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Poisson-Lie groups and complete integrability

I. DRINFELD BIGEBRAS, DUAL EXTENSIONS AND THEIR CANONICAL REPRESENTATIONS

by

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ABSTRACT. — This is the first part of a work on Poisson structures on Lie groups, complete integrability and Drinfeld quantum groups. In sections 1 and 2 we establish the algebraic preliminaries of the theory. Section 1 deals with a new kind of extension of Lie algebras, called *twilled extensions* (in French, *extensions croisées*), and with the special case of the *dual extensions* which are exactly the Drinfeld-Lie algebras. Section 2 deals with the exact Lie bigebras arising from the solutions of the classical and modified Yang-Baxter equations. The case of the non antisymmetric solutions (quasitriangular bigebras) is emphasized. In section 3 we study the equivariant one-forms and equivariant families of vector fields on a Lie group, and we introduce the notion of the *Schouten curvature.* In section 4 we prove the existence of the canonical representations of a twilled extension in the space of smooth functions on the Lie group factors, when these are connected and simply connected. Part II will study the Poisson and Lie-Poisson structures on groups, while Part III will connect this theory with that of the bihamiltonian structures and complete integrability.

RÉSUMÉ. — Nous exposons ici la première partie d'un travail sur les structures de Poisson sur les groupes de Lie, la complète intégrabilité, et les groupes quantiques de Drinfeld. Aux paragraphes 1 et 2, nous éta-

blissons les préliminaires algébriques de la théorie. Le paragraphe 1 traite d'une notion nouvelle d'extensions d'algèbres de Lie que nous appelons *extensions croisées* (en anglais, *twilled extensions*), et du cas particulier des *extensions duales*, qui sont exactement les bigèbres de Lie de Drinfeld. Le paragraphe 2 concerne les bigèbres de Lie exactes définies par des solutions de l'équation de Yang-Baxter classique ou modifiée. Le cas des solutions non antisymétriques (bigèbres quasitriangulaires) est étudié en détails. Au paragraphe 3, nous étudions les formes équivariantes et les familles de champs de vecteurs équivariantes sur un groupe de Lie, et nous introduisons la notion de *courbure de Schouten*. Au paragraphe 4, nous montrons l'existence des représentations canoniques d'une extension croisée dans les espaces de fonctions lisses sur les groupes de Lie facteurs, lorsque ceux-ci sont connexes et simplement connexes.

La deuxième partie sera consacrée à l'étude des structures de Poisson et de Lie-Poisson sur les groupes, tandis que la troisième partie établira la relation entre cette théorie et celle des structures bihamiltoniennes et de la complète intégrabilité.

INTRODUCTION

This article is the first part of a work on Poisson structures on Lie groups and their relationships with classical and quantum complete integrability. It expands in several directions the fundamental work of V. G. Drinfeld on Poisson groups and Lie bigebras. Sections 1 and 2 are algebraic in nature, although they are motivated by the differential-geometric concepts to be introduced later. In section 1 we introduce the *twilled extension* of two Lie algebras, a Lie algebra defined by a pair of Lie-algebra representations and Lie-algebra cocycles, a notion which generalizes that of a semi-direct product. The *Drinfeld-Lie bigebras* are obtained as *dual twilled extensions, i. e.*, twilled extensions in which the two Lie algebras are in duality.

Section 2 deals with the *exact Lie bigebras*. These are Lie bigebras defined by an exact cocycle. For a given Lie algebra $\mathfrak{g}$, and vector space $\mathfrak{h}$ isomorphic to the dual of $\mathfrak{g}$, we investigate the conditions under which a mapping r from $\mathfrak{h}$ to $\mathfrak{g}$ is a *Jacobian potential, i. e.*, it defines the structure of a left-exact dual twilled extension on $\mathfrak{g} \times \mathfrak{h}$. We show that a necessary and sufficient condition for r to be Jacobian is the ad-invariance of both its symmetric part and its *Schouten curvature*. When r is antisymmetric this condition reduces to the *generalized Yang-Baxter equation* introduced by Drinfeld in [*4*]. The case where the Schouten curvature of r vanishes is of special importance. We show that, for a potential with invertible ad-invariant symmetric part, this condition is equivalent to the *modified Yang-Baxter* *equation* introduced by Semenov-Tian-Shansky [*9*], and we call the potentials with invertible ad-invariant symmetric part whose Schouten curvature vanishes *quasitriangular potentials.* The corresponding Poisson structures on Lie groups and Lie algebras are the ingredients of the theory of completely integrable Hamiltonian systems (see [*9*] [*10*]) which we will discuss in parts II and III. When the potential is antisymmetric, the Schouten curvature reduces to the *Schouten bracket.* Therefore, in the antisymmetric case, the vanishing curvature condition reduces to the *classical Yang-Baxter equation* (or *classical triangle equation*). This equation was first introduced by Sklyanin [*12*] as the classical limit of the quantum Yang-Baxter equation. The classical Yang-Baxter equation was shown by Gelfand and Dorfman [*6*] to be the vanishing of a Schouten bracket, and Drinfeld [*4*] related it, as well as the generalized Yang-Baxter equation, to Poisson structures on Lie groups. The interpretation, which we introduce here, of the Jacobian condition in the non-antisymmetric case as the vanishing of a Schouten curvature has not been hitherto mentioned in the literature. In addition, we show at the end of this section that a triangular or quasitriangular potential defines an exact dual twilled extension which is isomorphic to the semi-direct product of two Lie algebras in duality.

In section 3, we consider a Lie group G, with Lie algebra $\mathfrak{g}$, acting on a Lie algebra $\mathfrak{h}$ by Lie algebra morphisms. We draw a parallel between equivariant one-forms on G with values in $\mathfrak{h}$ and equivariant families of vector fields on G parametrized by $\mathfrak{h}$. On the one hand, the Cartan curvature of an equivariant one-form is an equivariant two-form with values in $\mathfrak{h}$, and the vanishing of the Cartan curvature is equivalent to the existence of the *primitive.* On the other hand, the *Schouten curvature* of an equivariant family of vector fields parametrized by $\mathfrak{h}$ is an equivariant family of vector fields parametrized by $\mathfrak{h} \times \mathfrak{h}$, and the vanishing of the Schouten curvature is equivalent to the requirement that the family of vector fields constitute a representation of the Lie algebra $\mathfrak{h}$. If, in particular, the Lie algebra $\mathfrak{h}$ is Abelian, the Cartan curvature reduces to the exterior differential while the Schouten curvature reduces to the opposite of the Lie bracket, and the dual roles of these objects appear clearly. The values at the identity of these forms or vector fields determine them entirely. The corresponding algebraic objects—the coboundary and Cartan curvature of a linear mapping from $\mathfrak{g}$ to a $\mathfrak{g}$-module $\mathfrak{h}$, the Schouten bracket and Schouten curvature of a linear mapping from a $\mathfrak{g}$-module $\mathfrak{h}$ to $\mathfrak{g}$—are precisely those that appear in the study of the Jacobian potentials. An invertible Jacobian potential is the inverse of a linear mapping whose Cartan curvature is ad-invariant, and an invertible quasitriangular potential is the inverse of a linear mapping whose Cartan curvature vanishes. We give an example of such a linear mapping with vanishing Cartan curvature which arises in the theory of the integrability of the Toda lattice.

In section 4 we prove that, given connected and simply connected Lie groups, any twilled extension of their Lie algebras acts on the Lie groups, *i. e.*, admits a representation in the spaces of smooth functions on these Lie groups. This proof uses the results of section 3 to construct the group-cocycles associated with the given Lie-algebra cocycles. From the families of vector fields constructed in this section, the Poisson structures on Lie groups satisfying Drinfeld's property, *viz.*, group multiplication is a Poisson morphism, will in turn be constructed and studied. They will be applied to the theory of integrable systems in parts II and III.

1. TWILLED EXTENSIONS OF ALGEBRAS AND DRINFELD BIGEBRAS

This section deals with *twilled extensions* of Lie algebras. We use this term to emphasize the symmetric role played by the two Lie algebras. Particular cases of twilled extensions are the well-known *semi-direct products* (or *inessential extensions*) and the *Drinfeld bigebras* [*4*] [*5*] which are less familiar. A Lie algebra has a twilled extension structure if and only if it is the direct product of two vector subspaces which are Lie subalgebras.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be real or complex, finite-dimensional Lie algebras whose elements will be denoted by $x, y, z, \ldots$, and by $\xi, \eta, \zeta, \ldots$ respectively. We assume that each of these Lie algebras has a representation on the other. This means that we consider bilinear mappings $A : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ and $B : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}$ such that the partial linear maps $A_x : \mathfrak{h} \to \mathfrak{h}$ and $B_\xi : \mathfrak{g} \to \mathfrak{g}$, obtained from A and B by fixing their first arguments, obey the following conditions:

$$(1.1.1) \qquad [A_x, A_y] = A_{[x,y]},$$

$$(1.1.2) \qquad [B_\xi, B_\eta] = B_{[\xi,\eta]}.$$

Once these representations have been specified, we may seek the additional conditions under which the bracket

$$(1.2) \quad [(x_1, \xi_1), (x_2, \xi_2)]_{\mathfrak{k}} := ([x_1, x_2] + B_{\xi_1}(x_2) - B_{\xi_2}(x_1), [\xi_1, \xi_2] + A_{x_1}(\xi_2) - A_{x_2}(\xi_1))$$

defines a Lie-algebra-structure on the vector space $\mathfrak{k} := \mathfrak{g} \times \mathfrak{h}$. When the Jacobi identity for the antisymmetric bracket (1.2) is satisfied, we shall say that the Lie algebra thus defined is the *twilled extension* of $\mathfrak{g}$ and $\mathfrak{h}$. When the representation A (resp., B) vanishes, the bracket (1.2) reduces to the bracket of the semi-direct product of $\mathfrak{g}$ with $\mathfrak{h}$ (resp., of $\mathfrak{h}$ with $\mathfrak{g}$), which shows that the twilled extensions are a symmetric variant of the semi-direct products.

By imposing the Jacobi identity on the commutator (1.2), one readily obtains the additional necessary and sufficient conditions on the representations A and B:

$$(1.3.1)\quad \mathrm{A}_x([\xi,\eta]) = [\mathrm{A}_x\xi,\eta] + [\xi,\mathrm{A}_x\eta] - \mathrm{A}_{\mathrm{B}_\xi(x)}(\eta) + \mathrm{A}_{\mathrm{B}_\eta(x)}(\xi)$$

$$(1.3.2)\quad \mathrm{B}_\xi([x,y]) = [\mathrm{B}_\xi x, y] + [x,\mathrm{B}_\xi y] - \mathrm{B}_{\mathrm{A}_x(\xi)}(y) + \mathrm{B}_{\mathrm{A}_y(\xi)}(x).$$

The conditions may be rewritten as

$$(1.4.1)\qquad \mathrm{A}_{[\xi,\eta]} = (\mathrm{ad}_\xi \circ \mathrm{A}_\eta - \mathrm{A}_\eta \circ \mathrm{B}_\xi) - (\mathrm{ad}_\eta \circ \mathrm{A}_\xi - \mathrm{A}_\xi \circ \mathrm{B}_\eta)$$

$$(1.4.2)\qquad \mathrm{B}_{[x,y]} = (\mathrm{ad}_x \circ \mathrm{B}_y - \mathrm{B}_y \circ \mathrm{A}_x) - (\mathrm{ad}_y \circ \mathrm{B}_x - \mathrm{B}_x \circ \mathrm{A}_y)$$

by using the partial maps $\mathrm{A}_\xi : \mathfrak{g} \to \mathfrak{h}$ and $\mathrm{B}_x : \mathfrak{h} \to \mathfrak{g}$ obtained from the maps A and B by this time fixing their second arguments,

$$(1.5)\qquad \mathrm{A}(x,\xi) = \mathrm{A}_x(\xi) = \mathrm{A}_\xi(x), \qquad \mathrm{B}(\xi,x) = \mathrm{B}_\xi(x) = \mathrm{B}_x(\xi).$$

