ANNALES DE L'I. H. P., SECTION A

J. M. GAMBAUDO
S. VAN STRIEN
C. TRESSER

The periodic orbit structure of orientation preserving diffeomorphisms on D2 with topological entropy zero

Annales de l'I. H. P., section A, tome 50, nº 3 (1989), p. 335-356
<http://www.numdam.org/item?id=AIHPA_1989__50_3_335_0>

Ann. Inst. Henri Poincaré,

Vol. 49, n° 3, 1989, p. 335-356. *Physique théorique*

The periodic orbit structure of orientation preserving diffeomorphisms on D^2 with topological entropy zero

by

J. M. GAMBAUDO

Dept. de Math., Univ. de Nice, France

S. VAN STRIEN

Math. Dept., Techn. Univ. of Delft, the Netherlands

and

C. TRESSER

Dept. de Phys., Univ. de Nice, France

ABSTRACT. — A set of periodic orbits of a map on the disc is called "hereditarily rotation compatible" if the orbits are like those of rotations or repeated rotations above rotations. We prove C^1 orientation preserving embeddings of the disk have zero topological entropy if and only if its periodic set is of this form. Moreover a tree structure on this set is introduced: some periodic orbits imply the existence of others.

RÉSUMÉ. — L'ensemble des orbites périodiques d'une application du disque dans lui-même est appelé « héréditairement compatible à une rotation », si ces orbites sont semblables à celles des rotations ou de rotations successivement construites sur d'autres rotations.

On montre qu'un plongement C^1 du disque préservant l'orientation a une entropie topologique nulle si et seulement si l'ensemble de ses orbites périodiques est de cette forme.

De plus, on introduit dans cet ensemble une structure d'arbres. Certaines orbites périodiques impliquent l'existence d'autres orbites périodiques.

1. INTRODUCTION

In the last decade several papers appeared giving relations between the topological entropy of a map and the existence of periodic orbits. For continuous interval maps this relationship was first discovered by A. N. Sarkovski [Sar], see also for example [B.G.M.Y.]. In this case the existence of for example a periodic orbit of period three implies positive topological entropy. For orientation preserving homeomorphisms of the disc the situation is much less clear: rotations of the disc have topological entropy zero and can have arbitrary periods. Even so one can give necessary and sufficient conditions related to the "linking" of a periodic orbit which guarantee that it can occur as a periodic orbit of a homeomorphism of D^2 with zero entropy, *see* [Kob], [Bo 1].

In this paper we will consider orientation preserving diffeomorphisms of D^2 with zero entropy, but rather than just considering single orbits, as in [Kob] and [Bo 1], we will describe a tree structure on the set of all periodic orbits. This will enable us to give a complete description of all periodic structures realizable by orientation preserving diffeomorphisms of the two-disc with zero topological entropy. This description is rather geometric and in terms of a notion which we call "hereditarily rotation compatibility" and a "parent-child" relationship.

The simplest diffeomorphisms of the disc are rigid rotations. The main theorem in this paper states that the set of periodic orbits of a C^1 diffeomorphism of the disc with zero entropy has a "rotation compatible" structure, *i. e.*, it can be described by a tree structure on the set of periodic orbits so that the map is a composition of a number of "rotations built on to each other". In this tree the orbits which are lower in the tree circle around the mother orbits (like a discrete version of moons circling around planets). In Figure 1 we have drawn an example where the set of periodic orbits consists of three orbits of three consecutive generations. Later, when we give definitions, this picture will be made precise.

The result implies in particular that if one or more of the closed curves associated to periodic orbits via the suspension of f are too "linked" (we will give a precise definition below), then the topological entropy of f is positive, see *Fig.* 2. It is remarkable that this theorem gives necessary and sufficient conditions: the topological entropy of f is zero if and only if the set of periodic orbits are "hereditarily rotation compatible".

2. STATEMENT OF RESULTS

Let $D^2=\{(x, y);\ x^2+y^2<1\}$. By a disc in $D\subset D^2$ we mean the closed set bounded by a simple closed Jordan curve. Let f be a homeomorphism

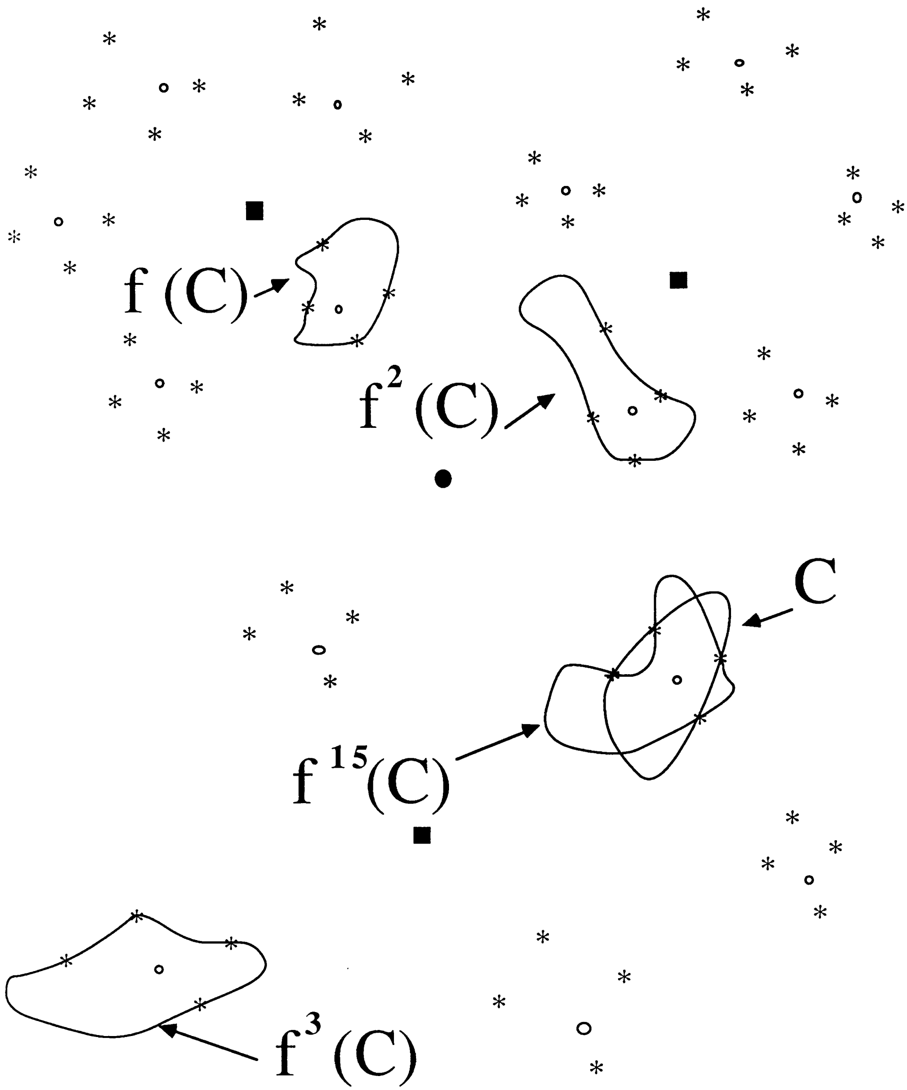

FIG. 1. — An hereditarily rotation compatible orbit of generation 3 and period $5 \times 3 \times 4$.

of D^2. The f-orbit A, $f(A)$, $f^2(A)$,... of some set $A \subset D^2$, is denoted by $(O(A); f)$ or, if no confusion can arise O(A). We say that the set A is *periodic* of *period n* if $f^n(A) = A$ and A, $f(A)$, ..., $f^{n-1}(A)$ are disjoint.

If $A \subset D^2$ and $f^n(A) = A$ then we denote the restriction of f^n to A simply by $f^n: A \to A$.

