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Quantum potential, uncertainty and the classical limit

by

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ABSTRACT. — We discuss generally and with reference to typical examples the classical limit of quantum mechanics, including equations of motion and the uncertainty relations, from the point of view of the causal interpretation. We show how universal criteria may be formulated in terms of sufficient conditions to be satisfied by the wavefunction in order that the classical results are recovered. These overcome the problems associated with limiting procedures such as $\hbar \to 0$ or high quantum numbers.

RÉSUMÉ. — Nous discutons de manière générale et exemples à l'appui la limite classique de la mécanique quantique, y compris les équations du mouvement et les relations d'incertitude, dans l'optique de l'interprétation causale. Nous montrons comment des critères universels peuvent être formulés en termes de conditions suffisantes que la fonction d'onde doit satisfaire afin de retrouver les résultats classiques. Ces critères surmontent les problèmes associés aux passages à la limite tels que $\hbar \to 0$ où celle des grands nombres quantiques.

1. INTRODUCTION

Any transformation in thought brings with it its dissenters and doubters. These may be reactionaries harking back to a system of ideas they feel comfortable with out of habit, but they may also prick the conscience of

the majority and remind them that perhaps there is a better way. All opposition contains a grain of truth. Throughout his long career in theoretical physics, Jean-Pierre Vigier has sought to put across in his work a very distinctive point of view. Challenging the most basic tenets of the orthodox interpretation of quantum mechanics, he has consistently argued that a materialist, causal interpretation of the theory is both possible and desirable. Because it has been ignored or misrepresented, this approach is still marginal in quantum physics. Yet the orthodoxy, with which we have lived since the 1920's, faces serious problems. Not least through the efforts of Jean-Pierre Vigier, the quest for a deeper explanation of quantum phenomena continues unabated. It is in the spirit of this debate that we dedicate this essay to Jean-Pierre.

The detailed relationship between classical and quantum mechanics is subtle and still not completely understood. It involves both conceptual and formal aspects. To illustrate the former we may note that in Bohr's interpretation of quantum mechanics the validity of classical concepts is already *presupposed* since, it is suggested, it is only in terms of these notions that we can unambiguously communicate the results of experiments in the quantum domain [*1*]. That is, classical physics must be considered as *prior* to quantum mechanics. Thus, in Bohr's approach, any procedure involving the second, formal, aspect, by which classical mechanics is recovered from quantum mechanics as a mathematical limit, can only be a demonstration of *consistency* with the already postulated epistemological relation between the two and not as a « derivation » of classical mechanics from quantum mechanics. Nevertheless, the discussion of this problem is usually carried on as if quantum mechanics is the more fundamental theory from which classical mechanics emerges when certain parameters naturally occurring in the theory are varied (e. g. $\hbar \to 0$ or $n \to \infty$ (high quantum numbers)). Such limits however have never been shown to have general validity and, indeed, as we shall see, do not always result in the physical behaviour expected from classical physics. (In fact, the limits $\hbar \to 0$ and $n \to \infty$ are not equivalent [*2*]). The question is therefore left open as to what constitutes a *universal* criterion for recovering classical mechanics as a limit of quantum mechanics.

The causal interpretation of quantum mechanics greatly clarifies and simplifies this problem [*3*]. It does not presuppose a « classical level » but rather deduces classical mechanics as a special case of quantum mechanics both conceptually and mathematically; it is thus more in harmony with the way in which most physicists think of the relation between the two theories. Moreover, the limiting process provides a universal criterion for when we can expect motions typical of the classical domain in terms of a condition to be satisfied by the wavefunction (vanishing quantum potential). It is thus not necessary to let constants such as $\hbar$ « vary ».

The purpose of this paper is to bring out the value of the quantum poten-

tial approach in discussing the classical limit for several typical examples. We include a discussion of the uncertainty relations, and briefly comment on spin $-$ 1/2 fields.