These conditions express the fact that A and B are one-cocycles on $\mathfrak{h}$ and $\mathfrak{g}$, respectively. In fact, the vector space Hom $(\mathfrak{g},\mathfrak{h})$ of linear maps from $\mathfrak{g}$ to $\mathfrak{h}$ may be regarded as an $\mathfrak{h}$-module, using the representation of $\mathfrak{h}$ on $\mathfrak{g}$ defined by B, and the adjoint representation of $\mathfrak{h}$. Similarly, Hom $(\mathfrak{h},\mathfrak{g})$ may be regarded as a $\mathfrak{g}$-module, using the representation defined by A. Then A, considered as a linear map from $\mathfrak{h}$ to Hom $(\mathfrak{g},\mathfrak{h})$, is a one-cocycle on $\mathfrak{h}$ with values in the $\mathfrak{h}$-module Hom $(\mathfrak{g},\mathfrak{h})$. Similarly, B is a one-cocycle on $\mathfrak{g}$ with values in Hom $(\mathfrak{h},\mathfrak{g})$. Therefore a twilled extension of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is defined by a pair of bilinear mappings $\mathrm{A}: \mathfrak{g}\times\mathfrak{h} \to \mathfrak{h}$ and $\mathrm{B}: \mathfrak{h}\times\mathfrak{g} \to \mathfrak{g}$ such that

i) the partial mapping A_x (resp., B_ξ) defines a representation of the Lie algebra $\mathfrak{g}$ (resp., $\mathfrak{h}$) on the Lie algebra $\mathfrak{h}$ (resp., $\mathfrak{g}$),

ii) the partial mapping B_x (resp., A_ξ) defines a one-cocycle on the Lie algebra $\mathfrak{g}$ (resp., $\mathfrak{h}$) with values in the $\mathfrak{g}$-module Hom $(\mathfrak{h},\mathfrak{g})$ defined by the representation A_x (resp., in the $\mathfrak{h}$-module Hom $(\mathfrak{g},\mathfrak{h})$ defined by the representation B_ξ).

In particular, if one of the two representations vanishes, say A, the cocycle conditions (1.3) reduce to

$$(1.6)\qquad \mathrm{B}_\xi([x,y]) = [\mathrm{B}_\xi x, y] + [x, \mathrm{B}_\xi y],$$

which expresses the fact that B_ξ is a derivation of the Lie algebra $\mathfrak{g}$. We thus see that twilled extensions correspond to one-cocycles in exactly the same way as semi-direct products correspond to derivations.

The problem of constructing a twilled extension is greatly simplified in the case when the given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ happen to be in duality, *i. e.*, they are related by a bilinear form $\langle\,,\,\rangle: \mathfrak{h}\times\mathfrak{g} \to \mathbb{R}$ that identifies each Lie algebra with the dual space of the other. In this case, we shall speak of a *dual twilled extension* or, for short, of a *dual extension.* For such a pair of Lie algebras, there is a natural choice for the representations A of $\mathfrak{g}$ on $\mathfrak{h}$ and B of $\mathfrak{h}$ on $\mathfrak{g}$, namely the coadjoint representations defined by

$$(1.7) \qquad A_x := - {}^t\mathrm{ad}_x =: \mathrm{ad}_x^*, \qquad B_\xi := - {}^t\mathrm{ad}_\xi =: \mathrm{ad}_\xi^*,$$

where the transpose is taken with respect to the given duality form. Under this choice, conditions (1.1) are automatically satisfied, while conditions (1.4) reduce to the *single* condition

$$(1.8) \quad \langle [\xi, \eta], [x, y] \rangle = \langle \mathrm{ad}_x^* \eta, \mathrm{ad}_\xi^* y \rangle + \langle \mathrm{ad}_y^* \xi, \mathrm{ad}_\eta^* x \rangle - (\langle \mathrm{ad}_x^* \xi, \mathrm{ad}_\eta^* y \rangle + \langle \mathrm{ad}_y^* \eta, \mathrm{ad}_\xi^* x \rangle),$$

yielding the constraint upon the Lie algebra structures on $\mathfrak{g}$ and $\mathfrak{h}$ and the duality form by means of which the dual extension $\mathfrak{k}$ is constructed. To obtain this result it suffices to remark that, when A and B are defined by (1.7), conditions (1.4) are the duals of each other. We shall also refer to *dual extensions* as *Drinfeld bigebras* (or *Lie bigebras*) [4] [5] since the two notions obviously coincide (See [7].)

Cocycles defined by the coadjoint representation of a Lie algebra structure, as in (1.7), are called *Jacobian cocycles.*

When condition (1.8) is satisfied, the Lie bracket (1.2) on the dual extension $\mathfrak{k}$ is the only Lie bracket for which the natural scalar product

$$(1.9) \qquad \langle (x_1, \xi_1), (x_2, \xi_2) \rangle_{\mathfrak{k}} = \langle \xi_1, x_2 \rangle + \langle \xi_2, x_1 \rangle$$

defined by the duality form is invariant under the adjoint action in $\mathfrak{k}$. This remark shows that *Manin triples* [5] are in one-to-one correspondence with dual twilled extensions.

2. EXACT DUAL EXTENSIONS AND YANG-BAXTER EQUATIONS

Left-(resp., *right-*) *exact twilled extensions* are naturally defined as those twilled extensions for which the one-cocycle B on $\mathfrak{g}$ (resp., the one-cocycle A on $\mathfrak{h}$) is exact. Here we shall treat the case of the left-exact dual extensions, or *exact Lie bigebras*, an object that is no longer self-dual. (The right-exact dual extensions may be treated in a similar fashion, by exchanging the roles of $\mathfrak{g}$ and $\mathfrak{h}$.)

Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be a vector space. Let A_x be a representation of the Lie algebra $\mathfrak{g}$ on $\mathfrak{h}$. With any element r in $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$, considered as a 0-cochain on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$, we associate a coboundary $\delta r : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ defined by

$$(2.1) \qquad \delta r(x) = \mathrm{ad}_x \circ r - r \circ A_x .$$

We shall assume that $\mathfrak{g}$ and $\mathfrak{h}$ are *in duality* and that $A_x = \mathrm{ad}_x^*$. We shall investigate the additional conditions on the 0-cochain r under which the exact one-cocycle δr is *Jacobian, i. e.*, defines the structure of a *left-exact dual twilled extension* on $\mathfrak{k} = \mathfrak{g} \times \mathfrak{h} \approx \mathfrak{g} \times \mathfrak{g}^*$, also called an *exact* (or a *coboundary*) *Lie-bigebra structure* on $\mathfrak{g}$. A 0-cochain r on $\mathfrak{g}$ with values in Hom $(\mathfrak{h}, \mathfrak{g})$, *i. e.*, a linear mapping from $\mathfrak{h} \approx \mathfrak{g}^*$ to $\mathfrak{g}$, or an element in $\mathfrak{g} \otimes \mathfrak{g}$, such that δr is Jacobian will be called a *Jacobian potential.* As we shall show below, the Jacobian condition is expressed most naturally in terms of the Schouten bracket and curvature of r, which we now define.

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a $\mathfrak{g}$-module that is not necessarily a Lie algebra. We denote the action of $\mathfrak{g}$ on $\mathfrak{h}$ by $x \to \mathrm{A}_x$. To each linear mapping $r: \mathfrak{h} \to \mathfrak{g}$, we associate its *Schouten bracket*, an antisymmetric bilinear mapping $[r, r]$ from $\mathfrak{h} \times \mathfrak{h}$ to $\mathfrak{g}$ defined by

$$[r, r](\xi, \eta) = r(\mathrm{A}_{r\xi}\eta - \mathrm{A}_{r\eta}\xi) - [r\xi, r\eta]. \tag{2.2}$$

If, in addition, we assume that $\mathfrak{h}$ is a Lie algebra and that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations, to each linear mapping $r: \mathfrak{h} \to \mathfrak{g}$ we can associate its *Schouten curvature*, an antisymmetric bilinear mapping K^r from $\mathfrak{h} \times \mathfrak{h}$ to $\mathfrak{g}$, defined by

$$\mathrm{K}^r(\xi, \eta) = [r, r](\xi, \eta) + r[\xi, \eta]_{\mathfrak{h}}. \tag{2.3}$$

Obviously, if $\mathfrak{h}$ is Abelian, the Schouten curvature reduces to the Schouten bracket. We shall see in section 3 that K^r is the algebraic version of the contravariant analogue of the Cartan curvature of Lie-algebra valued one-forms on a Lie group.

By considering the particular case where $\mathfrak{g}$ and $\mathfrak{h}$ are in duality, the action of $\mathfrak{g}$ on $\mathfrak{h}$ is the coadjoint action $\mathrm{A}_x = \mathrm{ad}^*_x$, and where the linear mapping r is *antisymmetric*, we shall show that our definition of the Schouten bracket extends the usual one because, in this case, we can identify $[r, r]: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ with an antisymmetric trilinear mapping

$$\langle [r, r] \rangle : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$$

defined by

$$\langle [r, r] \rangle (\xi_1, \xi_2, \xi_3) = \langle \xi_3, [r, r](\xi_1, \xi_2) \rangle. \tag{2.4}$$

Using the antisymmetry of r, we can write

$$\begin{aligned} \langle [r, r] \rangle (\xi_1, \xi_2, \xi_3) &= \langle \xi_3, r(\mathrm{ad}^*_{r\xi_1} \xi_2 - \mathrm{ad}^*_{r\xi_2} \xi_1) - [r\xi_1, r\xi_2] \rangle \\ &= \langle \xi_2, [r\xi_1, r\xi_3] \rangle - \langle \xi_1, [r\xi_2, r\xi_3] \rangle - \langle \xi_3, [r\xi_1, r\xi_2] \rangle \end{aligned}$$

whence the identity

$$\langle [r, r] \rangle (\xi_1, \xi_2, \xi_3) = - \oint \langle \xi_3, [r\xi_1, r\xi_2] \rangle. \tag{2.5}$$

(Here and below, $\oint$ denotes the sum over the circular permutations of the indices 1, 2, 3.)