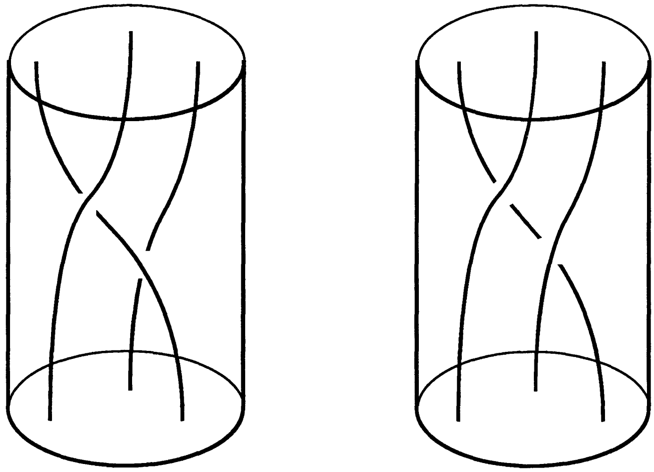

FIG. 2. — A map f with a suspension as drawn on the left *must* have positive topological entropy. The situation on the right does not imply anything about the topological entropy.

One can formalise the "linking" of a periodic orbit $O(p)$ of $f: D^2 \to D^2$ using braids or, equivalently, by the isotopy class of

$$f: D^2 - O(p) \to D^2 - O(p).$$

First we say when a periodic orbit is called rotation compatible.

Take a periodic region $A \subset D^2$ of period $n \geqq 1$. We say that $(O(A); f)$ is a *rotation compatible* f-orbit if

$$f^n, \mathrm{id}: D^2 - O(A) \to D^2 - O(A) \text{ are isotopic.}$$

Clearly a periodic orbit of a rotation of D^2 is rotation compatible. More generally, it will later turn out that A is rotation compatible iff there exists $n \geqq 1$ and g such that $f, g: D^2 - O(A) \to D^2 - O(A)$ are isotopic and such that

$$g: \quad D^2 - O(A) \to D^2 - O(A)$$

and

$$R: \quad D^2 - O(D_0) \to D^2 - O(D_0),$$

are conjugate, where R is a rotation of D^2 of order n (this means $R^n = \mathrm{id}$) and D_0 a periodic disc in D^2 (also of period n).

In order to define the notion of hereditarily rotation compatibility, we consider renormalizations. Roughly speaking, a renormalization is an iterate of a map restricted to some rotation compatible subset. Indeed, we say that $f: D^2 - O(p) \to D^2 - O(p)$ can be *renormalized* if the period of

$O(p)$ is the product of two integers r, $s>1$ and if there exists a periodic topological disc D_0 in D^2 of period r such that each of the discs $f^i(D_0)$ contains s points of $O(p)$ in its interior and

$$(O(D_0); f) \text{ is rotation compatible.}$$

If this holds then we say that $(O(p) \cap D_0; f^r | D_0)$ *renormalizes* $(O(p); f)$.

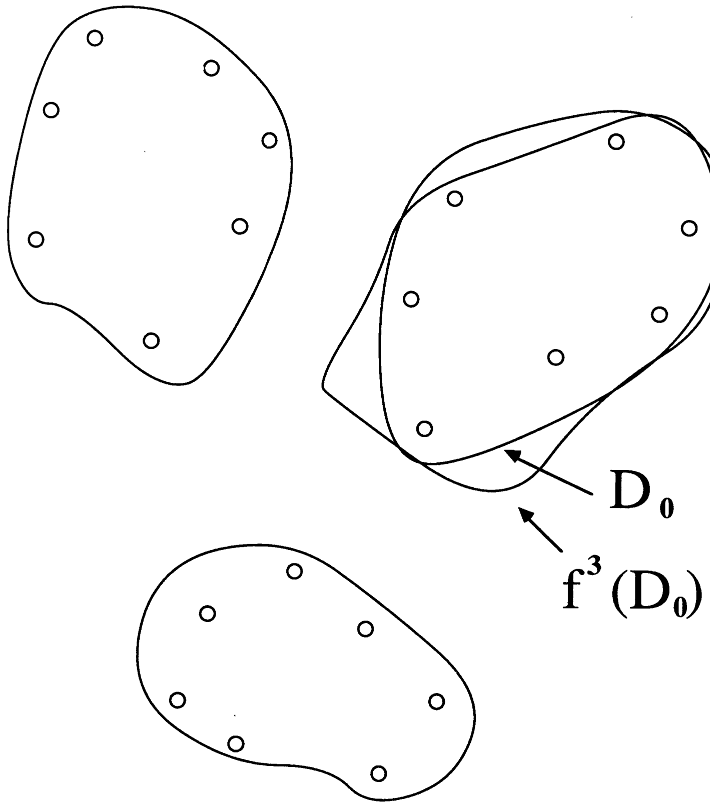

FIG. 3. – A renormalizable orbit.

Now we define inductively $(O(p); f)$ to be *hereditarily rotation compatible of generation k* as follows.

$(O(p); f)$ is hereditarily rotation compatible of generation 0 if p is a fixed point. It has generation 1 $(O(p); f)$ is rotation compatible.

The orbit $(O(p); f)$ is hereditarily rotation compatible of generation $k>1$ if there exists a homeomorphism g such that

$$f, g: D^2 - O(p) \to D^2 - O(p) \text{ are isotopic}$$

such that

– $(O(p); g)$ can be renormalized to $(O(p) \cap D_0; g^r | D_0)$. That is, up to isotopy f rotates a disc D_0 containing more than one point of $O(p)$ in its interior.

– $(O(p) \cap D_0; g^r | D_0)$ is hereditarily rotation compatible of generation $k-1$.

Let $h_{top}(f)$ be the topological entropy of f. The next result tells us that each of these periodic is hereditarily rotation compatible.

THEOREM A. – *Let f be an orientation preserving homeomorphism of* D^2. *If*

$$(*) \qquad h_{top}(f) = 0$$

then

(**) every periodic orbit of f is hereditarily rotation compatible

If f is a $C^{1+\epsilon}$ *diffeomorphism, then* (**) *is equivalent to* (**).

Let us describe these definitions in words. $(O(p); f)$ is rotation compatible it f acts as a rotation on $O(p)$. $(O(p); f)$ is hereditarily rotation compatible of generation k if there exist a map g in the isotopy class of $f: D^2 - O(p) \to D^2 - O(p)$ and a g-periodic disc D_0 containing at least two points of $O(p)$ in its interior such that g rotates this disc and such that the restriction of g^r to D_0 is hereditarily rotation compatible of generation $k-1$, see *Fig.* 2.

Theorem A is not difficult to prove and is not the main result of this paper. This theorem was essentially known by P. Boyland, T. Kobayashi and J. Smillie and others. For example, for the same class of mappings, P. Boyland [Bo 1] has proved that if $h(f) = 0$ and p is a periodic point of f whose period is an odd prime number then $O(p)$ is rotation compatible. T. Kobayashi, [Kob], proved a theorem equivalent to Theorem A, but his result is phrased in terms of a notion called graph link. This notion is defined using the language of three dimensional topology. J. Llibre and R. S. MacKay announced a similar result considering also orientation reversing diffeomorphisms.

Remark 1. – We should also point out the connection with a paper of J. Morgan. Let τ be the solid torus $D^2 \times S^1$. J. W. Morgan has proved that if $K \subset \tau$ is an attracting closed orbit for a Morse-Smale flow without singular points on τ, then K is an iterated torus knot. Here a knot $K \subset \tau$ is called an *iterated torus knot* if there is a sequence of solid tori $\tau \supset \tau_0 \supset \tau_1 \ldots \supset \tau_r$ such that the core of τ_0 is unknotted, the core of τ_i lies on a torus in τ_{i-1} which is parallel to $\partial\tau_{i-1}$ for $i \geqq 1$, and the core of τ_r is K. It is not hard to show that Theorem A implies Morgan's result for flows which are the suspension of Morse-Smale maps.

Remark 2. – We should note that the situation on other manifolds is more complicated. For example J. Smillie [Smi] shows that for surfaces with genus 145 there exists an isotopy class of diffeomorphisms which contains two maps f_i with topological entropy zero. f_1 has a periodic point of period 5 and f_2 a periodic point of period 7, but every map in this isotopy class which has periodic points of period 5 and of period 7 has positive entropy. Moreover this isotopy class contains no maps with zero entropy which have a fixed point.

3. A TREE STRUCTURE ON THE SET OF PERIODIC ORBITS

A much more precise description of the set of periodic points can be given. Indeed, there exists a partial ordering on the set of periodic orbits.

