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Geometric quantization of the MIC-Kepler problem via extension of the phase space

by

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ABSTRACT. — Kostant-Souriau's geometric quantization scheme is applied to the extended phase space of the modified Kepler problem. Thus we obtain the quantization of the magnetic charge and energy spectrum of the corresponding quantum problem.

RÉSUMÉ. — La quantification géométrique de Kostant-Souriau est appliquée au problème modifié de Kepler par l'extension de l'espace de phase. De cette manière, il est obtenu une quantification de charge magnétique et le spectre énergétique du problème quantique correspondant.

1. INTRODUCTION

Hamiltonian systems with symmetries are among the most interesting classical dynamical systems. Their natural description is in terms of symplectic geometry. The symmetries are presented by a Lie group G acting symplectically (canonically) on the symplectic (phase space) manifold (P, ω) and this results in the appearance of constraints. Factoring out the symmetries one gets the so called reduced phase space (P_μ, ω_μ) and reduced

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dynamics on it described by Marsden-Weinstein's [1] theorem (see Section 2 for details). Thus, each Hamiltonian system with symmetry has two symplectic faces: the initial and reduced phase spaces (P, ω) and (P_μ, ω_μ) respectively. Moreover there is no formal distinction between working on (P, ω) or (P_μ, ω_μ) at classical level. That is why we are free to reverse the procedure and to look at (P, ω) as an extension of (P_μ, ω_μ). But these two representations of one and the same mechanical system are not necessarily equivalent on quantum level. The problem to properly correlate the quantization of extended and reduced phase space is made intricate also by the existence of a variety of quantization schemes. The best choice seems to be the Kostant-Souriau's geometric quantization as this quantization procedure is suitable for arbitrary symplectic manifolds because it identifies the intrinsic geometry of the phase space with the objects infolved in ordinary quantization [2], [3].

Geometric quantization of the extended and reduced phase spaces has been proved [4] to be equivalent within the cotangent bundle category where the starting and reduced phase spaces are cotangent bundles provided with their standard symplectic structures and in the case when the symplectic manifold to be reduced is a compact Kaehler manifold [5]. The real situation in mechanics is somewhere between as when one starts with a cotangent bundle then after reduction obtains either a compact Kaehler manifold or a cotangent bundle whose symplectic for is not the canonical one.

Substantial progress towards clarifying the situation has already been made [6], but one of the problem is the lack of examples in which the corresponding physical systems are well understood. The purpose of this paper is to provide such a description in the case of the MIC-Kepler problem.

2. PRELIMINARIES

In this Section we collect the exact statements of reduction theorems and give a brief outline of geometric quantization program in the form we shall need later. Here we fix also notation and conventions we use.

THEOREM 1 ([1]). — Let (P, ω) be a symplectic manifold on which a Lie group G acts symplectically and $J : P \to \mathcal{G}^*$ be an Ad*-equivariant moment map. Assume that $\mu \in \mathcal{G}^*$ is a regular value of J and that the isotropy subgroup G_μ acts freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with the form ω_μ determined by $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \to P$ is the inclusion map. Let $H : P \to R$ be G-invariant: it induces a hamiltonian flow on P_μ whose Hamiltonian H_μ satisfies $H_\mu \circ \pi_\mu = H \cdot i_\mu$.

REMARK 1. — If the Hamiltonian system (P, ω, H) admits a symmetry group which commutes with G, then $(P_\mu, \omega_\mu, H_\mu)$ preserves this symmetry.

THEOREM 2 ([7]). — Let P be a cotangent bundle T^*M and G a one-parameter Lie group acting freely and properly on M. Let $M \to N = M/G$ be the induced principal fibre bundle and $\tilde{\alpha}$ – *a* connection one-form on it. The reduced manifold P_μ is symplectomorphic to T^*N endowed with a symplectic form given by the canonical one plus a « magnetic term » $\mu\tau_N^* d\tilde{\alpha}$ (where τ_N is the canonical projection $T^*N \to N$).

A thorough discussion concerning the reduction of symplectic manifolds and detailed examination of the classical examples from the modern point of view can be found in [8-10].