2. THE CLASSICAL LIMIT OF QUANTUM MECHANICS

All of our discussion is non-relativistic. Substituting $\psi = R \exp(iS/\hbar)$ into the Schrödinger equation yields

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + Q + V = 0 \tag{1}$$

and

$$\frac{\partial R^2}{\partial t} + \frac{1}{m}\nabla\cdot(R^2\nabla S) = 0 \tag{2}$$

where $Q = (-\hbar^2/2mR)(\nabla^2 R)$ is the quantum potential. According to the causal interpretation, eq. (1) is an equation of the Hamilton-Jacobi type which implies a new theory of motion: the phase function S is a physically effective field (coupling to R and hence Q via (2)) and not merely a mathematically convenient function of the kind employed in classical mechanics. Correspondingly, we have a new conception of matter: each individual « material system » comprises particle and field aspects, the latter being mathematically described by $\psi(\underline{x}, t)$. The particle trajectories are the solutions to the equation

$$m\dot{\underline{x}} = \nabla S\,|_{\underline{x}=\underline{x}(t)} \tag{3}$$

or of

$$m\ddot{\underline{x}} = -\nabla(V + Q)\,|_{\underline{x}=\underline{x}(t)} \tag{4}$$

once we have specified the initial position $\underline{x}_0 = \underline{x}(0)$. In this interpretation quantum mechanics is a causal theory of individuals which is not tied to any particular scale (micro or macro) and which is not essentially statistical in character. To ensure consistency with the results of quantum mechanics the subsidiary condition is imposed that R^2 is the probability density of a particle being at the point $\underline{x}$ at time t. An ensemble of particles is thus associated with the same ψ-field which only differ in their initial positions. Only one material system is ever present at any one time however.

If we set $Q = 0$ eq. (1) reduces to the classical Hamilton-Jacobi equation. It is convenient to work now in one dimension. For a stationary state of energy $-\partial S/\partial t = E$, with $E > V$, we can solve (1) and (2) (with $Q = 0$) to obtain an explicit expression for the wavefunction in terms of E and V:

$$\psi^{\pm}_{WKB}(x, t) = \frac{A_{\pm}}{[2m(E - V(x))]^{\frac{1}{2}}} \exp\frac{i}{\hbar}\left(\pm\int^{x}[2m(E - V(x))]^{\frac{1}{2}}dx - Et\right) \tag{5}$$

where $A_{\pm}$ is constant. This of course is the WKB wavefunction. Now the latter is usually obtained as the result of an asymptotic expansion of ϕ, where $\psi = e^{i\phi/\hbar}$, in terms of powers of $\hbar$: $\phi = \phi_0 + \hbar\phi_1 + \hbar^2\phi_2 + \ldots$. The criteria which allow us to neglect the terms in this expansion beyond ϕ_0 and ϕ_1 (and hence obtain the function (5)) are expressed in terms of the variation of the potential and the magnitude of the de Broglie wavelength as follows:

$$\frac{\hbar m(\partial V/\partial x)}{[2m(E - V(x))]^{3/2}} \ll 1 \tag{6}$$

$$\frac{m\hbar^2(\partial^2 V/\partial x^2)}{2[2m(E - V(x))]^2} \ll 1 \tag{7}$$

and so on for higher derivatives of V. What (6) means is that the fractional change in the wavelength within the distance of a wavelength is small compared with unity. This will be so when V is a slowly varying function of x and $E - V$ is not too small. Conditions (6) and (7) are consistent with the initial supposition $Q = 0$ since from them we can deduce that

$$\frac{-\hbar^2}{2m}(E - V(x))^{\frac{1}{4}} \frac{\partial^2}{\partial x^2}(E - V(x))^{-\frac{1}{4}} \ll E - V(x) \tag{8}$$

which will be recognised as the condition that the quantum potential evaluated from the amplitude of the wavefunction (5) is negligible in comparison with $E - V$.

We obtain the classical limit, then, when Q may be neglected in relation to the other relevant energies, and (8) states the condition to be satisfied by the external potential which is necessary for this. The important point is that the converse is not necessarily true—the conditions (6)—(8) do not always guarantee that Q will be negligible and hence on their own they do not constitute universal criteria for when we are approaching the classical limit. In other words, the short wavelength, high quantum numbers limit does not always result in classical behaviour. This is actually rather obvious and is easily demonstrated by simple examples (e. g. the superposition of two WKB wavefunctions (5)) but it has not been sufficiently stressed in the literature.