Definition of family relationship

$(O(q); f)$ is the *persistent parent* of $(O(p); f))$ if for each isotopy $f_t: D^2 - O(p) \to D^2 - O(p)$ there exists a curve $q_t \in D^2 - O(p)$ such that $q_0 = q$, and q_t is a periodic point of f_t of exactly period per(q) and such that

– there exist g_t isotopic to

$$f_t: D^2 - \{O(p) \cup O(q_t)\} \to D^2 - \{O(p) \cup O(q_t)\}$$

and a topological disc D_0^t, depending continuously on t, containing precisely one point of $(O(q_t); f_t)$ and at least two points of $O(p)$ in its interior, such that

– D_0^t is g_t-periodic of period $n = \# O(q)$

– $g_t^n | D_0^t$ is conjugate to a periodic rotation of D^2.

If these conditions only hold for f (instead for f_t) then $(O(q):f)$ is called the *parent* of $(O(p); f)$. For convenience we also call $(O(p); f)$ the *child* of $(O(q); f)$ if $(O(q); f)$ is the parent of $(O(p); f)$. A periodic orbit $(O(q); f)$ is an *ancestor* of a periodic orbit $(O(p); f)$ if there is a chain of periodic orbits

$$O_0 = (O(q); f), O_1, \ldots, O_n = (O(p); f)$$

so that O_i is a parent of O_{i+1}.

Example. – The orbit $(O(p); f)$ of period 15 in Figure 4 is the child of the orbit $(O(q); f)$ of period 5.

THEOREM B. – *Let f be a C^1 diffeomorphism of D^2 with topological entropy zero and isolated periodic points. Then every hereditarily rotation compatible orbit of generation k has a persistent hereditarily rotation compatible parent of generation $k-1$;*

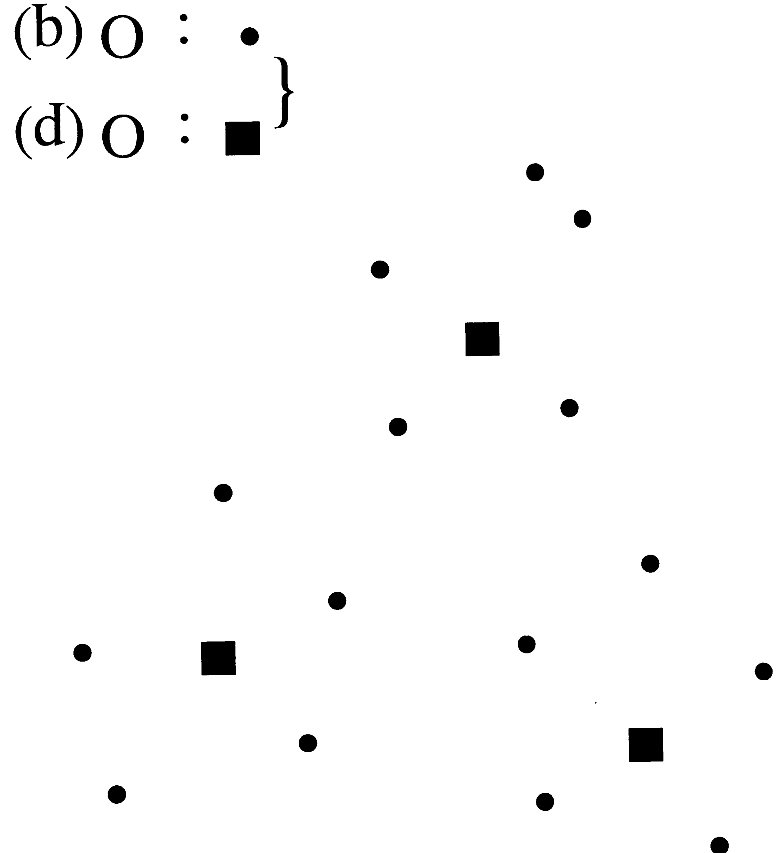

FIG. 4. – $(O(q); f)$ is the parent orbit of $(O(p); f)$.

This theorem is useful in particular for bifurcation problems. In a family of diffeomorphisms if a periodic orbit persists through a bifurcation then its parent orbit must also persist.

Finally we define a notion which expresses whether orbits can be considered spatially separate or not.

Definition of disjointness of orbits

Two orbits $(O(p); f)$ and $(O(q); f)$ are called *disjoint* if there are two disjoint simple closed curves C_1, C_2 bounding two disjoint discs D_1, D_2 containing $O(p)$ respectively $O(q)$ and such that

$$f(C_i) \text{ is isotopic to } C_i \text{ in } D^2 - \{O(p) \cup O(q)\} \text{ for } i=1, 2.$$

(In particular two fixed points are disjoint.)

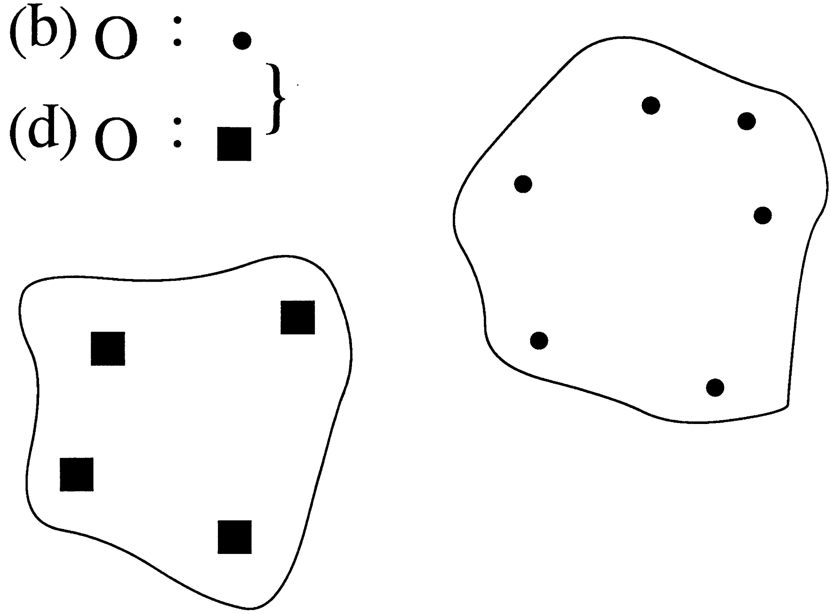

FIG. 5. — Two disjoint orbits.

O(p) and O(q) *lie nested* if there exist a map g in the isotopy class of $f: D^2 - O(p) \to D^2 - O(p)$ and g-periodic discs $D_2 \subset D_1$ such that $D_1 \setminus D_2$ is a g-periodic annulus, O(q) is contained in the orbit of the interior of this annulus and O(p) in the orbit of the interior of the disc D_2, *see* Figure 6.

THEOREM C. — *Let f be a C^1 diffeomorphism of D^2 with topological entropy zero and isolated periodic points. Then*

(*a*) *two orbits O_1 and O_2 may have a common child but then O_1 and O_2 have the same period and the same generation and any parent of O_1 is also a parent of O_2.*

(*b*) *Let O_1 and O_2 be periodic orbits and $O_1 \neq O_2$. Then there are three possibilities:*

they are disjoint, or

they lie nested and have a common ancestor, or

one of these orbits is the parent of the other orbit.

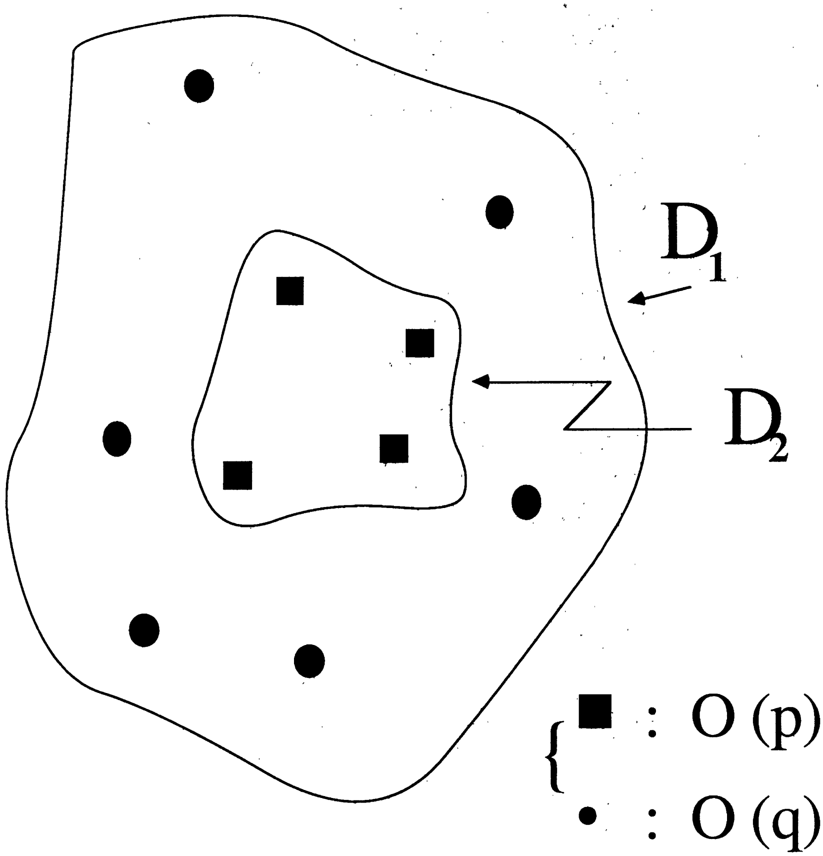

FIG. 6. – Two nested orbits.

