eta_{\mathbf{q}, \mathbf{p}} = [c_\beta(\zeta)]^{-1/2} \zeta_{\mathbf{q}, \mathbf{p}} \mid \zeta_{\mathbf{q}, \mathbf{p}} = U_W(\mathbf{q}, \mathbf{p}) \zeta, (\mathbf{q}, \mathbf{p}) \in \Gamma \}. \quad (4.19)$$

Then it can be seen that \mathfrak{G}_β is overcomplete in \mathcal{H}_W and moreover,

$$\int | \eta_{\mathbf{q}, \mathbf{p}} \rangle \langle \eta_{\mathbf{q}, \mathbf{p}} | d\mathbf{q} d\mathbf{p} = I_{\mathcal{H}}. \quad (4.20)$$

For this reason we shall call the family of vectors

$$\mathfrak{G}_\beta(T, \beta) = \bigcup_{\zeta \in \mathcal{A}_{(T, \beta)}} \mathfrak{G}_\beta(\zeta) \quad (4.21)$$

the set of *relativistic coherent states* on the *phase space* Γ . For each fixed ζ , the set $\mathfrak{G}_\beta(\zeta)$ will be called a *coherent section*.

Let $\mathbb{P}_\zeta = W_\zeta^\beta W_\zeta^{\beta*}$ be the projection operator onto the subspace \mathcal{H}_ζ of $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$, which is the image of \mathcal{H}_W under W_ζ^β . Then, there exists a reproducing kernel $K_\zeta: \Gamma \times \Gamma \rightarrow \mathbb{C}$ such that,

$$(4.22 a)$$

$$(i) \quad K_\zeta(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') = \langle \eta_{\mathbf{q}, \mathbf{p}} \mid \eta_{\mathbf{q}', \mathbf{p}'} \rangle$$

$$(ii) \quad (\mathbb{P}_\zeta \Phi)(\mathbf{q}, \mathbf{p}) = \int_\Gamma K_\zeta(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') \Phi(\mathbf{q}', \mathbf{p}') d\mathbf{q}' d\mathbf{p}',$$

$$\forall \Phi \in L^2(\Gamma, d\mathbf{q} d\mathbf{p}) \quad (4.22 b)$$

$$(iii) \quad \int_\Gamma K_\zeta(\mathbf{q}, \mathbf{p}; \mathbf{q}'', \mathbf{p}'') K_\zeta(\mathbf{q}'', \mathbf{p}''; \mathbf{q}', \mathbf{p}') d\mathbf{q}'' d\mathbf{p}''$$

$$= K_\zeta(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}'). \quad (4.22 c)$$

So far everything parallels the situation envisaged in Section II, and we try next to obtain orthogonality relations in the manner of (2.12). Indeed,

following the steps in the proof of Lemma 4.1 in the Appendix we easily establish the result:

THEOREM 4.4. — *The following relation holds for all $\zeta_1, \zeta_2 \in \mathcal{D}(H_0^{1/2})$ and all $\varphi_1, \varphi_2 \in \mathcal{H}_w$,*

$$\int_{\Gamma} \langle \varphi_1 | U_w(\mathbf{q}, \mathbf{p}) \zeta_1 \rangle \langle U_w(\mathbf{q}, \mathbf{p}) \zeta_2 | \varphi_2 \rangle d\mathbf{q} d\mathbf{p} \\ = 2\pi m^{-1} \{ \langle H_0^{1/2} \zeta_2 | H_0^{1/2} \zeta_1 \rangle \langle \varphi_1 | \varphi_2 \rangle \\ - \langle B H_0^{1/2} \zeta_2 | H_0^{1/2} \zeta_1 \rangle \langle \varphi_1 | B \varphi_2 \rangle \} \quad (4.23)$$

where B is the bounded self-adjoint operator given in (4.14). The second term on the RHS of (4.23) vanishes whenever $\zeta_1 = \zeta_2 \in \mathcal{A}_{(\Gamma, \beta)}$. ■

The form of the orthogonality relations in (4.23) ought to be compared to that in (2.12). First $\sqrt{2\pi m^{-1}} H_0^{1/2}$ plays a role analogous to that of the operator C . Then there appears an extra term, involving the operator B , and this has some very interesting implications. To understand this term better, let us begin by working out the Wigner transform for the representation U_w . Since $H_0^{-1/2}$ is a bounded operator, we may rewrite (4.23) as:

$$m/2\pi \int_{\Gamma} \langle \varphi_1 | U_w(\mathbf{q}, \mathbf{p}) H_0^{-1/2} \zeta_1 \rangle \langle U_w(\mathbf{q}, \mathbf{p}) H_0^{-1/2} \zeta_2 | \varphi_2 \rangle d\mathbf{q} d\mathbf{p} \\ = \langle \zeta_2 | \zeta_1 \rangle \langle \varphi_1 | \varphi_2 \rangle - \langle B \zeta_2 | \zeta_1 \rangle \langle \varphi_1 | B \varphi_2 \rangle, \quad (4.24)$$