The geometric quantization scheme associates to a phase space (X, Ω) a Hilbert space $\mathscr{H}$ of the quantum states and to a subalgebra of the smooth functions of X-quantum operators in $\mathscr{H}$. The Hilbert space $\mathscr{H}$ is built by the polarized section of the quantum line bundle $Q = L \otimes N_F^{1/2}$, where the bundle L is the prequantum line bundle and $N_F^{1/2}$ is the line bundle of half-forms normal to the polarization F (which is supposed to be invariant $[X_f, F] \subset F$).

If $\Psi = s \otimes v$, where $s \in \Gamma(L)$, $v \in \Gamma(N_F^{1/2})$, $\Psi \in \Gamma(Q)$ then one associates to the classical observable f a quantum operator $\hat{f}$ acting in $\mathscr{H}$ by

$$\hat{f}(\Psi) = (-iX_f - \Theta(X_f) + f)s \otimes v - is \otimes \mathscr{L}(X_f)v.$$

Here X_f is the Hamiltonian vector field generated by $f (i(X_f)\Omega = -df)$, Θ is the potential one form of $\Omega (\Omega = d\Theta)$ and $\mathscr{L}(X_f)$ is the Lie derivative with respect to X_f.

3. THE MIC-KEPLER PROBLEM

The MIC-Kepler problem [11] (see also [12]) is the Hamiltonian system

$$(T^*\dot{R}^3, \Omega_\mu, H_\mu) \tag{1}$$

where

$$\begin{aligned}
&T^*\dot{R}^3 = \{ (q, p) \in R^3 \times R^3 ; q \neq 0 \} \\
&\Omega_\mu = d\Theta + \sigma_\mu, \qquad \Theta = \Sigma p_j dq_j, \\
&\sigma_\mu = -\mu/(2|q|^3)\varepsilon_{ijk} q_i dq_j \Lambda dq_k, \qquad |q|^2 = q_1^2 + q_2^2 + q_3^2 = r^2 \\
&H_\mu = \frac{1}{2}|p|^2 - \frac{\alpha}{|q|} + \mu^2/2|q|^2, \qquad \alpha, \mu \in R, \alpha > 0.
\end{aligned} \tag{2}$$

This Hamiltonian system describes the motion of a charged particle in the presence of a Dirac monopole field $B_\mu = -\mu q/|q|^3$, a Newtonian potential $-\alpha/|q|$, and a centrifugal potential $\mu^2/2|q|^2$.

Here, we apply the geometric quantization (via extension) to the Hamiltonian system (1) which leads to the following theorem.

THEOREM 3. — The discrete spectrum (bound states) of the MIC-Kepler problem (1) (α and μ fixed) consists of the energy levels,

(3) $$E_N = -\alpha^2/2N^2, \quad N = |\mu| + 1, |\mu| + 2 \ldots$$

with multiplicities

(4) $$m(E_N) = N^2 - \mu^2 .$$

Theorem 3 will be proved in Sect. 5.

REMARK 2. — Prequantization of $(T^*\dot{R}^3, \Omega_\mu)$ selecting the integral symplectic forms produces immediately the magnetic charge quantization $\mu = 0, \pm \frac{1}{2}, \pm 1, \ldots$ (cf. Sect. 5). Unfortunately the geometric quantization scheme as described in Sect. 2 can not be applied because there is no way to dispense with the use of an invariant polarization.

REMARK 3. — The non-bijective transformations (extensions) have been applied with great success by Kibler and Negadi [*13*] to various problems in physics and chemistry (see also [*14*]).

