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Tensor fields defining a tangent bundle structure

Tensor fields defining
a tangent bundle structure

by

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ABSTRACT. — As a first step in the program of an algebraic formulation of Lagrangian Dynamics, a theorem, giving sufficient conditions in order for a manifold to carry a tangent bundle structure, is proven.

As examples, the electron in a monopole field and the free relativistic particle are considered.

RÉSUMÉ. — Comme point de départ, pour une formulation algébrique de la dynamique lagrangienne, on démontre un théorème donnant des conditions suffisantes afin qu'une variété puisse être considérée comme un fibré tangent.

On analyse, à titre d'exemple, l'électron dans un champ de monopole et la particule relativiste libre.

1. INTRODUCTION

The usual procedure to define a Lagrangian dynamics, consists in choosing, first, a configuration manifold and, then, in assigning a Lagrangian function on its tangent bundle. However, in several physically relevant instances, the situation is more involved.

Typical instances of this are constrained dynamics which play a pre-

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eminent role in the construction of theoretical frameworks for funda-
mental interactions. A consistent constrained dynamics corresponds, in
fact, to a redundant set of degrees of freedom and, both at merely classical
level for a well posed formulation of the initial data problem, and in order
to avoid meaningless divergent contributions to path-integral when inte-
grating on the orbits of gauge groups [1], this redundancy has somehow
to be tamed.

What should, in principle, be performed is a reduction procedure leading
to a reformulation without fictitious degrees of freedom.

The traditional setting in which this problem was first analysed, was
the hamiltonian one [2], although, with the growing relevance of the
Lagrangian formulation due to the general acceptance of path-integral
quantization, in its original Lagrangian form, as a working tool in gauge
field theories, this same question was also studied for Lagrangian dyna-
mics [3].

A related problem in this context is to realize the reduced dynamics as
a second order and, possibly, a Lagrangian one, the first natural step in
this direction obviously being the characterization of the reduced phase
space as a tangent bundle.

The aim of this paper is to state and prove a theorem (sect. 2) giving
sufficient conditions in order for a manifold to carry a tangent bundle
structure, which in the authors opinion, apart for its independent mathe-
matical relevance, can be of help in the aforementioned context. In section 3,
the electron in a monopole field and the free relativistic particle are consi-
dered.

As to terminology and notations, see ref. [4] for the general geometrical
setting and ref. [5] for tangent bundle geometry and second order dynamics.
By now there are also several papers dealing with constraints in the frame-
work of differential geometry [6].

2. THE TANGENT BUNDLE
AND ITS NATURAL TENSOR FIELDS [7]

It is worth to start with a brief review of some basic facts about tangent
bundle geometry.

A tangent bundle $\pi : TQ \rightarrow Q,$

being a vector bundle, is endowed with the dilation vector field

$$D \in \chi(TQ),$$

which is fiberwise defined: on a generic linear space W, the dilation field
is the generator of the flow

$$F : (t, w) \in R \times W \mapsto w \exp t \in W. \qquad (1)$$

A peculiar feature of tangent bundles is a $(1, 1)$ tensor field

$$v : x \in TQ \longmapsto v_x \in T_1^1 TQ \qquad (\pi_1^1 v = \mathrm{id}_{TQ}) \tag{2}$$

which, in terms of the corresponding endomorphisms:

$$v^\wedge x \in L(T_x Q ; T_x Q),$$

is naturally defined as follows.

If $w \in T_x TQ$, then $v^\wedge xw$ is the tangent vector in x to the curve

$$c : t \in R \longmapsto x + tT\pi w \tag{3}$$

where $T\pi : TTQ \to TQ$ is the tangent map of π.

If

$$(q^1, \ldots, q^k) : U \to R^k \tag{4}$$

is a local chart, and

$$(q^1, \ldots, q^k, u^1, \ldots, u^k) : TU \to R^{2k} \tag{5}$$

(where, with a harmless abuse of notation, the q's and their pull back are identified) is the corresponding tangent bundle chart, then the local expression for v and D is

$$v = dq^i \otimes \frac{\delta}{\delta u^i}, \qquad D = u^i \frac{\delta}{\delta u^i}, \tag{6}$$

where summation on i from 1 to k is implied.