Theorems B and C give a very precise picture: all periodic orbits of a diffeomorphism of D^2 can be organised in a tree: each orbit of period $n \geqq 2$ has a parent orbit.

It is not difficult to express the notion of rotation compatibility, child-parent relationship and disjointness in terms of braids. In particular there exists a finite algorithm to decide whether or not a particular orbit is hereditarily rotation compatible, *see* [Bi].

Remark. – Probably these theorems are also valid for C^0 homeomorphisms. The reason we confine ourself to C^1 diffeomorphisms is that we need a result from [A.F.] which is only proved for C^1 maps. It seems likely however that the results in [A.F.] are also valid for C^0 maps.

3. BACKGROUND

Let us introduce some notation: We say that two diffeomorphisms f, $g: \mathrm{F} \to \mathrm{F}$ of some topological space F are isotopic, if there exists a continuous arc $\mathrm{F}_t: \mathrm{F} \to \mathrm{F}$ of diffeomorphisms such that $\mathrm{F}_0 = f$, $\mathrm{F}_1 = g$. We abbreviate this as $f \simeq g$. (One can prove that two homotopic diffeomorphisms on surfaces are also isotopic.)

The main ingredient of our proof will be the following theorem of Thurston. Assume that F is an orientable surface with negative genus which is closed with boundary, or which is closed except that it has finite number of punctures. In this paper we consider homeomorphisms of D^2. Since D^2 is open, we can consider D^2 as $\mathrm{S}^2 - \{\infty\}$ and a homeomorphism of $f: \mathrm{D}^2 \to \mathrm{D}^2$ induces a homeomorphism of S^2. In this paper we will choose F equal to D^2 minus a finite set; if this finite set consists of at least two points then F has negative genus. Take the hyperbolic metric on F with constant curvature -1. A curve is *essential* if it is not isotopic to a point, nor is puncture or boundary parallel.

3.1. THEOREM Nielsen, Thurston [Th], *see* also [Ca]. — *Let* $f: \mathrm{F} \to \mathrm{F}$ *be an orientation preserving homeomorphism of a closed orientable surface (with possibly a finite number of punctures). Then (at least) one of the following occurs.*

1. *f is periodic; i. e. f^n is isotopic to the identity map for some n.*
2. *f is reducible;* i. e. *there exists a homeomorphism $g \simeq f$ such that g leaves some finite union of disjoint simple essential closed curves invariant.*
3. *there exists $h_f > 0$ such that for any map $g \simeq f$ one has $h_{\mathrm{top}}(g) \geqq h_f$. (In fact f is isotopic to a pseudo-Anosov map but we will not need to define this notion in this paper.)*

The next result describes the periodic case and tells us that if f is periodic then f is isotopic to an isometry:

3.2. THEOREM *See* Kerckhoff [Kerc]. — *Let* $f: \mathrm{F} \to \mathrm{F}$ *be an orientation preserving homeomorphism. If f^n is isotopic to id then f is isotopic to an isometry of* F.

Finally we will use the following theorem which will enable us to conclude that if one map has a periodic orbit then another map in the same isotopy class will have an "isotopic" periodic orbit. More precisely, let $f_0, f_1: \mathrm{F} \to \mathrm{F}$ be diffeomorphisms and p_0 and p_1 be fixed points of f_0 and f_1 respectively. Then (p_0, f_0) and (p_1, f_1) are in the same *Nielsen class* if there exist an isotopy f_t between f_0 and f_1 and a curve $c: [0, 1] \to \mathrm{F}$ joining p_0 and p_1 such that $(0, 1) \ni t \to f_t(c(t))$ is homotopic to $c(t)$ in $\mathrm{F} - \{p_0, p_1\}$. They are in the same *strong Nielsen class* if there exist such an isotopy f_t and a continuous curve $c(t)$ such that $f_t(c(t)) = c(t)$, for all $t \in [0, 1]$.

Let p be an isolated periodic point of period n. Then (p, f) is *unremovable* provided that for any g isotopic to f has a periodic point q such that (p, f) and (q, g) are in the same strong Nielsen class.

3.3. THEOREM (Asimov and Franks, [A.F.]). — *Let* $f: \mathrm{F} \rightarrow \mathrm{F}$ *be a diffeomorphism with a point p of period n which is the only point in its Nielsen class with respect to the map* f^n *and non-zero Lefschetz number,* $\mathrm{L}(p, f^n) \neq 0$. *Then* (p, f) *is unremovable.*

4. PROOF OF THEOREM A

Everywhere in the remainder of this paper we will assume that f is an diffeomorphism of the disc with zero topological entropy. The idea of the proof of Theorem A is simply to apply Theorem (3.1) on $f: \mathrm{D}^2 - \mathrm{O}(p) \rightarrow \mathrm{D}^2 - \mathrm{O}(p)$ and use induction in the reducible case.

4.1. LEMMA. — *Let* O *be a finite subset of* D^2 *and* $\mathrm{R}: \mathrm{D}^2 - \mathrm{O} \rightarrow \mathrm{D}^2 - \mathrm{O}$ *an isometry. Then* R *extends to a homeomorphism* $\tilde{\mathrm{R}}: \mathrm{D}^2 \rightarrow \mathrm{D}^2$ which is conjugate to a rotation of D^2.

Proof. — Since $\mathrm{R}: \mathrm{D}^2 - \mathrm{O} \rightarrow \mathrm{D}^2 - \mathrm{O}$ is an isometry, it can be extended to a homeomorphism $\tilde{\mathrm{R}}$ of D^2. The Lemma then follows from a theorem of Brouwer, see [Bro], [Kere] and [Ei], which says that if $g: \mathrm{D}^2 \rightarrow \mathrm{D}^2$ satisfies $g^n = \mathrm{id}$ then g is conjugate to a rotation.

Q.E.D.

Remark. — Since R is an isometry, one can prove this lemma also easily without using Brouwer's result: Denote p the fixed point of $\tilde{\mathrm{R}}: \mathrm{D}^2 \rightarrow \mathrm{D}^2$. Let $\mathrm{R}'(p): \mathrm{T}_p \mathrm{F} \rightarrow \mathrm{T}_p \mathrm{F}$ be the derivative of R at p. Identify $\mathrm{T}_p \mathrm{F}$ with R^2 and define $\mathrm{S}(x) = f'(p) x$. Then S is a linear rotation of D^2. Consider geodesics γ_θ in F through p having angle θ with some fixed line through p, where $\theta \in [0, \pi]$. These geodesics will either end in O or in $\partial \mathrm{D}^2$. Then the conjugacy h between R and S is defined by choosing an appropriate homeomorphism $h: \mathrm{D}^2 \rightarrow \mathrm{D}^2$ such that $h(\gamma_\theta)$ is contained in the line (through the origin $0 \in \mathrm{D}^2$ which has angle θ with the x-axis).

Q.E.D.