4. THE CONFORMAL KEPLER PROBLEM AND ITS REDUCTION

We start with the symplectic manifold

(5) $$T^*\dot{R}^4 = \{ (x, y) \in R^4 \times R^4, x \neq 0 \}$$

with the standard symplectic form

(6) $$\Omega = dy \Lambda dx = \sum_{j=1}^{4} dy_j \Lambda dx_j .$$

Next, we introduce three Hamiltonian functions on the phase space $(T^*\dot{R}^4, \Omega)$. First, the Hamiltonian of the conformal Kepler problem

(7) $$H = (|y|^2 - 8\alpha)/8|x|^2, \quad \alpha\text{-a fixed positive constant.}$$

Second, the Hamiltonian of a harmonic oscillator

(8) $$K = (|y|^2 + \lambda^2 |x|^2)/2, \quad \lambda\text{-an arbitrary positive constant,}$$

and third, a momentum Hamiltonian

$$M = \frac{1}{2}(x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3). \tag{9}$$

Obviously, we have

$$4 |x|^2 (H + \lambda^2/8) = K - 4\alpha, \tag{10}$$

which means that the energy hypersurfaces $H = E = -\lambda^2/8$ and $K = 4\alpha$ coincide. Moreover, the flows defined by the Hamiltonians H and K on the level sets

$$H^{-1}(E) \equiv K^{-1}(4\alpha), \tag{11}$$

coincide up to a monotonic change of parameter as there the corresponding Hamiltonian vector fields X_H and X_K satisfy:

$$4 |x|^2 \cdot X_H = X_K. \tag{12}$$

This have been observed by Iwai and Uwano [*11*] whose work has influenced substantially [*12*] and the present study. Following them we introduce the complex coordinates.

$$z_1 = \lambda(x_1 + ix_2) - i(y_1 + iy_2) \qquad z_2 = \lambda(x_3 + ix_4) - i(y_3 + iy_4)$$
$$z_3 = \lambda(x_1 - ix_2) - i(y_1 - iy_2) \qquad z_4 = \lambda(x_3 - ix_4) - i(y_3 - iy_4)$$

In these coordinates $T^*\dot{R}^4 = C^4 \backslash D$, where

$$D = \{ z \in \mathbb{C}^4 : z_1 = -\bar{z}_3, z_2 = -\bar{z}_4 \} \tag{13}$$

and the symplectic form Ω is a multiple of the standard Kaehler form on C^4

$$\Omega = \frac{i}{4\lambda} dz \Lambda d\bar{z} = \frac{i}{4\lambda} \Sigma dz_j \Lambda d\bar{z}_j. \tag{14}$$

The Hamiltonian functions K and M can also be easily expressed as follows

$$K = (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)/4 \tag{15}$$

and

$$M = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)/8\lambda \tag{16}$$

REMARK 4. — The Hamiltonians K and M as well as the symplectic form are well defined on the manifold

$$\dot{C}^4 = C^4 \backslash \{0\} \supset T^*\dot{R}^4. \tag{17}$$

We denote by K_t, M_s the flows of the Hamiltonian systems $(\dot{C}^4, \Omega, K)$, $(\dot{C}^4, \Omega, M)$.

LEMMA 1. — For any $z \in \dot{C}^4$, s, $t \in R$ we have:

$$K_t z = (e^{i\lambda t} z_1, e^{i\lambda t} z_2, e^{i\lambda t} z_3, e^{i\lambda t} z_4) \tag{18}$$

$$M_s z = (e^{is/2} z_1, e^{is/2} z_2, e^{-is/2} z_3, e^{-is/2} z_4). \tag{19}$$

In particular, the flows of all three Hamiltonians H, K, M commute where are defined. By lemma 1 the flow M_s defines a symplectic action of the circle group U(1) on the manifold $\dot{C}^4$. The moment map for this action is M itself. We note that the set D defined in (13) is invariant with respect to this U(1) action. Thus $T^*\dot{R}^4$ is also invariant, as well as the Hamiltonian H, and we may apply theorem 2 to reduce the Hamiltonian system (T^*R^4, Ω, H) with respect to the U(1) action (19). The result is the following proposition established by Iwai and Uwano [*11*].

PROPOSITION. — Let $\mu \in R$. Then

$$M^{-1}(\mu)/U(1) \cong T^*\dot{R}^3 \tag{20}$$

and the reduction of the form Ω and the Hamiltonian H give Ω_μ and H_μ, i. e. the result of the reduction is the MIC-Kepler problem (1).