This identity shows that $\langle [r, r] \rangle$ is indeed antisymmetric as a function of its three arguments

We now turn to the problem of determining the Jacobian potentials (not necessarily antisymmetric) on a Lie algebra $\mathfrak{g}$. Specifically, the problem is to find conditions on a linear mapping $r: \mathfrak{g}^* \to \mathfrak{g}$ under which the bracket $[\,,\,]^r = {}^t(\delta r)$ on $\mathfrak{g}^*$ that is explicitly given by

$$(2.6) \qquad [\xi, \eta]^r = -\left(\mathrm{ad}^*_{{}^t r\eta}\, \xi + \mathrm{ad}^*_{r\xi}\, \eta\right)$$

is antisymmetric and satisfies the Jacobi identity,

$$(2.7) \qquad [\xi, \eta]^r + [\eta, \xi]^r = 0$$

$$(2.8) \qquad [\xi_1, [\xi_2, \xi_3]^r]^r + [\xi_2, [\xi_3, \xi_1]^r]^r + [\xi_3, [\xi_1, \xi_2]^r]^r = 0\,.$$

Let us set $r = a + s$, ${}^t r = -a + s$, where $a = \frac{1}{2}(r - {}^t r)$ and $s = \frac{1}{2}(r + {}^t r)$ are the *antisymmetric* and *symmetric* parts of r, respectively. Then it is easily seen that condition (2.7) is equivalent to the ad-*invariance of the symmetric part* s of r,

$$(2.9) \qquad s \circ \mathrm{ad}^*_x = \mathrm{ad}_x \circ s\,.$$

In fact,

$$(2.10) \quad [\xi, \eta]^r + [\eta, \xi]^r = -\left(\mathrm{ad}^*_{(r+{}^t r)\eta}\xi + \mathrm{ad}^*_{(r+{}^t r)\xi}\eta\right) = -2\left(\mathrm{ad}^*_{s\eta}\xi + \mathrm{ad}^*_{s\xi}\eta\right).$$

If s is ad-invariant, $[\xi, \eta]^r$ depends only on a, and

$$(2.11) \qquad [\xi, \eta]^r = \mathrm{ad}^*_{a\eta}\xi - \mathrm{ad}^*_{a\xi}\eta\,.$$

We can now discuss the Jacobi identify, using the *Schouten bracket* $\langle [a, a] \rangle \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ of the antisymmetric part, a, of r. We shall prove the identity

$$(2.12) \qquad \oint \langle [\xi_3, [\xi_1, \xi_2]^r]^r, x \rangle = -\left(\mathrm{ad}_x \langle [a, a] \rangle\right)(\xi_1, \xi_2, \xi_3)\,.$$

In fact, using the Jacobi identity in $\mathfrak{g}$, and

$$\langle [\xi, \eta]^r, x \rangle = -\langle \xi, [a\eta, x] \rangle + \langle \eta, [a\xi, x] \rangle,$$

we obtain

$$\begin{aligned}
\oint \langle [\xi_3, [\xi_1, \xi_2]^r]^r, x \rangle &= \oint \langle \xi_3, \left[a\left(\mathrm{ad}^*_{a\xi_2}\xi_1 - \mathrm{ad}^*_{a\xi_1}\xi_2\right), x\right] \rangle \\
&\quad + \oint \langle \xi_1, [a\xi_2, [a\xi_3, x]] \rangle - \oint \langle \xi_2, [a\xi_1, [a\xi_3, x]] \rangle \\
&= -\oint \langle \mathrm{ad}^*_x\xi_3, a\left(\mathrm{ad}^*_{a\xi_2}\xi_1 - \mathrm{ad}^*_{a\xi_1}\xi_2\right) \rangle + \oint \langle \mathrm{ad}^*_x\xi_3, [a\xi_1, a\xi_2] \rangle \\
&= \oint \langle [a, a] \rangle (\xi_1, \xi_2, \mathrm{ad}^*_x \xi_3),
\end{aligned}$$

which proves (2.12) since $\langle [a, a] \rangle$ is antisymmetric. Formula (2.12) shows that the Jacobi identity for $[\,,\,]^r$ is equivalent to the ad-invariance of $[a, a]$. Thus we can state

PROPOSITION 2.1. — *An element r of $\mathfrak{g} \otimes \mathfrak{g}$ is a Jacobian potential if and only if the symmetric part s of r, and the Schouten bracket $[a, a]$ of the antisymmetric part of r, are* ad-*invariant.*

The condition $\mathrm{ad}_x \langle [a, a] \rangle = 0$ is called the *generalized Yang-Baxter equation* (GYB). The condition $\langle [a, a] \rangle = 0$, which obviously implies (GYB), is called the *classical Yang-Baxter equation* (CYB).

By the above result, the set of Jacobian potentials on $\mathfrak{g}$ can be regarded as a trivial fibre bundle P, contained in $\mathfrak{g} \otimes \mathfrak{g}$, whose base is the vector space of ad-invariant symmetric elements of $\mathfrak{g} \otimes \mathfrak{g}$, and whose fibre is the set of solutions of the generalized Yang-Baxter equation. Our aim is to introduce an important subclass of Jacobian potentials, the *quasitriangular potentials*, those with vanishing Schouten curvature when $\mathfrak{h} \approx \mathfrak{g}^*$ is equipped with the Lie bracket which we shall now define.

Each mapping $s : \mathfrak{h} \to \mathfrak{g}$ which is invertible or zero defines a Lie-algebra structure on $\mathfrak{h}$

$$(2.13) \qquad [\xi, \eta]_s := -2s^{-1}[s\xi, s\eta],$$

if s is invertible and

$$(2.13') \qquad [\xi, \eta]_s := 0, \qquad \text{if} \quad s = 0.$$

The numerical factor is introduced for convenience. When s is ad-invariant, ad_x^* is a derivation of $(\mathfrak{h}, [\,,\,]_s)$, and the bilinear mapping $[s, s] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ satisfies

$$(2.14) \qquad [s, s](\xi, \eta) = [s\xi, s\eta]$$

and is ad-invariant.

A potential will be called *regular* if it has an *invertible*, ad-*invariant symmetric part*. Let r be a *regular* or *antisymmetric* potential whose symmetric part we denote by s. When $\mathfrak{h} \approx \mathfrak{g}^*$ is equipped with the Lie bracket $[\,,\,]_s$, the Schouten curvature (2.3) of r is given by

$$(2.15) \qquad \mathrm{K}^r = [a, a] - [s, s].$$

In fact, using the ad-invariance of s, we obtain

$$(2.16) \qquad [\xi, \eta]_s = -2\mathrm{ad}_{s\xi}^* \eta = \mathrm{ad}_{s\eta}^* \xi - \mathrm{ad}_{s\xi}^* \eta,$$

whence

$$\begin{aligned}
K^r(\xi, \eta) &= r(\mathrm{ad}^*_{r\xi}\eta - \mathrm{ad}^*_{r\eta}\xi) - [r\xi, r\eta] + r[\xi, \eta]_s \\
&= r(\mathrm{ad}^*_{r\xi}\eta - \mathrm{ad}^*_{r\eta}\xi) - [r\xi, r\eta] - 2r\,\mathrm{ad}^*_{s\xi}\eta \\
&= a(\mathrm{ad}^*_{a\xi}\eta - \mathrm{ad}^*_{a\eta}\xi) - [a\xi, a\eta] - [s\xi, s\eta] \\
&\quad + r(\mathrm{ad}^*_{s\xi}\eta - \mathrm{ad}^*_{s\eta}\xi - 2\,\mathrm{ad}^*_{s\xi}\eta) \\
&\quad + (s\,\mathrm{ad}^*_{a\xi}\eta - [a\xi, s\eta]) - (s\,\mathrm{ad}^*_{a\eta}\xi - [a\eta, s\xi]) \\
&= [a, a](\xi, \eta) - [s\xi, s\eta].
\end{aligned}$$

Therefore $[a, a]$ is ad-invariant if and only if K^r is ad-invariant. Thus,

PROPOSITION 2.2. — *Let r be a regular or antisymmetric potential. Then r is Jacobian if and only if the Schouten curvature K^r of r is* ad*-invariant.*

Clearly the *regular potentials* with *vanishing Schouten curvature* are an important subclass of Jacobian potentials. They will be called the *quasitriangular potentials.*

We shall now show that the quasitriangular potentials with preassigned, invertible, ad-invariant symmetric part s are nothing other than the solutions of the *modified Yang-Baxter equation* (MYB) in the sense of Semenov-Tian-Shansky [*9*] [*10*] [*11*]. In fact, if we set

$$R = a \circ s^{-1}, \tag{2.17}$$

the condition $K^r = 0$, *i. e.*,

$$[a, a](\xi, \eta) - [s\xi, s\eta] = 0,$$

can be written as

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -[x, y]. \tag{2.18}$$

Therefore, utilizing the ad-invariant scalar product s, we see that *the condition $K^r = 0$ is the modified Yang-Baxter equation for $R = a \circ s^{-1}$.* This condition plays an important role in the theory of completely integrable systems [*9*] [*10*] [*11*]. This condition had been independently introduced in [*8*] as the *pseudococycle condition* where it figured in the integration of the finite Toda lattice equations.

If s is scaled by a numerical factor β in $\mathbb{R}$, $s' = \beta s$ defines the Lie bracket on $\mathfrak{h}$,

$$[\xi, \eta]_{s'} = \beta[\xi, \eta]_s. \tag{2.19}$$

The condition $K^r = 0$ is scaled accordingly to

$$[a, a](\xi, \eta) - \beta^2[s\xi, s\eta] = 0, \tag{2.20}$$

and the case $\beta = 0$, where $r = a$ is antisymmetric and $\mathfrak{h} \approx \mathfrak{g}^*$ is Abelian, is nothing other than GYB. The modified Yang-Baxter equation is scaled to

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -\beta^2[x, y]. \tag{2.21}$$

But the case $\beta = 0$ is to be excluded in (2.21) since, for $s = 0$, R is not defined.

We have borrowed the term « quasitriangular » from Drinfeld's article [5], where it was defined in a seemingly very different way. We shall now relate our vanishing curvature condition to equation (7) of [5]. In Drinfeld's notation, for r in $\mathfrak{g} \otimes \mathfrak{g}$, $[r^{12}, r^{13}]$ is the element ${}^{+}\mathrm{f}\ \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ which satisfies

$$[r^{12}, r^{13}](\xi_1, \xi_2, \xi_3) = \langle \xi_1, [{}^t r(\xi_2), {}^t r(\xi_3)] \rangle .$$

Similarly,

$$[r^{12}, r^{23}](\xi_1, \xi_2, \xi_3) = \langle \xi_2, [r\xi_1, {}^t r\xi_3] \rangle ,$$
$$[r^{13}, r^{23}](\xi_1, \xi_2, \xi_3) = \langle \xi_3, [r\xi_1, r\xi_2] \rangle .$$

Therefore, when $s = \frac{1}{2}(r + {}^t r)$ is ad-invariant,

$$([r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}])(\xi_1, \xi_2, \xi_3) = \langle \xi_3, \mathrm{K}^r(\xi_2, \xi_1) \rangle . \tag{2.22}$$

In fact, K^r can be also written as

$$\mathrm{K}^r(\xi, \eta) = -\left(r\left(\mathrm{ad}^*_{{}^t r\xi}\eta + \mathrm{ad}^*_{r\eta}\xi\right) + [r\xi, r\eta]\right). \tag{2.23}$$

Equation (2.22) shows that the ad-invariance of K^r, *i. e.*, the condition that r be a Jacobian potential, is equivalent to condition (6) of Drinfeld [5], and that the vanishing of K^r, *i. e.*, the condition that r be a quasitriangular potential, is equivalent to condition (7) of Drinfeld in the same paper. So a quasitriangular potential as defined above determines a quasitriangular Lie bigebra structure on $\mathfrak{g}$, in the sense of Drinfeld. (But we observe that a quasitriangular potentiel in our sense is necessarily regular.)

When r is antisymmetric,

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = -\langle [a, a] \rangle , \tag{2.24}$$

and Drinfeld's conditions (6) and (7) therefore reduce to GYB and CYB, respectively, as expected.

We shall now summarize the various classes of potentials that we have introduced and their relationships with the various definitions to be found in the literature. Let us denote by S the star-shaped region of invertible or zero symmetric elements of $\mathfrak{g} \otimes \mathfrak{g}$. Proposition 2.2 shows that, in the trivial fibre bundle P of all Jacobian potentials, the set P′ of regular or antisymmetric Jacobian potentials is defined by

$$\mathrm{P}' = \{ r \in \mathfrak{g} \otimes \mathfrak{g} ; r + {}^t r \in \mathrm{S}, r + {}^t r \text{ ad-invariant and } \mathrm{K}^r \text{ ad-invariant} \} .$$

For each ad-invariant symmetric element s of S, the fibre P_s of P (or P′) over s is

$$\mathrm{P}_s = \{ r \in \mathfrak{g} \otimes \mathfrak{g} ; r + {}^t r = s \text{ and } \mathrm{K}^r \text{ ad-invariant} \} .$$

In particular, by proposition 2.1, the fibre P_0 over $s = 0$ consists of the antisymmetric potentials r whose Schouten brackets are ad-invariant,

$$\mathrm{P}_0 = \{ r \in \mathfrak{g} \otimes \mathfrak{g} ; r + {}^t r = 0 \text{ and } [r, r] \text{ ad-invariant} \} .$$

As mentioned above, an element of P_0 is a solution of the *generalized Yang-Baxter equation* (GYB). The exact Lie bigebras defined by antisymmetric Jacobian potentials have been called *Lie-Sklyanin* [*bigebras*] in [*7*].

