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# Operatorial quantization of dynamical systems subject to constraints. A further study of the construction

by

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ABSTRACT. — An investigation is carried out of some formal aspects of the operatorial method developed earlier for generalized canonical quantization of dynamical systems subject to constraints. Within the generalized Hamiltonian approach the master equation is obtained for the effective action, as well as an equation that expresses an analog of the basic postulate of covariant quantization. The generating of the operator gauge algebra of the most general form is considered both in the canonical and the Wick realization of the ghost sector, with various most common types of the normal orderings used. The explicit form of the corresponding structural relations is also obtained. The phenomenon of quantum deformation of the structural relations is studied on general ground. It is shown, for instance, that the central extension of the Virasoro algebra for a string in the critical dimension appears naturally as an effect of quantum deformation of the involution relations corresponding to the Wick normal form. The local analysis is performed of the existence problem of solutions to generating equations of the operator gauge algebra. It is shown that the solution of quantum equations for the symbols of the gauge algebra generating operators can be locally constructed as formal series in powers of the Planck constant, provided that the corresponding classical solution exists and is locally Abelized (this is surely the case for a gauge algebra possessing reducibility of any finite stage). Thus it is shown that, once there is no topological obstacles against the global continuation of solutions for the symbols from a set of overlapping local open vicinities, one can built solutions to the operatorial generating equations of the gauge algebra in the form of formal series in powers of the Planck constant. The formal property of Abelian factorizability is established for these solutions, which makes a basis for the separation of physical and non-physical degrees of freedom at the operatorial level. With the use of this separation it is shown that the dependence of any physical operator on nonphysical degrees of freedom can be completely included in the so-called doublet component that does not contribute into physical matrix elements. It is also shown that the singlet component extracted from the evolution operator that only depends upon operators responsible for the physical degrees of freedom, is itself a unitary operator.

RÉSUMÉ. — On étudie quelques aspects de la méthode opératorielle précédemment développée pour la quantification canonique généralisée des systèmes dynamiques contraints. Dans le cadre de l'approche hamiltonienne généralisée, on obtient l'équation maîtresse pour l'action effective, ainsi qu'une équation exprimant un analogue du postulat fondamental de la quantification covariante. On considère la génération de l'algèbre opératorielle de jauge la plus générale en utilisant la réalisation canonique du secteur de fantômes, celle de Wick, ainsi que les types les plus courants d'ordre normal. On obtient aussi la forme explicite des relations structurelles correspondantes. On étudie de façon générale le phénomène de déformation quantique des relations structurelles. On montre, par exemple, que l'extension centrale de l'algèbre de Virasoro pour une corde en dimension critique apparaît naturellement comme un effet de déformation quantique des relations d'involution correspondant à la forme normale de Wick. On analyse l'existence locale des solutions des équations qui engendrent l'algèbre opératorielle de jauge. On montre que la solution des équations quantiques pour les symboles des générateurs de l'algèbre de jauge peut être construite localement comme une série formelle en puissances de la constante de Planck à condition que la solution classique correspondante existe et soit localement abélienne (ce qui est toujours le cas si l'algèbre de jauge est réductible à tout ordre fini. Cela prouve (s'il n'y a pas d'obstacles topologiques pour passer du local au global) que l'on peut construire des solutions aux équations opératorielles engendrant l'algèbre de jauge sous la forme d'une série formelle en puissances de la constante de Planck. On établit la propriété formelle de factorisabilité abélienne de ces solutions, ce qui est fondamental pour séparer, au niveau opératoriel, les degrés de liberté physiques des degrés non physiques. A l'aide de cette séparation on montre que la dépendance d'un opérateur physique quelconque vis-à-vis des degrés de liberté non physiques est complètement incluse dans la composante « doublet » qui ne contribue pas aux éléments de matrice physiques. On montre aussi que la composante singulet de l'opérateur d'évolution (laquelle ne dépend que des opérateurs correspondants aux degrés de liberté physiques) est elle-même un opérateur unitaire.

## INTRODUCTION

In our previous papers [1]-[6], the operatorial version of the generalized method for canonical quantization of dynamical systems subject to constraints of general form was, in the main, constructed. The most important aspects of the process of the operatorial quantization as well as of the construction of the unitarizing Hamiltonian within the extended phase space concept [7], [8] were considered in those papers. We shall not restate here the basic components of this concept, but shall merely recollect the sequence of the main steps in the procedure of the general canonical quantization itself. Simultaneously, we shall present some motivations and explanations concerning the therminology we keep to in what follows.

