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ABSTRACT. — A general definition of an implementation of a Jordan isomorphism between two von Neumann algebras in terms of two partial isometries is proposed. It is shown how different implementations are connected with each other. The concept of a standard implementation is introduced and proven to be unique. The existence of the standard implementation is constructively demonstrated by the application of the general modular formalism. The implementation of groups of Jordan isomorphisms is investigated and the composition of the implementing operators are derived as well as some continuity properties.

RÉSUMÉ. — Nous proposons une définition générale pour l'implémentation d'un isomorphisme de Jordan entre deux algèbres de Von Neumann utilisant deux isométries partielles. Nous montrons comment différentes implémentations sont reliées entre elles.
Nous introduisons un concept d'implémentation standard et nous montrons qu'elle est unique. Nous montrons constructivement l'existence de l'implémentation standard en appliquant le formalisme modulaire général. Nous étudions l'implémentation de groupes d'automorphismes de Jordan, en particulier la composition des opérateurs d'implémentation et des propriétés de continuité.

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I. INTRODUCTION

The (spatial) implementation of Jordan isomorphism between arbitrary (and not necessarily σ-finite) von Neumann algebras is of both mathematical and physical interest. Before describing the physical motivation let us first have a look on the mathematical background in the literature and on the results in this paper.

The problem of implementing a *-isomorphisms between von Neumann algebras by means of a unitary operator has been treated by Dixmier [Dix] for special cases and has been solved generally in [Ha]. In the latter work the technique of standard representations has been generalized to non-σ-finite von Neumann algebras. This method has also been employed in [DV] to implement an arbitrary completely positive map between von Neumann algebras in terms of a partial isometry. This rather general investigation does not, however, cover Jordan isomorphisms. In [Co] and [BR] Jordan isomorphisms between σ-finite von Neumann algebras in (cyclic) standard representations have been unitarily implemented, but only in a weak form: the implementation formula is valid only inside scalar products with vectors of the self-dual cone. (The operator identity in [BR], p. 222 does not conform to our Definition 1.1 below, but could be adapted to it.) Another route has been followed in [Ro1], where von Neumann algebras were considered which have a cyclic (but not necessarily separating) vector defining a state invariant under the Jordan automorphism α. The implementation of α is effectuated by means of two partial isometries. The one corresponding to the *-homomorphic part of α is (complex) linear, and the other is anti-linear and is connected with the anti-homomorphic part of α. (Observe that the definition of the Hermitian adjoint W* of an anti-linear operator W differs slightly from that of a linear operator, whereas the defining relation for partial isometries is identical to that in the linear case.)

Since quite generally a Jordan isomorphism decomposes into a homomorphic and an anti-homomorphic part (cf. Appendix A), one expects the implementation in terms of a pair of operators to be the appropriate general form. Taking into account that the implementing operators fix the homomorphic and anti-homomorphic parts and thus specify the decomposing projection (cf. Appendix A), one is led to the following definition:

1.1. DEFINITION. — Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and α a Jordan isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$ with decomposing central projection $E_2 \in \mathcal{Z}(\mathcal{M}_2)$. The pair of operators (V, W) is said to implement (α, E_2) if

i) V and W are partial isometries acting on $\mathcal{H}_1$ with range in $\mathcal{H}_2$, V linear, W antilinear;

ii)
$$VV^* = E_2, \qquad V^*V = \alpha^{-1}(E_2),$$
$$WW^* = E_2^\perp, \qquad W^*W = \alpha^{-1}(E_2^\perp) ;$$

iii)
$$\alpha(A) = VAV^* + WA^*W^* \qquad \text{for all} \quad A \in \mathcal{M}.$$

(Here $\mathcal{Z}(\mathcal{M})$ denotes the centre of $\mathcal{M}$; $E^\perp := 1 - E$; we shall use the notation $E_1 := \alpha^{-1}(E_2)$.)

This definition is the starting point for the present investigation. In Section II we shall show that the properties of the implementing operators, as listed in the definition, are already implied by a smaller set of assumptions.

We also show, that an implementation for (α, E_2) leads to an implementation for (α^{-1}, E_1) and to all other implementations for (α, E_2') with E_2' an arbitrary decomposing projection. Further properties are worked out as a preparation for the construction theorem in Section III. Especially, the notion of a standard implementation is introduced and shown to be unique.

In Section III the techniques of [Ro1] are adapted to construct standard implementations for σ-finite von Neumann algebras. These are extended to larger von Neumann algebras (the commutants of the centers) and pieced together to give a standard implementation also for the non-σ-finite case.

In Section IV we treat groups of Jordan automorphisms. We derive composition laws for the implementing operators and connect various continuity properties with each other.

Two Appendices provide some known results on Jordan isomorphisms and on the general modular formalism in standard representations.

In order to discuss the physical motivation for the implementation procedure let us recall that Jordan isomorphisms constitute a natural generalization of the concept of a Wigner symmetry in traditional quantum mechanics. More specifically, one can show that any bijective transformation of the formalism of C*-algebraic quantum theory ([HK], [E]), which acts in a dual manner in both the observable algebra and the state space (and thus leaves the expectation values invariant) gives rise to a Jordan isomorphism and an affine transformation, respectively [RR], [Ri]. A Jordan transformed observable algebra is, therefore, as suitable as the original one on principal grounds. In many models one has to restrict the physical relevant states to a subfolium of the whole state space and correspondingly to extend the original C*-algebra to the closure in the weak topology given by the folium. If the folium is non-separable in the norm topology, a typical situation for a global non-equilibrium theory, the corresponding von Neumann algebra is not σ-finite. In any case, the symmetry transformations in the Heisenberg picture are Jordan isomorphisms of von Neumann algebras. In traditional quantum theory there

are known solely *-automorphisms or anti-automorphisms (the latter for the time inversion T or the PTC-transformation), since 1 and 0 are the only central projections. In nonrelativistic many-body physics, where the thermodynamic limit is performed at least implicitly, there apparently arise examples for genuine Jordan isomorphisms.

Now, it is a general experience in physics, that in order to construct a symmetry transformation and to calculate with it, one needs an explicit implementation.

II. GENERAL RESULTS

By J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$ we denote the set of Jordan isomorphisms from $\mathcal{M}_1$ onto $\mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The following lemma is a simple consequence of Definition 1.1.

2.1. LEMMA. — Let $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$ with decomposing projection E_2. Then $E_1 = \alpha^{-1}(E_2) \in \mathcal{Z}(\mathcal{M}_1)$ is a decomposing projection for α^{-1}. If (V, W) implement (α, E_2), then (V^*, W^*) implement (α^{-1}, E_1).

The first statement is known, the proof of the second one is obvious because Def. 1.1 ii) implies $VW^* = 0$ and $V^*W = 0$.   □

For the explicit construction of an implementation it is useful to specify a subset of properties which imply all assumption of Definition 1.1.

2.2. PROPOSITION. — Consider $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$ and a pair (V, W) of bounded real-linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ fulfilling the following conditions:

i) There are projections $E_1, E_1' \in \mathcal{M}_1$ such that $VE_1 = V$, $WE_1' = W$, and $E_1 E_1' = 0$;

ii) $\alpha(A) = VAV^* + WA^*W^*$ for all $A \in \mathcal{M}_1$.

Then $E_2 = \alpha(E_1)$ is a central decomposing projection for α, and (V, W) implement (α, E_2).

(Given a bounded real-linear operator A, A* is defined with the help of the unique decomposition of A into a (complex-)linear and an antilinear part: $A = A_1 + A_2$, $A^* = A_1^* + A_2^*$, where $A_1 = \frac{1}{2}(A - iAi)$, $A_2 = \frac{1}{2}(A + iAi)$.)