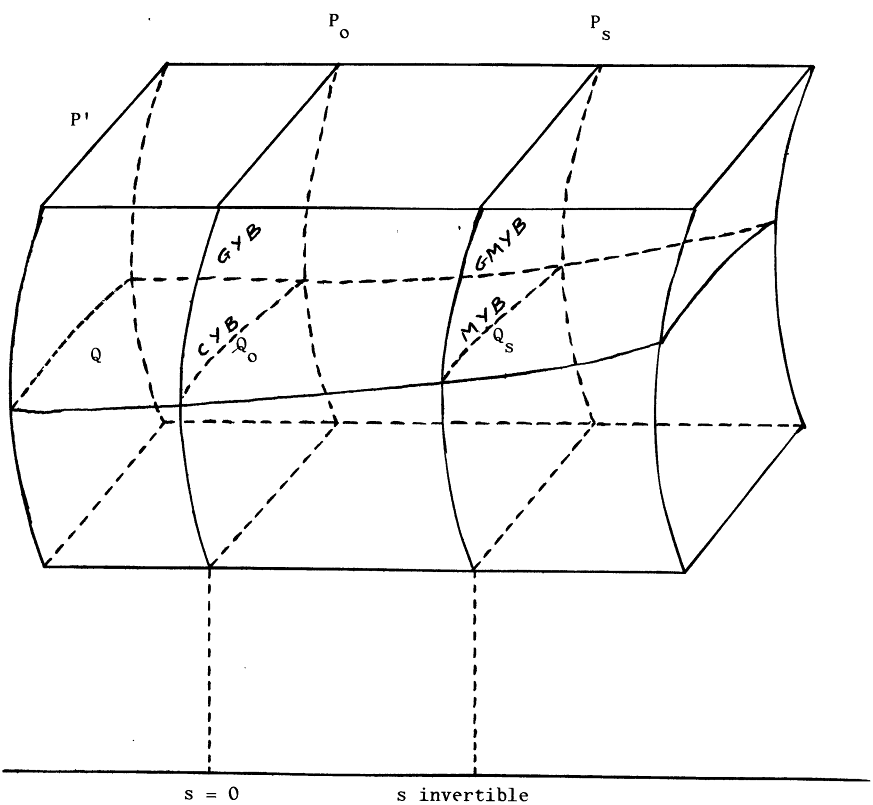
In the fibre bundle P′, we have identified the important subset of the regular or antisymmetric Jacobian potentials with vanishing Schouten curvature

$$Q = \{\, r \in \mathfrak{g} \otimes \mathfrak{g} \,;\, r + {}^t r \in S,\; r + {}^t r \text{ ad-invariant and } K^r = 0 \,\}.$$

If s is invertible, the intersection Q_s of Q with P_s is the set of *quasitriangular potentials* with preassigned ad-invariant symmetric part s. We have shown that the potentials in Q_s are in one-to-one correspondence with the solutions of the *modified Yang-Baxter equation* (MYB) relative to s. They define the *quasitriangular Lie bigebras*, which are therefore in one-to-one correspondence with the *Lie-Semenov algebras* (See [*5*], [*9*], [*1*].)

If $s = 0$, the intersection Q_0 of Q with P_0 is the set of antisymmetric potentials with vanishing Schouten bracket,

$$Q_0 = \{\, r \in \mathfrak{g} \otimes \mathfrak{g} \,;\, r + {}^t r = 0 \text{ and } [r, r] = 0 \,\},$$



i. e., the solutions of the *classical Yang-Baxter equation* (CYB) or *triangle equation.* They can be called *triangular potentials* and they define the *triangular Lie bigebras* [*5*].

In view of the existing terminology we shall say that a potential in P_s is a *solution of the generalized modified Yang-Baxter equation* (GMYB) relative to s.

Many examples of Lie bigebras and their quantization have already been given by Drinfeld [*4*] [*5*]. In the examples that we borrow in the list below, we have recast the first two in our language of dual twilled extensions.

i) The trivial left-exact dual extension is defined by the potential $r = 0$ on $\mathfrak{g}$, which is of course Jacobian and even triangular. Whence $B = \delta r = 0$ and $\mathfrak{g}^*$ is Abelian. The corresponding Poisson structure on G, the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, is trivial.

ii) The trivial right-exact dual extension defined by the potential $\rho = 0$ on $\mathfrak{g}^*$, whence $A = \delta\rho = 0$ and $\mathfrak{g}$ is Abelian. The corresponding Poisson structure on the Abelian group $\mathfrak{g}^*$ is the Lie-Poisson structure.

iii) By the Whitehead lemma, any semi-simple Lie bigebra is exact.

iv) In the non-Abelian, two-dimensional Lie algebra $\mathfrak{g}$ with the basis e_1, e_2 and the Lie bracket $[e_1, e_2] = \alpha e_2, \alpha \in \mathbb{R}$, there exist Jacobian cocycles $B : \mathfrak{g} \to \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g})$ defined by $B(e_1) = 0$, $B(e_2) = \beta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\beta \in \mathbb{R}$, which define on $\mathfrak{g}^*$ the Lie brackets

$$[e_1^*, e_2^*] = \beta e_2^* .$$

Those cocycles are not exact unless $\beta = 0$.

v) Let us consider $\mathfrak{g} = sl(2, \mathbb{C})$, with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\mathfrak{g}^*$ with the dual basis H^*, X^{+*}, X^{-*}. Since

$$\operatorname{tr} \operatorname{ad} H \operatorname{ad} H = 8, \qquad \operatorname{tr} \operatorname{ad} X^+ \operatorname{ad} X^- = \operatorname{tr} \operatorname{ad} X^- \operatorname{ad} X^+ = 4,$$

the Killing form of $\mathfrak{g}$ considered as a linear mapping k from $\mathfrak{g}^*$ to $\mathfrak{g}$ has the matrix $\frac{1}{8}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

We consider $r = s + a$, where s has the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ and a has the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$. Since s is a multiple of the Killing form, s is ad-invariant. Moreover $K^r = [a,a] - [s,s]$ vanishes since

$$\begin{aligned}[a,a](X^{+*},X^{-*}) &= -4H = [sX^{+*},sX^{-*}],\\ [a,a](H^*,X^{+*}) &= -4X^- = [sH^*,sX^{+*}],\\ [a,a](H^*,X^{-*}) &= 4X^+ = [sH^*,sX^{-*}].\end{aligned}$$

Thus r is a quasitriangular potential on $\mathfrak{g}$.

The coboundary of r is $B = \delta r : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}^*,\mathfrak{g})$, where $\delta r(H) = 0$, $\delta r(X^+)$ has the matrix $\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\delta r(X^-)$ has the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.

When we identify $\mathrm{Hom}(\mathfrak{g}^*,\mathfrak{g})$ with $\mathfrak{g}\otimes\mathfrak{g}$, we obtain

$$\begin{aligned} s &= 8k = H\otimes H + 2(X^+\otimes X^- + X^-\otimes X^+),\\ a &= 2(X^+\otimes X^- - X^-\otimes X^+),\\ r &= H\otimes H + 4X^+\otimes X^-,\end{aligned}$$

$$\delta r(H)=0,\quad \delta r(X^+)=2(X^+\otimes H - H\otimes X^+),\quad \delta r(X^-)=2(X^-\otimes H - H\otimes X^-).$$

This is the simplest particular case of the formula for arbitrary semi-simple Lie algebras and Kac-Moody algebras given by Drinfeld in [*5*].

vi) When $\mathfrak{g}$ is an arbitrary Lie bigebra, $\mathfrak{k} = \mathfrak{g}\times\mathfrak{g}^*$ is a quasitriangular Lie bigebra. In fact, on the dual extension Lie algebra $\mathfrak{k}$ there exists a canonical quasitriangular potential $m : \mathfrak{k}^* \to \mathfrak{k}$ defined by

$$m(\xi, x) = (x, 0). \tag{2.25}$$

The quasitriangular Lie bigebra $\mathfrak{k}$ is called the *double* of the Lie bigebra $\mathfrak{g}$. (See [*5*], [*1*]). For instance, taking the double of the two-dimensional Lie bigebra of example *iv*) will furnish an example of a four-dimensional quasi-triangular Lie bigebra.

In the rest of this section we shall consider the case where a dual twilled extension is a semi-direct product.

Quasitriangular Lie bigebras and semi-direct products.

Let r be a Jacobian potential in $\mathfrak{g}\otimes\mathfrak{g}$. Then $\mathfrak{h}\approx\mathfrak{g}^*$ is a Lie algebra $\mathfrak{h}_r$, with the Lie bracket

$$[\xi,\eta]^r = -\left(\mathrm{ad}^*_{{}^t r\eta}\,\xi + \mathrm{ad}^*_{r\xi}\,\eta\right) = \mathrm{ad}^*_{a\eta}\,\xi - \mathrm{ad}^*_{a\xi}\,\eta, \tag{2.26}$$

and therefore, as shown by formula (2.1),

$$\mathrm{ad}^*_\xi\, x = \left(\mathrm{ad}_x\circ r - r\circ\mathrm{ad}^*_x\right)\xi. \tag{2.27}$$

Let $\mathfrak{k}^{((r))}$ be the corresponding left-exact dual extension, with the Lie bracket

$$[(x,\xi),(y,\eta)]_{\mathfrak{k}} = \left([x,y] + \mathrm{ad}^*_\xi\, y - \mathrm{ad}^*_\eta\, x,\ [\xi,\eta]^r + \mathrm{ad}^*_x\,\eta - \mathrm{ad}^*_y\,\xi\right). \tag{2.28}$$

This Lie bracket can also be written as

$$(2.29)\quad [(x,\xi),(y,\eta)]_{\mathfrak{k}} = \big([x - r\xi, y - r\eta] - [r\xi, r\eta] + r\,(\mathrm{ad}_x^*\,\eta - \mathrm{ad}_y^*\,\xi),\ [\xi,\eta]^r + \mathrm{ad}_x^*\,\eta - \mathrm{ad}_y^*\,\xi\big).$$

When r is regular or antisymmetric, $\mathfrak{k}$ is also a Lie algebra $\mathfrak{k}_s$, with the Lie bracket defined by (2.13) or (2.13'),

$$(2.30)\qquad [\xi,\eta]_s = -\,2\,\mathrm{ad}_{s\xi}^*\,\eta = \mathrm{ad}_{s\eta}^*\,\xi - \mathrm{ad}_{s\xi}^*\,\eta\,.$$

Since s is ad-invariant, ad_x^* is a derivation of both $\mathfrak{h}_s$ and $\mathfrak{h}_{-s}$, and $\mathfrak{k}$ is a *semi-direct product Lie algebra* $\mathfrak{k}_{((s))}$ with the Lie bracket

$$(2.31)\qquad [(x,\xi),(y,\eta)]^{((s))} = \big([x,y],\ -\ [\xi,\eta]_s + \mathrm{ad}_x^*\,\eta - \mathrm{ad}_y^*\,\xi\big).$$

We shall show that when r is triangular or quasitriangular, $\mathfrak{k}^{((r))}$ and $\mathfrak{k}_{((s))}$ are isomorphic Lie algebras. More precisely,

PROPOSITION 2.3. — *The linear mapping $u:(x,\xi)\in\mathfrak{k}\to(x-r\xi,\xi)\in\mathfrak{k}$ is a Lie algebra isomorphism from $\mathfrak{k}^{((r))}$ to the semi-direct product $\mathfrak{k}_{((s))}$ if and only if the Schouten curvature of r vanishes.*

In fact

$$(2.32)\qquad u\,[(x,\xi),(y,\eta)]^{((r))} - [u(x,\xi),u(y,\eta)]_{((s))} = (\mathrm{K}^r(\xi,\eta),0).$$

To prove this formula, we use the identity

$$[\xi,\eta]^r + [\xi,\eta]_s = \mathrm{ad}_{r\eta}^*\,\xi - \mathrm{ad}_{r\xi}^*\,\eta$$

and its consequence

$$-\,\mathrm{K}^r(\xi,\eta) = [r\xi, r\eta] + r\,[\xi,\eta]^r\,.$$

Whence

$$\begin{aligned} &u\,[(x,\xi),(y,\eta)]^{((r))} - [u(x,\xi),u(y,\eta)]_{((s))}\\ &\quad = \big([x-r\xi, y-r\eta] - [r\xi,r\eta] - r\,[\xi,\eta]^r,\ [\xi,\eta]^r + \mathrm{ad}_x^*\,\eta - \mathrm{ad}_y^*\,\xi\big)\\ &\quad - \big([x-r\xi, y-r\eta],\ -\ [\xi,\eta]_s + \mathrm{ad}_{x-r\xi}^*\,\eta - \mathrm{ad}_{y-r\eta}^*\,\xi\big) = (\mathrm{K}^r(\xi,\eta),0)\,,\end{aligned}$$

as claimed.