4.2. LEMMA. — *Let* $f: \mathrm{D}^2 \rightarrow \mathrm{D}^2$ *be a topological diffeomorphism and* $\mathrm{O}(p)$ *be a periodic orbit of period n. Assume that* $f: \mathrm{D}^2 - \mathrm{O}(p) \rightarrow \mathrm{D}^2 - \mathrm{O}(p)$ *is periodic. Then*

1. $f^n: \mathrm{D}^2 - \mathrm{O}(p) \rightarrow \mathrm{D}^2 - \mathrm{O}(p)$ *and* $id: \mathrm{D}^2 - \mathrm{O}(p) \rightarrow \mathrm{D}^2 - \mathrm{O}(p)$ *are isotopic;*
2. f *is isotopic to an isometry of* $\mathrm{D}^2 - \mathrm{O}(p)$ *(where we take the hyperbolic metric on* $(\mathrm{O}(p); f)$ *if* $n \geqq 2$ *and the usual metric if* $n = 1$*);*

3. $D^2-O(p) \to D^2-O(p)$ *is rotation compatible;*

Proof. — Since $f: D^2-O(p) \to D^2-O(p)$ is periodic, according to Theorem (3.2) this map is isotopic to an isometry R of $D^2-O(p)$. According to Lemma (4.1), R extends to a map $\tilde{R}: D^2 \to D^2$ which is conjugate to a rotation on D^2. Since $O(p)$ has period n it follows that $\tilde{R}^n$ fixes each point of $O(p)$ and it follows that $\tilde{R}^n = id$. This proves 1, 2 and 3.

Q.E.D.

4.3. LEMMA. — *Assume that* $f: D^2 \to D^2$ *is a homeomorphism of* D^2 *with zero topological entropy. Then every periodic orbit* $O(p)$ *of* f *is hereditarily rotation compatible.*

Proof. — Take a periodic orbit of period n. If $n \in \{1, 2\}$ then $O(p)$ is trivially rotation compatible. So let us assume that we have proved by induction that every periodic orbit of period $<n$ is rotation compatible. We will apply the theorems of section 3 of $F = D^2-O(p)$. So take the hyperbolic metric on $D^2-O(p)$. Since $h(f)=0$, Theorem (3.1) implies that only one of the following two cases can occur:

Case 1. — $f: D^2-O(p) \to D^2-O(p)$ is periodic. Then Lemma (4.1) gives that $(O(p); f)$ is rotation compatible.

Case 2. — $f: D^2-O(p) \to D^2-O(p)$ is reducible. Then there exist $r \geqq 1$, a simple essential closed curve C_0 and a map g in the isotopy class of $f: D^2-O(p) \to D^2-O(p)$ such that

$$g^r(C_0) = C_0,$$

and

$$C_i = g(C_0), \qquad i = 0, 1, \ldots, r-1 \text{ are pairwise disjoint.}$$

For each $i = 1, 2, \ldots, r$, $C_i \simeq (C_0)$ $(\text{rel}\, D^2-O(p))$ is an essential simple curve and therefore there exists $s>1$ such that for each $i=0, \ldots, r-1$ the curve C_i bounds a disc D_i containing precisely s points (where $1<s<n$ because C_i is simple) and each point of $O(p)$ is contained in one of the discs $D_0, D_1, \ldots, D_{r-1}$. Take $x \in O(p) \cap D_0$. Since $f^r(C_0)$ is isotopic to C_0 in $D^2-O(p)$, it follows that all of the points $x, f^r(x), f^{2r}(x), \ldots,$ are contained in D_0. From this and since each of the discs D_i contains $1<s<n=\text{per}(O(x))$ points one has $n = s \times r$. Notice that the discs D_i, $0 \leqq i < r$, are pairwise disjoint.

Choose a 'periodic' essential simple curve C_0 in $D^2-O(p)$ as above such that there is no periodic essential simple curve C'_0 bounding more points of $O(p)$ than C_0. Choose r and g corresponding to C_0.

We claim that $(O(D_0\}; g)$ is rotation compatible. Indeed, consider $g: D^2-O(D_0) \to D^2-O(D_0)$. This map cannot be isotopic to a pseudo-Anosov map. Indeed, otherwise from Theorem (3.1) one gets that for every essential curve τ in $D^2 - O(D_0)$ one has $g^n(\tau) \not\simeq g^m(\tau)$ for all $n \neq m$. Then $f: D^2 - O(p) \to D^2 - O(p)$ would also have this property and therefore be isotopic to a pseudo-Anosov map. This is impossible since the topological entropy of f is zero. Also $g: D^2 - O(D_0) \to D^2 - O(D_0)$ cannot be reducible, since otherwise there would be an essential simple curve C_0' in $D^2 - O(D_0) \subset D^2 - O(p)$ which is periodic up to isotopy and bounding a disc containing strictly more points of $O(p)$ than $D_0 \cap O(p)$, contradicting the choice of C_0. It follows from Thurston's Theorem (3.1) that g : $D^2 - O(D_0) \to D^2 - O(D_0)$ must be periodic and therefore there exists $n \geqq 1$ such that

$$g^n,\ id: D^2 - O(D_0) \to D^2 - O(D_0) \text{ are isotopic.}$$

As in Lemma (4.1) it follows that $(O(D_0); g)$ is rotation compatible. In particular $g^r | D_0$: $D_0 - O(p) \to D_0 - O(p)$ renormalizes $f: D^2 - O(p) \to D^2 - O(p)$. Since $O(p) \cap D_0$ is a periodic orbit of $g^r | D_0$ of period $1 < s < n$ and since $h(g^r | D_0) = 0$ it follows from the induction assumption that $(O(p) \cap D_0; g^r | D_0)$ is hereditarily rotation compatible. Therefore by definition $(O(p); f)$ is hereditarily rotation compatible.

Q.E.D.

4.4. Lemma. — *Let f be an orientation preserving $C^{1+\varepsilon}$ diffeomorphism of D^2 with isolated periodic points. If every periodic orbit of f is hereditarily rotation compatible then f has topological entropy zero.*

Proof. — If $h(f) > 0$ then a result of A. Katok [Ka] implies that there exists a periodic orbit $O(p)$ of saddle-type with a transversal homoclinic intersection. Therefore there exists a rectangle $R \subset D^2$ and $n > 0$ such that $f^n | R$ has a horseshoe. It is well known that there are many periodic orbits in a horseshoe which are not rotation compatible, *see* [H. W.], [Fr], [Bo 1]. For example $f^n | R$ has a periodic orbit $O(p)$ of 5 as in Figure 7. Taking the curve C as drawn in this figure it follows that $f^n(C)$ is not isotopic to C in $D^2 - O(p)$. Hence $O(p)$ is not rotation compatible. Since 5 is a prime number this orbit can also not be hereditarily rotation compatible.

5. PROOF OF THEOREM B

5.1. Lemma. — *Let $f: D^2 \to D^2$ be a C^1 diffeomorphism and $O(p)$ be a periodic orbit of period n. Assume that $f: D^2 - O(p) \to D^2 - O(p)$ is periodic. Either p is a fixed point or there exists a fixed point q of f which is a parent orbit of $(O(p); f)$.*

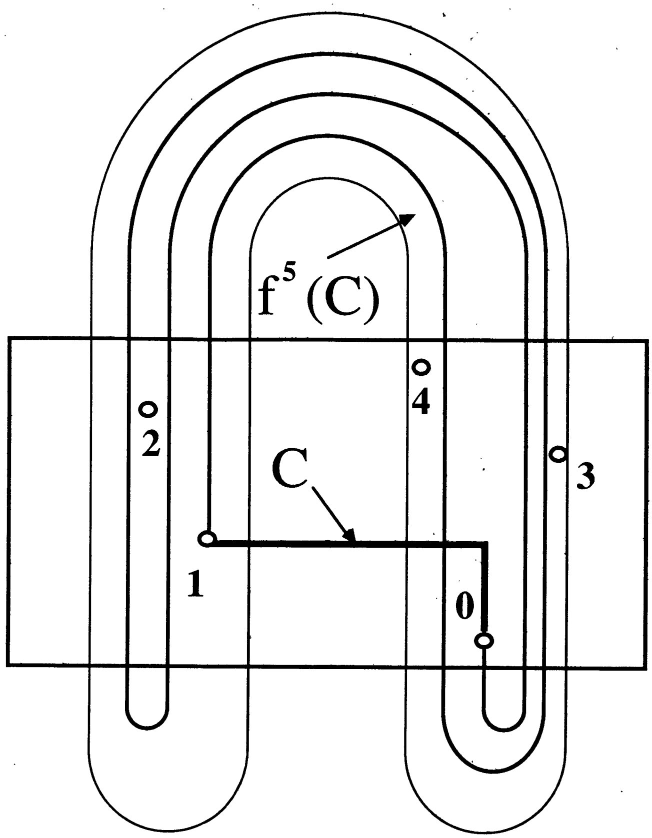

FIG. 7. — A periodic orbit of period 5 in the horseshoe map which generates positive entropy.