Proof. — We have $V^* = E_1 V^*$, $WE_1 = 0$ and $E_1' V^* = 0$, consequently

$$\alpha(AE_1) = VAE_1 V^* + WE_1 A^* W^* = VAV^*, \qquad (1)$$

$$\alpha(AE_1') = VAE_1' V^* + WE_1' A^* W^* = WA^* W^*. \qquad (2)$$

With $A = \mathbb{1}$ we get

$$\alpha(E_1) = VV^* =: E_2, \qquad \alpha(E_1') = WW^* =: E_2'. \qquad (3)$$

E_2 and E_2' are projections because α maps projections on projections; furthermore, condition *ii*) with $A = \mathbb{1}$ implies

$$E_2 + E_2' = \mathbb{1}, \qquad (4)$$

and, consequently,

$$E_1 + E_1' = \mathbb{1}. \qquad (5)$$

Because of $\alpha(AE_1) = VAV^* = VE_1AV^* = \alpha(E_1A)$, $A \in \mathcal{M}_1$ commutes with E_1, hence $E_1 \in \mathscr{Z}(\mathcal{M}_1)$. This gives in turn $E_1' = E_1^\perp \in \mathscr{Z}(\mathcal{M}_1)$ and $E_2, E_2^\perp \in \mathscr{Z}(\mathcal{M}_2)$ since α maps $\mathscr{Z}(\mathcal{M}_1)$ onto $\mathscr{Z}(\mathcal{M}_2)$.

α is complex-linear, thus taking $A = \lambda \mathbb{1}$ in (1), $\lambda \in \mathbb{C}$, we find

$$\lambda VV^* = \lambda\alpha(E_1) = \alpha(\lambda E_1) = V\lambda V^*,$$

i. e.
$$\lambda V = V\lambda \qquad (6)$$

holds on $V^*\mathscr{H}$. The set $V^*\mathscr{H}$ is not *a priori* known to be a (complex-)linear space since V in only assumed to be real-linear; but using (6) on $V^*\mathscr{H}$ repeatedly one easily checks that it holds on the complex linear hull of $V^*\mathscr{H}$.

Now we want to demonstrate that (6) is valid on $(V^*\mathscr{H})^\perp$ too. To this end we show that $(V^*\mathscr{H})^\perp \subset \operatorname{Ker} V$: we use the decomposition $V = V_1 + V_2$ into a linear and an antilinear part; if $\eta \in (V^*\mathscr{H})^\perp$ it follows that $\big((V_1^* + V_2^*)\xi, \eta\big) = 0$ for all $\xi \in \mathscr{H}$; replacing ξ by $i\xi$ we conclude that $(V_1^*\xi, \eta) = (\xi, V\eta) = 0$ and $(V_2^*\xi, \eta) = (V_2\eta, \xi) = 0$ for arbitrary $\xi \in \mathscr{H}$, hence $(\xi, (V_1 + V_2)\eta) = 0$, whence $V\eta = 0$, $\eta \in \operatorname{Ker} V$. $\eta \in (V^*\mathscr{H})^\perp$ implies $\lambda\eta \in (V^*\mathscr{H})^\perp$, thus $V\lambda\eta = 0 = \lambda V\eta$.

Therefore, (6) holds on all of $\mathscr{H}$, V is a linear operator.

Now we can infer in the usual manner from (3) and the fact that E_2 is a projection that V is a partial isometry; and, because of $VE_1 = V$ and $VE_1^\perp = VE_1' = 0$, E_1 is its initial projection. Reasoning along the same line we find that W is an antilinear partial isometry, $W^*W = E_1^\perp$.

It remains to be shown that E_2 is a decomposing projection. In view of eq. (1) and $V^*V = E_1 \in \mathscr{Z}(\mathcal{M}_1)$ it follows for all $A, B \in \mathcal{M}_1$ that $\alpha(AB)E_2 = \alpha(ABE_1) = VAV^*VBV^* = \alpha(AE_1)\,\alpha(BE_1) = \alpha(A)\,\alpha(B)E_2$, and analogously,

$$\alpha(AB)E_2^\perp = \alpha(ABE_1^\perp) = WB^*W^*WA^*W^* = \alpha(BE_1^\perp)\,\alpha(AE_1^\perp) = \alpha(B)\,\alpha(A)E_2^\perp\,.$$

$\square$

2.3. PROPOSITION. — Let be given $\alpha \in \operatorname{J-Iso}(\mathcal{M}_1, \mathcal{M}_2)$ and an implementation (V, W) of (α, E_2). Another pair (V', W') of real linear operators from $\mathscr{H}_1$ into $\mathscr{H}_2$ also implements (α, E_2), if and only if there is a unitary operator $U' \in \mathcal{M}_1'$ such that

$$V' = VU'E_1 \qquad \text{and} \qquad W' = WU'E_1^\perp. \qquad (7)$$

where we have again denoted $E_1 = \alpha^{-1}(E_2)$.

Proof. — *i*) Let V', W' be of the form (7). Then their right supports are E_1

and $E_1^\perp$, respectively. Since U', E_1 and $E_1^\perp$ commute with all $A \in \mathcal{M}_1$ we obtain

$$V'AV'^* + W'A^*W'^* = VAV^* + WA^*W^* = \alpha(A).$$

Proposition 2.2 shows, that (V', W') implement (α, E_2).

ii) Let (V', W') implement (α, E_2). Then for $A \in \mathcal{M}_1$

$$\alpha(AE_1) = VAE_1V^* = V'AE_1V'^*.$$

Multiplication by V^* from the left and by V' from the right gives, with $U_1 := V^*V'$, $AU_1 = U_1A$ for all $A \in \mathcal{M}_1$; thus $U_1 \in \mathcal{M}_1'$. Furtheron, the equations

$$U_1U_1^* = V^*V'V'^*V = V^*E_2V = \alpha^{-1}(E_2) = E_1,$$
$$U_1^*U_1 = V'^*VV^*V' = V'^*E_2V' = \alpha^{-1}(E_2) = E_1,$$

where we used Lemma 2.1, show that U_1 is a unitary operator in $E_1\mathcal{H}_1$. Setting $U_2 := W^*W'$ we get from the equation

$$\alpha(AE_1^\perp) = WA^*E_1^\perp W^* = W'A^*E_1^\perp W'^*, \qquad A \in \mathcal{M}_1,$$

that $U_2 \in \mathcal{M}_1'$. Since U_2 is unitary in $E_1^\perp\mathcal{H}_1$,

$$U' := U_1 + U_2$$

is a unitary operator in $\mathcal{H}_1$ and an element of $\mathcal{M}_1'$. The equations (7) are directly verified. $\qquad \square$

Let us now vary the decomposing projection. From now on we assume that the von Neumann algebras $\mathcal{M}$, $\mathcal{M}_1$, ... under consideration are given in standard representation with conjugations J, J_1, J_2, ... and self-dual cones $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, ..., respectively (i. e. in hyper-standard representation in the notation of ([SZ]). We do not assume the existence of a cyclic and separating vector in general.

2.4. PROPOSITION. — Let E_2 and E_2' be arbitrary decomposing projections of $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$. Given an implementation (V, W) of (α, E_2), the following procedure yields an implementation (V', W') of (α, E_2'):

i) If $E_2' < E_2$, denote $F_2 := E_2 - E_2'$, $F_1 := \alpha^{-1}(F_2)$, $E_1' := \alpha^{-1}(E_2')$ and define

$$V' := VE_1', \qquad W' := W + J_2VF_1, \tag{8}$$

where J_2 is the conjugation of the standard von Neumann algebra $\mathcal{M}_2$.

ii) If $E_2' > E_2$, denote $F_2 := E_2^\perp - E_2'^\perp$, $F_1 := \alpha^{-1}(F_2)$, $E_1'^\perp := \alpha^{-1}(E_2'^\perp)$ and define

$$V' := V + J_2WF_1, \qquad W' := WE_1'^\perp.$$

iii) The general case is treated by a combination of ii) and i) with the maximal decomposing projection $E_{2\,max}$ as an intermediate.

Proof. — We show that V' and W' of eq. (8) fulfil the requirements of

Proposition 2.2; the reasoning for case *ii*) runs analogously; *iii*) is then obvious.