It is often stated that classical mechanics is recovered in the limit of high quantum numbers and indeed this was the view of Einstein [*4*]. Schiff [*5*] states: « The agreement between classical and quantum probability densities improves rapidly with increasing n » (for the harmonic oscillator). Yet this is only true for a restricted class of states. The basic problem with limiting procedures such as $\hbar \to 0$ or $n \to \infty$ is that the wavefunction depends on these parameters and so the quantum potential does not necessarily vanish when the limit is taken (even though Q contains the factor $\hbar^2$). As we have seen the latter is necessary in order to recover the classical

Hamilton-Jacobi equation. As a result, we do not always find in the limit that the accessible particle motions are those which obtain for the given potential V in the corresponding classical problem. We shall see this explicitly in the next section. The implications for the Principle of Equivalence of gravitation and inertia in the quantum domain have been discussed elsewhere [6].

To summarize, the distinction between quantum and classical mechanics does not lie in the finiteness of Planck's constant in the one and its absence in the other, or in a particular range of values taken by quantum numbers, but in the value of the quantum potential. The vanishing of the latter provides a universal and generally applicable criterion for the classical limit; in those regions where it is not negligible we obtain typical quantum effects ([1]). In the causal interpretation there is no need to introduce a « cut » between quantum and classical levels and it is clearly evident that quantum physics is not tied just to the microscopic level.

3. STATIONARY STATES

We have seen that the classical Hamilton-Jacobi equation is recovered when $Q \sim 0$ in (1). If this condition is satisfied the S field decouples from the R field and the theory reduces to the classical statistical mechanics of a single particle. We shall now study some typical examples where Q does not become negligible when $\hbar \to 0$ and/or $n \to \infty$.

In a stationary state, $\psi(\underline{x}, t) = u(\underline{x}) \exp(-iEt/\hbar)$ and the energy $-\partial S/\partial t = E$ is a constant of the motion. In addition, the quantum potential and the velocity field $\nabla S/m$ are independent of time. In fact, since the energy eigenstates are often real functions, the associated velocity field is often zero. The particle is at rest where one would classically expect it to move since the quantum force $(-\nabla Q)$ cancels the classical force $(-\nabla V)$.

3.1. Particle in a Box.

In one dimension the eigenfunctions for a particle trapped between infinitely high walls placed at $x = \pm a$ are

$$u_n(x) = \frac{1}{\sqrt{a}} \sin Kx, \qquad K = \frac{n\pi}{2a}, \qquad n \in \mathbb{Z}. \tag{9}$$

This function possesses a distribution of nodes. The higher the quantum number n the *more* nodes will be present and hence the more positions

([1]) To be absolutely precise we should say that $Q \sim 0$ is a *sufficient* condition for the classical limit, it sometimes happens that we obtain classically expected individual motions even when Q is not negligible. Cf. § 4.1.

inaccessible to the particle. The particle motion implied by the causal interpretation is consistent with nodes because the spatial part of ψ, i. e. (9), is real so that $\underline{v} = \nabla S/m = 0$. The energy entirely resides in quantum potential energy:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2 = Q\,. \tag{10}$$

There is no limit that we can perform which would result in the motion expected from classical mechanics in this situation, i. e. uniform motion backwards and forwards with equal likelihood.

The wavefunction (9) in fact predicts that if we perform a *measurement* to determine the momentum then we will obtain as results either $\pm \hbar K$, with equal probability. But to do this we have to remove the barriers and this implies a new wavefunction and quantum potential and consequently different particle motion. If at time $t = 0$ we suddenly remove the walls, the wavefunction at time t is given by

$$\psi(x, t) = \frac{1}{2\pi\sqrt{a}}\int_{-\infty}^{\infty} dk\left(\frac{\sin(K + k)a}{K + k} - \frac{\sin(K-k)a}{K - k}\right) \times \\ \times \exp\{\, i(kx - \hbar K^2 t/2m)\,\}\,. \tag{11}$$

That is, two separating packets begin to form peaked around $k = \pm K$ in k-space. The maximum in phase occurs where $\partial S/\partial k = 0$ which yields for the motion of the centres of the packets $x = \pm \hbar K t/m$. The particle enters one of the packets with approximate velocity $\pm \hbar K/m$. The results of a momentum measurement are thus causally but not directly related to the actual momentum of the system (in this case, zero) prior to the operation of measurement. In fact the latter is treated as a particular application of the theory and does not possess the fundamental significance it is afforded in the usual interpretation.