Therefore quasitriangular and triangular exact dual extensions are in fact semi-direct products and, in particular, any triangular exact dual extension is isomorphic to the semi-direct product of $\mathfrak{g}$ with the Abelian Lie algebra $\mathfrak{g}^*$, defined by the coadjoint representation.

Setting

$$\lambda(\xi) = (r\xi,\xi)\,,\qquad i(\xi) = (0,\xi)\,,$$
$$\mu(x,\xi) = x - r\xi\,,\qquad j(x,\xi) = x\,,$$

we obtain the following commutative diagram of linear mappings, with two exact sequences:

$$\begin{tikzcd}
& & \mathfrak{k}^{((r))} \arrow[dd, "u"] \arrow[dr, "\mu"] & & \\
0 \arrow[r] & \mathfrak{h}_{-s} \arrow[ur, "\lambda"] \arrow[dr, "i"'] & & \mathfrak{g} \arrow[r] & 0 \\
& & \mathfrak{k}_{((s))} \arrow[ur, "j"'] & &
\end{tikzcd}$$

While i and j are Lie-algebra morphisms in all cases, the linear mappings u, λ and μ are Lie-algebra morphisms if and only if $\mathrm{K}^r = 0$. In fact

$$[\lambda(\xi), \lambda(\eta)]^{((r))} - \lambda[\xi, \eta]_{-s} = (\mathrm{K}^r(\xi, \eta), 0),$$

and

$$[\mu(x, \xi), \mu(y, \eta)] - \mu[(x, \xi), (y, \eta)]^{((r))} = -\mathrm{K}^r(\xi, \eta),$$

and the property for u was proved above.

This diagram shows that, when the Schouten curvature of r vanishes, the exact sequence of Lie algebra morphisms

$$0 \to \mathfrak{h}_{-s} \xrightarrow{\lambda} \mathfrak{k}^{((r))} \xrightarrow{\mu} \mathfrak{g} \to 0$$

is an *inessential extension* [2] of $\mathfrak{g}$ by $\mathfrak{h}_{-s}$, with $\{ (x, 0) \in \mathfrak{k} ; x \in \mathfrak{g} \} \approx \mathfrak{g}$ a Lie subalgebra of $\mathfrak{k}^{((r))}$ complementary to the ideal

$$\operatorname{Im} \lambda = \operatorname{Ker} \mu = \{ (r\xi, \xi) \in \mathfrak{k} ; \xi \in \mathfrak{h} \},$$

which is equivalent to the canonical inessential extension

$$0 \to \mathfrak{h}_{-s} \xrightarrow{i} \mathfrak{k}_{((s))} \xrightarrow{j} \mathfrak{g} \to 0,$$

defined by the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{h}$.

3. CARTAN AND SCHOUTEN CURVATURE FORMS

To continue our study of twilled extensions we need some results concerning equivariant one-forms with values in a Lie algebra, and equivariant families of vector fields parametrized by a Lie algebra, and defined on a Lie group. They will be reviewed briefly in this section.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on a second Lie group H by means of a family of morphisms $\mathrm{C}_g : \mathrm{H} \to \mathrm{H}$. This action induces an action of G on the Lie algebra $\mathfrak{h}$ of H which will be denoted by $\mathrm{A}_g : \mathfrak{h} \to \mathfrak{h}$. The corresponding infinitesimal action will be denoted by $\mathrm{A}_x : \mathfrak{h} \to \mathfrak{h}$, with $x \in \mathfrak{g}$. A typical example is the case where $\mathrm{H} = \mathrm{G}$ and $\mathrm{C}_g(h) = g \cdot h \cdot g^{-1}$. A one-form $\omega : \mathfrak{X}(\mathrm{G}) \to \mathfrak{h}$, defined on G and taking values in $\mathfrak{h}$, is said to be equivariant with respect to the actions L_g by left-translations, and A_g of G on itself and on $\mathfrak{h}$ if

$$\omega(TL_g(X)) = A_g \circ \omega(X) \tag{3.1}$$

for every vector field $X \in \mathfrak{X}(G)$. It is obvious that equivariant one-forms are completely defined by their value at the identity e of G. Indeed, the value ω_g at the point g is related to the value ω_e at the identity by the relation

$$\omega_g = A_g \circ \omega_e \circ dL_g(e)^{-1} \tag{3.2}$$

which can also be written as

$$\omega_g(l_x) = A_g \circ \omega_e(x) \tag{3.3}$$

where l_x is the left-invariant vector field on G defined by an element x in $\mathfrak{g}$.

For any one-form ω, and for an equivariant one-form in particular, there is an associated *Cartan curvature* two-form

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]_{\mathfrak{h}} . \tag{3.4}$$

If the group G is connected and simply connected, the vanishing of Ω is equivalent to the existence of a *primitive function* (or *primitive*) for ω [*3*] which is defined as the unique function $\sigma : G \to H$ such that

$$\sigma(e_G) = e_H \tag{3.5}$$

$$\sigma(g)^{-1} \circ d\sigma(g) = \omega_g \tag{3.6}$$

where the composition on the left-hand side denotes the action on $d\sigma(g)$ of the differential at $\sigma(g)$ of the left translation by $\sigma(g)^{-1}$.

Our aim in this section is to state and prove the equivariance property of the curvature form Ω and the primitive function σ of an equivariant one-form. We must first prove that Ω fulfills the relation

$$\Omega(TL_g(X), TL_g(Y)) = A_g \circ \Omega(X, Y) \tag{3.7}$$

or equivalently,

$$\Omega_g(l_x, l_y) = A_g \circ \Omega_e(x, y) , \tag{3.8}$$

where Ω_g and Ω_e denote the values of Ω at the points g and e of G. This is accomplished by observing that

$$\begin{aligned}(d\omega)_g(l_x(g), l_y(g)) &= l_x(\omega(l_y))(g) - l_y(\omega(l_x))(g) - \omega_g([l_x, l_y]) \\ &= l_x(A_g\omega_e(y)) - l_y(A_g\omega_e(x)) - A_g\omega_e[x, y] \\ &= A_g(A_x\omega_e(y) - A_y\omega_e(x) - \omega_e[x, y])\end{aligned}$$

and that

$$[\omega_g(l_x), \omega_g(l_y)] = [A_g\omega_e(x), A_g\omega_e(y)]_{\mathfrak{h}} = A_g[\omega_e(x), \omega_e(y)]_{\mathfrak{h}} .$$

These equalities show that (3.8) is valid when Ω_e is given by

$$(3.9) \quad \Omega_e(x, y) = A_x\omega_e(y) - A_y\omega_e(x) - \omega_e[x, y] + [\omega_e(x), \omega_e(y)]_{\mathfrak{h}}.$$

We therefore obtain

PROPOSITION 3.1. — *The Cartan curvature* $\Omega = d\omega + [\omega, \omega]$ *of the equivariant one-form* ω *defined by* $\omega_e : \mathfrak{g} \to \mathfrak{h}$ *is the equivariant two-form defined by* $\Omega_e : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$, *where*

$$(3.10) \qquad \Omega_e(x, y) = \delta\omega_e(x, y) + [\omega_e(x), \omega_e(y)]_{\mathfrak{h}},$$

where $\delta\omega_e$ *is the coboundary of* ω_e *in the cohomology of the Lie algebra* $\mathfrak{g}$ *with values in the* $\mathfrak{g}$*-module* $\mathfrak{h}$.

This result shows that, for equivariant one-forms, the process of computing the curvature form Ω is purely algebraic. Ω_e will be called the *Cartan curvature* of the linear mapping ω_e.

Example. — The linear space of sequences, $x = (x_n)_{n\in\mathbb{Z}}$, of real or complex matrices of a given order constitutes a real or complex Lie algebra $\mathfrak{g}$ with the Lie bracket defined by

$$[x, y] = ([x_n, y_n])_{n\in\mathbb{Z}}.$$

This Lie algebra is infinite-dimensional. We consider the linear mapping $\omega_e : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\omega_e(x) = (x_{n+1} - x_n)_{n\in\mathbb{Z}}.$$

Its Cartan curvature vanishes because

$$\begin{aligned}(\Omega_e(x, y))_n = {} & [x_n, y_{n+1} - y_n] - [y_{n+1} - y_n, x_n] \\ & - ([y_n, x_{n+1} - x_n] - [x_n, y_{n+1} - y_n]) - ([x_{n+1}, y_{n+1}] - [y_{n+1}, x_{n+1}]) \\ & + ([x_n, y_n] - [y_n, x_n]) + ([x_{n+1} - x_n, y_{n+1} - y_n] - [y_{n+1} - y_n, x_{n+1} - x_n]) = 0.\end{aligned}$$

We shall now prove that the primitive σ of an equivariant one-form ω, with vanishing Cartan curvature Ω satisfies the condition

$$(3.11) \qquad \sigma(g' \cdot g) = \sigma(g') \cdot C_{g'}\sigma(g).$$

To prove this identity we introduce the auxiliary one-forms

$$(3.12) \qquad \omega'_g = \omega_{g'g} \circ dL_{g'}(g)$$

and

$$(3.13) \qquad \omega''_g = A_{g'} \circ \omega_g$$

depending parametrically on a given element g' of G. As a consequence of the vanishing of the curvature Ω of ω, the curvature forms of ω' and ω'' also vanish for any $g' \in G$. Indeed, let us show that the functions

$$(3.14) \qquad \sigma'(g) = \sigma(g' \cdot g)$$

and

$$\sigma''(g) = \sigma(g') \cdot C_{g'}\sigma(g) \tag{3.15}$$

are primitives of ω' and ω'', respectively, satisfying the additional condition

$$\sigma'(e) = \sigma''(e) = \sigma(g'). \tag{3.16}$$

The relation, $\sigma'(g)^{-1} \circ d\sigma'(g) = \omega'_g$, follows from the chain rule, while the second relation, $\sigma''(g)^{-1} \circ d\sigma''(g) = \omega''_g$, follows from

$$\begin{aligned} d\sigma''(g) &= dL_{\sigma(g')}(C_{g'}\sigma(g)) \circ dC_{g'}(\sigma(g)) \circ d\sigma(g) \\ &= dL_{\sigma(g')}(C_{g'}\sigma(g)) \circ dL_{C_{g'}(\sigma(g))}(e) \circ A_{g'} \circ dL_{\sigma(g)}(e)^{-1} \circ d\sigma(g) \\ &= dL_{\sigma''(g)}(e) \circ A_{g'} \circ \omega_g \end{aligned}$$

where we have used the identity

$$dC_g(h) \circ dL_h(e) = dL_{C_g(h)}(e) \circ A_g$$

which is valid for any Lie group morphism $C_g : H \to H$. Finally, it suffices to observe that $\omega' = \omega''$ if ω is equivariant. Then the relation (3.11) that was sought follows from the uniqueness of the primitive of a form with vanishing curvature on a connected Lie group [*3*].