Now assume $E_2' < E_2$. E_2', F_2 and $E_2^\perp$ are pairwise orthogonal central projections with sum 1; the same holds true of E_1', F_1 and $E_1^\perp$. Evidently, E_1' is the right support of V', and that of W' is given by $E_1^\perp + F_1 = 1 - E_1'$; E_1' and $1 - E_1'$ fulfil condition *i*) of Proposition 2.2.
Now we calculate

$$V'AV'^* + W'A^*W'^* = VE_1'AE_1'V^* + WA^*W^* + J_2VF_1A^*F_1V^*J_2 +$$
$$+ \text{ two vanishing terms} \quad (9)$$

(because $W = WE_1^\perp$ and $F_1E_1^\perp = 0$).

The third term of the r. h. s. equals $J_2\alpha(A^*F_1)J_2 = J_2\alpha(A)^*F_2J_2$ (remember $F_1 = F_1E_1$ and eq. (1)). By definition of F_2, we have $F_2 \le E_{2c}$, where E_{2c} is the projection onto that subspace of $\mathscr{H}_2$ on which all commutators vanish, cf. eq. (A4) of Appendix A. Therefore, $\alpha(A)^*F_2 \in \mathscr{Z}(\mathscr{M}_2)$, see eq. (A5), and thus

$$J_2\alpha(A)^*F_2J_2 = F_2\alpha(A) = \alpha(AF_1)E_2 = VAF_1V^*.$$

Insertion into (9) then shows that the r. h. s. equals $VAV^* + WA^*W^* = \alpha(A)$.

$\square$

Propositions 2.3 and 2.4 show that any implementation of $\alpha \in$ J-Iso($\mathscr{M}_1$, $\mathscr{M}_2$) may be reached by starting from a special one. Thus, our main task will be to construct a distinguished implementation, where E_2 is maximal and V and W are fixed by supplementary conditions. To this end we make use of the modular formalism in standard representations.

Let J be the conjugation of $\mathscr{M}$; we shall use the following notation:

$$j(A) := JAJ, \qquad A \in \mathscr{B}(\mathscr{H}). \quad (10)$$

To $\alpha \in$ J-Iso$(\mathscr{M}_1, \mathscr{M}_2)$ we can associate a Jordan isomorphism $\tilde{\alpha}$ of $\mathscr{M}_1'$ onto $\mathscr{M}_2'$, and as a corollary of our main theorem, a common extension to a Jordan isomorphism of $\mathscr{Z}(\mathscr{M}_1)'$ onto $\mathscr{Z}(\mathscr{M}_2)'$. But first we need a preliminary extension lemma:

2.5. LEMMA. — Let $\alpha \in$ J-Iso$(\mathscr{M}_1, \mathscr{M}_2)$ with decomposing projection E_2.

i) Define $\tilde{\alpha}: \mathscr{M}_1' \to \mathscr{M}_2'$ by $\tilde{\alpha} := j_2 \circ \alpha \circ j_1$, then $\tilde{\alpha} \in$ J-Iso$(\mathscr{M}_1', \mathscr{M}_2')$ with decomposing projection E_2.

ii) If (V, W) implement (α, E_2), then (J_2VJ_1, J_2WJ_1) implement $(\tilde{\alpha}, E_2)$.

iii) α and $\tilde{\alpha}$ have a common extension $\bar{\alpha}$ to $\overline{\mathscr{M}}_1 := $ L. H. $\mathscr{M}_1\mathscr{M}_1'$; $\bar{\alpha}$ is a Jordan isomorphism from $\overline{\mathscr{M}}_1$ onto $\overline{\mathscr{M}}_2 := $ L. H. M. $\mathscr{M}_2\mathscr{M}_2'$ with decomposing projection E_2.

Proof. — *i*) Trivially, $\tilde{\alpha}(\mathscr{M}_1') = \mathscr{M}_2'$, and α and $\tilde{\alpha}$ coincide on $\mathscr{Z}(\mathscr{M}_1)$. Any $A' \in \mathscr{M}_1'$ can be written as $A' = j_1(A)$ with $A \in \mathscr{M}_1$; for A, $B \in \mathscr{M}_1$ we have $\tilde{\alpha}(j_1(A)j_1(B))E_2 = j_2(\alpha(AB))E_2 = j_2(\alpha(A) \alpha(B)E_2) = \tilde{\alpha}(j_1(A))\tilde{\alpha}(j_1(B))E_2$ since $j_2(E_2) = E_2$. Analogously, it follows that $\tilde{\alpha}(j_1(A)j_1(B))E_2^\perp = \tilde{\alpha}(j_1(B))\tilde{\alpha}(j_1(A))E_2^\perp$.

ii) J_2VJ_1 and J_2WJ_1 fulfil *i*) and *ii*) of Def. 1.1 because of $j_i(E_i) = E_i$, $i = 1, 2$. Furthermore, for $A' = j_1(A) \in \mathcal{M}'_1$, we have

$$\tilde{\alpha}(A') = j_2(\alpha(A)) = J_2(VAV^* + WA^*W^*)J_2 = (J_2VJ_1)j_1(A)(J_2VJ_1)^* +$$
$$+ (J_2WJ_1)j_1(A)^*(J_2WJ_1)^*.$$

iii) If $A \in \mathcal{M}_1$, $A' \in \mathcal{M}'_1$, we define $\bar{\alpha}(AA') = \alpha(A)\tilde{\alpha}(A')$. This is a bijection of $\mathcal{M}_1\mathcal{M}'_1$ onto $\mathcal{M}_2\mathcal{M}'_2$. Linear extension is possible because $\Sigma A_i A'_i = 0$ implies: there exist $Z_{ik} \in \mathcal{Z}(\mathcal{M}_1)$ such that $\displaystyle\sum_i Z_{ik}A_i = 0$ for all k and $\displaystyle\sum_k Z_{ik}A'_k = A'_i$ for all i (see for instance [SZ, E.3.11]), hence

$$\bar{\alpha}\left(\sum_i A_iA'_i\right) := \sum_i \alpha(A_i)\tilde{\alpha}(A'_i) = \sum_{i,k} \alpha(A_i)\tilde{\alpha}(Z_{ik}A'_k) =$$
$$= \sum_{i,k} \alpha(A_i)\alpha(Z_{ik})\tilde{\alpha}(A'_k) = \sum_k \alpha\left(\sum_i A_iZ_{ik}\right)\tilde{\alpha}(A'_k) = 0.$$

The Jordan property and the fact that E_2 is a decomposing projection are straightforward. □

Let ω_ζ denote the vector state on $\mathcal{M}_1$ resp. $\mathcal{M}_2$, given by $\zeta \in \mathcal{H}_1$ or $\mathcal{H}_2$. $\alpha \in \text{J-Iso}(\mathcal{M}_1, \mathcal{M}_2)$ is an ultraweakly continuous map, thus $\omega_\xi \circ \alpha^{-1}$, $\xi \in \mathcal{P}_1$, is a normal state on $\mathcal{M}_2$. Hence there exists a unique $v(\xi) \in \mathcal{P}_2$ such that

$$\omega_\xi \circ \alpha^{-1} = \omega_{v(\xi)}, \qquad \xi \in \mathcal{P}_1. \tag{11}$$

2.6. LEMMA. — $v : \mathcal{P}_1 \to \mathcal{P}_2$, defined by eq. (11), is a bijective, norm preserving, norm continuous map.

Proof. — Given $\xi' \in \mathcal{P}_2$, there is a unique $v'(\xi') \in \mathcal{P}_1$ with $\omega_{\xi'} \circ \alpha = \omega_{v'(\xi')}$; (11) implies $\omega_{v(\xi)} \circ \alpha = \omega_\xi$, therefore, v has an inverse $v^{-1} = v'$.

(11) applied to $A = \mathbb{1}$ yields $\| \xi \| = \| v(\xi) \|$, thus v is norm preserving.