It can be easily shown that v and D satisfy the following properties

I) Ker $v^\wedge P = $ Im $v^\wedge P$ for every $P \in TQ$

II) Im $v^\wedge = \chi_{\mathrm{ver}}(TQ)$

III) $N_v = 0$

IV) $D \in \chi_{\mathrm{ver}}(TQ)$

V) $L_D v = -v.$

where $v^\wedge$ denotes the endomorphism of $\chi(TQ)$ corresponding to v, $\chi_{\mathrm{ver}}(TQ)$ the vertical subalgebra of $\chi(TQ)$ and N_v the Nijenhuis tensor of v

$$N_v(\theta, X, Y) \equiv \langle \theta, [L_v{}^\wedge Xv)^\wedge - v^\wedge (LXv)^\wedge]Y \rangle. \tag{7}$$

Because of property II), v is usually called the vertical endomorphism of TQ.

In the following it will be shown how the existence of a tensor field v, and a complete vector field D on a manifold M, which fulfil properties I), with M substituted for TQ, III), IV) with the range of the endomorphism replacing the vertical algebra, and V), allows to state that the considered manifold can be endowed with a unique tangent bundle structure whose dilation field and vertical endomorphism respectively are v and D. This can be proved if a further requirement on D is satisfied, which essentially grants that all integral curves of D lie on unstable manifolds of points belonging to the zero section of the implied tangent bundle).

We shall prove the following

THEOREM. — Let M be a C^2 manifold, v and D respectively a $(1, 1)$ tensor field and a complete vector field on M which fulfil the following hypotheses

1) Ker $v^\wedge P = $ Im $v^\wedge P$ for every $P \in M$

2) $N_v = 0$

3) $D \in$ Im $v^\wedge$

4) $L_D v = -v$

5) $\lim\limits_{t \to -\infty} e^{tD} P$ exists for every $P \in M$,

where

$$e^{\cdot D} : (t, P) \in R \times M \mapsto e^{tD} P \in M \qquad (8)$$

denotes the flow of D.

Under such hypotheses M has a unique tangent bundle structure (i. e. a sub-atlas of its maximal atlas being a maximal tangent bundle atlas) whose dilation operator is D and whose vertical endomorphism is v.

For the dynamical applications alluded to before, involving degenerate lagrangians and reduction procedures, the manifold M is to be identified with the reduced phase space while v and D could be the projections onto M (if they are well defined) of the corresponding canonical objects on the tangent bundle one started with. A couple of examples will be given in the following to clarify this point.

The theorem will be now proved in two steps: first hypotheses 1) through 4) will be used to show that the manifold is locally endowed with a uniquely defined tangent bundle structure; then chart domains in the neighbourhood of singular points of D will be extended all along integral curves of D itself (the flow $e^{\cdot D}$, when applied to a neighbourhood of a singular point generates the whole corresponding fibre). Hypothesis 5) will then grant that the so built charts give the looked for atlas. The proof will be unusually detailed in such a way that the construction procedure for the tangent bundle structure can be easily applied by any interested reader to his or her own needs.

Proof. — Since the rank of every finite-dimensional linear operator is both equal to the dimension of the range and the codimension of the kernel, condition 1) implies

$$\dim (M) = 2 \dim (\text{Ker } v^\wedge P) = 2 \dim (\text{Im } v^\wedge P) \equiv 2k \quad \text{for every } P \in M. \qquad (9)$$

K vector fields $e_1, \ldots, e_k$ can, then, be chosen in a suitable open neighbourhood U of an arbitrary point $P \in M$ so to give a basis of Im $v^\wedge Q$ for all $Q \in U$, by which v is locally written

$$v = \Sigma_{i=1}^k e_i \otimes \theta^i \qquad (10)$$

where $\theta^1, \ldots, \theta^k$ are local 1-forms which, by 1), vanish on the e's, i.e.

$$\langle \theta^i, e_j \rangle = 0; \qquad i, j = 1, \ldots, k \tag{11}$$

and are linearly independent,

$$\theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^k = 0, \tag{12}$$

all over U, since otherwise Im $v^{\wedge}$ would be somewhere smaller than the linear span of $e_1, \ldots, e_k$. Incidentally, $(\theta^1, \ldots, \theta^k)$ is a basis of the image and of the kernel of the transposed endomorphism $v^{\wedge T}$.