Proof. — We may assume that $O(p)$ is not a fixed point. Let R be the isometry of $D^2-O(p)$ isotopic to $f: D^2-O(p) \to D^2-O(p)$ and let $\tilde{R}: D^2 \to D^2$ be the extension of R. Let p' be the (unique) fixed point of $\tilde{R}$. Since $O(p)$ is not a fixed point, one has $p' \notin O(p)$ and p' is a fixed point of R.

Moreover, since $\tilde{R}$ is conjugate to a rotation there exists a simple closed curve $\tilde{\Gamma}$ in D^2 which goes through each of the points of $O(p)$, such that $\tilde{\Gamma}$ bounds a disc containing p' in its interior and such that $\tilde{R}(\tilde{\Gamma})=\tilde{\Gamma}$. Write $\tilde{\Gamma} \subset D^2-(O(p) \cup \{p'\})$ as the disjoint union of n simple arcs $\tilde{\gamma}_i$ in D^2 with endpoints in $O(p)$. Let γ_i be the geodesic (w.r.t. to the hyperbolic metric in $D^2-(O(p) \cup \{p\})$ isotopic to $\tilde{\gamma}_i$ in $D^2-(O(p) \cup \{p\})$ and let $\Gamma = \bigcup \gamma_i$. Since an isometry maps geodesics onto geodesics, $R(\Gamma)=\Gamma$. Γ is a simple closed curve and bounds a disc which contains the fixed point p' of R in its interior.

As we remarked before, p' is the unique fixed point of R. In particular this fixed point is the only fixed point in its Nielsen class. Since R is an orientation preserving isometry, the Lefschetz number $L(p', R^n)$ is equal to one. It follows from Theorem (3.3) that (p', R) is unremovable. Hence there is an isotopy f_t between $R=f_0$ and $f=f_1$ and a continuous curve $c:[0,1]\to D^2-O(p)$ such that $f_t(c(t))=c(t)$. In particular for $q=c(1)$ one has $f(q)=q$. Take the hyperbolic metric d^t (resp. d) on $D^2\setminus O(p)\cup\{c(t)\})$ [resp. $D^2-O(p)$] and let i^t : $D^2\setminus(O(p)\cup\{c(t)\}\to D^2\setminus O(p)$ be the canonical inclusion. Notice that $c(t)\in D^2-O(p)$ for all $t\in[0,1]$ and that $c(t)$ depends continuously on t. Therefore we can choose, for each $t\in[0,1]$, a geodesic γ_i^t in $D^2\setminus(O(p)\cup\{c(t)\})$ (with respect to d^t) isotopic to γ_i in $D^2-O(p)$ and depending continuously on t. Then $\Gamma^t=\bigcup\gamma_i^t$ is a simple closed curve in $D^2\setminus(O(p)\cup\{c(t)\}$. Let $\tilde{\Gamma}^t=i^t(\Gamma^t)\subset D^2-O(p)$. Γ^t depends continuously on t [in terms of the metric d on $D^2-O(p)$]. Since $c(t)\notin\Gamma^t$ for each $t\in[0,1]$ this implies that for each $t\in[0,1]$, Γ^t is a simple closed curve containing each of the points p_i such that the disc D^t bounded by Γ^t contains the point $c(t)$ in its interior. Since Γ^t is isotopic to $\Gamma^0=\Gamma$ and f_t isotopic to R, $f_t(\Gamma^t)$ is isotopic to Γ^t in $D^2-O(p)$.

Q.E.D.

5.2. LEMMA. — *Assume that f: $D^2\to D^2$ is a C^1 diffeomorphism of D^2 with zero topological entropy. Then every hereditarily rotation compatible periodic orbit has a persistent parent.*

Proof. — We show by induction that a periodic orbit $(O(p); f)$ of $f: D^2\to D^2$ which is hereditarily rotation compatible of generation k has a persistent parent. For $k=1$ this statement is proved in Lemma (4.2). So assume that this induction statement is proved for all hereditarily rotation compatible orbits of generation $\leqq k$ and assume that $(O(p); f)$ is hereditarily rotation compatible of generation $k+1$. By definition there exist a map g in the isotopy class of f: $D^2-O(p)\to D^2-O(p)$ and a g-periodic disc D_0 of period r such that $(O(p)\cap D_0; g^r|D_0)$ is hereditarily rotation compatible of generation k. Consider the periodic orbit $O(p)\cap D_0$ of $g^r|D_0$ as a periodic orbit of g^r: $D^2\to D^2$. Then $(O(p)\cap D_0; g^r)$ is also hereditarily rotation compatible of generation k and therefore $(O(p)\cap D_0; f^r)$ is hereditarily rotation compatible of generation k. It follows from the induction assumption that $(O(p)\cap D_0; f^r)$ has a persistent parent orbit. This orbit is persistent under isotopies of f^r : $D^2-(O(p)\cap D_0)\to D^2-(O(p)\cap D_0)$ and therefore under isotopies of f: $D^2-O(p)\to D^2-O(p)$. This means that there exist $q\in D_0$, $s\in\mathbb{N}$ and simple closed curves $\tilde{\Gamma}_{ir}$, $i=0, 1, \ldots, s-1$, such that q

is a persistent periodic point of f^r of period s, $\# (O(p) \cap D_0) = r_0 \times s$, for each $i = 0, 1, \ldots, s-1$, $\tilde{\Gamma}_{ir}$ bounds a disc containing $q_{ir} = f^{ir}(q)$ and r_0 points of $O(p) \cap D_0$ and such that $f^r(\tilde{\Gamma}_{ir})$ is isotopic to $\tilde{\Gamma}_{(i+1)r}$ in $D^2 - \{O(p) \cup O(q)\}$. We can choose these curves $\tilde{\Gamma}_{ir}$ such that $\tilde{\Gamma}_{ir} \subset D_0$. Then $\tilde{\Gamma}_{ir}$ bounds precisely r_0 points of $O(p)$. For $k = 0, 1, \ldots, r-1$, choose $\Gamma_{ir+k} = f^k(\Gamma_{ir})$, and $q_i = f^i(q)$. It follows that $per(p) = r \times \#(O(p) \cap D_0) = r \times r_0 \times s = r_0 \times per(q)$. Furthermore Γ_i is a simple curve, bounding r_0 points of $O(p)$ and q_i and $f(\Gamma_i)$ is isotopic to Γ_{i+1} in $D^2 - \{O(p) \cup O(q)\}$. This proves that $(O(q); f)$ is a persistent parent orbit of $(O(p); f)$. This finishes the proof of Lemma (5.2).

Q.E.D.

6. PROOF OF THEOREM C

6.1. LEMMA. — *Assume that $f: D^2 \to D^2$ is a C^1 diffeomorphism of D^2 with zero topological entropy. Then the following statements hold.*

1. *If two orbits $O(p)$ and $O(p')$ have a common child $O(q)$ then $O(p)$ and $O(p')$ have the same period, the same generation and any parent of $O(p)$ is also a parent of $O(p')$.*
2. *If two orbits $O(p)$ and $O(p')$ are disjoint then they have no common parent $O(q)$.*
3. *If two orbits $O(p)$ and $O(p')$ are disjoint and have a common child $O(q)$ then $O(p)$ and $O(p')$ are both fixed points.*

Proof. — Consider three periodic orbits $O(p)$, $O(p')$ and $O(q)$ as in 1, 2 or 3. The Lemma is trivially true if $\#(O(p) \cup O(p') \cup O(q)) = 3$. So let us assume that we have proved by induction that statements 1 and 2 are true for all periodic orbits with

$$\#(O(p) \cup O(p') \cup O(q)) < n.$$

Consider

$$f: \ D^2 - \{O(p) \cup O(p') \cup O(q)\} \to D^2 - \{O(p) \cup O(p') \cup O(q)\}.$$

Since $h(f) = 0$, according to Thurston's theorem there are at most two possibilities.

Case 1. — f is isotopic to a periodic map and therefore isotopic to a map

$$g: \ D^2 - \{O(p) \cup O(p') \cup O(q)\} \to D^2 - \{O(p) \cup O(p') \cup O(q)\}$$

which extends continuously to a map on D^2 which is conjugate to a rotation of D^2. In this case statements 1, 2 and 3 follow immediately.