The norm continuity of v follows from

$$\| v(\xi) - v(\xi') \|^2 \leq \| \omega_{v(\xi)} - \omega_{v(\xi')} \| = \| \omega_\xi \circ \alpha^{-1} - \omega_{\xi'} \circ \alpha^{-1} \|$$
$$= \| \omega_\xi - \omega_{\xi'} \| \leq \| \xi + \xi' \| \, \| \xi - \xi' \|.$$

(The second equality is due to $\| \alpha(A) \| = \| A \|$, as for the inequalities see e. g. [SZ, Prop. 10.24].) □

Now let us assume that (α, E_2) is implemented by (V, W), and, consequently, (α^{-1}, E_1) by (V^*, W^*). Then

$$U := V + W \tag{12}$$

is a real linear operator from $\mathscr{H}_1$ onto $\mathscr{H}_2$ fulfilling

$$U^*U = 1, \qquad UU^* = 1 \tag{13}$$

and

$$(\zeta, \alpha^{-1}(A)\zeta) = (U\zeta, AU\zeta), \qquad A \in \mathscr{M}_2, \quad \zeta \in \mathscr{H}_1. \tag{14}$$

Here U^* is defined by $U^* = V^* + W^*$. Eq. (14) holds because $(W\zeta, AV\zeta) = 0$ due to $V = E_2V$, $E_2 \in \mathscr{Z}(\mathscr{M}_2)$, and $E_2W = 0$. If $U\mathscr{P}_1 \subset \mathscr{P}_2$, then (11) and (14) imply

$$v(\xi) = U\xi, \quad \xi \in \mathscr{P}_1. \tag{15}$$

This motivates the following definition.

2.7. DEFINITION. — An implementation (V, W) of (α, E_2) fulfilling $(V + W)\mathscr{P}_1 \subset \mathscr{P}_2$ will be called a standard implementation.

We note in passing, that the condition $U\mathscr{P}_1 \subset \mathscr{P}_2$ is equivalent to $U\mathscr{P}_1 = \mathscr{P}_2$ and equivalent to $V\mathscr{P}_1 = E_2\mathscr{P}_2$, $W\mathscr{P}_1 = E_2^\perp\mathscr{P}_2$.

The standard implementation will be seen to be unique (Lemma 2.9 below), it is the canonical implementation mentioned in the introduction. If $\mathscr{M}_1$ is separable, U defined by (12) equals Connes' U_α ([Co, Theorem 3.2]).

In the above setting with $U\mathscr{P}_1 \subset \mathscr{P}_2$, V and W obviously satisfy

$$\left. \begin{aligned} &\text{Ker } V = E_1^\perp \mathscr{H}_1, \\ &V \lceil_{E_1\mathscr{H}_1} = \text{linear extension of } v \lceil_{E_1\mathscr{P}_1}, \end{aligned} \right\} \tag{16}$$

$$\left. \begin{aligned} &\text{Ker } W = E_1 \mathscr{H}_1, \\ &W \lceil_{E_1^\perp\mathscr{H}_1} = \text{antilinear extension of } v |_{E_1^\perp\mathscr{P}_1}. \end{aligned} \right\} \tag{17}$$

There is a partial converse:

2.8. LEMMA. — Given (α, E_2), let $v: \mathscr{P}_1 \to \mathscr{P}_2$ be defined by eq. (11). Then

$$v(\lambda\xi) = \lambda v(\xi) \qquad \text{for} \quad \lambda > 0. \tag{18}$$

If v happens to be an affine map, define V and W by eqs. (16) and (17). Then

i) V and W are well-defined linear resp. antilinear operators;

ii) $(V + W)\xi = v(\xi)$, and $(V + W)\mathscr{P}_1 \subset \mathscr{P}_2$;

iii) V and W are partial isometries with initial (resp. final) projections E_1 and $E_1^\perp$ (resp. E_2 and $E_2^\perp$).

Proof. — Eq. (18) follows from

$$\omega_{v(\lambda\xi)} = \omega_{\lambda\xi} \circ \alpha^{-1} = \lambda^2 \omega_\xi \circ \alpha^{-1} = \lambda^2 \omega_{v(\xi)} = \omega_{\lambda v(\xi)}.$$

V and W are well-defined because v is a bijection. ii) is trivial. The proof of iii) will be omitted because we do not need it in the sequel; it is a consequence of the main theorem and the following lemma. □

2.9. LEMMA. — Let (V, W) and $(\tilde{V}, \tilde{W})$ both implement (α, E_2), further-more, assume $U \mathscr{P}_1 \subset \mathscr{P}_2$ and $\tilde{U} \mathscr{P}_1 \subset \mathscr{P}_2$. Then we have $V = \tilde{V}$ and $W = \tilde{W}$. Hence the standard implementation—if existing—is unique.

Proof. — This is a direct consequence of eqs. (15), (16) and (17).   □

III. CONSTRUCTION THEOREM

Now we want to prove

3.1. — THEOREM. — Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be standard von Neumann alge-bras. Any $\alpha \in \text{J-Iso}(\mathscr{M}_1, \mathscr{M}_2)$ has a standard implementation, which is unique if the decomposing projection E_2 is specified.

The uniqueness follows from Lemma 2.9.

3.2. COROLLARY. — $\alpha \in \text{J-Iso}(\mathscr{M}_1, \mathscr{M}_2)$ and $\tilde{\alpha} \in \text{J-Iso}(\mathscr{M}_1', \mathscr{M}_2')$ defined in Lemma 2.2 have a common extension to $\bar{\alpha} \in \text{J-Iso}(\mathscr{Z}(\mathscr{M}_1)', \mathscr{Z}(\mathscr{M}_2)')$.

Proof. — Let (V, W) denote the standard implementation of (α, E_2). $V \mathscr{P}_1 = E_2 U \mathscr{P}_1 \subset \mathscr{P}_2$ implies $V J_1 = J_2 V$ on $\mathscr{P}_1$, and by linear extension, on $\mathscr{H}_1$; hence $J_2 V J_1 = V$; analogously, $J_2 W J_1 = W$. Remember that $(J_2 V J_1, J_2 W J_1)$ implement $\tilde{\alpha}$. According to [Ro1, Theorem 3.3] $A \mapsto V A V^* + W A^* W^*$ defines a Jordan isomorphism of $\{ E_1 \}'$ onto $\{ E_2 \}'$ with decomposing projection E_2; denote its restriction to $\mathscr{Z}(\mathscr{M}_1)'$ by $\bar{\alpha}$. Then $\bar{\alpha} \lceil_{\text{L.H.} \mathscr{M}_1 \mathscr{M}_1'}$ equals $\bar{\alpha}$ of Lemma 2.5, $\bar{\alpha}$ is thus an extension of α and $\tilde{\alpha}$; furthermore, $\bar{\alpha}(\text{L. H. } \mathscr{M}_1 \mathscr{M}_1') = \text{L. H. } \mathscr{M}_2 \mathscr{M}_2'$, and consequently, $\bar{\alpha}(\mathscr{Z}(\mathscr{M}_1)') = \mathscr{Z}(\mathscr{M}_2)'$ by ultraweak continuity of $\bar{\alpha} \in \text{J-Iso} \{ E_1 \}', \{ E_2 \}'$ (cf. [Ro2]).   □

As a first step in the proof of the Theorem we state the result for σ-finite algebras:

3.3. PROPOSITION. — Let $\mathscr{M}_1, \mathscr{M}_2$ be σ-finite standard von Neumann algebras. Then (α, E_2) with $\alpha \in \text{J-Iso}(\mathscr{M}_1, \mathscr{M}_2)$ has a standard implemen-tation.

This can be inferred from e. g. [Co, Th. 3.2]: it can be demonstrated that $(E_2 U_\alpha, E_2^\perp U_\alpha)$ yield a standard implementation of (α, E_2) where U_α is defined in the cited theorem; we prefer to give a constructive proof.