Now property 2) implies, which would be very easy to show, that Im $v^{\wedge}$ is an involutive distribution,

$$[\operatorname{Im} v^{\wedge}, \operatorname{Im} v^{\wedge}] \subset \operatorname{Im} v^{\wedge}, \tag{13}$$

and this in turn is equivalent to

$$d\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^k = 0. \tag{14}$$

Relation (14) implies (Frobenius theorem) that k real functions $x^1, \ldots, x^k$ on an open neighbourhood $V \subset U$ of P can be found such that in V

$$\theta^i = \Sigma_{j=1}^k a_j^i dx^j; \qquad i = 1, \ldots, k \tag{15}$$

where $\| a_j^i \|_{i,j=1,\ldots,k} : V \to \operatorname{GL}(n, \mathbb{R})$. If

$$f_j \equiv \Sigma_{i=1}^k e_i a_j^i \tag{16}$$

then in V

$$v = \Sigma_{j=1}^k f_j \otimes dx_j, \tag{17}$$

where, as a consequence of eqs. (11) and (13),

$$\langle dx^i, f_j \rangle = 0 = \langle dx^i, [f_j, f_h] \rangle; \qquad i, j, h = 1, \ldots, k. \tag{18}$$

On the other hand, if $g_1, \ldots, g_k$ are k vector fields, in an appropriate open neighbourhood $W \subset V$ of P, such that $(f_1, \ldots, f_k, g_1, \ldots, g_k)$ is a local frame in W and

$$\langle dx^i, g_j \rangle = \delta_j^i, \tag{19}$$

then, if θ is a generic 1-form in W,

$$\begin{aligned}
N(\theta, g_i, g_h) &= \langle \theta, [(L_{v^{\wedge}g_i} v)^{\wedge} - v^{\wedge}(L_{g_i} v)^{\wedge}] g_h \rangle \\
&= \langle \theta, [(L_{f_i} v)^{\wedge} - v^{\wedge} \Sigma_{l=1}^k ([g_i, f_l] \otimes dx^l + f_l \otimes d\langle dx^l, g_i \rangle)^{\wedge}] g_h \rangle \\
&= \langle \theta, \Sigma_{l=1}^k ([f_i, f_l] dx^l + f_l d\langle dx^l, f_i \rangle - \Sigma_{j=1}^k \langle dx^j, [g_i, f_l] \rangle f_j dx^l) g_h \rangle \\
&= \langle \theta, [f_i, f_h] \rangle,
\end{aligned} \tag{20}$$

where eqs. (18) and (19) were repeatedly used, so that

$$N_v = 0 \Rightarrow [f_j, f_h] = 0 \quad \text{for every } j, h = 1, \ldots, k. \tag{21}$$

By eq. (21) k real functions $x^{k+1}, \ldots, x^{2k} : U' \subset W \to \mathbb{R}$ on a suitable open neighbourhood U' of P exist such that

$$(x^1, \ldots, x^{2k}) : U' \to \mathbb{R}^{2k}$$

is a local coordinate system and in U'

$$f_j = S_j(x^1, \ldots, x^k)\delta/\delta x^{j+k}.$$ (22)

If a coordinate transformation

$$y^j = x^j$$
$$y^{j+k} = F^j(x^1, \ldots, x^k, x^{j+k}); \quad j = 1, \ldots, k$$ (23)

is now considered, the local expression of v becomes

$$v = \Sigma_{j=1}^k S_j \frac{\delta F^j}{\delta x^{j+k}} \frac{\delta}{\delta y^{j+k}} \otimes dy^j \cdot$$ (24)

where the S^j's nowhere vanish, v having constant rank, so that, if $F^j = x^{j+k}/S_i$, the canonical form

$$v = \Sigma_{j=1}^k \delta/\delta y^{j+k} \otimes dy^j$$ (25)

is obtained.