a relation which is now valid for all ζ_1, ζ_2 and $\varphi_1, \varphi_2 \in \mathcal{H}_w$. The first term in the RHS of (4.24) vanishes whenever $\zeta_1 \perp \zeta_2$, while the second term is zero any time $B \zeta_2 \perp \zeta_1$. In particular if $\zeta_1 = \zeta_2 = \zeta$ and if $H_0^{-1/2} \zeta$ satisfies (4.16), then this term vanishes. As in the last section, we can now define a Wigner transform \mathcal{W} , initially on all Hilbert-Schmidt operators ρ on \mathcal{H}_w of the form $\rho = |\varphi\rangle\langle\zeta|$, $\varphi, \zeta \in \mathcal{H}_w$. Thus, for all such ρ ,

$$(\mathcal{W} \rho)(\mathbf{q}, \mathbf{p}) = \sqrt{m/2\pi} \operatorname{tr} [U_w(\mathbf{q}, \mathbf{p})^* \rho H_0^{-1/2}] \quad (4.25)$$

and we then use a continuity argument to extend \mathcal{W} to a linear map $\mathcal{W}: \mathcal{B}_2(\mathcal{H}_w) \rightarrow L^2(\Gamma, d\mathbf{q} d\mathbf{p})$. Thus, for every $\rho \in \mathcal{B}_2(\mathcal{H}_w)$, (4.24) assumes the form

$$\langle \mathcal{W} \rho_1 | \mathcal{W} \rho_2 \rangle_{L^2(\Gamma)} = \langle \rho_1 | \rho_2 \rangle_{\mathcal{B}_2(\mathcal{H}_w)} - \langle \rho_1 | D \rho_2 \rangle_{\mathcal{B}_2(\mathcal{H}_w)} \quad (4.26)$$

where D is the bounded linear positive operator on $\mathcal{B}_2(\mathcal{H}_w)$:

$$D\rho = B\rho B, \quad \forall \rho \in \mathcal{B}_2(\mathcal{H}_w). \quad (4.27)$$

Using the methods of Reference 19, the following result can now be proved.

LEMMA 4.5. — *The range \mathcal{R}_w of the linear map \mathcal{W} is dense in $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ and coincides with the domain of the operator \hat{H}_0 :*

$$(\hat{H}_0 \Phi)(\mathbf{q}, \mathbf{p}) = (\mathbf{p}^2 + m^2)^{1/2} \Phi(\mathbf{q}, \mathbf{p}). \quad (4.28)$$

The inverse map \mathcal{W}^{-1} is unbounded, while \mathcal{W} satisfies the norm estimate:

$$1 < \|\mathcal{W}\| < \sqrt{2}. \quad \blacksquare \tag{4.29}$$

On \mathcal{R}_w , let us introduce the new scalar product

$$\langle \Phi_1 | \Phi_2 \rangle_w = \langle \Phi_1 | \Phi_2 \rangle_{L^2(\Gamma)} + \langle \Phi_1 | D_w \Phi_2 \rangle_{L^2(\Gamma)} \tag{4.30}$$

where $D_w = \mathcal{W}^{-1} * D \mathcal{W}^{-1}$. Explicitly,

$$\langle \Phi_1 | \Phi_2 \rangle_w = m^{-2} \int_{\Gamma} \overline{\Phi_1(\mathbf{q}, \mathbf{p})} P_0 p_0^{-1} (P_0 p_0 + \mathbf{P} \cdot \mathbf{p}) \Phi_2(\mathbf{q}, \mathbf{p}) d\mathbf{q} d\mathbf{p}, \tag{4.31}$$

where the operators \mathbf{P} and P_0 are self-adjoint in $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ and are defined as

$$\left. \begin{aligned} \mathbf{P} &= -i \partial/\partial \mathbf{q} \\ P_0 &= \sqrt{\mathbf{P}^2 + m^2}. \end{aligned} \right\} \tag{4.32}$$

Clearly D_w is an unbounded self-adjoint operator in $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$, with domain $\mathcal{R}_w = \mathcal{D}(\mathcal{W}^{-1})$, and the scalar product (4.30) corresponds to its graph norm. Thus the latter turns \mathcal{R}_w into a Hilbert space, that we denote by $\mathcal{H}(D_w)$, and we get the rigged Hilbert space [25]:

$$\mathcal{H}(D_w) \subset L^2(\Gamma, d\mathbf{q} d\mathbf{p}) \subset \mathcal{H}(D_w)^*, \tag{4.33}$$

where $\mathcal{H}(D_w)^* \equiv \mathcal{H}(D_w^{-1})$ is the dual of $\mathcal{H}(D_w)$ with respect to the inner product of $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$. The interest of the triplet (4.33) lies in the fact that the orthogonality relation in (4.26) becomes now:

$$\langle \mathcal{W} \rho_1 | \mathcal{W} \rho_2 \rangle_w = \langle \rho_1 | \rho_2 \rangle_{\mathcal{B}_2(\mathcal{H}_w)}, \quad \forall \rho_1, \rho_2 \in \mathcal{B}_2(\mathcal{H}_w). \tag{4.34}$$

Thus the Wigner map is an isometry from $\mathcal{B}_2(\mathcal{H}_w)$ into $\mathcal{H}(D_w)$.

Again, one ought to note that if $\rho_1, \rho_2 \in \mathcal{H}_w \otimes \zeta$, where $\zeta \in \mathcal{A}(\Gamma, \beta)$, *i. e.* ζ is admissible, then,

$$\langle \mathcal{W} \rho_1 | \mathcal{W} \rho_2 \rangle_w = \langle \mathcal{W} \rho_1 | \mathcal{W} \rho_2 \rangle_{L^2(\Gamma)}. \tag{4.35}$$

A complete decomposition of the representation \tilde{U} generated by U_w on $\mathcal{B}_2(\mathcal{H}_w)$ in the manner of Equations (2.17)-(2.20) can be undertaken. The results are similar to those obtained in Reference 19, except that now, with the new scalar product (4.30), the phase space representation turns out to be globally unitary.