Proof of 3.3. — i) Since $\mathscr{M}_1$ is assumed to be σ-finite and standard, there is a cyclic and separating vector $\xi_1 \in \mathscr{P}_1$, $\| \xi_1 \| = 1$. We choose $\xi_2 = v(\xi_1) \in \mathscr{P}_2$, v given by eq. (11), and define V and W by

$$\left. \begin{array}{ll} V A \xi_1 = \alpha(A) E_2 \xi_2, & A \in \mathscr{M}_1 ; \\ W A \xi_1 = \alpha(A^*) E_2^\perp \xi_2, & A \in \mathscr{M}_1 . \end{array} \right\} \qquad (19)$$

V and W are well-defined because ξ_1 is separating, they are densely defined because ξ_1 is cyclic, and bounded since α is an isometry, and thus can be extended to $\mathcal{H}_1$. The latter assertion follows from

$$\| V A \xi_1 \|^2 + \| W A \xi_1 \|^2 = \| \alpha(A) E_2 \xi_2 \|^2 + \| \alpha(A^*) E_2^\perp \xi_2 \|^2$$

$$= (\xi_2, \alpha(A^*)\alpha(A) E_2 \xi_2) + (\xi_2, \alpha(A)\alpha(A^*) E_2^\perp \xi_2) = (\xi_2, \alpha(A^*A)\xi_2) = (\xi_1, A^*A\xi_1),$$

the last equality being a consequence of eq. (11) and the choice of ξ_2.

ii) V and W are partial isometries implementing (α, E_2). The proof of this statement can be copied from [Ro1, proof of Th. 3.1], with obvious modifications; $\omega_\Omega \circ \alpha = \omega_\Omega$ of [Ro1] is to be replaced by eq. (11): $\omega_{\xi_2} \circ \alpha = \omega_{\xi_1}$. In particular we get

$$V^* A \xi_2 = \alpha^{-1}(A) E_1 \xi_1, \qquad W^* A \xi_2 = \alpha^{-1}(A^*) E_1^\perp \xi_2, \qquad A \in \mathcal{M}_2 \qquad (20)$$

iii) The following argument shows that $\xi_2 \in \mathcal{P}_2$ is separating and hence cyclic for $\mathcal{M}_2$: For self-adjoint $A \in \mathcal{M}_2$, $A\xi_2 = 0$ implies

$$0 = \| A\xi_2 \|^2 = \omega_{v(\xi_1)}(A^2) = \omega_\xi(\alpha^{-1}(A)^2) = \| \alpha^{-1}(A)\xi_1 \|^2,$$

hence $\alpha^{-1}(A) = 0$, whence $A = 0$. If $A \in \mathcal{M}_2$ is arbitrary, consider its polar decomposition $A = \tilde{V} |A|$; $A\xi_2 = 0$ implies $\tilde{V}^*\tilde{V} |A| \xi = 0$, thus $|A| \xi = 0$, consequently $|A| = 0$ and $A = 0$.

Now let Δ_i denote the modular operator of $\mathcal{M}_i$ and $S_i = J_i \Delta_i^{1/2}$, $i = 1, 2$. Using $S_i A \xi_i = A^* \xi_i$, $A \in \mathcal{M}_i$ and the definition of V by eq. (19) one finds, for $A \in \mathcal{M}_1$, $S_2 V A \xi_1 = E_2 \alpha(A^*)\xi_2 = V S_1 A \xi_1$, i.e.

$$V S_1 \subset S_2 V \qquad (21)$$

since $\mathcal{M}_1 \xi_1$ is a core of S_1. In the same manner, eq. (20) implies

$$V^* S_2 \subset S_1 V^*. \qquad (22)$$

Combining (21) and (22) we conclude that $V\Delta_1 \subset \Delta_2 V$ and, consequently,

$$V J_1 = J_2 V. \qquad (23a)$$

Similarly it follows that

$$W J_1 = J_2 W. \qquad (23b)$$

Now let us prove $V\mathcal{P}_1 \subset \mathcal{P}_2$. Besides eq. (23a) we use the following formulae, which follow from what has been shown in part *ii*) of this proof: $V = VE_1$, $V^*V + W^*W = 1$, $VAW^* = 0$ for $A \in \mathcal{M}_1$. Eq. (20) implies $V^*\xi_2 = E_1\xi_1$ and, together with (19), $VAV^* = \alpha(A)E_2$. Therefore, since $\mathcal{P}_1 = \overline{\{ Aj_i(A)\xi_i; A \in \mathcal{M}_i \}}$, we have

$$V A j_1(A)\xi_1 = V A j_1(A) E_1 \xi_1 = V A (V^*V + W^*W) J_1 A J_1 V^* \xi_2$$
$$= V A V^* J_2 V A V^* J_2 \xi_2 = \alpha(A) j_2(\alpha(A)) E_2 \xi_2 \in \mathcal{P}_2,$$

whence $V\mathcal{P}_1 \subset \mathcal{P}_2$.

Analogously, it is shown that $W\mathcal{P}_1 \subset \mathcal{P}_2$. $\qquad \square$

Now we can demonstrate Theorem 3.1. Let $\bar\alpha$ denote the extension of α to L. H. $\mathcal{M}_1\mathcal{M}_1'$ as defined in Lemma 2.5.

There exists an increasing directed family $\{\,F_n, n \in I\,\}$ of σ-finite commuting projections in $\mathcal{M}_1$, such that $\bigvee_{n\in I} F_n = \mathbb{1}_1$, i. e. s-lim $F_n = \mathbb{1}_1$ (see e. g. [SZ, 10.24 Lemma 1]). We define the following quantities:

$$Q_{1n} := F_n j_1(F_n) \in \text{L. H. } \mathcal{M}_1\mathcal{M}_1' \tag{24a}$$

$$Q_{2n} := \alpha(F_n) j_2(\alpha(F_n)) \in \text{L. H. } \mathcal{M}_2\mathcal{M}_2', \tag{24b}$$

$$\mathcal{M}_{in} := Q_{in}\mathcal{M}_i Q_{in}, \qquad i = 1, 2. \tag{24c}$$

$\mathcal{M}_{1n}$ and $\mathcal{M}_{2n}$ are considered as von Neumann algebras acting on $Q_{1n}\mathcal{H}_1$ and $Q_{2n}\mathcal{H}_2$, respectively. Clearly, we have

$$Q_{2n} = \bar\alpha(F_n)\bar\alpha(j_1(F_n)) = \bar\alpha(Q_{1n}). \tag{25}$$

Let $z(\,.\,)$ denote the central support, then because of

$$z(F_{1n}) = z(Q_{1n}) \tag{26}$$

we have $Q_{1n} \neq 0$, for all $n \in I$, moreover,

$$s\text{-lim } Q_{1n} = \mathbb{1}_1, \tag{27a}$$

and, due to (25),

$$s\text{-lim } Q_{2n} = \mathbb{1}_2. \tag{27b}$$

Furthermore, (26) implies $\mathcal{M}_{1n} \simeq F_n\mathcal{M}_1 F_n$ and $\mathcal{M}_{2n} \simeq \alpha(F_n)\mathcal{M}_2\alpha(F_n)$, therefore, the algebras $\mathcal{M}_{in}$ are σ-finite for all $n \in I$, $i = 1, 2$. They are standard von Neumann algebras with conjugations $Q_{in}J_i$ and natural cones $\mathscr{P}_{in} \equiv Q_{in}\mathscr{P}_i$.