First of all, proceeding from the nature of the constraints inherent in the theory, one should establish the structure of the complete set of canonical operator pairs in the extended phase space. As it is known [9], in the most general case all the constraints can be divided into those of first and second class. The first-class constraints generate gauge symmetries, while the ones of the second class just carry out the reduction of the phase space of the system, with the symplectic structure retained. When second-class constraints are present in the system one can always turn them into first-class constraints by defining new degrees of freedom [5], [6]. So, the first-class constraints are, in a way, a more fundamental concept. Hence, in what follows we shall adhere to the position that we are handling first class constraints alone. In the simplest case of irreducible (i. e. linearly-independent) first-class constraints, the complete set of canonical operator pairs in the extended phase space includes, according to ref. [1], the dynamical variables of the original phase space, the dynamically active Lagrange multipliers to constraints and ghosts, and the ghost dynamical variables. In a more general situation, when the first-class constraints are reducible (i. e. linearly dependent), the complete set of canonical operator pairs includes, in principle, analogous groups of dynamical variables. In fact, however, it has a more complicated structure described in detail in ref. [4].

In what follows we keep to a terminology somewhat different from the one used in our previous papers [1-6]. First, we give up the division of the manifold of the operator-valued phase variables into the minimum and auxiliary sectors. Second, we adhere to the principle of discrimination using the ghost number. To be more exact the complete set of canonical pairs of operators is divided in two main sectors: that of the original variables, and the ghost sector. The original variable sector is defined as the one including all the canonical pairs with zero ghost number, while the ghost sector includes all the other canonical pairs.

Technically, the most difficult step in the generalized canonical quantization is to find the fermion and boson generating operators of the gauge algebra. Universal generating equations for these operators were found in our paper [1]. Solutions to these equations are sought for, in the general case, in the form of normally ordered series in powers of ghost operators. The coefficients in these series only depend on the operators of the original dynamical variables, and are structural operators of the gauge algebra. The lowest structural coefficients in the expansions of the fermion and boson generating operators are identified, respectively, with the constraints of the theory and its original Hamiltonian. The basic equations for generating operators of the gauge algebra lead to a set of recurrence equations for the structural operators. These recurrence equations make a sequence of coupled structural relations of the gauge algebra, where every relation provides the fulfilment of the necessary compatibility conditions for previous ones. Therefore, in the lowest order in ghost operators, involution relations appear for constraints and the Hamiltonian. In the next order, the lowest Jacobi relations arise, that provide the fulfilment of the necessary compatibility conditions in the set of involution relations. Later when we go to higher orders in ghosts, the higher Jacobi relations arise in sequence, etc.

The process of generating the operator gauge algebra outlined above is characterized by two main features. The first is that the explicit form of the structural relations depend on a special choice of normal ordering of ghosts in expansions of the generating operators. The second feature is due to the phenomenon of quantum deformation of the structural relations. This phenomenon, first found in our work [1], means that essentially quantum terms appear in the structural relations and that they cannot be removed by any alteration of the ordering in the product of the structural operators provided the ordering of ghosts in the generating operators is fixed. Entirely unpredictable classically, as they are, the contributions responsible for the quantum deformation of structural relations are, nevertheless, quite necessary for providing the algebraic compatibility in the operator domain. A special example of quantum deformation in the involution relations is provided by the central extension of the Virasoro algebra for a string in the critical dimension.

After the solution for the generating operators of the gauge algebra has been found, an expression for the total unitarizing Hamiltonian in the extended phase space is readily constructed using the universal formula of ref. [1]. Apart from the generating operators, already found, this formula for the unitarizing Hamiltonian includes another necessary ingredient, the so-called gauge fermion operator. This operator generates gauge conditions, necessary for removal of the degeneracy. When duly defined, the physical dynamics does not depend on any special choice of the gauge fermion.

Now, that we have, in the main, recalled the schedule of the generalized canonical quantization, we are in a position to formulate the main objects of the present paper. They pursue a more profound investigation of the above operatorial construction with the emphasis on those of its aspects that are related to general properties of the generating equations of the gauge algebra. Now we shall give necessary explanations and, at the same time, describe in detail the organization of the present paper.

In section 1 we study some general consequences of the operatorial dynamical description of gauge systems in the extended phase space. We formulate the Heisenberg operator equations of motion induced by the total unitarizing Hamiltonian in presence of external sources, introduced in a special way. With the use of generating equations of the gauge algebra in the representation that depends on external sources, we obtain an operatorial version of the Ward relations for the gauge system of the most general type, as a consequence of equations of motion. Then we define an effective action for the system and derive for it the so-called master equation from the operatorial Ward relation. We also obtain here an equation that expresses an analog to the main postulate of the covariant quantization scheme accepted in the Lagrangian formalism [*10-12*].