Case 2. — f is reducible. Then there exist $r \geqq 1$, and a simple essential closed curve C_0, and a map g in the isotopy class of

$$f: \quad D^2 - \{O(p) \cup O(p') \cup O(q)\} \to D^2 - \{O(p) \cup O(p') \cup O(q)\}$$

such that $g^r(C_0) = C_0$, and such that the curves $C_i = g(C_0)$, $i = 0, 1, \ldots, r-1$ are pairwise disjoint. There exists $s > 1$ such that for each $i = 0, \ldots, r-1$ the curve C_i bounds a disc D_i containing precisely s points of $O(p) \cup O(p') \cup O(q)$ (where $1 < s < n$ because C_i is simple). Choose a "periodic" essential simple curve C_0 in $D^2 - \{O(p) \cup O(p') \cup O(q)\}$ as above such that there is no periodic essential simple curve C'_0 bounding more points of $O(p) \cup O(p') \cup O(q)$ than C_0. Choose r and g corresponding to C_0.

Precisely as in Lemma (4.3) it follows that $(O(D_0); g)$ is rotation compatible and that $g: D^2 - O(D_0) \to D^2 - O(D_0)$ is periodic. So we may assume that $g: D^2 - O(D_0) \to D^2 - O(D_0)$ continuously extends to a map R on D^2 which is conjugate to a rotation of D^2. Therefore, the following properties are satisfied:

(i) if one of the orbits $O(p)$, $O(p')$ or $O(q)$, say $O(p)$, is in $D^2 - O(D_0)$ and has children then $O(p)$ is the (unique) fixed point of R and O(D) has period $r \geqq 2$;

(ii) if $O(q)$ is in $D^2 - O(D_0)$ and $O(p)$ is a parent of $O(q)$ then $r = 1$ (*i. e.* $O(D_0) = D_0$), $O(p)$ is a fixed point and $O(p) \subset D_0$;

(iii) if one of the orbits $O(p)$, $O(p')$ or $O(q)$ is in $D^2 - O(D_0)$ then this orbit cannot be disjoint from one of the other orbits $O(p)$, $O(p')$ or $O(q)$ unless $R = id$;

(iv) $O(q)$ is the parent of $O(p)$ then $O(q) \subset O(D_0)$ and $r = 1$ [*i. e.* $O(D_0) = D_0$] and $O(q)$ is a fixed point.

Let us first assume that both orbits $O(p)$, $O(p')$ are in $D^2 - O(D_0)$. So from (i), if $O(p)$ and $O(p')$ have a common child then $O(p) = O(p')$ is a fixed point. This proves 1. From (iii) they cannot be disjoint unless $R = id$. So statements 2 and 3 are trivially true.

Next assume that (precisely) one of the orbits $O(p)$, $O(p')$, say $O(p)$, is contained in $D^2 - O(D_0)$. If $O(p)$ and $O(p')$ have a common child, then from (ii) above, $r = 1$ and $O(p)$, $O(p') \subset O(D_0)$, a contradiction. This proves 1 in this case. From (iii) $O(p)$ and $O(p')$ cannot be disjoint unless $R = id$ and therefore 2 and 3 are trivially satisfied.

So we may assume that $O(p)$ and $O(p')$ both have at least one point in common with D_0. Then $O(p)$, $O(p') \subset O(D_0)$. First assume that $O(q) \cap O(D_0) = \varnothing$. Then if $O(p)$ and $O(p')$ are parents of $O(q)$ then, from (i), $O(D_0) = D_0$, and $O(p)$, $O(p')$ must both be fixed points. This proves 1 in this case. So assume $O(p)$ and $O(p')$ are disjoint. Then $r = 1$ and any parent of these orbits must be contained in $O(D_0)$. This proves that 2 is trivially satisfied. If $O(q)$ is a child of $O(p)$, $O(p')$ then from (iv), $O(p)$, $O(p')$ are both fixed points.

So we are left with the case that $O(p)$, $O(p')$ and $O(q)$ all have some points in $O(D_0)$. Then $O(p)\cup O(p')\cup O(q)\subset \bigcup_{i=0}^{r-1} O(D_0)$. Since C_0 is an essential curve, $\#\{(O(p)\cup O(p')\cup O(q))\cap D_0\}<n$ and it follows from the induction assumption that $O(p)\cap D_0$, $D(p')\cap D_0$ and $O(q)\cap D_0$, satisfy statements 1, 2 and 3 of the Lemma. The Lemma follows easily.

Q.E.D.

6.2. LEMMA. — *Assume that* $f: D^2\to D^2$ *is a* C^1 *diffeomorphism of* D^2 *with zero topological entropy. If two periodic orbits* $O(p)$ *and* $O(q)$ *are not disjoint then either:*

(i) *one of the orbits is an ancestor of the other or*

(ii) *they lie nested and have a common ancestor.*

Proof. — Let us prove this Lemma by induction on $\#(O(p)\cup O(q))$. If $\#(O(p)\cup O(q))=2$ then $O(p)$ and $O(q)$ are both fixed points and they are disjoint by definition. So assume that the statement of the Lemma holds for all maps and all periodic orbits $O(p)$ and $O(q)$ with $\#(O(p)\cup O(q))<n$. Consider a map f and periodic orbits $O(p)$ and $O(q)$ such that $\#(O(p)\cup O(q))=n$.

Consider

$$f:\quad D^2-(O(p)\cup O(q))\to D^2-(O(p)\cup O(q)).$$

Then there are two possibilities for f.

Case 1. — f is irreducible. Let us show that one of the orbits $O(p)$, $O(q)$ is the persistent parent of the other. Indeed, since $h(f)=0$ and f is irreducible, f is periodic. So f is isotopic to an isometry R. It follows from Lemma (4.1) that R extends uniquely to a map $\tilde{R}$ on D^2 which is conjugate to a rotation. In particular either the orbits $O(p)$ and $O(q)$ of $\tilde{R}$ have the same period and one of these orbits is inside the other, or one of the orbits consists just of one point. The first case is impossible since by assumption $f: D^2-\{O(p)\cup O(q)\}\to D^2-\{O(p)\cup O(q)\}$ is irreducible. In the second case one of these orbits is a fixed point and a parent of the other orbit.

Case 2. — f is reducible. Then there exist a map g which is isotopic to $f: D^2-\{O(p)\cup O(q)\}\to D^2-\{O(p)\cup O(q)\}$, $r\in N$ and a simple essential closed curve C in $D^2-\{O(p)\cup O(q)\}$ such that $g^r(C)=C$ and such that C, $g(C)$, ..., $g^{r-1}(C)$ are disjoint. Since C is essential it bounds at least two and not all of the points of $O(p)\cap O(q)$. Choose a periodic essential simple curve C such that there is no other periodic essential simple curve C′ which bounds fewer points of $O(p)\cup O(q)$ and let g and $r\in N$ correspond to C. Let $C_i=g^i(C)$. Consider two subcases.

Case 2A. — C_0 bounds points of both $O(p)$ and of $O(q)$. Since $g^r(C_0)=C_0$ it follows then that each point of $O(p)\cup O(q)$ is contained in one of the discs $D_0, D_1, \ldots, D_{r-1}$ bounded by $C_0, C_1, \ldots, C_{r-1}$. Since C_0 is essential, it cannot bound all points of $O(p)\cup O(q)$ and therefore $r\geqq 2$. Let us consider $(O(p)\cap D_0; d^r|D_0)$ and $(O(q)\cup D_0; g^r|D_0)$. The choice of C (bounding a minimal number of points of $O(p)\cup O(q)$) implies that g^r: $D_0-\{O(p)\cup O(q)\}\to D_0-\{O(p)\cup O(q)\}$ is irreducible. From Case 1 one gets that one of the orbits $(O(p)\cap D_0; g^r|D_0)$, $(O(q)\cap D_0; g^r|D_0)$ is the persistent parent of the other. Hence, as before, one of the orbits of $(O(p); f)$, $(O(q); f)$ is the persistent parent of the other.