Now consider $\alpha_n : \mathcal{M}_{1n} \to \mathcal{M}_{2n}$, given by

$$\alpha_n(Q_{1n}AQ_{1n}) = Q_{2n}\alpha(A)Q_{2n}. \tag{28}$$

Consequently, $\alpha_n = \bar\alpha\,\lceil_{\mathcal{M}_{1n}}$ if the operators are considered as acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\alpha_n \in \text{J-Iso}\,(\mathcal{M}_{1n}, \mathcal{M}_{2n})$ with decomposing projections $E_{2n} \equiv E_2\,\lceil_{Q_{2n}\mathcal{H}_2}$; and $\alpha_n^{-1}(E_{2n}) = E_{1n} = E_1\,\lceil_{Q_{1n}\mathcal{H}_1}$. By abuse of notation we shall also consider $Q_{in}AQ_{in}$ and E_{in} as operators on $\mathcal{H}_i$, replacing E_{in} by $E_{in}Q_{in} = Q_{in}E_i$. Since $\mathcal{M}_{in}$ are σ-finite, we can apply Proposition 3.2: there exist standard implementations (V_n, W_n) of (α_n, E_{2n}). Consequently, with $U_n = V_n + W_n$,

$$\omega_\xi \circ \alpha_n^{-1} = \omega_{v_n(\xi)}, \qquad \xi \in \mathscr{P}_{1n} \tag{29}$$

holds on $\mathcal{M}_{2n}$, where

$$v_n(\xi) = U_n\xi, \qquad \xi \in \mathscr{P}_{1n}, \tag{30}$$

cf. eqs. (11) and (15), and $v_n(\xi) \in \mathscr{P}_{2n}$. Now consider $\mathscr{P}_{1n} = Q_{1n}\mathscr{P}_1$ as a

subset of $\mathscr{P}_1$, and $\xi \in \mathscr{P}_{1n} \subset \mathscr{P}_1$; according to eq. (11) we have for arbitrary $A \in \mathscr{M}_2$

$$(v(\xi), Av(\xi)) = (\xi, \alpha^{-1}(A)\xi) = (Q_{1n}\xi, \alpha^{-1}(A)Q_{1n}\xi) =$$
$$= (\xi, \bar{\alpha}^{-1}(Q_{2n})\alpha^{-1}(A)\bar{\alpha}^{-1}(Q_{2n})\xi) = (\xi, \bar{\alpha}^{-1}(Q_{2n}AQ_{2n})\xi) = (\xi, \alpha_n^{-1}(Q_{2n}AQ_{2n})\xi).$$

(Here we have made use of eq. (A2) of Appendix A.)

Using eqs. (29) and (30) we finally get

$$(v(\xi), Av(\xi)) = (Q_{2n}v_n(\xi), AQ_{2n}v_n(\xi)), \qquad A \in \mathscr{M}_2 ,$$

and because $v(\xi)$ and $Q_{2n}v_n(\xi) = v_n(\xi)$ are in $\mathscr{P}_2$ we conclude from the uniqueness of the vectors that

$$v(\xi) = v_n(\xi) = U_n\xi, \qquad \xi \in \mathscr{P}_{1n}. \tag{31}$$

In view of eqs. $(27a, b)$ and the continuity of v (compare Lemma 2.6), this shows that $v(\xi)$ is an affine map. Hence we can define V and W by eqs. (16) and (17) such that $(V + W)\mathscr{P}_1 \subset \mathscr{P}_2$, see Lemma 2.8. With $U = V + W$ we have $v(\xi) = U\xi$; if $\xi \in E_1\mathscr{P}_1$, then $E_{1n}\xi = Q_{1n}E_1\xi \in \mathscr{P}_{1n}$ and $\xi = \lim E_{1n}\xi$; since $V = UE_1$ and $V_n = U_nE_{1n}$, eq. (31) implies $V\xi = \lim V_n\xi$, and by linear extension to $\mathscr{H}_1$,

$$V = s\text{-}\lim V_n . \tag{32a}$$

In the same way it follows that

$$W = s\text{-}\lim W_n . \tag{32b}$$

V_n and W_n have initial (resp. final) projections E_{1n} and $E_{1n}^\perp$ (resp. E_{2n} and $E_{2n}^\perp$); therefore, V and W are partial isometries with initial (resp. final) projections $E_1 = s\text{-}\lim E_{1n}$ and $E_1^\perp$ (resp. $E_2 = s\text{-}\lim E_{2n}$ and $E_2^\perp$).

The same arguments leading to eqs. $(32a, b)$ yield strong convergence of V_n^* and W_n^*, respectively, to V* resp. W* if we start from α_n^{-1} instead of α_n.

It remains to be shown that $\alpha(A) = VAV^* + WA^*W^*$, $A \in \mathscr{M}_1$. According to eq. $(27a)$, $A = w\text{-}\lim Q_{1n}AQ_{1n}$; using $(32a, b)$ we get

$$VAV^* + WA^*W^* = w\text{-}\lim (V_nQ_{1n}AQ_{1n}V_n^* + W_nQ_{1n}A^*Q_{1n}W_n^*) =$$
$$= w\text{-}\lim \alpha_n(Q_{1n}AQ_{1n}) ;$$

due to eqs. (28) and $(27b)$, the latter expression equals $w\text{-}\lim Q_{2n}\alpha(A)Q_{2n} = \alpha(A)$.

This concludes the proof of Theorem 3.1.

IV. GROUPS OF JORDAN AUTOMORPHISMS

This section is devoted to an extension of Propositions 3.2 and 3.4 of [Ro1]: we are interested in representations of an arbitrary group G

by Jordan automorphisms α_g, $g \in G$, of a standard von Neumann algebra $\mathscr{M}$ (with $\alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2}$, $\alpha_e = id$, g_1, $g_2 \in G$, $e = $ identity of G) and their standard implementations. We use the same notation as in the preceding sections, putting $\mathscr{M}_1 = \mathscr{M}_2 = \mathscr{M}$ and dropping the indices of $\mathscr{H}$, $\mathscr{P}$, J, etc.

We fix the decomposing projection E_g of α_g by requiring that E_g be maximal; then we have

$$E_{gg'} = E_g \alpha_g(E_{g'}) + E_g^\perp \alpha_g(E_{g'}^\perp) \tag{33}$$

and

$$E_{g^{-1}} = \alpha_g^{-1}(E_g) \tag{34}$$

(cf. [Ro1, Lemma 2.3 and eq. (6)].

Now let (V_g, W_g) denote the standard implementation of (α_g, E_g), which we have shown to exist for all $g \in G$, and $U_g := V_g + W_g$.

4.1. PROPOSITION. — Under the assumptions stated above, the following relations hold for g, $g' \in G$:

$$V_g^* = V_{g^{-1}}, \qquad W_g^* = W_{g^{-1}}, \tag{35}$$

$$U_{gg'} = U_g U_{g'}, \tag{36}$$

$$V_{gg'} = V_g V_{g'} + W_g W_{g'}, \tag{37}$$

$$W_{gg'} = V_g W_{g'} + W_g V_{g'}. \tag{38}$$

Proof. — *i*) (V_g^*, W_g^*) is the standard implementation of $(\alpha_g^{-1}, \alpha_g^{-1}(E_g))$, see Lemma 2.1; on the other hand, $V_{g^{-1}}, W_{g^{-1}}$ standard-implement $(\alpha_{g^{-1}}, E_{g^{-1}})$. The relation $\alpha_{g^{-1}} = \alpha_g^{-1}$ and eq. (34) together with the uniqueness of the standard implementation imply (35).

ii) According to eqs. (11) and (15) we have

$$\omega_{U_{gg'}\xi} = \omega_\xi \circ \alpha_{gg'}^{-1} = \omega_\xi \circ \alpha_{g'}^{-1} \circ \alpha_g^{-1} = \omega_{U_{g'}\xi} \circ \alpha_g^{-1} = \omega_{U_g U_{g'}\xi}$$

(because $U_g \mathscr{P} \subset \mathscr{P}$), hence (36) holds.

iii) Eqs. (37) and (38) are consequences of eq. (36):

Since $E_g = V_g V_g^*$ and $V_g^* W_g = 0$, it follows that

$$V_g = E_g U_g, \qquad W_g = E_g^\perp U_g. \tag{39}$$

Using (33) and (36) we get

$$V_{gg'} = E_{gg'} U_{gg'} = [E_g \alpha_g(E_{g'}) + E_g^\perp \alpha_g(E_{g'}^\perp)]U_{gg'} =$$
$$= [E_g(V_g E_{g'} V_g^* + \dots) + E_g^\perp(\dots + W_g E_{g'} W_g^*)](V_g + W_g)U_{g'}.$$

Because of (39), the terms indicated by dots do not contribute, and $V_g = E_g V_g$, etc.; hence $V_{gg'} = V_g E_{g'}(V_g^* V_g)U_{g'} + W_g E_{g'}(W_g^* W_g)U_{g'}$. Since E_g, $V_g^* V_g$ and $W_g^* W_g$ are central projections, they commute and eq. (37) follows. Similar arguments yield eq. (38).   □

It can be checked that the r. h. s. of eqs. (37) and (38) yield an implementation of $(\alpha_{gg'}; E_{gg'})$, irrespective of whether (V_g, W_g) and $(V_{g'}, W_{g'})$ are stan-

dard implementations. But to relate two implementations of $(\alpha_{gg'}; E_{gg'})$, uniqueness is necessary; this is why we assumed that the implementations are standard.