3.2. **Harmonic Oscillator.**

Consider another one-dimensional problem with $V(x) = (1/2)m\omega^2 x^2$. The eigenfunctions are given by:

$$u_n(x) = \sqrt{\frac{(m\omega \mid \hbar)}{\sqrt{\pi} 2^n n\,!}}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \tag{12}$$

where H_n is the Hermite polynomial of order n. We can calculate the quantum potential for the nth eigenstate by means of the formula:

$$Q = -\frac{\hbar\omega}{2}\,\frac{\dfrac{\partial^2}{\partial\xi^2}\left(H_n(\xi)e^{-\frac{1}{2}\xi^2}\right)}{H_n(\xi)e^{-\frac{1}{2}\xi^2}} \tag{13}$$

which yields:

$$Q = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{1}{2}m\omega^2x^2. \tag{14}$$

Once again $v = \nabla S/m = 0$ so that the total energy of the system comprises just the oscillator and quantum potential (the latter cancels the former to yield the familiar quantized oscillator energy).

We cannot deduce by taking the limits $\hbar \to 0/n \to \infty$ any of the features characteristic of a classical oscillator. The velocity stays zero, the quantum potential remains finite and the probability density derived from (12) is quite distinct from the classical probability density which is, from (5),

$$\rho_l = (1 \mid \pi)(x_0^2 - x^2)^{-\frac{1}{2}} \tag{15}$$

where x_0 is the amplitude of the motion. The function (15) follows the local average of $\mid u_n \mid^2$ for high n but the latter possesses nodes which are incompatible with oscillatory motion.

4. OSCILLATING PACKETS

4.1. One Packet.

We now consider a case where the quantum analogue of a classically conserved system admits solutions which imply non-conservative motion due to a time-dependent quantum potential. It is particularly instructive since it provides an example of how precisely the motion expected in the classical case is obtained for the quantum particle motion even through Q is not negligible. It is well known that we can construct a non-dispersive Gaussian shaped wave packet by an appropriate superposition of the stationary wavefunctions (12):

$$\psi(x, t) = \sum_{n=0}^{\infty} A_n u_n(x) e^{-iE_n t/\hbar}.$$

If we choose $A_n = \left(\xi_0^n e^{-\frac{1}{4}\xi_0^2}/(2^n n!)^{\frac{1}{2}}\right)$ then the packet is centered around ξ_0 $\left(\xi_0 = \alpha x_0 = \sqrt{\frac{m\omega}{\hbar}}\, x_0\right)$ at $t = 0$ and [5]:

$$\psi(\xi, t) = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\frac{1}{2}(\xi - \xi_0 \cos\omega t)^2 - \frac{i}{2}\left(\omega t + 2\xi\xi_0 \sin\omega t - \frac{1}{2}\xi_0^2 \sin 2\omega t\right)}. \tag{16}$$

The position probability density reads:

$$\mid \psi(\xi, t) \mid^2 = \frac{\alpha}{\pi^{1/2}} e^{-(\xi - \xi_0 \cos\omega t)^2}. \tag{17}$$

The wavefunction is thus non-dispersive and its centre oscillates between the points $x = \pm x_0$.

We now consider the individual energies associated with this state:

$$E_{KIN} = \frac{1}{2} m\omega^2 x_0^2 \sin^2 \omega t \tag{18}$$

$$V = \frac{1}{2} m\omega^2 x^2 \tag{19}$$

$$Q = -\frac{1}{2} m\omega^2 (x - x_0 \cos \omega t)^3 + \frac{1}{2}\hbar\omega \tag{20}$$

which add up to give the total energy

$$-\frac{\partial S}{\partial t} = \frac{1}{2}\hbar\omega + m\omega^2 x x_0 \cos \omega t - \frac{1}{2} m\omega^2 x_0^2 \cos 2\omega t. \tag{21}$$

The trajectory of a particle guided by this packet is given by the solution to

$$\frac{1}{\hbar} m\dot{x} = \frac{1}{\hbar}\frac{\partial S}{\partial x} = -\alpha^2 x_0 \sin \omega t. \tag{22}$$