Moreover, the energy-momentum manifold

$$\mathcal{M}(\lambda, \mu) = \{ (x, y) \in T^*\dot{R}^4 \mid K = 4\alpha, M = \mu \} \tag{21}$$

is mapped by π_μ onto the energy hypersurface $H_\mu = -\lambda^2/8$ ($\lambda = \sqrt{-8E}$) of the MIC-Kepler problem. In order that $\mathcal{M}(\lambda, \mu)$ be non-empty, λ and μ must obey:

$$\lambda \mid \mu \mid \leqslant 2\alpha. \tag{22}$$

In what follow we shall assume that $\lambda \mid \mu \mid < 2\alpha$ holds. What is happening in the case of $\lambda \mid \mu \mid = 2\alpha$ is that this energy level of the reduced system coincides with the minimal value $-\alpha^2/2\mu^2$ of the potential $U_\mu(v) = \mu^2/2r^2 - \alpha/r$ and consequence the energy-momentum manifold $\mathfrak{M}(2\alpha/\mid \mu \mid, \mu)$ consists of all the point of equilibrium.

5. QUANTIZATION

If we choose our polarization F to be spaned by the antiholomorphic directions $\left\{ \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \frac{\partial}{\partial \bar{z}_3}, \frac{\partial}{\partial \bar{z}_4} \right\}$ and adapted to it potential one-form $\Theta = -\frac{i}{4\lambda} \bar{z} dz$ we find that Ψ can be written in the form:

$$\Psi = \varphi(z)(dz_1 \Lambda dz_2 \Lambda dz_3 \Lambda dz_4)^{1/2}$$

where φ is a holomorphic function. The essential idea of Dirac's method of quantization in the presence of constraints in the extended phase space is that they must be enforced quantum mechanically if they have not been eliminated classically. Since the constraints specifying the energy-momentum manifold $\mathcal{M}(\lambda, \mu)$ are given by $\mathrm{K} = 4\alpha$ and $\mathrm{M} = \mu$, it follows that the physically admissible quantum states are those which belong to the subspace $\mathcal{H}_\mu$ of $\mathcal{H}$ defined by

$$\mathcal{H}_\mu = \{ \Psi \in \mathcal{H} \mid \hat{\mathrm{K}}\Psi = 4\alpha\Psi, \hat{\mathrm{M}}\Psi = \mu\Psi \}.$$

Now, we have

$$\hat{\mathrm{K}}\Psi = \lambda\left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_4} + 2 \right)\varphi \otimes \nu$$
$$= \lambda(\mathcal{N} + 2)\Psi = 4\alpha\Psi. \quad \mathcal{N} = 0, 1, 2, \ldots$$

and

$$\hat{\mathrm{M}}\Psi = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4} \right)\varphi \otimes \nu = \mu\Psi$$

where φ in a homogeneous polynomial of degree $\mathcal{N} \geqslant 0$ in z's and $\nu = (dz_1 \Lambda dz_2 \Lambda dz_3 \Lambda dz_4)^{1/2}$. Introducing $\mathrm{N} = \mathcal{N}/2 + 1$ and solving

$$2\mathrm{N}\sqrt{-8\mathrm{E}} = 4\alpha$$

one gets (3) as well

$$n_1 + n_2 + n_3 + n_4 = 2\mathrm{N} - 2 \tag{24}$$

$$n_1 + n_2 - n_3 - n_4 = 2\mu, \quad n_i \geqslant 0, \quad i = 1, 2, 3, 4. \tag{25}$$

which is equivalent to:

$$n_1 + n_2 = \mathrm{N} + \mu - 1 = \mathcal{N}_1 \tag{26}$$

$$n_3 + n_4 = \mathrm{N} - \mu - 1 = \mathcal{N}_2. \tag{27}$$

By (25) the magnetic charge μ is quantized according Dirac's prescription

$$\mu = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2} \ldots . \tag{28}$$

Combined (26) and (27) ensure that $\mathrm{N} = |\mu| + 1, |\mu| + 2, \ldots$

To find the multiplicities $m(\mathrm{E_N})$ we remark that φ reduces to a product $\varphi_1(z_1, z_2)\varphi_2(z_3, z_4)$ of homogeneous polynomials in two variables of degree $\mathcal{N}_1$, $\mathcal{N}_2$ respectively. The dimensionality of the Hilbert space $\mathcal{H}_{\mu,\mathrm{N}}$ is then

$$m(\mathrm{E_N}) = (\mathcal{N}_1 + 1)(\mathcal{N}_2 + 1) = \mathrm{N}^2 - \mu^2 .$$

This proves theorem 5.