Now let G be a topological group. We then may ask whether continuity of α_g with respect to g implies continuity of V_g and $W_{g'}$ and *vice versa*. The answer is in the following theorem; actually, the group property is not needed.

4.2. THEOREM. — Let $\alpha : G \rightarrow$ J-Aut $(\mathcal{M})$ be a map of the topological space G into the set of Jordan automorphisms of a standard von Neumann algebra $\mathcal{M}$, and let α_g^* denote the predual of α_g.
Then the following assumptions are equivalent:

(A$_1$) $g \mapsto \alpha_g^* \omega_\xi$ is norm continuous for all $\xi \in \mathscr{P}$.
(A$_2$) $g \mapsto V_g$ and $g \mapsto W_g$ are weakly continuous.

If one of the assumptions holds, it follows that

i) $g \mapsto \alpha_g(A)$ is ultrastrongly continuous for all $A \in \mathcal{M}$;
ii) $g \mapsto E_g$ is ultrastrongly continuous.
iii) $g \mapsto V_g$ and $g \mapsto W_g$ are ultrastrongly continuous, the same holds of $g \mapsto V_g^*$ and $g \mapsto W_g^*$.

If G is a locally compact group, condition (A$_1$) is equivalent to

(A$_1'$) $g \mapsto \omega_\xi(\alpha_g(A))$ is continuous for all $A \in \mathcal{M}$, $\xi \in \mathscr{P}$,

compare [Ha, proof of Cor. 3.6].

Proof. — The structure of the proof is as follows: (A$_1$) $\Rightarrow$ i) $\Rightarrow$ ii), (A$_1$) $\Rightarrow$ (A$_1$) $\wedge$ ii) $\Rightarrow$ (A$_2$) $\Rightarrow$ (A$_2$) $\wedge$ ii) $\Rightarrow$ iii) $\Rightarrow$ (A$_1$).

Assume (A$_1$), then i) and ii) follow from [Ro2, Lemma (22)]. The cited results are formulated for one-parameter groups of automorphisms; checking the proofs one easily realises that they hold for arbitrary topological groups; moreover, the group property is not needed at all. Property iii) will be seen to follow from (A$_2$); let us demonstrate that (A$_1$) implies (A$_2$). Norm continuity of $g \mapsto \alpha_g^* \omega_\xi = \omega_\xi \circ \alpha_g$ implies norm continuity of $g \mapsto \omega_{U_g \xi} = \omega_\xi \circ \alpha_g^{-1}$:
Since $U_g \xi \in \mathscr{P}$ for $\xi \in \mathscr{P}$, we have

$$\| \omega_\xi(B) - \omega_{U_g \xi}(\alpha_{g'}(B)) \| = \| \omega_{U_g \xi}(\alpha_g(B)) - \omega_{U_g \xi}(\alpha_{g'}(B)) \| < \varepsilon \| B \|$$

for all B, provided $g' \in U_{g,\varepsilon}$ where $U_{g,\varepsilon}$ is a suitable neighbourhood of g. Put $B = \alpha_{g'}^{-1}(A)$, then $\| B \| = \| A \|$ and

$$\| (\omega_\xi \circ \alpha_{g'}^{-1})(A) - \omega_{U_g \xi}(A) \| < \varepsilon \| A \|.$$

Thus $g \mapsto \omega_{U_g \xi}$ is norm continuous.
Using $\| U_g \xi - U_{g'} \xi \|^2 \leq \| \omega_{U_g \xi} - \omega_{U_{g'} \xi} \|$ (see e. g. [SZ, Proposi-

tion 10.24]) we conclude that $g \mapsto U_g$ is strongly continuous on $\mathscr{P}$, and thus on $\mathscr{H}$. Consequently, $g \mapsto V_g$ is strongly continuous:

$$\| (V_g - V_{g'})\eta \| = \| (E_g U_g - E_{g'} U_{g'})\eta \|$$
$$\leq \| (E_g - E_{g'})U_g\eta \| + \| E_{g'}(U_g - U_{g'})\eta \| \to 0 \quad (40)$$

as $g' \mapsto g$ for all $\eta \in \mathscr{H}$ (because of property ii)).

In the same manner, strong continuity of $g \to W_g$ follows, therefore, (A_2) holds.

Now assume (A_2). We introduce $\hat{U}_g := V_g + JW_g$. As is easily checked, $\hat{U}_g$ is a unitary operator. By assumption, $g \mapsto \hat{U}_g$ is weakly, and, due to unitarity, strongly continuous; the same holds true of $g \mapsto \hat{U}_g^*$. (Note that the reasoning leading from weak to strong continuity for unitaries and projections does not hold in general for isometries.) Using $E_g = V_g V_g^* = \hat{U}_g V_g^*$ and the weak continuity of V_g^* we arrive at weak, and hence strong, continuity of $g \mapsto E_g$:

$$| (\eta_1, (E_g - E_{g'})\eta_2) | = | (\eta_1, \hat{U}_g(V_g^* - V_{g'}^*)\eta_2) + (\eta_1, (\hat{U}_g - \hat{U}_{g'})V_g^*\eta_2) | \leq$$
$$\leq | (\hat{U}_g^*\eta_1, (V_g^* - V_{g'}^*)\eta_2) | + \| (\hat{U}_g^* - \hat{U}_{g'}^*)\eta_1 \| \, \| \eta_2 \| \to 0 \quad \text{as} \quad g' \to g$$

for all $\eta_1, \eta_2 \in \mathscr{H}$. Hence again property ii) holds; applying the same argument as in (40) to $V_g = E_g \hat{U}_{g'}$ we conclude that $g \mapsto V_g$ and, likewise, $g \mapsto W_g$ are strongly continuous. This implies the first part of iii) because V_g and W_g are of norm 1. Starting from $V_g^* = \hat{U}_g E_g, W_g^* = \hat{U}_g E_g^\perp$, we can reason analogously in order to get strong, and thus ultrastrong, continuity of $g \mapsto W_g^*$; i.e. iii) holds. Now it is easy to show (A_1). (V_g^*, W_g^*) implement α_g^{-1}, and $g \mapsto U_g^* = V_g^* + W_g^*$ is strongly continuous; therefore we have $\omega_\xi \circ \alpha_g = \omega_{U_g^*\xi}, \xi \in \mathscr{P}$, and

$$| \omega_\xi(\alpha_g(A)) - \omega_\xi(\alpha_{g'}(A)) | = | (U_g^*\xi, A U_g^*\xi) - (U_{g'}^*\xi, A U_{g'}^*\xi) | \leq$$
$$\leq \| A^* U_g^*\xi \| \, \| (U_g^* - U_{g'}^*)\xi \| + \| (U_g^* - U_{g'}^*)\xi \| \, \| A U_{g'}^*\xi \| \leq$$
$$\leq 2 \| A \| \, \| \xi \| \, \| (U_g^* - U_{g'}^*)\xi \| \to 0 \quad \text{as} \quad g' \to g,$$

thus $g \mapsto \omega_\xi \circ \alpha_g$ is norm continuous. $\qquad \square$

APPENDIX A

Jordan-Isomorphisms.

Let $\mathcal{M}_i$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_i$, $i = 1, 2$, respectively. A map $\alpha: \mathcal{M}_1 \to \mathcal{M}_2$ is a Jordan isomorphism if α is linear, *-preserving, bijective and fulfilling $\alpha(AB + BA) = \alpha(A)\alpha(B) + \alpha(B)\alpha(A)$ for all A, B $\in \mathcal{M}_1$.