At $t = 0$, $\dot{x} = 0$. The solution is

$$x(t) = x_0 \cos \omega t + c \tag{23}$$

where c is a constant which measures the distance of the particle from the centre of the packet: $x(0) = x_0 + c$. The particle therefore performs a simple harmonic motion of amplitude x_0 about a centre point $x = c$. Now this is precisely the motion of a classical oscillator in the potential $V = (1/2)m\omega^2(x - c)^2$, with equation of motion

$$\ddot{x} + \omega^2 x = \omega^2 c. \tag{24}$$

With initial conditions $x(0) = x_0 + c$ and $\dot{x}(0) = 0$, the solution to (24) is (23). Eq. (24) should be compared with the quantum equation of motion (4):

$$\ddot{x} = -\omega^2 x_0 \cos \omega t. \tag{25}$$

It follows that when $c = 0$, so that the quantum particle is at the centre of the packet, the particle in the state (16) behaves precisely as a classical particle in the same potential $V = \frac{1}{2} m\omega^2 x^2$. Yet this result is obtained in a situation where the quantum potential is far from being negligible. As is clear from (19) and (20) part of Q cancels V and it is the variable remainder that brings about the S*hm*.

Is there nevertheless a limit in which Q becomes negligible? If we let $\hbar \to 0$, then from (20) we see that

$$Q \to -\frac{1}{2} m\omega^2 (x - x_0 \cos \omega t)^2. \tag{26}$$

The resulting quantum potential gives rise to exactly the same motions as were obtained with a finite $\hbar$ indeed the latter has no bearing at all on the motion and only appears in a constant zero point contribution to the total energy (via Q). Eq. (26) thus provides a striking example of how $Q \not\to 0$ as $\hbar \to 0$. To be sure, as $\hbar \to 0$ the width of the packet, $\Delta x = (\hbar/2m\omega)^{\frac{1}{2}}$ decreases, but, as with the harmonic oscillator stationary states (§ 3.2), there does not seem to be any simple limit in which we can make Q small in relation to the other energies in the Hamilton-Jacobi equation and recover the classical probability density (15) from (17).

4.2. **Interference of packets.**

To conclude our discussion of the particle in a harmonic oscillator potential we shall consider the case of the superposition of two packets of the form (16), centred about $x = \pm x_0$ respectively at $t = 0$. The particle motions resulting from the interference of the packets are qualitively distinct from the classical-type motion that we found in § 4.1. We have:

$$\psi(\xi, t) = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\frac{1}{2}(\xi^2 + \xi_0^2 \cos^2 \omega t) + \frac{i}{4}\xi_0^2 \sin 2\omega t} 2 \cosh [\xi \xi_0 e^{-i\omega t}] \tag{27}$$

with a position probability density:

$$|\psi(\xi, t)|^2 = \frac{\alpha}{\pi^{1/2}} e^{-(\xi^2 + \xi_0^2 \cos^2 \omega t)} 2 [\cosh (2\xi\xi_0 \cos \omega t) + \cos (2\xi\xi_0 \sin \omega t)].$$

It is convenient to restrict ourselves to the case where the centres of the packets coincide ($\cos \omega t = 0$). Then (27) has nodes at the points $x = (\alpha^2 x_0)^{-1}(2n + 1)\pi/2$ and the quantum potential

$$Q = -\frac{\hbar\omega}{2} (\xi^2 - \xi_0^2 - 1 + 2\xi\xi_0 \tan \xi\xi_0)$$

becomes singular there. This result is typical of interference phenomena (although Q is not always singular at nodes as we saw in § 3) and is irreducible; we cannot perform a limiting procedure that would lead to $Q \to 0$.

The case of a many-body system in a harmonic oscillator potential has been discussed elsewhere [7].

5. MEAN VALUES AND THE UNCERTAINTY RELATIONS

Having established a sufficient criterion which enables us to pass from the quantum to the classical *equations of motion*, one might ask whether we can arrive at the classical limit of other aspects of the quantum formalism

by a similar limiting procedure—for example, the operator structure, and the commutation relations as manifested through mean values and the uncertainty relations. In fact these concepts are closely related with the quantum potential since the latter arises (along with $(\nabla S)^2/2m$) from the kinetic energy operator in the Schrödinger equation. Is then the vanishing of Q sufficient to imply classical mean values and uncertainty relations, or are we compelled to attempt to apply a limiting procedure based on $\hbar \to 0$ (which we already know to be problematic from our results above)?