This result exhibits the fundamental difference between the nonrelativistic situation and the relativistic one, already noticed in Reference 19. In the former case, the operator \mathcal{W}^{-1} (which corresponds to the map Θ of Ref. 19) is bounded, in fact unitary. In the present context, however, \mathcal{W}^{-1} is unbounded, so that $\mathcal{H}(D_w) \neq L^2(\Gamma)$. Now the difficulty encountered in Reference 19 becomes clear. The representation \hat{U} induced in $L^2(\Gamma)$ by \tilde{U} , via \mathcal{W} , is not unitary, because the norm of $L^2(\Gamma)$ is not the correct

one. Now, since \mathcal{D}_w is invariant under \hat{U} , as follows from (the equivalent of) the relation (2.18), we may consider the restriction $\hat{U} \upharpoonright \mathcal{D}_w$. Then, if one replaces the $L^2(\Gamma)$ norm by the graph norm of D_w , everything falls in place and the phase space representation becomes globally unitary, as it should, but in $\mathcal{H}(D_w)$ instead of $L^2(\Gamma)$.

An interesting side question is the physical significance, if any, of the larger space $\mathcal{H}(D_w)^\times$, which consists of continuous linear functionals over $\mathcal{H}(D_w)$. Perhaps it could play a rôle in the analysis of relativistic measurements, as suggested in Reference 26. This makes sense, since the unitary representation $\hat{U} \upharpoonright \mathcal{H}(D_w)$ may be transported to $\mathcal{H}(D_w)^\times$ by duality, so that the active and passive interpretations are indeed possible.

V. A RELATIVISTIC WEYL TRANSFORM

The Wigner transform, defined by the extension of

$$(\mathcal{W} \rho)(\mathbf{q}, \mathbf{p}) = \sqrt{m/2\pi} \operatorname{tr} [U_w(\mathbf{q}, \mathbf{p})^* \rho H_0^{-1/2}] \quad (5.1)$$

may be used to construct a relativistic Weyl transform in analogy with a similar transform which is used in non-relativistic statistical mechanics ([13]-[15]). We start by defining a relativistic symplectic Fourier transform.

On $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$, consider the operator \hat{H}_0 in (4.28) and construct the set of functions $f: \mathcal{M}(1, 1) \times \mathcal{V}_m^+ \rightarrow \mathbb{C}$ defined by

$$f(q, p) = (e^{-i\hat{H}_0 t} \Phi)(\mathbf{q}, \mathbf{p}), \quad \Phi \in L^2(\Gamma, d\mathbf{q} d\mathbf{p}) \quad (5.2)$$

where, of course, $q = (t, \mathbf{q})$, $p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$ and $\mathcal{M}(1, 1)$ is the Minkowski space (see Appendix). Since, for fixed q_0 , $f(p, q)$ defines a function on $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$, we shall denote the set of functions (5.2) again by $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$, for we shall only be concerned with fixed values of q_0 . The relativistic symplectic Fourier transform \tilde{f} of a function f is then defined as:

$$\tilde{f}(q, p) = (2\pi)^{-1} \int_{q'_0 = \text{const}} e^{-i(q \cdot p' - p \cdot q')} f(q', p') d\mathbf{q}' d\mathbf{p}' \quad (5.3)$$

$$f(q, p) = (2\pi)^{-1} \int_{q'_0 = \text{const}} e^{-i(q \cdot p' - p \cdot q')} \tilde{f}(q', p') d\mathbf{q}' d\mathbf{p}'. \quad (5.4)$$

For fixed q_0, q'_0 , the norms of f and \tilde{f} , considered as functions in $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ satisfy

$$\|f\|^2 = \|\tilde{f}\|^2. \quad (5.5)$$

Since the representation operators $U_w(a, \Lambda_p)$ in (4.4) leave the domain of H_0 in (4.12) invariant, it follows that the function $f: \mathcal{M}(1, 1) \times \mathcal{V}_m^+ \rightarrow \mathbb{C}$

defined by

$$f(q, p) = \sqrt{m/2\pi} \operatorname{tr} [U_w(a, \Lambda_p)^* \rho H_0^{-1/2}] \tag{5.6}$$

is an element of \mathcal{R}_w for all rank one operators ρ . Moreover, it is easy to see that

$$f(q, p) = (e^{-iH_0 t} g)(\mathbf{q}, \mathbf{p}), \tag{5.7}$$

where $g = \mathcal{W} \rho$.