From hypothesis 3) the dilation operator can locally be written in this coordinates, as

$$D = \Sigma_{i=1}^k D^i(y^1, \ldots, y^{2k})\delta/\delta y^{i+k},$$ (26)

hypothesis 4) implying

$$\delta D^i/\delta y^{j+k} = \delta_j^i; \quad i, j = 1, \ldots, k,$$ (27)

i. e. $$D^j = y^{j+k} + d^j(y^1, \ldots, y^k); \quad j = 1, \ldots, k.$$ (28)

If the coordinate transformation in U'

$$u^j = y^{j+k} + d^j(y^1, \ldots, y^k)$$
$$q^j = y^j; \quad j = 1, \ldots, k$$ (29)

is performed, v and D simultaneously assume their canonical form:

$$v = \Sigma_{j=1}^k \frac{\delta}{\delta u^j} \times dq^j; \quad D = \Sigma_{j=1}^k u^j \frac{\delta}{\delta u^j}.$$ (30)

It is easily checked that coordinate transformations, sending canonical coordinates into canonical ones, that is preserving the canonical form both of v and D, just are tangent bundle coordinate transformations, that is the ones of the form:

$$\tilde{q}^j = t^j(q^1, \ldots, q^k)$$
$$\tilde{u}^j = \Sigma_{h=1}^k u^h \delta t^j/\delta q^h.$$ (31)

By the above construction, a globally and uniquely defined tangent bundle atlas on M is then exhibited, apart for checking that the chart domains can be extended in such a way to have the u^j's ranging over the whole real line.

The completeness assumption for D and hyp. 5) are now going to be

used to exhibit chart domains of the form $\pi^{-1}(\mathrm{I})$, I being an open set in the « base manifold » of singular points of D and

$$\pi : \mathrm{P} \in \mathrm{M} \mapsto \lim_{t \to -\infty} e^{t\mathrm{D}} \mathrm{P} \tag{32}$$

the canonical projection of the deduced tangent bundle structure.

In order to get a true tangent bundle atlas, consider now a canonical chart constructed according to the above procedure, whose domain $\mathrm{U_P}$ is an open neighbourhood of the generic singular point P of D. A smaller neighbourhood $\mathrm{V_P} \subset \mathrm{U_P}$ can obviously be considered such that the image of $\mathrm{V_P}$, by the considered canonical chart is the cartesian product

$$(q^1, \ldots, q^k)(\mathrm{V_P}) \times \mathrm{B}_r \tag{33}$$

where $(q^1, \ldots, q^k)(\mathrm{V_P})$ denotes the image of $\mathrm{V_P}$ by

$$(q^1, \ldots, q^k) : \mathrm{Q} \in \mathrm{U_P} \mapsto (q^1(\mathrm{Q}), \ldots, q^k(\mathrm{Q})) \in \mathrm{R}^k$$

and is assumed to be star-shaped, while B_r denotes the zero centred ball with a sufficiently small radius r.

It will be now shown how the local coordinate system

$$(q^1, \ldots, q^k, u^1, \ldots, u^k) : \mathrm{V_P} \mapsto \mathrm{R}^{2k}, \tag{34}$$

where, once again, explicit notational reference to the restriction operation is omitted, can be extended to a proper tangent bundle chart, i. e. one in which the ball B_r in (33) is replaced by R^k.

To this end, if the local 1-forms

$$dq^1, \ldots, dq^k, du^1, \ldots, du^k \tag{35}$$

are considered, then by the coordinate expression of D in (30):

$$\mathrm{L_D} dq^i = 0, \quad \mathrm{L_D} dui = dui; \quad i = 1, \ldots, k. \tag{36}$$

By this a unique extension of 1-forms (35) from $\mathrm{V_P}$ to

$$\mathrm{W_P} \equiv \{ m \in \mathrm{M} \mid \lim e^{t\mathrm{D}} m \in \mathrm{V_P} \} . \tag{37}$$

is defined if equations

$$\mathrm{L_D} a^i = 0 ; \quad a^i \mid_{\mathrm{V_P}} = dq^i$$
$$\mathrm{L_D} b^i = b^i ; \quad b^i \mid_{\mathrm{V_P}} = du^i \qquad i = 1, \ldots, k \tag{38}$$

are required to be satisfied in $\mathrm{W_P}$. On the other hand eqs. (38) by taking exterior derivative give

$$\mathrm{L_D} da^i = 0_{\_} ; \quad da^i \mid_{\mathrm{V_P}} = 0$$
$$\mathrm{L_D} db^i = db^i ; \quad db^i \mid_{\mathrm{V_P}} = 0 \qquad i = 1, \ldots, k \tag{39}$$

and, by uniqueness of solution of these equations, the extended 1-forms

are closed. Exactness follows by Poincare lemma since W_P is homotopic to V_P and V_P is star-shaped.