It turns out that the vanishing of the quantum potential on its own is not sufficient to establish the classical limit of the uncertainty relations, but that a condition of a similar nature can be formulated in terms of the stress tensor associated with the ψ-field.

Let us begin by considering mean values. According to classical statistical mechanics (in the Hamilton-Jacobi formalism) the mean momentum and energy of a free particle, over an ensemble, are given by

$$\langle \underline{p} \rangle = \int R^2 \nabla S d\underline{x} \tag{28}$$

$$\left\langle \frac{1}{2m} \underline{p}^2 \right\rangle = \int R^2 [\nabla S)^2/2m] d\underline{x} \tag{29}$$

respectively, where R^2 is the probability density (R being the amplitude of the classical wavefunction). In quantum mechanics we expect the expression (28) for the mean momentum to be the same, but (29) will be modified since a (classically) free particle has quantum potential energy. That is, the mean energy of a (classically) free quantum mechanical system will be given by:

$$\int R^2 \left(\frac{1}{2m} (\nabla S)^2 + Q \right) d\underline{x}. \tag{30}$$

Now it is hardly surprising that (28) and (30) are just the usual quantum mechanical expectation values for the operators $\hat{\underline{p}} = -i\hbar\nabla$ and $\hat{\underline{p}}^2/2m$, respectively. And so we see that the quantum mechanical definition of averages, $\langle \hat{A} \rangle = \langle \psi, \hat{A}\psi \rangle$ where $\hat{A}$ is some non-multiplicative operator, has in these cases a simple physical meaning: it is equivalent to taking an ensemble average according to the classical prescription, but taking care to include contributions of purely quantum mechanical origin.

This result can be easily extended to the case where $\hat{A}$ is an arbitrary polynomial function of $\hat{\underline{x}}$ and $\hat{\underline{p}}$. Physically, polynomials in $\hat{\underline{p}}$ higher than the second order do not appear to be relevant. Nevertheless, should they prove to be useful, we will find contributions to the mean values of such operators analogous to Q, in addition to the classical terms (for $\hat{\underline{p}}^n$ the latter is $(\nabla S)^n$, etc.). And once again the reason for the appearance of the non-classical contributions to the ensemble average comes about

from the properties of the ψ-field, and the mean value defined in this way is mathematically equivalent to the usual definition of $\langle \hat{A} \rangle$ given above.

The classical limit of quantum mechanical expectation values is then readily established: this simply follows when we are able to neglect the extra local contributions to the ensemble average which are characteristic of quantum mechanics (in (30) this is Q).

If the light of these remarks, let us turn to the uncertainty relations. The key point is that these are deductions from the *tensorial* relations:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \qquad i, j = 1, 2, 3. \tag{31}$$

Defining

$$\left.\begin{aligned}(\Delta\hat{x}_i)^2 &= \langle \hat{x}_i^2 \rangle - \langle \hat{x}_i \rangle^2 \\ (\Delta\hat{p}_i)^2 &= \langle \hat{p}_i^2 \rangle - \langle \hat{p}_i \rangle^2\end{aligned}\right\} \tag{32}$$

where in rectangular Cartesian coordinates, with $\psi = Re^{iS/\hbar}$,

$$\langle \hat{x}_i^2 \rangle = \int x_i^2 |\psi|^2 d\underline{x} = \int R^2 x_i^2 d\underline{x} \tag{33}$$

$$\langle \hat{p}_i^2 \rangle = -\hbar^2 \int \psi^* \partial_i^2 \psi d\underline{x} = \int R^2 (\partial_i S)^2 d\underline{x} - \hbar^2 \int R^2 (\partial_i^2 R/R) d\underline{x} \tag{34}$$

the Heisenberg relations are

$$(\Delta\hat{x}_i)^2 (\Delta\hat{p}_j)^2 \geqslant (\hbar/2)^2 \delta_{ij}. \tag{35}$$

Now $(\Delta\hat{x}_i)^2$ and $(\Delta\hat{p}_i)^2$ are each components (lying on the diagonal) of rank 2 tensor quantities and we might expect therefore that a sufficient condition to obtain the classical limit of (35) is that we can neglect some tensor quantity constructed from ψ. We shall now see that this is so—the proof of (35) tacitly assumes that certain tensor components are finite.