Let $\mathcal{F} : L^2(\Gamma, d\mathbf{q} d\mathbf{p}) \rightarrow L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ denote the (unitary) symplectic Fourier transform operator, i. e.,

$$\mathcal{F} f = \tilde{f}, \quad f \in L^2(\Gamma, d\mathbf{q} d\mathbf{p}). \tag{5.8}$$

We define the relativistic Weyl transform Θ_w as the operator

$$\Theta_w = \mathcal{F} \mathcal{W}. \tag{5.9}$$

Let $\mathcal{R}_\theta \subset L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ be the range of this operator. From Lemma 4.5 and Equations (5.3)-(5.4) we easily establish:

LEMMA 5.1. — The range \mathcal{R}_θ of the linear map Θ_w is dense in $L^2(\Gamma, d\mathbf{q} d\mathbf{p})$ and coincides with the domain of the operator P_0 defined in (4.32). ■

On \mathcal{R}_θ , we introduce the scalar product:

$$\langle \Psi_1 | \Psi_2 \rangle_\theta = \langle \Psi_1 | \Psi_2 \rangle_{L^2(\Gamma)} + \langle \Psi_1 | D_\theta \Psi_2 \rangle_{L^2(\Gamma)} \tag{5.10}$$

where [cf. (4.30)] $D_\theta = \mathcal{F}^{-1} * D_w \mathcal{F}^{-1}$. An explicit computation yields:

$$\begin{aligned} \langle \Psi_1 | \Psi_2 \rangle_\theta &= m^{-2} \int_\Gamma \overline{\Psi_1(\mathbf{q}, \mathbf{p})} \\ &\quad \times P_0^{-1} p_0 (P_0 p_0 + \mathbf{P} \cdot \mathbf{p}) \Psi_2(\mathbf{q}, \mathbf{p}) d\mathbf{q} d\mathbf{p}. \end{aligned} \tag{5.11}$$

Moreover, with this scalar product \mathcal{R}_θ becomes a Hilbert space $\mathcal{H}(\Theta_w)$, and from (4.34) we get,

$$\langle \Theta_w \rho_1 | \Theta_w \rho_2 \rangle_\theta = \langle \rho_1 | \rho_2 \rangle_{\mathcal{B}_2(\mathcal{H}_w)}, \quad \forall \rho_1, \rho_2 \in \mathcal{B}_2(\mathcal{H}_w). \tag{5.12}$$

Thus, as in the non-relativistic case, the relativistic Weyl transform is the symplectic Fourier transform of the Wigner transform and acts as an isometry between $\mathcal{B}_2(\mathcal{H}_w)$ and $\mathcal{H}(\Theta_w)$. The inverse Weyl transform $\Theta_w^{-1} : \mathcal{H}(\Theta_w) \rightarrow \mathcal{B}_2(\mathcal{H}_w)$ is also easily computed. We collect all these results into the following theorem.

THEOREM 5.2. — *The relativistic Weyl transform*

$$\Theta_w : \mathcal{B}_2(\mathcal{H}_w) \rightarrow \mathcal{H}(\Theta_w),$$

given by the relation:

$$(\Theta_w \rho)(q, p) = \sqrt{m/2\pi} \int_{q_0 = \text{const}} e^{-i(q \cdot p' - p \cdot q')} \times \text{tr} [U_w(\mathbf{q}, \mathbf{p})^* \rho H_0^{-1/2}] dq' dp', \quad (5.13)$$

(defined initially on rank one operators and then extended by continuity to $\mathcal{B}_2(\mathcal{H}_w)$) is a Hilbert space isometry. The inverse map $\Theta_w^{-1}: \mathcal{H}(\Theta_w) \rightarrow \mathcal{B}_2(\mathcal{H}_w)$ is given by

$$\Theta_w^{-1} \Psi = \sqrt{m/2\pi} \int_{q_0 = \text{const}} U_w(\mathbf{q}, \mathbf{p}) H_0^{-1/2} (X \tilde{\Psi})(q, p) dq dp, \quad (5.14)$$

where $\tilde{\Psi}$ is the symplectic Fourier transform of Ψ , and

$$X = m^{-2} P_0 p_0^{-1} (P_0 p_0 + \mathbf{P} \cdot \mathbf{p}) \quad (5.15)$$

is a self-adjoint operator on $L^2(\Gamma, dq dp)$ with domain \mathcal{R}_w . ■

Let us point out some interesting features of the relativistic Weyl transform (5.13)-(5.14). First, in (5.14), $\tilde{\Psi}$ belongs to \mathcal{R}_w , *i.e.* it is in the domain of \tilde{H}_0 in (4.28). Thus, Θ_w^{-1} associates, to any vector $\psi \in L^2(\Gamma, dq dp)$ which satisfies

$$\int \mathbf{p}^2 |\psi(\mathbf{q}, \mathbf{p})|^2 dq dp < \infty, \quad (5.16)$$

a Hilbert-Schmidt operator on \mathcal{H}_w . Moreover, looking at it in this way, $\Theta_w^{-1}: \Psi \in \mathcal{H}(\Theta_w) \mapsto \rho \in \mathcal{B}_2(\mathcal{H}_w)$ is also the inverse of the Wigner transform (5.1). Secondly, if we make the (very heuristic) non-relativistic approximation, $p_0 \sim P_0 \sim H_0 \sim m \gg \mathbf{p}, \mathbf{P}$, then (5.1), (5.13) and (5.14) collapse to their non-relativistic counterparts ([13]-[16]).

VI. FINAL COMMENTS

We end this paper with some comments regarding related work, already existing or in progress. First, our definition of a coherent section (4.17)-(4.20) is a generalization of the notion of a family of coherent states, introduced by Perelomov [2,26]. In the latter case, H in (3.1) would simply correspond to the stability subgroup, up to a multiplicative phase factor, of a fixed vector $\zeta \in \mathcal{H}$, satisfying (3.3). However, the subgroup T of $\mathcal{P}_+^\uparrow(1, 1)$, which we use in (4.6), is definitely not such a stability subgroup for any of the admissible vectors ζ in (4.15), (4.16) that we consider.