The $2k$ 1-forms solving equations (38) are then the exterior derivatives of $2k$ functions which can be chosen to be extensions from V_P to W_P of the coordinate functions in (34). The so defined function

$$(q^1, \ldots, q^k, u^1, \ldots, u^k) : W_P \rightarrow R^{2k} \qquad (40)$$

is C^2 by construction and is invertible since different u coordinates on the same integral curve of D and different x coordinates respectively imply being different points on the same curve and being on different integral curves. On the other hand eqs. (27) imply that

$$L_D dq^1 \wedge \ldots \wedge dq^k \wedge du^1 \wedge \ldots \wedge du^k$$
$$= K dq^1 \wedge \ldots \wedge dq^k \wedge du^1 \wedge \ldots \wedge du^k \quad (41)$$

and then that $dq^1 \wedge \ldots \wedge dq^k \wedge du^1 \wedge \ldots \wedge du^k$ nowhere vanish, by which the inverse of (40) is C^2.

The function (40) is, then, a chart whose range is the cartesian product

$$(q^1, \ldots, q^k)(V_P) \times R^k \qquad (42)$$

as implied by eqs. (38) and completeness of D. If the vector fields

$$\delta/\delta q^1, \ldots, \delta/\delta q^k, u^1 \delta/\delta u^1, \ldots, u^k \delta/\delta u^k \qquad (43)$$

are considered, they give a frame on W_P apart for singular points of D, whose elements are Lie transported along integral curves of D, as can easily be checked.

The expression of D in this frame is then a linear combination with constant coefficients, which implies that D has canonical form as in (30) all over W_P. As to v, the same conclusion can now be derived by condition 4) and the assumed validity of eqs. (30) in V_P.

Since the above procedure leads to the construction of a proper tangent bundle chart in the neighbourhood of every singular point of D, in which both v and D assume their canonical form, then by condition 5) the proof is therefore accomplished.

3. EXAMPLES

A couple of examples will now be used to illustrate the given theorem in the realm of second order dynamics on tangent bundles.

For what follows it is worth to remark that, while projectability for a vector field X, with respect to an involutive distribution K generated by $X_1, \ldots, X_H$, just means

$$(L_{X_j} X)_P \in K_P; \qquad P \in M, \quad j = 1, \ldots, H,$$

for a (1, 1) tensor field T it will be here checked by means of the corresponding endomorphisms $T^\wedge P$, in terms of which projectability conditions read

$$T^\wedge PK_P \in K_P \quad \text{and} \quad \text{Im}\,(L_{X_j}T)^\wedge P \subset K^\wedge P; \qquad P \in M, \quad j = 1, \ldots, H.$$

A. Electron in a monopole field [8].

It is well known that equations of motion for an electrically charged particle in a magnetic monopole field do not admit a global Lagrangian description if the configuration manifold Q is identified with the tridimensional euclidean space deprived of the singular point of the magnetic field. If $Q = R^3 - \{0\}$, equations of motion read

$$\begin{aligned} dx^i/dt &= u^i \\ du^i/dt &= n\varepsilon^i_{jk}x^3u^k/r^3 \end{aligned} \qquad (44)$$

where (x^1, x^2, x^3) are orthonormal coordinates an Q, $(x^1, x^2, x^3, u^1, u^2, u^3)$ are the corresponding tangent bundle coordinates an TQ, n is the product of electric and magnetic charge divided by the mass of the electrically charged particle. The obstruction to a global Lagrangian description comes from the field strength 2-form just being closed but not exact, which forces to define the gauge 1-form on a simply connected open submanifold of Q (Dirac string). A way out of this problem is to enlarge the configuration manifold to $SU(2) \times R$ and to define an $T(SU(2) \times R)$ a globally Lagrangian dynamics which projects to the original one.