One usually proves (35) by applying the Schwarz inequality:

$$\int |f|^2 d\underline{x} \int |g|^2 d\underline{x} \geqslant \left| \int f^* g d\underline{x} \right|^2 \tag{36}$$

where f, g are complex functions and it is assumed that the integrals converge. In one dimension, the proof proceeds [8] by letting $f = x\psi$, $g = p\psi$, so that the right hand side of (36) becomes $\left| \int \psi^* xp\psi dx \right|^2$, and writing $xp = (1/2)(xp + px) + (1/2)i\hbar$. To bring out the assumptions implicit in this proof, we shall demonstrate (35) in a slightly different way.

We assume without loss of generality that $\langle \hat{x}_i \rangle = \langle \hat{p}_i \rangle = 0$. Multiplying (33) by the last term on the right hand side of (34) we have:

$$-\hbar^2 \int x_i^2 \mid \psi \mid^2 d\underline{x} \int \mathrm{R}\partial_j^2 \mathrm{R} d\underline{x} \tag{37}$$

$$= \hbar^2 \int \mid x_i \psi \mid^2 d\underline{x} \int (\partial_j \mathrm{R})^2 d\underline{x} \text{ (by parts)} = \hbar^2 \int \mid x_i \psi \mid^2 d\underline{x} \int \mid \psi \partial_j \log \mathrm{R} \mid^2 d\underline{x} \tag{38}$$

$$\geqslant \hbar^2 \left| \int x_i \psi^* \psi \partial_j \log \mathrm{R} d\underline{x} \right|^2 = (\hbar^2/4) \left| \int x_i \partial_j \mathrm{R}^2 d\underline{x} \right|^2 \tag{39}$$

$$= (\hbar/2)^2 \delta_{ij} \tag{40}$$

by parts and using $\int \mathrm{R}^2 d\underline{x} = 1$. In going from lines (38) to (39) we have used (36) with $f = x_i$ and $g = \psi \partial_j \log \mathrm{R}$. It is easy to see also that

$$\int \mid x_i \psi \mid^2 d\underline{x} \int \mathrm{R}^2 (\partial_j \mathrm{S})^2 d\underline{x} \geqslant 0. \tag{41}$$

Combining the results (40) and (41) we then recover (35).

This proof assumes that the integral $\left(\hbar^2 \int \mathrm{R}\partial_j^2 \mathrm{R} d\underline{x} \right)$ is finite. If it is not, then the proof breaks down and (35) reduces to (41), the classical relations. To see the *local* significance of this assumption, we write down the stress tensor of the Schrödinger field in terms of the R and S fields, as derived from the Lagrangian [9]:

$$\mathrm{T}_{ij} = (1/m)\mathrm{R}^2 \partial_i \mathrm{S} \partial_j \mathrm{S} + \theta_{ij} \tag{42}$$

where

$$\theta_{ij} = \frac{\hbar^2}{2m} (2\partial_i \mathrm{R} \partial_j \mathrm{R} - \delta_{ij} (\mathrm{R}\nabla^2 \mathrm{R} + (\nabla \mathrm{R})^2)). \tag{43}$$

Clearly, the first term on the right hand side of (42) represents the classical stress tensor density of a free ensemble of particles, and (43) contains the quantum effects. The mean stress tensor is then easily shown to be

$$\int \mathrm{T}_{ij} d\underline{x} = (1/m) \int \mathrm{R}^2 \partial_i \mathrm{S} \partial_j \mathrm{S} d\underline{x} - (\hbar^2/m) \int \mathrm{R}\partial_{ij} \mathrm{R} d\underline{x} \tag{44}$$

by parts. Comparing (44) with (37) we see that a sufficient covariant local condition for the validity of the above proof of the uncertainty relations is that part of the quantum stress tensor be non-vanishing:

$$(1/\mathrm{R}^2)\theta_{ij} \neq 0. \tag{45}$$

(We have multiplied by the factor $1/\mathrm{R}^2$ since θ_{ij} is a density).

If this condition is not satisfied then T_{ij} reduces to its classical form and, as we have said, we obtain (44) ([2]).