Secondly, group orbits \mathcal{G} of the type $\mathcal{G} = \{\zeta_g = U(g)\zeta \mid g \in G\}$, for a continuous representation U of G on \mathcal{H} , and fixed $\zeta \in \mathcal{H}$, have remarkable geometrical properties. For example, if $\zeta \in \mathcal{H}$ is a fixed analytic vector [29],

the orbit \mathfrak{G} carries, in a natural way, a degenerate symplectic (*i. e.* presymplectic) structure [30]. The corresponding moment map J_ζ is computed to be:

$$J_\zeta: \zeta_g \in \mathfrak{G} \mapsto \langle \zeta_g \mid \Gamma(\cdot) \zeta_g \rangle \in \mathfrak{g}^*,$$

where \mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of G , and Γ is the representation of \mathfrak{g} on \mathcal{H} obtained from U via Stone's theorem. The image of \mathfrak{G} under J_ζ is an orbit \mathfrak{G}^* of the coadjoint action [23] of G on \mathfrak{g}^* , and as such is naturally a symplectic homogeneous space of G . If $H \subset G$ is defined by $h \in H \Leftrightarrow U(h)\zeta = \zeta$, then $\mathfrak{G} \cong G/H$. Moreover, defining $K \subset G$ by $k \in K \Leftrightarrow \text{Ad}_k^* J_\zeta(\zeta) = \langle U(k)\zeta \mid \Gamma(\cdot) U(k)\zeta \rangle = J_\zeta(\zeta)$, one has $\mathfrak{G}^* \cong G/K$ and $H \subset K$. Under appropriate conditions, such as in Equations (4.15), (4.16) (*see* [31], [32] for details), and specially when G has the form of a semi-direct product, generalized coherent states can be constructed, labeled by the points in \mathfrak{G}^* . Thus, square integrability mod (K, β) of the representation U would follow once these conditions are met by any analytic vector ζ .

Next, we have restricted our considerations in this paper to the Poincaré group in 1-space and 1-time dimensions. However, in view of the analysis carried out in Reference 19, for the standard Wigner representations, for mass $\neq 0$ and spin $j=0, 1, 2, 3, \dots$, of the usual Poincaré group in 3-space and 1-time dimensions, it is clear that an exactly analogous theory of square integrability could be built in those cases as well. Of course, the subgroup H would then have to be $T \otimes \text{SO}(3)$, of all time translations and space rotations. The admissible vectors would have to satisfy the additional condition of being invariant under $\text{SO}(3)$. However, the $\mathcal{P}_+^{\uparrow}(1, 1)$ case brings out all the interesting features related to square integrability without the formalism getting too involved due to the presence of the additional rotation variables.

Finally comes the question as to which groups the present method is applicable. It is symptomatic that both the Poincaré and the Galilei groups, for which the same approach was pursued in Reference 18, have the structure of a semi-direct product $G = T \wedge S$, where the normal subgroup T is a vector group (the translations). In fact general results have been obtained for an arbitrary group of this type, assuming in addition that S is a semisimple Lie group ([31], [32]). As one can expect from the discussion above, the crucial ingredient is the symplectic structure of the orbits in the representation space. Further work in this direction is in progress.

APPENDIX

A.1. The 1 + 1 dimensional Poincaré group

Let $\mathcal{P}_+^\dagger(1, 1)$ denote the Poincaré group in 1-space + 1-time dimensions. It acts on the Minkowski space $\mathcal{M}(1, 1)$ whose points we denote by $x = (x_0, \mathbf{x})$, $x_0 = t$, $\mathbf{x} \in \mathbb{R}$ (we take units such that $\hbar = c = 1$). The metric is $\eta = \text{diag}(1, -1)$. We denote the elements of $\mathcal{P}_+^\dagger(1, 1)$ by $g = (a, \Lambda)$, where $a = (a_0, \mathbf{a})$ is a space-time translation and Λ is a Lorentz boost, which necessarily has the form,

$$\Lambda = (\Lambda_\mu^\nu) = \begin{pmatrix} \gamma & -\mathbf{v}\gamma \\ -\mathbf{v}\gamma & \gamma \end{pmatrix}, \quad \gamma = (1 - \mathbf{v}^2)^{-1/2}, \quad (\text{A.1})$$

where $\mathbf{v} \in \mathbb{R}$ denotes a velocity. Introducing the forward mass hyperbola (see Eqs. (4.1), (4.2)):

$$\mathcal{V}_m^+ = \{(p_0, \mathbf{p}) \in \mathbb{R}^2 \mid p_0 > 0, p_0^2 - \mathbf{p}^2 = m^2\} \quad (\text{A.2})$$

for some generic mass m , we rewrite Λ as:

$$\Lambda = \Lambda_p = m^{-1} \begin{pmatrix} p_0 & \mathbf{p} \\ \mathbf{p} & p_0 \end{pmatrix}, \quad p = m(\gamma, -\mathbf{v}\gamma) \in \mathcal{V}_m^+. \quad (\text{A.3})$$