REMARK 5. — Looking at the MIC-Kepler problem as an one-parameter family of deformations of the Kepler problem Bates [*15*] exhibits its 0(4, 2) dynamical symmetry and arrived at the same results. On the other side, it turns out that the MIC-Kepler problem and the Taub-NUT problem with negative mass parameter are « hiddenly » simplectomorphic systems with identical degeneracies [16].

REMARK 6. — The Hilbert space $\mathcal{H}_{\mu,N}$ carries the most general $(N + \mu - 1/2, N - \mu - 1/2)$ irreducible representation of SO(4). The wave functions within are labeled by four quantum numbers wich are eigenvalues of a complete set of commuting operators — e. g. — H (energy), M (magnetic charge) L_3 (third projection of the generalized angular momentum) and J_3 (third projection of the generalized Runge-Lenz vector). When $\mu = 0$ one reproduces the well known results for Hydrogen atom.

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REFERENCES

[1] J. MARSDEN, A. WEINSTEIN, Reduction of Symplectic Manifolds with Symmetry. *Rep. Math. Phys.*, vol. **5**, 1974, p. 121-130.
[2] B. KOSTANT, Quantization and Unitary Representations, *Lecture Notes in Mathematics*, vol. **170**, 1970, p. 87-208.
[3] J. M. SOURIAU, *Structure des Systèmes Dynamiques*, Dunod, Paris, 1970.
[4] M. PUTA, On the Reduced Phase Space of a Cotangent Bundle. *Lett. Math. Phys.*, t. **8**, 1984, p. 189-194.
[5] V. GUILLEMIN, S. STERNBERG, Geometric Quantization and Multiplicities of Group Representations. *Invent. Math.*, t. **67**, 1982, p. 515-538.
[6] M. GOTAY, Constraints, Reduction and Quantization. *J. Math. Phys.*, t. **27**, 1986, p. 2051-2066.
[7] M. KUMMER, On the Construction of the Reduced Phase Space of a Hamiltonian System with Symmetry. *Indiana Univ. Math. J.*, t. **30**, 1981, p. 281-291.
[8] R. ABRAHAM, J. MARSDEN, *Foundation of Mechanics*. Benjamin, Mass. 1978.
[9] G. MARMO, E. SALETAN, A. SIMONI, B. VITALE, Dynamical Systems. A Différential Geometric Approach to Symmetry and Reduction, Wiley, Chichester, 1985.
[10] P. LIBERMANN, C.-M. MARLE, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.
[11] T. IWAI, Y. UWANO, The Four-Dimensional Conformal Kepler Problem Reduces to the Three-Dimensional Kepler Problem with a Centrifugal Potential and Dirac's Monopole Field. Classical Theory. *J. Math. Phys.*, t. **27**, 1986, p. 1523-1529.
[12] I. MLADENOV, V. TSANOV, Geometric Quantisation of the MIC-Kepler Problem. *J. Phys. A. Math. Gen.*, t. **20**, 1987, p. 5865-5871.
[13] M. KIBLER, T. NEGADI, The Use of Nonbijective Canonical Transformations in Chemical Physics. *Croatica Chem. Acta*, t. **57**, 1984, p. 1509-1523.

[*14*] L. Davtyan, L. Mardoyan, G. Pogosyan, A. Sissakian, V. Ter-Antonyan, Generalized KS Transformation: From Five-Dimensional Hydrogen Atom to Eight-Dimensional Isotrope Oscillator. *J. Phys. A. Math. Gen.*, t. **20**, 1987, p. 6121-6125.
[*15*] L. Bates, Ph. D. Thesis, University of Calgary, 1988.
[*16*] B. Cordani, L. Feher, P. Horvathy, Monopole Scattering Spectrum from Geometric Quantisation. *J. Phys. A. Math. Gen.*, t. **21**, 1988, p. 2835-2837.

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