As with the vanishing of the quantum potential, it cannot be guaranteed that $(1/R^2)\theta_{ij}$ becomes negligible when $\hbar \to 0$ indeed, both $\Delta\hat{x}_i$ and $\Delta\hat{p}_i$ depend on $\hbar$. We conclude that *it is not necessary, or necessarily consistent, to assume that* $\hbar \to 0$ *in order to achieve the result (41).*

The origin of the statistical correlations between $\Delta\hat{x}_i$ and $\Delta\hat{p}_i$ may thus be considered to be locally due to the distribution of stresses in the ψ-field. In fact, the existence of a relation between $\Delta\hat{x}_i$ and $\Delta\hat{p}_i$, which becomes apparent over an ensemble of trials, comes about because the field (ψ) which guides each individual particle also enters into the definition of mean values. We have seen that the physical aspect of the field which is relevant to the determination of mean values here is the stress tensor. As regards the particle motion the latter has precisely the effect of the quantum force since, as is easily shown,

$$\partial_j\theta_{ij} = R^2\partial_i Q .$$

The Heisenberg relations thus reflect statistically deviations from classical motion due to a quantum force and have nothing to do with the issue of whether or not a particle actually possesses simultaneously well-defined position and momentum variables.

We have focussed attention on the Heisenberg relations because of their historical importance. It should be noted however that these are only one possible deduction from (31) and they are not always the most physically useful correlation relations; one may derive many other types of « uncertainty » relations [*10*].

6. SPIN-1/2 AND QUANTUM FIELDS

The point of view advocated in this paper is that all characteristic quantum effects are mediated by the quantum potential. In a series of papers we have shown how this approach can be extended to include spin-1/2 phenomena [*11*], [*12*]. This is best developed in the angular coordinate representation of the underlying quantum theory [*12*], [*13*]. A quantum system with spin comprises an extended rotator whose orientation is specified

([2]) The energy density (T_{00}), momentum density (T_{i0}) and energy current density (T_{0i}) of the Schrödinger field are already identical to their classical counterparts and there is no need to perform a limit [*9*].

by the Euler angles, together with a wave whose evolution determines the time development of the angles. The wave is defined on SU(2) but the physical motions take place on SO(3). The spin-1/2 character of the wave manifests itself in the particle motion via a quantum torque (the rotational analogue of the quantum force for translational motion of the centre of mass)—when the latter is negligible the theory reduces to the classical statistical mechanics of a rotating body. Despite all the statements in the literature to the contrary, there *is* therefore a classical analogue of spin-1/2; the causal interpretation provides a way of continuously passing from one to the other.

Finally we point out that this approach has been extended to embrace quantum fields. In the Fermi case, the Jordan-Wigner algebra is reformulated as an algebra of spin-1/2 angular momentum generators which are realized in the angular coordinate representation [*13*]. The quantized field is then represented as an infinite set of rotators whose orientations evolve under the action of the superwavefunction Ψ (the Fermi analogue of the representation of a Bose field as a collection of oscillators). The Clifford algebraic structure of the anticommutation relations is translated into physical effects on the motion of the rotators via a quantum torque constructed from Ψ. Once again the theory has a clearly defined classical analogue: when the torque is negligible we obtain a collection of independent rotators performing a Larmor-type precession. The reader is referred to the references for further details.

7. CONCLUSION

The essence of the correspondence principle is that classical mechanics should emerge in the $\hbar \to 0/n \to \infty$ limit of quantum mechanics. But it has never been demonstrated that this will always be so. And indeed these limits cannot be used as universal criteria for the classical limit since it cannot be guaranteed that the quantum potential will always vanish and hence that classically expected behaviour will result.

We conclude that the universal sufficient criterion for the classical limit of quantum mechanical motions is the vanishing of the quantum potential. Planck's constant is then always a fixed number, and what varies is the relative magnitude of energies in the Hamilton-Jacobi equation. Other aspects of the quantum formalism, such as the uncertainty relations, may be treated by a similar process. This procedure also makes clear how the superposition principle is essentially quantum mechanical in character. When Q is negligible the (now classical) Hamilton-Jacobi equation and equation of conservation combine to yield a *non-linear* equation for the propagation of ψ.

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