The product law in $\mathcal{P}_+^\dagger(1, 1)$ is

$$(a', \Lambda_{p'}) (a'', \Lambda_{p''}) = (a' + \Lambda_{p'} a'', \Lambda_{p'} \Lambda_{p''}), \quad (\text{A.4})$$

where the matrix

$$\Lambda_{p'} \Lambda_{p''} = \Lambda_{\Lambda_{p'} p''} = \Lambda_{\Lambda_{p''} p'} = \Lambda_p \quad (\text{A.5})$$

corresponds to the momentum

$$p = m^{-1} (p'_0 p''_0 + \mathbf{p}' \cdot \mathbf{p}'', p'_0 \mathbf{p}'' + \mathbf{p}' p''_0). \quad (\text{A.6})$$

The elements Λ_p of the Lorentz group act transitively on \mathcal{V}_m^+ according to (4.3) and the corresponding invariant measure on \mathcal{V}_m^+ is easily seen to be dk/k_0 .

We consider now the subgroup T of time translations of $\mathcal{P}_+^\dagger(1, 1)$ and the two coset spaces,

$$\Gamma_l = \mathcal{P}_+^\dagger(1, 1)/T, \quad \Gamma_r = T \backslash \mathcal{P}_+^\dagger(1, 1). \quad (\text{A.7})$$

Points in both Γ_l and Γ_r can be parametrized by $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$, and the map $\beta: \Gamma_{l,r} \rightarrow \mathcal{P}_+^\dagger(1, 1)$ defined by

$$\beta(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), \Lambda_p), \quad p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}) \quad (\text{A.8})$$

is a Borel section for both Γ_l and Γ_r . To establish this it is only necessary to note that an arbitrary element $(q, \Lambda_p) \in \mathcal{P}_+^\dagger(1, 1)$ may be written either

as:

$$(q, \Lambda_p) = ((0, \mathbf{q} - q_0 p_0^{-1} \mathbf{p}), \Lambda_p) ((m q_0 p_0^{-1}, \mathbf{0}), \mathbf{I}) \tag{A.9}$$

according to $\mathcal{P}_+^\dagger(1, 1)/\mathbf{T}$, or as:

$$(q, \Lambda_p) = ((q_0, \mathbf{0}), \mathbf{I}) ((0, \mathbf{q}), \Lambda_p) \tag{A.10}$$

according to $\mathbf{T} \setminus \mathcal{P}_+^\dagger(1, 1)$. The left and right actions of $\mathcal{P}_+^\dagger(1, 1)$ on Γ_l and Γ_r , respectively, are then the following.

On Γ_b , we have,

$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}') = (a, \Lambda_k)(\mathbf{q}, \mathbf{p}) \tag{A.11}$$

$\forall (a, \Lambda_k) \in \mathcal{P}_+^\dagger(1, 1)$, where

$$\mathbf{q}' = (\mathbf{a} + m^{-1} k_0 \mathbf{q}) - (a_0 + m^{-1} \mathbf{k} \cdot \mathbf{q})(k_0 \mathbf{p} + \mathbf{k} p_0)(k_0 p_0 + \mathbf{k} \cdot \mathbf{p})^{-1} \tag{A.12 a}$$

$$\mathbf{p}' = m^{-1}(k_0 \mathbf{p} + \mathbf{k} p_0). \tag{A.12 b}$$

Similarly, on Γ_r , $\forall (a, \Lambda_k) \in \mathcal{P}_+^\dagger(1, 1)$,

$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}') = (\mathbf{q}, \mathbf{p})(a, \Lambda_k), \tag{A.13}$$

where now

$$\mathbf{q}' = \mathbf{q} + m^{-1}(\mathbf{p} a_0 + p_0 \mathbf{a}), \tag{A.14 a}$$

$$\mathbf{p}' = m^{-1}(k_0 \mathbf{p} + \mathbf{k} p_0). \tag{A.14 b}$$

A straightforward computation then shows that the measure

$$d\mu_l(\mathbf{q}, \mathbf{p}) = d\mathbf{q} d\mathbf{p} \tag{A.15}$$

is invariant on Γ_l under the action (A.11)-(A.12), while the measure

$$d\mu_r(\mathbf{q}, \mathbf{p}) = d\mathbf{q} d\mathbf{p}/p_0 \tag{A.16}$$

is invariant on Γ_r under (A.13)-(A.14). We emphasize again that, whereas the coset spaces Γ_b, Γ_r are equal, the two invariant measures $d\mu_l$ and $d\mu_r$ are different.

A.2. Proof of Lemma 4.1

We only prove the finiteness of $I_l(\varphi, \zeta)$, $\forall \zeta \in \mathcal{D}(H_0^{1/2})$ and $\forall \varphi \in \mathcal{H}_w$. The proof of the finiteness of $I_r(\varphi, \zeta)$ is entirely analogous. Indeed,

$$\begin{aligned} I_l(\varphi, \zeta) &= \int_{\Gamma} |f_{\varphi, \zeta}(\mathbf{q}, \mathbf{p})|^2 d\mu_l(\mathbf{q}, \mathbf{p}) \tag{A.17} \\ &= \int_{\mathbb{R}^2} |\langle U_w(\mathbf{q}, \mathbf{p}) \zeta | \varphi \rangle|^2 d\mathbf{q} d\mathbf{p}. \end{aligned}$$