To be specific let

$$\pi_H : SU(2) \to S^2$$

denote the canonical projection of the U(1) Hopf bundle based on S^2. Then the implied projection

$$T(SU(2) \times R) \to T(S^2 \times R)$$

is the tangent map of

$$\pi_H \times \text{id}_R : SU(2) \times R \to S^2 \times R, \qquad (45)$$

the dynamics on $T(SU(2) \times R)$ having a global Lagrangian description with Lagrangian function given by

$$L = (\dot{r}^2 + r^2\dot{y}^j\dot{y}_j)/2 + ni\,\text{tr}\,\sigma^3s^{-1}\dot{s}. \qquad (46)$$

Here r denotes the R coordinate, s the generic element of SU(2), and y^1, y^2, y^3 are orthonormal coordinates on S^2 ($y^jy_j = 1$), σ^3 being the usual Pauli matrix. (For simplicity SU(2) is here identified with its usual matrix representation.)

An alternative more natural approach to obtain the reduced dynamics

from the one an T(SU(2) × R) would be to project with respect to the invo-
lutive distribution of the kernel of the Lagrangian 2-form

$$\Omega_L \equiv d\theta_L \equiv dv^{\wedge T} dL; \tag{47}$$

here

$$v^{\wedge T} : X^*(M) \to X^*(M) \tag{48}$$

denotes the endomorphism of the 1-form module defined by

$$\langle v^{\wedge T}\theta, X \rangle \equiv v(\theta, X) = \langle \theta, v^{\wedge} X \rangle; \qquad \theta \in X^*(M), \quad X \in X(M). \tag{49}$$

It can be shown that this kernel is the linear span of the tangent lift X_3^T
of the generator X^3 of U(1) in the Hopf bundle, i. e. of the flow

$$(t, s) \in R \times SU(2) \to s \exp it\sigma^3 \in SU(2) \tag{50}$$

and of the vertical field $V_3 \equiv v^{\wedge} X_3^T$. (If e^{-X} is the flow of $X \in X(Q)$ than the
tangent lift $X^T \in X(TQ)$ of X is the generator of the flow $\exp \cdot X^T$ where

$$\exp tX^T : TQ \to TQ \tag{51}$$

is the tangent map of

$$\exp tX : Q \to Q. \tag{52}$$

If in local coordinates

$$X = X^j(q)\delta/\delta q^j, \tag{53}$$

then in the corresponding tangent bundle coordinates

$$X^T = X^j(q)\delta/\delta q^j + \delta X^j/\delta q^i u^i \delta/\delta u^j. \tag{54}$$

In fact, since tangent lifts leave v invariant and X_3^T is a symmetry field
for L by inspection, then

$$iX_3^T\Omega_L = iX_3^T d\theta_L = -diX_3^T\theta_L. \tag{55}$$

On the other hand

$$\theta_L = \dot{r}dr + r^2\dot{y}^j dy_j + ni \operatorname{tr} \sigma^3 s^{-1} ds \tag{56}$$

and then

$$iX_3^T\theta_L = ni \operatorname{tr} \sigma^3 iX_3^T s^{-1} ds. \tag{57}$$

but $i_{X_3^T} s^{-1} ds$ is a constant matrix (in intrinsic terms it is a constant element
of the Lie algebra $su(2)$) since $s^{-1}ds$ is the Maurer-Cartan 1-form and X_3^T
is left-invariant, it being a generator of the right action of SU(2) on itself.
This, by eq. (55), proves that X_3^T belongs to the kernel of Ω_L; this is also
true for V_3, since it can be shown in general that

$$\Omega_L(v^{\wedge} X, Y) = \Omega_L(v^{\wedge} Y, X); \quad X, Y \in X(TQ). \tag{}$$

In the present case the two generators of the kernel of the Lagrangian
2-form X_3^T, V_3 preserve the vertical endomorphism, i. e. $L_X v = 0$ for X repre-
senting both fields, by which v is projectable. Moreover the dilation vector
field is projectable because the tangent lift of any vector field on Q commutes
with D and property V) of v, D implies that $L_D v^{\wedge} X^T = -v^{\wedge} X^T$.

Thus one could conclude *a priori* that, if the quotient manifold, $w.r.$ to the considered distribution, does exist with smooth projection, the projections of v and D provide such a manifold with a tangent bundle structure, the theorem hypotheses being then all trivially satisfied.

In the present example one can explicitly show that the quotient manifold does exist. As a matter of fact the kernel of the Lagrangian 2-form is spanned by the complete vector fields X_3^T and V_3; the two of them span the Lie algebra of TU(1) [9] and the integral manifolds of the distribution are the orbits of the action of TU(1) on TSU(2). Then, by the functoriality of T (the tangent functor) it follows that TSU(2)/TU(1) exists (it being diffeomorphic with T(SU(2)/U(1)).