In section 2 the structure of the basic formula for the unitarizing Hamiltonian is discussed and the notion of the gauge-independent physical dynamics is defined. In addition, we discuss there the connection between the generalized Hamiltonian approach and the covariant quantization postulate in terms of the Feynman integral in configuration space.

In section 3 the process of generating a gauge algebra is considered using the normal-ordered expansion of the corresponding generating equations in powers of ghost operators. Explicit forms of structural relations are found here both for the canonical and Wick realizations of the ghost sector, which correspond to the most common ways of the normal ordering of ghosts in expansions of generating operators. Relations are given between the structural operators corresponding to different types of normal ordering of ghosts, as well as formulas that express the transformation laws of the structural operators under the Hermite conjugation. General properties of the structural relations are discussed and the essentially quantum terms due to the quantum deformation effect are pointed out. In that section the general definition of the ghost number operator is also given, as well as its explicit form in the canonical and Wick realizations of the ghost sector.

In section 4 the existence problem for solutions of the generating equations of the operator gauge algebra is studied as a quantization problem. First, we list some facts and relations necessary for what follows, that concern the correspondence problem between a symbol and its operator. Next, we accomplish a formal expansion of the equations for symbols of the gauge algebra generating operators in powers of the Planck constant. After that, the main theorem is proved using the induction method, that states that the solution for the symbols of the generating operators can be locally constructed as formal series in powers of the Planck constant provided the corresponding classical solution exists and is locally Abelizable (the latter is the case for gauge algebra of any stage of reducibility). If, besides, there are no topological obstacles against the global continuation of the germs of symbols from the system of overlapping local open vicinities, then one can state also that there exists a solution of operatorial generating equations of the gauge algebra as formal series in powers of the Planck constant.

When proving the basic theorem we use the local Abelizability of classical solutions as the property of Abelian factorizability of the classical counterpart of the fermion generating operator. It is this property, that provides the possibility of canonical reparametrization that effectively realizes, at the classical level, local separation of physical and nonphysical degrees of freedom. This leads next to the possibility of extracting the pure singlet component, that depends on physical degrees of freedom alone, from any classical observable, while the dependence on all nonphysical degrees of freedom is concentrated in the so-called doublet component that has a pure gauge origin.

That section is completed by a consideration of quasiclassical expansion of the Weyl symbol of the unitarizing Hamiltonian, as well as of path integral representation in the extended phase space, for the Weyl symbol of the evolution operator.

In section 5 we proceed from the fact that the formal solution of operatorial generating equations has been already constructed in the form of power series in the Planck constant. We establish for this solution the property of Abelian factorizability of the generating fermion operator, analogous to the corresponding classical property. Then a canonical reparametrization is considered that performs the effective separation of physical and nonphysical degrees of freedom at the operator level. In analogy with the classical situation, this enables one to decompose the operator of any physical observable into the sum of its singlet and doublet components. Here the singlet component to be extracted only depends on operators of physical degrees of freedom, while the doublet component entirely absorbs the dependence on operators of nonphysical degrees of freedom and thus does not contribute into physical matrix elements. By employing this decomposition we establish the important fact that the extracted singlet component of the evolution operator is unitary. That section is completed by considering the extraction of the singlet component of a physical state.

In the last section 6 we come back to analysing the quantum deformation phenomenon in the involution relations, confining ourselves to the case, most important in practice, of rank-1 irreducible theories. We obtain an explicit form of the quantum deformation of the involution relations corresponding to every type of normal ordering of ghosts in generating operators considered. The section is completed by a consideration of the central extension of the Virasoro algebra understood as an effect of the quantum deformation of the involution relations, corresponding to the Wick normal form.

Finally, let us explain some basic notation to be used henceforth. Operators are considered to be primary objects and are not marked by any hats thereby. As usual, the Grassmannian parity of an operator A is designated as $\varepsilon(A)$, while the ghost number, as gh (A). The supercommutator of operators A and B is also defined in the usual way as

$$[A, B] \equiv AB - BA(-1)^{\varepsilon(A)\varepsilon(B)}.$$

C-numerical functions defined in a classical phase space (including the phase variables themselves) are marked with a tilde, e. g. $\tilde{A}(\tilde{\Gamma})$. The Planck constant is set equal to unity throughout the paper, except for the formulas where it serves as a formal expansion parameter and is therefore left to be explicitly present. Other notations will be clear from the context.