Now, by (4.4) and (4.10),

$$\langle U_w(\mathbf{q}, \mathbf{p}) \zeta | \varphi \rangle = \int_{\mathcal{V}_m^+} e^{i \mathbf{k} \cdot \mathbf{q}} \overline{\zeta(\Lambda_p^{-1} \mathbf{k})} \varphi(k) d\mathbf{k}/k_0.$$

Since by virtue of (A.2), the functions ζ and φ are really functions of the single variable \mathbf{k} , we may use (A.6) to write,

$$\langle U_{\mathbf{w}}(\mathbf{q}, \mathbf{p}) \zeta | \varphi \rangle = \int_{\mathbb{R}} e^{i\mathbf{k}\cdot\mathbf{q}} \overline{\zeta(m^{-1}(-k_0\mathbf{p} + \mathbf{k}p_0))} \varphi(k) d\mathbf{k}/k_0$$

whence

$$\begin{aligned} I_1(\varphi, \zeta) &= \int_{\mathbb{R}^2} d\mathbf{q} d\mathbf{p} \int_{\mathbb{R}} d\mathbf{k}/k_0 \int_{\mathbb{R}} d\mathbf{k}'/k'_0 e^{i(\mathbf{k}\mathbf{k}')\cdot\mathbf{q}} \\ &\quad \times \overline{\zeta(m^{-1}(-k_0\mathbf{p} + \mathbf{k}p_0))} \zeta(m^{-1}(-k'_0\mathbf{p} + \mathbf{k}'p_0)) \overline{\varphi(\mathbf{k})} \varphi(\mathbf{k}'). \end{aligned} \quad (\text{A.18})$$

Replacing now the integral over \mathbf{q} of $\exp i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{q}$ by $2\pi\delta(\mathbf{k}-\mathbf{k}')$ and performing the $d\mathbf{k}'$ integration, we obtain

$$I_1(\varphi, \zeta) = 2\pi \int_{\mathbb{R}} d\mathbf{p} \int_{\mathbb{R}} d\mathbf{k}/k_0^2 |\zeta(m^{-1}(-k_0\mathbf{p} + \mathbf{k}p_0))|^2 |\varphi(\mathbf{k})|^2. \quad (\text{A.19})$$

Using Fubini's theorem to perform the $d\mathbf{p}$ integration in (A.19) first, which we do by first changing variables,

$$\mathbf{p} \mapsto \mathbf{p}' = m^{-1}(-k_0\mathbf{p} + \mathbf{k}p_0) \quad (\text{A.20})$$

so that,

$$\left. \begin{aligned} \mathbf{p} &= m^{-1}(k_0\mathbf{p}' - \mathbf{k}p'_0) \\ d\mathbf{p} &= -(p' \cdot k/m p'_0) d\mathbf{p}' \end{aligned} \right\} \quad (\text{A.21})$$

we obtain

$$I_1(\varphi, \zeta) = 2\pi/m \int_{\mathbb{R}} d\mathbf{k}/k_0 \int_{\mathbb{R}} d\mathbf{p}'/p'_0 (p' \cdot k/k_0) |\zeta(\mathbf{p}')|^2 |\varphi(\mathbf{k})|^2. \quad (\text{A.22})$$

Since $p' \cdot k = p'_0 k_0 - \mathbf{p}' \cdot \mathbf{k}$, the RHS of (A.22) can be written as

$$I_1(\varphi, \zeta) = I_1(\varphi, \zeta) + I_2(\varphi, \zeta)$$

where

$$I_1(\varphi, \zeta) = 2\pi/m \|\varphi\|^2 \int_{\mathbb{R}} |\zeta(\mathbf{p})|^2 d\mathbf{p} \quad (\text{A.23})$$

and

$$I_2(\varphi, \zeta) = -2\pi/m \int_{\mathbb{R}} |\varphi(\mathbf{k})|^2 (\mathbf{k}/k_0) d\mathbf{k}/k_0 \int_{\mathbb{R}} |\zeta(\mathbf{p})|^2 (\mathbf{p}/p_0) d\mathbf{p}. \quad (\text{A.24})$$

Clearly, $I_1(\varphi, \zeta)$ is finite iff $\zeta \in \mathcal{D}(\mathbf{H}_0^{1/2})$ [compare with (4.12)]. On the other hand, since $|\mathbf{k}/k_0| < 1$ and $|\mathbf{p}/p_0| < 1$, it follows that, if $\zeta \in \mathcal{D}(\mathbf{H}_0^{1/2})$, then $I_2(\varphi, \zeta)$ also converges. ■

A.3. Proof of Theorem 4.2

From the proof of Lemma 4.1 above, we see that for fixed $\zeta \in \mathcal{D}(H_0^{1/2})$, if

$$\langle BH_0^{1/2} \zeta | H_0^{1/2} \zeta \rangle = \int_{\mathbb{R}} \mathbf{p} |\zeta(\mathbf{p})|^2 d\mathbf{p}/p_0 = 0, \quad (\text{A.25})$$

then $I_2(\varphi, \zeta) = 0, \forall \varphi \in \mathcal{H}_w$. Hence

$$I_1(\varphi, \zeta) = \int_{\mathbb{R}^2} |\langle U_w(\mathbf{q}, \mathbf{p}) \zeta | \varphi \rangle|^2 d\mathbf{q} d\mathbf{p} = I_1(\varphi, \zeta) = c_p(\zeta) \|\varphi\|^2,$$

from (A.23), whence the result. ■

ACKNOWLEDGEMENTS

S. T. Ali acknowledges the kind hospitality of the Institut de Physique Théorique, Louvain-la-Neuve, where part of the work was performed, as well as travel grants from the Accords Culturels Belgique-Québec (C.G.R.I., Communauté Française de Belgique). It is a pleasure also to thank S. De Bièvre, J.-P. Gazeau, K. H. Gröchenig, A. Grossmann and V. S. Varadarajan for stimulating discussions about the various topics involved in this paper.