We shall now study equivariant families of vector fields on G parametrized by $\mathfrak{h}$, a concept which is dual to that of one-forms taking their values in $\mathfrak{h}$. To construct such a family, we consider a linear mapping $p : \mathfrak{h} \to \mathfrak{g}$ which we use to define, for each ξ in $\mathfrak{h}$, a vector field X_ξ on G according to

$$X_\xi(g) = dL_g(e) \circ p \circ A_{g^{-1}}(\xi). \tag{3.17}$$

In this way we obtain a family of vector fields parametrized by the algebra $\mathfrak{h}$ with the equivariance property

$$TL_g(X_\xi) = X_{A_g\xi} \tag{3.18}$$

with respect to the actions L_g and A_g of G. Conversely, it is easily seen that any family of vector fields satisfying (3.18) can be put into the form of (3.17) for $p : \mathfrak{h} \to \mathfrak{g}$ defined by $p(\xi) = X_\xi(e)$. This justifies the name of *equivariant family of vector fields parametrized by* $\mathfrak{h}$ that we have given to (3.17).

The main property of families of equivariant vector fields is that of being closed under Lie brackets. The Lie bracket of X_ξ and X_η is given by

$$[X_\xi, X_\eta](g) = -\, dL_g(e) \circ [p, p](A_{g^{-1}}\xi, A_{g^{-1}}\eta) \tag{3.19}$$

where

$$[p, p](\xi, \eta) := p(A_{p\xi}\eta - A_{p\eta}\xi) - [p(\xi), p(\eta)] \tag{3.20}$$

is the *Schouten bracket* of p, where p is considered to be a one-form on $\mathfrak{h}$ with values in $\mathfrak{g}$. In fact, from the relation

$$\begin{aligned} \mathrm{TL}_g[\mathrm{X}_\xi, \mathrm{X}_\eta] &= [\mathrm{TL}_g(\mathrm{X}_\xi), \mathrm{TL}_g(\mathrm{X}_\eta)] \\ &= [\mathrm{X}_{\mathrm{A}_g\xi}, \mathrm{X}_{\mathrm{A}_g\eta}], \end{aligned}$$

we see that the value of the commutator at the identity determines the commutator at each point g,

$$[\mathrm{X}_\xi, \mathrm{X}_\eta](g) = d\mathrm{L}_g(e)([\mathrm{X}_{\mathrm{A}_{g^{-1}}(\xi)}, \mathrm{X}_{\mathrm{A}_{g^{-1}}(\eta)}](e)).$$

Since

$$\mathrm{X}_\xi(g) = l_{p\circ \mathrm{A}_{g^{-1}}(\xi)}(g),$$

the chain rule yields

$$[\mathrm{X}_\xi, \mathrm{X}_\eta](g) = [l_{p\circ \mathrm{A}_{g^{-1}}(\xi)}, l_{p\circ \mathrm{A}_{g^{-1}}(\eta)}](g) + l_x(g) - l_y(g),$$

where x (resp., y) is the value at e of the Lie derivative of the $\mathfrak{g}$-valued function, $g \to p \circ \mathrm{A}_{g^{-1}}(\eta)$ (resp., $p \circ \mathrm{A}_{g^{-1}}(\xi)$), by the vector field X_ξ (resp., X_η). In particular, for $g = e$,

$$(3.21) \qquad [\mathrm{X}_\xi, \mathrm{X}_\eta](e) = [p(\xi), p(\eta)] - p(\mathrm{A}_{p(\xi)}(\eta) - \mathrm{A}_{p(\eta)}(\xi)).$$

This proves (3.19).

The *Schouten curvature* of the equivariant family of vector fields X indexed by $\mathfrak{h}$ is the family of vector fields $\mathscr{K}$ indexed by $\mathfrak{h} \times \mathfrak{h}$,

$$(3.22) \qquad \mathscr{K}(\xi, \eta) := -[\mathrm{X}_\xi, \mathrm{X}_\eta] + \mathrm{X}_{[\xi,\eta]_\mathfrak{h}}.$$

If the equivariant family of vector fields X is defined by $p : \mathfrak{h} \to \mathfrak{g}$, then $\mathscr{K}$ is the equivariant family of vector fields defined by $\mathrm{K}^p : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$, where

$$\mathrm{K}^p(\xi, \eta) := [p, p](\xi, \eta) + p([\xi, \eta]_\mathfrak{h}).$$

The bilinear mapping, K^p, is called the *Schouten curvature* of p.

This result shows that the process of computing the Lie bracket is purely algebraic, and that it allows us to deal easily with the question of determining when the vector fields X_ξ yield a representation of the algebra $\mathfrak{h}$ over G. Indeed, the relation

$$(3.23) \qquad [\mathrm{X}_\xi, \mathrm{X}_\eta] = \mathrm{X}_{[\xi,\eta]_\mathfrak{h}}$$

is valid if and only if the Schouten curvature of p vanishes.

When p, as a linear map from $\mathfrak{h}$ into $\mathfrak{g}$, is invertible, we introduce the equivariant one-form defined by

$$(3.24) \qquad \omega_e = p^{-1},$$

and we compute the coboundary, $\delta\omega_e$, and the curvature, Ω_e. We obtain

PROPOSITION 3.2. — *If* $p^{-1} = \omega_e$, *then*

$$(3.25) \qquad \delta\omega_e(x, y) = \omega_e([p, p](\omega_e(x), \omega_e(y))),$$

and

$$\Omega_e(x, y) = \omega_e(\mathrm{K}^p(\omega_e(x), \omega_e(y))) \,. \tag{3.26}$$

The results of this section show that the Schouten bracket is the exact counterpart, for equivariant families of vector fields, of the exterior differential for forms, and that the Schouten curvature for vector fields is the analogue of the Cartan curvature for forms. Therefore this section provides the differential-geometric constructions which justify the algebraic definitions of section 2. The results of this section regarding the existence of the primitive will be used in the next section.

4. CANONICAL ACTIONS OF TWILLED EXTENSIONS ON THE FACTOR LIE GROUPS

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. When G and H are connected and simply connected, any twilled extension $\mathfrak{k}$ of $\mathfrak{g}$ and $\mathfrak{h}$ acts canonically on both G and H, *i. e.*, it admits representations in the spaces of smooth functions over G and H. This property is a consequence of the relation between twilled extensions and one-cocycles that was explained in section 1. In view of their importance, we shall now study these actions in detail. It is sufficient to construct the action on G because that on H is defined analogously. Let us denote the elements of G by $g, g', \ldots$

The basic tools for constructing the action of $\mathfrak{k}$ on G are the bilinear mappings $\mathrm{A} : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ and $\mathrm{B} : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}$ that have already been discussed. Let us recall that these mappings satisfy the following four conditions:

$$[\mathrm{A}_x, \mathrm{A}_y] = \mathrm{A}_{[x,y]} \tag{4.1}$$

$$(\mathrm{ad}_x \circ \mathrm{B}_y - \mathrm{B}_y \circ \mathrm{A}_x) - (\mathrm{ad}_y \circ \mathrm{B}_x - \mathrm{B}_x \circ \mathrm{A}_y) = \mathrm{B}_{[x,y]} \tag{4.2}$$

$$\mathrm{A}_x([\xi, \eta]) - [\mathrm{A}_x\xi, \eta] - [\xi, \mathrm{A}_x\eta] = \mathrm{A}_{\mathrm{B}_x(\eta)}(\xi) - \mathrm{A}_{\mathrm{B}_x(\xi)}(\eta) \tag{4.3}$$

$$[\mathrm{B}_\xi, \mathrm{B}_\eta] = \mathrm{B}_{[\xi,\eta]} \,. \tag{4.4}$$

The first two conditions imply that A defines a representation of $\mathfrak{g}$ on $\mathfrak{h}$, and that B defines a one-cocycle of $\mathfrak{g}$ with values in Hom $(\mathfrak{h}, \mathfrak{g})$. We assume that the Lie-algebra representation A is the differential of a group representation of G on $\mathfrak{h}$, which, in this section, we denote by the same letter, $\mathrm{A}_g : \mathfrak{h} \to \mathfrak{h}$, for g in G. When G is connected and simply connected, a uniquely determined Lie-group representation corresponds to each Lie-algebra representation.

Let us now show that, out of the Lie-algebra one-cocycle B and the action A, we can construct a *one-cocycle* of G with values in Hom $(\mathfrak{h}, \mathfrak{g})$.

Here the action of G on the vector space $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ is defined by

$$(4.5) \qquad g \cdot m = \mathrm{Ad}_g \circ m \circ \mathrm{A}_{g^{-1}},$$

for $m \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$. We consider B to be a linear mapping from $\mathfrak{g}$ to the G-module $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ considered as an Abelian Lie algebra. We apply the results of section 3 in the case of an Abelian G-module. Since B is a one-cocycle of $\mathfrak{g}$, the Cartan curvature of the equivariant one-form β on G with values in $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ defined by B vanishes. Therefore, if G is connected and simply connected, β has a unique primitive, b, satisfying

$$(4.6) \qquad b_e = 0.$$

Furthermore, in the case at hand, identity (3.11) takes the form

$$(4.7) \qquad b_{gg'} = b_g + \mathrm{Ad}_g \circ b_{g'} \circ \mathrm{A}_{g^{-1}}.$$

These two relations express the fact that the mapping b which we have constructed is a one-cocycle of G with values in $\mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$. By definition, b satisfies the differential equation

$$(4.8) \qquad db(g)(l_x(g)) = \mathrm{Ad}_g \circ \mathrm{B}_x \circ \mathrm{A}_{g^{-1}}.$$

We now state the following *basic identity* relating the one-cocycle b_g on the group G to the one-cocycle B_x of the Lie algebra $\mathfrak{g}$,

$$(4.9) \qquad \mathrm{Ad}_g \circ \mathrm{B}_{\mathrm{Ad}_{g^{-1}} x} \circ \mathrm{A}_{g^{-1}} - \mathrm{B}_x = \mathrm{ad}_x \circ b_g - b_g \circ \mathrm{A}_x.$$

This identity is obtained by computing the partial derivatives with respect to g and g' of the cocycle condition (4.7), at $g = e$ or $g' = e$,

$$db(g') \circ d\mathrm{R}_{g'}(e)(x) = \mathrm{B}_x + \mathrm{ad}_x \circ b_{g'} - b_{g'} \circ \mathrm{A}_x,$$
$$db(g) \circ d\mathrm{L}_g(e)(x) = \mathrm{Ad}_g \circ \mathrm{B}_x \circ \mathrm{A}_{g^{-1}}.$$

and then by eliminating the differential of b between these two relations.