Case 2B. — One of the orbits $O(p)$ or $O(q)$, say $O(q)$, has no points inside D_0. Since C_0 is essential, C_0 bounds some of the points of $O(p)$ and therefore all of the points of $O(p)$ are contained in $D_0, D_1, \ldots, D_{r-1}$. Identify the discs D_i to a point $\tilde{r}_i$. More precisely, $x\sim y$ if and only if there exists i such that $x, y\in D_i$. Let $\pi: D^2\to D^2/_\sim$ be the natural projection and

$$\tilde{g}:\quad D^2/_\sim\to D^2/_\sim$$

be the map corresponding to g with these discs identified. Notice that g^r: $D_0\setminus O(p)\to D_0\setminus O(p)$ is irreducible. So, from the Lemma (5.1), $(O(p)\cap D_0; g^r|D_0)$ has a fixed point $r_0\in D_0$ as a persistent parent. As before, this implies that $(O(r_0); g)$ is the persistent parent of $(O(p); g)$. Since C_0 is essential it bounds at least two points of $O(p)$ and therefore $\#(O(\tilde{r}_0)\cup O(q))<\#(O(p)\cup O(q))$. Hence we can apply the induction assumption and there are four possibilities.

(i) $(O(\tilde{r}_0); \tilde{g})$ and $(O(q); \tilde{g})$ are disjoint. Then there exist curves $\tilde{C}'$ and $\tilde{C}''$ in $D^2-(O(D_0)\cup O(q))/_\sim\cong D^2-(O(\tilde{r}_0)\cup O(q))$ bounding all points of respectively $O(\tilde{r}_0)$, $O(q)$ and such that $\tilde{g}(\tilde{C}')\simeq C'$, $\tilde{g}(\tilde{C}'')\simeq\tilde{C}''$ [rel $D^2-O(\tilde{r}_0)\cup O(q))$]. Hence the curves $C'=\pi^{-1}(\tilde{C}')$ and $C''=\pi^{-1}(C'')$ in $D^2-(O(D_0)\cup O(q))$ corresponding to $\tilde{C}'$, $\tilde{C}''$ in $D^2-(O(D_0)\cup O(q))/_\sim$ bound all points of respectively $O(p)$, $O(q)$ and $g(C')\simeq C'$, $g(C'')\simeq C''$ [rel $D^2-(O(p)\cup O(q))$]. This implies that $(O(p); f)$ and $(O(q); f)$ are also disjoint.

(ii) $(O(\tilde{r}_0); \tilde{g})$ is the ancestor of $(O(q); \tilde{g})$. Since $(O(\tilde{r}_0); \tilde{g})$ is an ancestor of $(O(q); \tilde{g})$, the orbit $(O(r_0); g)$ is also an ancestor of $(O(q); g)$. In particular there exists a simple closed essential curve C' in $D^2-(O(D_0); g)$ such that the disc D' bounded by C' contains C_0 in its interior, such that $O(q)\subset\bigcup_{i=0}^{r-1} f^i(D')$ and such that $f^r(C')\simeq C'$ [rel $D^2-(O(q)\cup O(p))$]. Since $O(q)$ has no points inside D_0, and therefore $O(q)$ is contained in the orbit of the annulus $D'\setminus D$. It follows that $(O(p); f)$ and $(O(q); f)$ lie nested.

(iii) $(O(q); \tilde{g})$ is the ancestor of $(O(\tilde{r}_0); \tilde{g})$. Since $(O(q); \tilde{g})$ is an ancestor of $(O(\tilde{r}_0); \tilde{g})$, the orbit $(O(q); g)$ is also an ancestor of $(O(r_0); g)$. From this, and since $(O(r_0), g)$ is the parent of $(O(p); g)$, one gets that $(O(q); g)$ is a (persistent) ancestor of $(O(p); g)$. Hence $(O(q); f)$ is a parent of $(O(p); f)$.

(iv) The orbits $(O(\tilde{r}_0); \tilde{g})$, $(O(q); \tilde{g})$ lie nested and have a common ancestor. Call this common ancestor $(O(P); \tilde{g})=(O(P); g)$. This implies that $O(P); g)$ is the common ancestor of $(O(r_0); g)$ and $(O(q); g)$. Since $(O(r_0); g)$ is the parent of $(O(p); g)$ this implies that $(O(P); g)$ is the common father of $(O(p); g)$ and $(O(q); g)$. Since $(O(\tilde{r}_0); \tilde{g})$ and $(O(q); \tilde{g})$ lie nested the orbits $(O(p); f)$ and $(O(q); f)$ also lie nested.

Q.E.D.

ACKNOWLEDGMENT

This work was done during a visit by J. M. Gambaudo to the Technical University of Delft and finished during a visit in November 1987 by all three authors to the City University of New York on an invitation of D. Sullivan. We would like to thank D. Sullivan, Ph. Boyland and J. Smillie for some helpful remarks.

REFERENCES

[A.F.] D. ASIMOV and J. FRANKS, *Unremovable Closed Orbits*, in *Geometric Dynamics*, J. PALIS Ed., *Lecture Notes in Math.*, No. **1007**, 1983, pp. 22-29, (1983).

[B.G. M.Y.] L. BLOCK, J. GUCKENHEIMER, M. MISIUREWICZ and L. S. YOUNG, Periodic Orbits and Topological Entropy of One-Dimensional Maps, *Lecture Notes in Math.*, No. **819**, 1980, pp. 18-34.

[Bi] J. BIRMAN, Braids, Links, and Mapping Class Groups, *Annals of Mathematics studies*, Princeton University Press, 1975.

[Bl.F.] P. BLANCHARD and J. FRANKS, Dynamical Complexity of Orientation Reversing Diffeomorphisms of Surfaces, *Inventiones math.*, Vol. **62**, 1980, pp. 333-339.

[Bo 1] P. BOYLAND, *Braid Types and a Topological Method of Proving Positive Entropy*, preprint, 1987.

[Bo 2] P. BOYLAND, An Analog of Sharkovski's. Theorem for Twist Maps, *Summer Conference on Hamiltonian Dynamical Systems*, proceeding A.M.S.-S.I.A.M. 1987.

[Bro] L.E.J. BROUWER, Über die periodischen transformationen der Kugel, *Math. Ann.*, Vol. **80**, 1919, pp. 39-41.

[Ca] A. CASSON, *Automorphisms of Surfaces After Nielsen and Thurston*, lecture notes by S. Bleirer.

[Ei] S. EILENBERG, Sur les transformations périodiques de la surface de sphere, *Fund. Math.*, Vol. **22**, 1934, pp. 28-41.

[F.L.P.] A. FATHI, F. LAUDENBACH and V. PEANARU, Travaux de Thurston sur les surfaces, *Astérisque*, Vol. **66-67**, 1979.

[Fr] D. FRIED, Entropy and Twisted Cohomology, *Topology*, Vol. **25**, 1986, pp. 455-470.

[Ha] M. HANDEL, *Zero Entropy Surface Diffeomorphisms*, preprint, 1986.

[H.W.] P. HOLMES and R. WILLIAMS, Knotted Periodic Orbits in Suspension of Smale's Horseshoe: Torus Knots and Bifurcation Sequences, *Arch. Rat. Mech. Anal.*, Vol. **90**, 1985, pp. 115-194.

[Ka] A. KATOK, Lyapounov Exponents, Entropy and Periodic Orbits for Diffeomorphisms, *Publ. Math. I.H.E.S.*, Vol. **51**, 1980, pp. 137-173.

[Kerc] S. P. KERCKHOFF, The Nielsen Realization Problem, *Annals of Math.*, Vol. **117**, 1983, pp. 235-265.

[Kere] B. KEREKJARTO, Sur la structure des transformations topologiques des surfaces en elles-mêmes, *L'Enseignement Math.*, Vol. **35**, 1936, pp. 297-316.

[Kob] T. KOBOYASHI, Links of Homeomorphisms of a Disc and Topological Entropy, *Invent. math.*, Vol. **80**, 1985, pp. 153-159.

[Mor] J. W. MORGAN, Non-Singular Morse-Smale Flows on 3-Dimensional Manifolds, *Topology*, Vol. **18**, 1978, pp. 41-53.

[Sar] A. N. SARKOVSKII, Coexistence of Cycles of a Continuous Map of a Line Into Itself, *Ukr. Math.*, Vol. **16**, 1964, pp. 61-71.

[Smi] J. SMILLIE, Periodic Points of Surface Homeomorphisms with Zero Entropy, *Ergod. Th. & Dynam. Sys.*, Vol. **3**, 1983, pp. 315-334.

[Th] W. THURSTON, Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry, *Bull. A.M.S.*, Vol. **6**, 1982, pp. 357-381.

(*Manuscrit reçu le* 12 *septembre* .)