PROPOSITION. — Let $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$.

i) α is order preserving.

ii) α is an isometry.

iii) There is a projection E_2 in the centre $\mathcal{Z}(\mathcal{M}_2)$ of $\mathcal{M}_2$ such that for all A, B $\in \mathcal{M}_1$

$$\alpha(AB) = \alpha(A)\alpha(B)E_2 + \alpha(B)\alpha(A)(1 - E_2). \tag{A1}$$

For a proof compare e. g. [BR, Th. 3.2.3] where also the following useful identity can be found:

$$\alpha(ABA) = \alpha(A)\alpha(B)\alpha(A), \qquad A, B \in \mathcal{M}_1. \tag{A2}$$

If $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$, then $\alpha^{-1} \in$ J-Iso $(\mathcal{M}_2, \mathcal{M}_1)$. This statement together with the above Proposition, implies that α is a positive normal map, hence α is continuous with respect to the ultraweak topologies in $\mathcal{M}_1$ and $\mathcal{M}_2$, cf. [Dix, Th. I.4.2]. (In order to draw this conclusion, α has to be defined on a von Neumann algebra; therefore, the implication is not possible in Lemma 2.5.)

A central projection fulfilling (A1) is called a decomposing projection; in general, it is not unique.

LEMMA. — Consider $\alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_2)$.

i) $\alpha(\mathcal{Z}(\mathcal{M}_1)) = \mathcal{Z}(\mathcal{M}_2)$.

ii) There exists a unique maximal decomposing projection.

iii) Let $\beta \in$ J-Iso $(\mathcal{M}_2, \mathcal{M}_3)$ and let E_α, E_β be decomposing projections of α and β, respectively. Then $\beta \circ \alpha \in$ J-Iso $(\mathcal{M}_1, \mathcal{M}_3)$ with decomposing projection

$$E_{\beta \circ \alpha} = E_\beta \beta(E_\alpha) + (1 - E_\beta)\beta(1 - E_\alpha).$$

Maximality of E_α and E_β implies maximality of $E_{\beta \circ \alpha}$.

The complete proof can be found in [Ro1], let us indicate the proof of ii). Define E_{2c} to be the projection onto

$$E_{2c}\mathcal{H} = \mathcal{H}_{2c} := \{ \eta \in \mathcal{H}_2 ; (AB - BA)\eta = 0 \quad \text{for all} \quad A, B \in \mathcal{M}_2 \}. \tag{A3}$$

The distinction between morphism and antimorphism is meaningless on $\mathcal{H}_{2c}$. It can be shown that $E_{2c} \in \mathcal{Z}(\mathcal{M}_2)$; thus it is possible to enlarge any given decomposing projection $E_2: E_{2\max} := E_2 \vee E_{2c}$; and E_2 is maximal iff $E_2 \geq E_{2c}$.

The set of all decomposing projections of α is given by

$$\{ E_2 = E_{2\max} - E_{2c}F ; F \in \mathcal{Z}(\mathcal{M}_2) \}. \tag{A4}$$

Finally let us note a simple consequence of (A3):

$$\text{If} \quad E_0 \leq E_{2c}, \quad \text{then} \quad AE_0 \in \mathcal{Z}(\mathcal{M}_2), \qquad A \in \mathcal{M}_2. \tag{A5}$$

(Let B $\in \mathcal{M}_2$, then $BAE_0 = BAE_0E_{2c} = AE_0BE_{2c} = AE_0B$, hence $AE_0 \in \mathcal{M}_2'$; on the other hand, $AE_0 \in \mathcal{M}_2$.)

APPENDIX B

Standard representations.

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$. $\mathcal{M}$ is called a standard von Neumann algebra if there exists a conjugation $J : \mathcal{H} \rightarrow \mathcal{H}$ and a self-dual cone $\mathcal{P} \subset \mathcal{H}$ (i. e. $(\zeta, \xi) \geq 0$ for all $\xi \in \mathcal{P}$ holds iff $\zeta \in \mathcal{P}$) such that:

$i)$ $\qquad\qquad$ $J\mathcal{M}J = \mathcal{M}'$, $\qquad$ and $\quad JAJ = A^*$ $\quad$ for $\;A \in \mathcal{Z}(\mathcal{M})$;

$ii)$ $\qquad\qquad\qquad\qquad$ $\xi \in \mathcal{P}$ $\;$ implies $J\xi = \xi$;

$iii)$ $\qquad\qquad\qquad$ $A(JAJ)\mathcal{P} \subset \mathcal{P}$ $\;$ for all $A \in \mathcal{M}$.

THEOREM. — Any W*-algebra has a faithful standard representation.

It is useful to recall the basic ingredients for the constructive proof of this assertion, cf. e. g. [SZ]. The starting point is the fact that there exists a normal faithful semi-finite weight w mapping the positive part $\mathcal{M}_+$ of $\mathcal{M}$ into $[0, + \infty]$. There is an associated hereditary cone

$$\mathcal{D} := \{ A \in \mathcal{M}_+ \,;\, w(A) < \infty \},$$

and a left ideal

$$\mathcal{N} := \{ A \in \mathcal{M} \,;\, A^*A \in \mathcal{D} \}. \qquad (B1)$$

$\mathcal{N}$ can be considered as a pre-Hilbert space, denoted by $\eta(\mathcal{N})$, with the scalar product

$$(\eta(A), \eta(B)) := w(A^*B), \qquad A, B \in \mathcal{N}.$$

The completion of $\eta(\mathcal{N})$ is denoted by $\mathcal{H}_w$. The formula

$$\pi_w(A)\eta(B) := \eta(AB) \qquad (B2)$$

yields a faithful normal representation of $\mathcal{M}$ in $\mathcal{H}_w$. It can be shown that $\mathcal{A} := \eta(\mathcal{N}^* \cap \mathcal{N}) \subset \mathcal{H}_w$ is a generalized Hilbert algebra with multiplication $\eta(A)\eta(B) = \eta(AB)$ and conjugation $\eta^*(A) = \eta(A^*)$, $\mathcal{A}$ being dense in $\mathcal{H}_w$.

There is an antilinear map $S_0 : \mathcal{A} \rightarrow \mathcal{H}_w$, defined by

$$S_0\eta(B) := \eta(B^*), \qquad B \in \mathcal{N}^* \cap \mathcal{N}. \qquad (B3)$$

The closure S of S_0 exists, its polar decomposition being $S = J\Delta^{1/2}$. J is the desired conjugation; $\mathcal{P}$ is given by the closure of $\{ \pi_w(A)J\eta(A); A \in \mathcal{N}^* \cap \mathcal{N} \}$.

With J and $\mathcal{P}$ introduced in this manner, $\pi_w(\mathcal{M})$ is a standard representation of $\mathcal{M}$.

If $\mathcal{M}$ is σ-finite, w may be chosen as a faithful normal state on $\mathcal{M}$ defining a cyclic and separating vector $\xi_0 \in \mathcal{H}_w$ for $\pi_w(\mathcal{M})$. Then one has $\mathcal{D} = \mathcal{M}_+$ and $\mathcal{N} = \mathcal{M} = \mathcal{N}^* \cap \mathcal{N}$, $\eta(A) = \pi_w(A)\xi_0$, and $\mathcal{P} = \overline{\{ \pi_w(A)J\pi_w(A)\xi_0 ; A \in \mathcal{M} \}}$.

The most important property is given by the following proposition.

PROPOSITION. — The mapping $\xi \in \mathcal{P} \mapsto \omega_\xi = (\xi, \pi_w(.)\xi) \in \mathcal{M}_*^+$ defines a homeomorphism between the self-dual cone $\mathcal{P}$ and the set $\mathcal{M}_*^+$ of normal positive functionals on $\mathcal{M}$, both sets being equipped with their respective norm topologies.

For the proofs of these facts consult [SZ] or [Ha].

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