In the constructions that we have prescribed, we have used the mappings A_x and B_x, defined over $\mathfrak{g}$, which fulfill conditions (4.1-4), to arrive at the linear mappings A_g and b_g, defined on G, that fulfill the conditions

$$(4.10) \qquad \mathrm{A}_{gg'} = \mathrm{A}_g \circ \mathrm{A}_{g'}, \qquad \mathrm{A}_e = \mathrm{Id}.$$
$$(4.11) \qquad b_{gg'} = b_g + \mathrm{Ad}_g \circ b_{g'} \circ \mathrm{A}_{g^{-1}}, \qquad b_e = 0.$$

We observe that we have used only conditions (4.1) and (4.2) in this construction. In fact, (4.10) is the group-theoretical form of condition (4.1), while (4.11) is the group-theoretical form of (4.2). Consequently, A_g and b_g satisfy *additional conditions which characterize the group one-cocycles associated with twilled extensions*, namely,

$$(4.12) \qquad \mathrm{A}_g[\mathrm{A}_{g^{-1}}\xi, \mathrm{A}_{g^{-1}}\eta] - [\xi, \eta] = \mathrm{A}_\xi \circ b_g(\eta) - \mathrm{A}_\eta \circ b_g(\xi)$$

$$(4.13) \qquad -[b_g(\xi), b_g(\eta)] + b_g(\mathrm{A}_{b_g(\xi)}\eta - \mathrm{A}_{b_g(\eta)}\xi) = b_g([\xi, \eta]) - \mathrm{B}_\xi \circ b_g(\eta) + \mathrm{B}_\eta \circ b_g(\xi).$$

Condition (4.12) expresses the fact that the group cocycle b_g measures how much the action A_g differs from an automorphism of the Lie algebra $\mathfrak{h}$, while in the corresponding condition (4.3), the Lie-algebra cocycle B_x measures how much the infinitesimal action A_x differs from a derivation of the Lie algebra $\mathfrak{h}$. To prove (4.12), let us introduce the mapping $\sigma : G \to L^2(\mathfrak{h}, \mathfrak{h})$, where $L^2(\mathfrak{h}, \mathfrak{h})$ is the linear space of bilinear mappings from $\mathfrak{h} \times \mathfrak{h}$ to $\mathfrak{h}$, defined by

$$(4.14) \quad \sigma_g(\xi, \eta) := A_g[A_{g^{-1}}\xi, A_{g^{-1}}\eta] - [\xi, \eta] + A_\eta \circ b_g(\xi) - A_\xi \circ b_g(\eta).$$

We observe that σ fulfills the cocycle conditions

$$(4.15) \qquad \sigma_e = 0$$

$$(4.16) \qquad \sigma_{gg'}(\xi, \eta) = \sigma_g(\xi, \eta) + A_g \circ \sigma_{g'}(A_{g^{-1}}\xi, A_{g^{-1}}\eta)$$

as a result of the cocycle condition on b_g and of identity

$$(4.17) \qquad A_g \circ A_x = A_{\mathrm{Ad}_g x} \circ A_g$$

which is valid for any representation A_g of G. Then a simple computation shows that the derivative Σ of σ_g at the identity e,

$$(4.18) \qquad \Sigma := d\sigma(e),$$

i. e., the Lie-algebra cocycle corresponding to the group-cocycle σ, is determined by

$$(4.19) \quad \Sigma_x(\xi, \eta) = A_x[\xi, \eta] - [A_x\xi, \eta] - [\xi, A_x\eta] + A_\eta \circ B_x\xi - A_\xi \circ B_x\eta$$

and vanishes because of (4.3). Also, by (4.7) and (4.17), the differential of σ at g satisfies

$$d\sigma(g)(x)(\xi, \eta) = A_g \circ \Sigma_x(A_{g^{-1}}\xi, A_{g^{-1}}\eta),$$

and therefore the differential of σ vanishes everywhere on G. Since G is connected and σ vanishes at the identity, σ itself must vanish on G.

The second condition (4.13) can be rewritten, using the Schouten bracket $[b_g, b_g]$ of the linear mapping b_g from the $\mathfrak{g}$-module $\mathfrak{h}$ (for the action A_x) to the Lie algebra $\mathfrak{g}$, given by formula (3.20), and the coboundary δb_g of b_g considered as a one-cocycle of the Lie algebra $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathfrak{g}$ (for the action B_ξ). In fact, (4.13) is equivalent to

$$(4.20) \qquad [b_g, b_g] + \delta b_g = 0.$$

In order to prove formula (4.13), we consider the mapping $f : G \times \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ defined by

$$(4.21) \quad f(g, \xi, \eta) := B_\xi \circ b_g(\eta) - B_\eta \circ b_g(\xi) - b_g([\xi, \eta]) + b_g(A_\eta \circ b_g(\xi) - A_\xi \circ b_g(\eta)) - [b_g(\xi), b_g(\eta)].$$

The derivative of f with respect to g is given by

$$
\begin{aligned}
(4.22)\quad & r_x(f)(g, \xi, \eta) \\
&= \mathrm{Ad}_g \circ \{((\mathrm{Ad}_{g^{-1}} \circ \mathrm{B}_\xi \circ \mathrm{Ad}_g) \circ \mathrm{B}_{\bar{x}}\bar{\eta} - (\mathrm{Ad}_{g^{-1}} \circ \mathrm{B}_\eta \circ \mathrm{Ad}_g) \circ \mathrm{B}_{\bar{x}}\bar{\xi}) \\
&\quad - (\mathrm{B}_{\bar{x}} \circ \mathrm{A}_{g^{-1}} [\mathrm{A}_g\bar{\xi}, \mathrm{A}_g\bar{\eta}]) - (\mathrm{B}_{\bar{x}} \circ \mathrm{A}_{\bar{\eta}} \circ b_{g^{-1}}\bar{\xi} + b_{g^{-1}} \circ \mathrm{A}_{\bar{\eta}} \circ \mathrm{B}_{\bar{x}}\bar{\xi}) \\
&\quad + (\mathrm{B}_{\bar{x}} \circ \mathrm{A}_{\bar{\xi}} \circ b_{g^{-1}}\bar{\eta} + b_{g^{-1}} \circ \mathrm{A}_{\bar{\xi}} \circ \mathrm{B}_{\bar{x}}\bar{\eta}) - ([\mathrm{B}_{\bar{x}}\bar{\eta}, b_{g^{-1}}\bar{\xi}] + [b_{g^{-1}}\bar{\eta}, \mathrm{B}_{\bar{x}}\bar{\xi}])\},
\end{aligned}
$$

where r_x is the right-invariant vector field defined by x in $\mathfrak{g}$, and where

$$(4.23)\qquad \bar{\xi} := \mathrm{A}_{g^{-1}}\xi, \qquad \bar{\eta} := \mathrm{A}_{g^{-1}}\eta, \qquad \text{and} \qquad \bar{x} := \mathrm{Ad}_{g^{-1}}\, x.$$

To obtain (4.22) we have used the relations (4.8), (4.7) and (4.17) in the following forms:

$$
\begin{aligned}
(4.24)\quad (db(g) \circ r_x(g))(\xi) = \mathrm{Ad}_g \circ \mathrm{B}_{\bar{x}}\bar{\xi}, \qquad & \mathrm{Ad}_{g^{-1}}\, b_g(\xi) = -\, b_{g^{-1}}(\bar{\xi}), \\
& \mathrm{A}_{g^{-1}} \circ \mathrm{A}_x\eta = \mathrm{A}_{\bar{x}}\bar{\eta}.
\end{aligned}
$$

In the notations of (4.23), the *basic identity* (4.9), relating the one-cocycle b_g of the group G to the one-cocycle B_x of the algebra $\mathfrak{g}$, can be written

$$(4.25)\qquad \mathrm{B}_{\bar{\xi}}\bar{x} = \mathrm{Ad}_{g^{-1}} \circ \mathrm{B}_\xi x - [\bar{x}, b_{g^{-1}}\bar{\xi}] + b_{g^{-1}}\mathrm{A}_{\bar{\xi}}\bar{x}.$$

From (4.22) and (4.23) we obtain

$$
\begin{aligned}
(4.26)\quad r_x(f)(g, \xi, \eta) = {} & \mathrm{Ad}_g \circ \mathrm{B}_{\bar{x}} \circ (\mathrm{A}_{\bar{\xi}} \circ b_{g^{-1}}(\bar{\eta}) - \mathrm{A}_{\bar{\eta}} \circ b_{g^{-1}}(\bar{\xi}) + [\bar{\xi}, \bar{\eta}] \\
& - \mathrm{A}_{g^{-1}} \circ [\mathrm{A}_g\bar{\xi}, \mathrm{A}_g\bar{\eta}]) + \mathrm{Ad}_g \circ ([\mathrm{B}_{\bar{\xi}}, \mathrm{B}_{\bar{\eta}}](\bar{x}) - \mathrm{B}_{[\bar{\xi},\bar{\eta}]}(\bar{x})).
\end{aligned}
$$

The derivative of f vanishes on G because of identities (4.12) and (4.4). This implies that $f \equiv 0$ since f vanishes at the identity e and G is connected.

Let us summarize the preceding discussion.

PROPOSITION 4.1. — *(Properties of the Group-Cocycles Associated with Twilled Extensions). Let* G *and* H *be connected and simply connected Lie groups with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$. *Suppose that* $\mathrm{A}_x : \mathfrak{h} \to \mathfrak{h}$ *and* $\mathrm{B}_\xi : \mathfrak{g} \to \mathfrak{g}$ *are representations of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *and of* $\mathfrak{h}$ *on* $\mathfrak{g}$ *which lend* $\mathfrak{g} \times \mathfrak{h}$ *the structure of a twilled extension, i.e., that* A_x *and* B_ξ *satisfy conditions (4.2) and (4.3) in addition to (4.1) and (4.4). Let* A_g *and* B_h *be the group representations whose differentials are* A_x *and* B_ξ. *Then there exists a unique one-cocycle* b *of the group* G *with values in* Hom $(\mathfrak{h}, \mathfrak{g})$, *and a unique one-cocycle* a *of the group* H *with values in* Hom $(\mathfrak{g}, \mathfrak{h})$ *which satisfy, respectively,*

$$(4.27)\qquad db(g)(v) = \mathrm{Ad}_g \circ \mathrm{B}_{d\mathrm{L}_{g(e)^{-1}}(v)} \circ \mathrm{A}_{g^{-1}}, \quad g \in \mathrm{G}, \quad v \in \mathrm{T}_g\mathrm{G}, \quad b_e = 0,$$

$$(4.28)\qquad da(h)(w) = \mathrm{Ad}_h \circ \mathrm{A}_{d\mathrm{L}_{h(e)^{-1}}(w)} \circ \mathrm{B}_{h^{-1}}, \quad h \in \mathrm{H}, \quad w \in \mathrm{T}_h\mathrm{H}, \quad a_e = 0.$$

The solution b of (4.27) satisfies the one-point relation (4.9) and the two-point cocycle condition (4.7), while the solution a of (4.28) satisfies the analogous relations. The integrability conditions for equation (4.27) for b are (4.1) and (4.3), while the integrability conditions for equation (4.28) for a are (4.4) and (4.2). Furthermore, (4.4) and (4.2) imply (4.12) and (4.13), while (4.1) and (4.3) imply the analogous conditions on a.

Once the one-cocycle $b : \mathrm{G} \to \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ has been constructed, the action of the twilled extension $\mathfrak{k}$ on G is readily obtained by introducing the vector fields on G,

$$r_x(g) := d\mathrm{R}_g(e)(x), \qquad x \in \mathfrak{g}, \quad g \in \mathrm{G}, \tag{4.29}$$

$$\varphi_\xi(g) := d\mathrm{R}_g(e) \circ b_g(\xi), \qquad \xi \in \mathfrak{h}, \quad g \in \mathrm{G}, \tag{4.30}$$

where R_g is the right-translation defined by g. We shall show that they satisfy the commutation relations

$$[r_x, r_y] = -\, r_{[x,y]}, \tag{4.31}$$

$$[r_x, \varphi_\xi] = r_{\mathrm{B}_x(\xi)} - \varphi_{\mathrm{A}_x(\xi)} \tag{4.32}$$

and

$$[\varphi_\xi, \theta_n] = -\, \varphi_{[\xi,\eta]}, \tag{4.33}$$

so that the vector fields

$$\varphi_{(x,\xi)} := -\,(r_x + \varphi_\xi), \qquad (x, \xi) \in \mathfrak{k} \tag{4.34}$$

yield an action of $\mathfrak{k}$ on G.