REFERENCES

- [1] J. R. KLAUDER and B. S. SKAGERSTAM, *Coherent States - Applications in Physics and Mathematical Physics*, World Scientific, Singapore, 1985.
- [2] A. PERELOMOV, *Generalized Coherent States and Their Applications*, Springer, Berlin, 1986.
- [3] J. DIXMIER, *Les C*-algèbres et leurs représentations*, 2^e éd., Gauthier-Villars, Paris, 1969.
- [4] S. A. GAAL, *Linear Analysis and Representation Theory*, Springer, Berlin, 1973.
- [5] M. DUFLO and C. C. MOORE, *J. Funct. Anal.*, Vol. **21**, 1976, p. 209.
- [6] A. CAREY, *Bull. Austral. Math. Soc.*, **15**, 1976, p. 1; *Rep. Math. Phys.*, Vol. **14**, 1978, p. 247.
- [7] A. GROSSMANN, J. MORLET and T. PAUL, *J. Math. Phys.*, Vol. **26**, 1985, p. 2473; *Ann. Inst. H. Poincaré*, Vol. **45**, 1986, p. 293.
- [8] M. FLENNED-JENSEN, *Ann. of Math.*, Vol. **111**, 1980, p. 253.
- [9] H. SCHLICHTKRULL, *Invent. Math.*, Vol. **68**, 1982, p. 497.
- [10] S. T. ALI, *Rivista del Nuovo Cim.*, Vol. **8**, # 11, 1985, p. 1.
- [11] E. PRUGOVECKI, *Stochastic Quantum Mechanics and Quantum Spacetime*, D. Reidel, Dordrecht, 1984.
- [12] S. T. ALI and H. D. DOEBNER, Quantization, topology and ordering, in *Lect. Notes in Physics*, Vol. **278**, Springer, Berlin, 1987.
- [13] A. GROSSMANN, G. LOUPIAS and E. M. STEIN, *Ann. Inst. Fourier*, Grenoble, Vol. **18**, 1968, p. 343.
- [14] I. DAUBECHIES and A. GROSSMANN, *J. Math. Phys.*, Vol. **21**, 1980, p. 2080.

- [15] I. DAUBECHIES, A. GROSSMANN and J. REIGNIER, *J. Math. Phys.*, Vol. **24**, 1983, p. 239.
- [16] J. C. T. POOL, *J. Math. Phys.*, Vol. **7**, 1966, p. 66.
- [17] S. T. ALI and E. PRUGOVECKI, *Acta Appl. Math.*, Vol. **6**, 1986, p. 1.
- [18] S. T. ALI and E. PRUGOVECKI, *Acta Appl. Math.*, Vol. **6**, 1986, p. 19.
- [19] S. T. ALI and E. PRUGOVECKI, *Acta Appl. Math.*, Vol. **6**, 1986, p. 47.
- [20] R. KUNZE, *Pacific J. Math.*, Vol. **53**, 1974, p. 465.
- [21] A. BOREL, *Représentations des Groupes Localement Compacts; Lect. Notes in Mathematics*, Vol. **276**, Springer, Berlin, 1972.
- [22] H. MOSCOVICI, *Commun. Math. Phys.*, Vol. **54**, 1977, p. 63.
- [23] A. A. KIRILLOV, *Elements of the Theory of Representations*, Springer, Berlin, 1976.
- [24] H. MOSCOVICI and A. VERONA, *Ann. Inst. H. Poincaré*, Vol. **29**, 1978, p. 139.
- [25] H. SCUTARU, *Lett. Math. Phys.*, **2**, 1977, p. 101.
- [26] A. M. PERELOMOV, *Commun. Math. Phys.*, Vol. **26**, 1972, p. 222 (reprinted in Ref. 1); *ibid.*, Vol. **44**, 1975, p. 197.
- [27] I. M. GEL'FAND and N. YA. VILENKIN, *Generalized Functions*, Vol. **4**, Academic Press, New York, 1964.
- [28] J.-P. ANTOINE, *J. Math. Phys.*, Vol. **10**, 1969, pp. 53, 2276.
- [29] E. NELSON, *Annals of Math.*, Vol. **70**, 1959, p. 570.
- [30] P. BONA, *Czech. J. Phys.*, Vol. **B33**, 1983, p. 837.
- [31] S. T. ALI and S. DE BIÈVRE, in *XVIth International Colloquium on Group-Theoretical Methods in Physics (Proceedings, Varna, 1987); Lecture Notes in Physics*, Vol. **313**, p. 201; H.-D. DOEBNER, J.-D. HENNIG, T. D. PALEV Eds., Springer, Berlin, 1988.
- [32] S. DE BIÈVRE, *J. Math. Phys.*, vol. **30**, 1989, p. 1401.
Toronto, 1988 (preprint).

(Manuscript received November 29, 1988.)