It is worth to remark by the way that the Lagrangian function is not projectable, because, although it is invariant under X_3^T, it is not invariant under V_3.

B. Free relativistic particle.

Now the configuration space is taken to be R^4 endowed with Minkowski metric; the considered degenerate Lagrangian is

$$L = - (\dot{u}_\mu \dot{u}^\mu)^{1/2}$$

and the kernel of the Lagrangian 2-form is spanned by

$$\Gamma \equiv u^\mu \delta/\delta x^\mu, \qquad D = u^\mu \delta/\delta u^\mu,$$

the x's and u's once again denoting tangent bundle coordinates. In this case the dilation field D obviously projects onto the zero vector field while it can be easily shown that the vertical endomorphism v is not projectable. (This easily follows by considering that Im $(L_\Gamma v)^\wedge$ and Im $(L_D v)^\wedge$ = Im $v^\wedge$ are not included in the involutive algebra generated by Γ and v),

Thus in this case the quotient manifold will not inherit a tangent bundle structure.

Conclusions and final remarks.

It should be stressed that the present theorem, when applied to the reduction problem, does not allow to exclude that the quotient manifold carries suitable tangent bundle structure. It just implies, in some cases as in application B), that the possible tangent bundle structure is not inherited by projection. Just to be specific, consider on $S^1 \times R$, a possible reduced phase manifold, the vector field $\delta/\delta r$ and the vertical endomorphism $e^{-r} d\theta \times \delta/\delta r$ where r, θ denote polar coordinates.

Then, although « algebraic conditions » 1) through 4) are satisfied, these tensor fields do not fulfil global topological assumptions, i. e. completeness and condition 5), and our theorem does not allow for a tangent bundle structure associated with these tensor fields. On the other hand, the considered

manifold can obviously be endowed with a tangent structure, since it is diffeomorphic to TS^1. Clearly tensor fields associated with this tangent bundle structure do not coincide with previous one.

Thus, from our theorem, it is clear that a tangent bundle structure is a joint property of our starting manifold and the two tensor fields we are considered

We shall mention, at this point, that a theorem which is the analogue of the present one in the realm of cotangent bundles was proved in ref. [10] (see also ref. [11]).

In it, the existence of a 1-form, with symplectic exterior derivative, is postulated; by it the canonical 2-form, which plays a role similar to the vertical endomorphism, and the dilation vector field are constructed.

It is easily seen that, in close analogy to the assumptions of the present theorem, the existence of the dilation field and of the 2-form, satisfying suitable requirements, could be equivalently assumed. A remarkable difference in the genuinely topological assumptions is, on the contrary, of a substantial nature.

To construct a (unique) cotangent bundle structure, in fact, not only hypothesis 5) like here has to be made, but also it is necessary to assume the set of singular points of the dilation field to be a submanifold of half the dimension of the manifold, which could be deduced here. This in a sense is not surprising since the vertical endomorphism carries with itself an exhaustive information on the bundle structure, it defining vertical subspaces, while the symplectic form gives a less stringent information.

It is also worth to remark that generalized versions of Lagrangian dynamics can be formulated on manifolds which are endowed with a so called quasi-tangent structure [12]. The present theorem gives sufficient conditions for such a q.-t. structure to reduce to an ordinary tangent one.

As for the possibility to prove our theorem under weaker assumptions, it should be remarked that counterexamples can easily be given if one of them is removed. It is under investigation if hypothesis 1) can be weakened by requiring that Ker $v^\wedge$ = Im $v^\wedge$ where $v^\wedge$ is globally considered as a vector field module endomorphism.

This point is connected with a purely algebraic version of Lagrangian dynamics which is presently being developed [paper in preparation]. It is formulated in the realm of derivation algebras on rings endowed with the algebraic version of the vertical endomorphism and the dilation field. If in particular the algebra is the one of derivations on the ring of C^∞ real functions on a C^∞ manifold, then the presented theorem allows to reconstruct a Lagrangian dynamics in the usual sense.

A different characterization of tangent bundle structure on a manifold M has been provided by Crampin and Thompson [13]. The first part of their paper, i. e. how to construct a local tangent bundle structure, is close to our presentation.

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