Commutation relation (4.29) is well known. To prove (4.30) we recall that (4.29) implies that

$$\mathrm{L}_{r_x}(\rho_\mu) = \rho_{{}^t\mathrm{ad}_x \mu} \tag{4.35}$$

for any right-invariant one-form ρ_μ defined on G, where $\mu \in \mathfrak{g}^*$. Then, from the relation

$$\langle \rho_\mu, \varphi_\xi \rangle (g) = \langle \mu, b_g(\xi) \rangle, \tag{4.36}$$

which is equivalent to definition (4.30), we obtain

$$\begin{aligned} \langle \rho_\mu, \mathrm{L}_{r_x}(\varphi_\xi) \rangle (g) &= -\langle \mathrm{L}_{r_x}(\rho_\mu), \varphi_\xi \rangle (g) + \langle \mu, (db(g) \circ r_x(g))(\xi) \rangle \\ &\overset{(4.35)}{=} -\langle \rho_{{}^t\mathrm{ad}_x \mu}, \varphi_\xi \rangle (g) + \langle \mu, (db(g) \circ r_x(g))(\xi) \rangle \\ &\overset{(4.8)}{=} -\langle \rho_{{}^t\mathrm{ad}_x \mu}, \varphi_\xi \rangle (g) + \langle \mu, \mathrm{Ad}_g \circ \mathrm{B}_{\mathrm{Ad}_{g^{-1}} x} \circ \mathrm{A}_{g^{-1}} \xi \rangle \\ &= \langle \mu, \mathrm{Ad}_g \circ \mathrm{B}_{\mathrm{Ad}_{g^{-1}} x} \circ \mathrm{A}_{g^{-1}} \xi - \mathrm{ad}_x \circ b_g(\xi) \rangle \\ &\overset{(4.9)}{=} \langle \mu, \mathrm{B}_x(\xi) - b_g \circ \mathrm{A}_x(\xi) \rangle \\ &= \langle \rho_\mu, r_{\mathrm{B}_x(\xi)} - \varphi_{\mathrm{A}_x}(\xi) \rangle (g), \end{aligned} \tag{4.37}$$

proving (4.32). Finally, to prove (4.33) we first observe that (4.32) implies that

$$\langle \mathrm{L}_{\varphi_\xi}(\rho_\mu), r_x \rangle = -\langle \rho_\mu, \mathrm{L}_{\varphi_\xi}(r_x) \rangle = \langle \rho_\mu, r_{\mathrm{B}_x(\xi)} - \varphi_{\mathrm{A}_x(\xi)} \rangle. \tag{4.38}$$

Then, taking the Lie derivative of (4.36) along the vector field φ_η we obtain

$$
\begin{aligned}
(4.39)\quad \langle \rho_\mu, \mathrm{L}_{\varphi_\eta}(\varphi_\xi) \rangle (g) &= \langle \mu, (db(g) \circ \varphi_\eta(g))(\xi) \rangle - \langle \mathrm{L}_{\varphi_\eta}(\rho_\mu), \varphi_\xi \rangle (g) \\
&\overset{(4.8)}{=} \langle \mu, \mathrm{Ad}_g \circ \mathrm{B}_{\mathrm{Ad}_{g^{-1}} b_g(\eta)} \circ \mathrm{A}_{g^{-1}} \xi \rangle + \langle \rho_\mu, \varphi_{\mathrm{A}_{b_g(\xi)}(\eta)} - r_{\mathrm{B}_{b_g(\xi)}(\eta)} \rangle \\
&\overset{(4.9)}{=} \langle \mu, \mathrm{B}_{b_g(\eta)}(\xi) + \mathrm{ad}_{b_g(\eta)} \circ b_g(\xi) - b_g \circ \mathrm{A}_{b_g(\eta)}(\xi) + b_g \circ \mathrm{A}_{b_g(\xi)}(\eta) - \mathrm{B}_{b_g(\xi)}(\eta) \rangle \\
&\overset{(4.13)}{=} \langle \mu, b_g([\xi, \eta]) \rangle \\
&= \langle \rho_\mu, \varphi_{[\xi,\eta]} \rangle (g),
\end{aligned}
$$

proving (4.33). The last statement about the commutation relations for the vector fields $\varphi_{(x,\xi)}$ follows from

$$
\begin{aligned}
(4.40)\qquad [\varphi_{(x,\xi)}, \varphi_{(y,\eta)}] &= [r_x, r_y] + [\varphi_\xi, r_y] + [r_x, \varphi_\eta] + [\varphi_\xi, \varphi_\eta] \\
&= -r_{([x,y] + \mathrm{B}_y(\xi) - \mathrm{B}_x(\eta))} - \varphi_{([\xi,\eta] + \mathrm{A}_x(\eta) - \mathrm{A}_y(\xi))} \\
&= \varphi_{([x,y] + \mathrm{B}_y(\xi) - \mathrm{B}_x(\eta), [\xi,\eta] + \mathrm{A}_x(\eta) - \mathrm{A}_y(\xi))} \\
&= \varphi_{[(x,\xi),(y,\eta)]_{\mathfrak{k}}}.
\end{aligned}
$$

We summarize these results in the following proposition.

PROPOSITION 4.2. — *(Canonical Representations of Twilled Extensions). Let* G *and* H *be connected and simply connected Lie groups equipped with infinitesimal actions* $\mathrm{A}_x : \mathfrak{h} \to \mathfrak{h}$ *and* $\mathrm{B}_\xi : \mathfrak{g} \to \mathfrak{g}$ *that lend* $\mathfrak{k} = \mathfrak{g} \times \mathfrak{h}$ *the structure of a twilled extension. We denote the corresponding group-cocycles defined in 3.1 by* $b : \mathrm{G} \to \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$ *and by* $a : \mathrm{H} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$*. Then the twilled extension* $\mathfrak{k} = \mathfrak{g} \times \mathfrak{h}$ *acts canonically on* G *(resp., on* H*) by the vector fields on* G *(resp.,* H*),*

$$
\begin{aligned}
\varphi_{(x,\xi)}(g) &= -\,d\mathrm{R}_g(e)(x + b_g\xi) = -\,(r_x + \varphi_\xi)(g), &\quad g \in \mathrm{G}, \\
(\text{resp., } \psi_{(x,\xi)}(h) &= -\,d\mathrm{R}_h(e)(\xi + a_h x) = -\,(r_\xi + \psi_x)(h), &\quad h \in \mathrm{H}),
\end{aligned}
$$

and there is a canonical representation of the twilled extension $\mathfrak{k}$ *in the space of smooth functions over each of the factor Lie groups.*

If $\mathfrak{k}$ is a left-exact dual twilled extension defined by a Jacobian potential $p : \mathfrak{h} \to \mathfrak{g}$, then, by definition, $\mathrm{B} = \delta p$, *i. e.*, by (2.1),

$$
(4.41)\qquad \mathrm{B}_x = \mathrm{ad}_x \circ p - p \circ \mathrm{A}_x,
$$

where $\mathrm{A}_x = \mathrm{ad}_x^*$. Therefore

$$
(4.42)\qquad b_g = \mathrm{Ad}_g \circ p \circ \mathrm{A}_{g^{-1}} - p,
$$

where $\mathrm{A}_g = \mathrm{Ad}_g^*$ and, by (4.30),

$$
(4.43)\qquad \varphi_\xi(g) = d\mathrm{L}_g(e) \circ p \circ \mathrm{A}_{g^{-1}}(\xi) - d\mathrm{R}_g(e) \circ p(\xi).
$$

Let X_ξ be the equivariant family of vector fields on G parametrized by $\mathfrak{h}$ defined by p as in (3.17). Then

$$
(4.44)\qquad \varphi_\xi = \mathrm{X}_\xi - r_{p\xi}
$$

and, by definition (4.34),

$$\varphi_{(x,\xi)} = -(r_{x-p\xi} + X_\xi). \tag{4.45}$$

We conclude this study of the representation of the algebra $\mathfrak{k}$ on the group G with a final remark regarding the transformation properties of the vector fields $\varphi_{(x,\xi)}$ with respect to the left-translations on G. As is well known, the vector fields r_x are Ad-equivariant, *i. e.*, they satisfy the transformation law

$$dL_g(g') \circ r_x(g') = r_{\mathrm{Ad}_g x}(gg'). \tag{4.46}$$

There is no such simple transformation law for the vector fields φ_ξ. Because of the b_g's cocycle property they satisfy the more complicated transformation law

$$dL_g(g') \circ \varphi_\xi(g') + dR_{g'}(g) \circ \varphi_{A_g\xi}(g) = \varphi_{A_g\xi}(gg'), \tag{4.47}$$

as can be easily shown from the identity

$$\begin{aligned} (4.48)\quad & dL_g(g') \circ \varphi_\xi(g') + dR_{g'}(g) \circ \varphi_{A_g\xi}(g) - \varphi_{A_g\xi}(gg') \\ &= dL_g(g') \circ dR_{g'}(e) \circ b_{g'}\xi + dR_{g'}(g) \circ dR_g(e) \circ b_g \circ A_g\xi - dR_{gg'}(e) \circ b_{gg'} \circ A_g\xi \\ &= dR_{gg'}(e) \circ \mathrm{Ad}_g \circ b_{g'}\xi + dR_{gg'}(e)(b_g \circ A_g\xi - b_{gg'} \circ A_g\xi) \\ &= dR_{gg'}(e) \circ (\mathrm{Ad}_g \circ b_{g'} \circ A_{g^{-1}} + b_g - b_{gg'}) \circ A_g\xi . \end{aligned}$$

This transformation law can be given a geometric meaning by introducing the multiplication map $\pi : G \times G \to G$ defined by $\pi(g, g') = gg'$. We shall have to invoke this transformation law (4.47) in parts II and III of this article where we shall call it the *Drinfeld property* of the vector fields φ_ξ.

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REFERENCES

[1] R. AMINOU and Y. KOSMANN-SCHWARZBACH, Bigèbres de Lie, doubles et carrés. *Annales Inst. Henri Poincaré*, série A *(Physique théorique)*, t. **49**, n° 4, 1988, p. 461-478.

[2] N. BOURBAKI, *Groupes et algèbres de Lie*, *Chapitre 1*, *Algèbres de Lie*, Hermann, Paris, 1960.

[3] N. BOURBAKI, *Groupes et algèbres de Lie*, *Chapitres 2 et 3*, Hermann, Paris, 1972.

[4] V. G. DRINFELD, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations. *Soviet Math. Dokl.*, t. **27**, no. 1, 1982, p. 68-71.

[5] V. G. DRINFELD, Quantum groups. *Proceedings Int. Congress Math.* (Berkeley, 1986), Amer. Math. Society, 1988.

[6] I. M. GELFAND and I. Ya. DORFMAN, Hamiltonian operators and the classical Yang-Baxter equation. *Funct. Anal. Appl.*, t. **16**, no. 4, 1982, p. 241-248.

[7] Y. KOSMANN-SCHWARZBACH, Poisson-Drinfeld groups, in *Topics in Soliton theory and exactly solvable nonlinear equations*, M. Ablowitz, B. Fuchssteiner and M. Kruskal, eds., World Scientific, Singapore, 1987.

[8] F. MAGRI, *Pseudocociclo di Poisson e strutture PN gruppale, applicazione al reticolo di Toda*, unpublished manuscript, Milan, 1983.

[9] M. A. SEMENOV-TIAN-SHANSKY, What is a classical r-matrix? *Funct. Anal. Appl.*, t. **17**, no. 4, 1983, p. 259-272.

[10] M. A. SEMENOV-TIAN-SHANSKY, Dressing transformations and Poisson group actions. *Publ. RIMS* (Kyoto University), t. **21**, 1985, p. 1237-1260.

[11] M. A. SEMENOV-TIAN-SHANSKY, Classical r-matrices, Lax equations, Poisson-Lie groups and dressing transformations, in *Field theory, quantum gravity and strings, II*, H. J. de Vega and N. Sanchez, eds., *Lecture notes in physics*, Springer Verlag, Berlin, t. **280**, 1987, p. 174-214.

[12] E. K. SKLYANIN, Quantum version of the method of inverse scattering problem. *Journal of Soviet Math.*, t. **19**, no. 5, 1982, p. 1546-1596.

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