## 1. BASIC RELATIONS OF THE OPERATORIAL APPROACH. MASTER EQUATION FOR THE EFFECTIVE ACTION

The fundamental objects within the operatorial method for quantization of gauge systems are the fermion operator $\Omega$ that generates the gauge algebra of constraints, and the boson unitarizing Hamiltonian H. These operators are governed by the following universal equations

$$[\Omega, \Omega] = 0, \qquad \Omega = \Omega^{\dagger}, \qquad \varepsilon(\Omega) = 1, \qquad \mathrm{gh}\,(\Omega) = 1, \qquad (1.1)$$

$$[H, \Omega] = 0, \qquad H = H^{\dagger}, \qquad \varepsilon(H) = 0, \qquad \mathrm{gh}\,(H) = 0. \qquad (1.2)$$

Physical states of the theory are specialized by the condition

$$\Omega \,|\, \Phi \rangle = 0. \qquad (1.3)$$

Denote the whole set of operator-valued dynamical variables of the extended phase space of the system as $\Gamma$. Their time evolution in a certain interval $(t_i, t_f)$ is determined by the standard equations of motion with the Hamiltonian H subject to (1.2)

$$i\partial_t \Gamma = [\Gamma, H], \qquad t_i < t < t_f, \qquad (1.4)$$

and operators $\Gamma(t_i)$ for initial data.

First of all we have from (1.4) in virtue of (1.2) that

$$i\partial_t \Omega = [\Omega, H] = 0, \qquad (1.5)$$

i. e. the operator $\Omega$ is conserved in time. Thus condition (1.3) specializing physical states, if imposed at the initial time, remains unchanged at every later time. On the other hand, equation (1.1) precludes obtaining further restrictions on the vector $|\Phi\rangle$, by repeatedly applying the operator $\Omega$ since (1.1) is equivalent to the requirement that the operator $\Omega$ be nilpotent, $\Omega^2 = 0$.

Using the solution $\Gamma(t)$ of the equations of motion (1.4) consider the generating operator $Z(t, t_i)$ obeying the following equation

$$i\partial_t Z(t, t_i) = (-J(t)\cdot\Gamma(t) + \langle\Gamma^*(t)\rangle\cdot[\Gamma(t),\Omega]i)Z(t, t_i), \quad (1.6)$$

$$Z(t, t_i)|_{t=t_i} = 1. \quad (1.7)$$

Here $J(t)$ are classical external sources of the operators $\Gamma(t)$. The functions denoted as $\langle\Gamma^*(t)\rangle$ are classical external sources to variations of the operators $\Gamma(t)$ generated by the operator $\Omega$. The dot $(\cdot)$ in products like $J\cdot\Gamma$ or $\langle\Gamma^*\rangle\cdot\Gamma$ denotes henceforth the contraction over the corresponding indices of the multiplied quantities. The external sources J and $\langle\Gamma^*\rangle$ are of opposite statistics

$$\varepsilon(J) = \varepsilon(\Gamma), \qquad \varepsilon(\langle\Gamma^*\rangle) = \varepsilon(\Gamma) + 1. \quad (1.8)$$

Using a solution $Z(t, t_i)$ of the problem (1.6), (1.7), let us map each operator $A(t)$ in the initial $\Gamma$-representation whose time evolution is determined by eq. (1.4) into an operator $A'(t)$ taken in a new representation determined by the external sources J and $\langle\Gamma^*\rangle$:

$$A(t) \mapsto A'(t) \equiv Z^{-1}(t, t_i)A(t)Z(t, t_i). \quad (1.9)$$

In virtue of (1.7) both representations coincide at the initial time

$$A'(t_i) = A(t_i). \quad (1.10)$$

The basic equations (1.1), (1.2), evidently, retain their form in the new representation

$$[\Omega', \Omega'] = 0, \qquad [H', \Omega'] = 0. \quad (1.11)$$

For the operators $\Gamma'(t)$ that correspond to $\Gamma(t)$ in the sense of the definition (1.9) we obtain with the help of (1.4), (1.6) the following equations of motion in the new representation

$$i\partial_t\Gamma' = [\Gamma', H' - J\cdot\Gamma' + \langle\Gamma^*\rangle\cdot[\Gamma', \Omega']i]. \quad (1.12)$$

Using (1.11) one obtains from this the following equation for the operator $\Omega'$, instead of the conservation law (1.5)

$$i\partial_t\Omega' = [J\cdot\Gamma', \Omega'], \qquad \Omega'(t_i) = \Omega. \quad (1.13)$$

Note now, that the following relation holds true owing to (1.6) and (1.7)

$$\frac{\delta_l Z(t_f, t_i)}{\delta\langle\Gamma^*(t)\rangle} = Z(t_f, t_i)[\Gamma'(t), \Omega'(t)]. \quad (1.14)$$

Eq. (1.14) enables us to represent (1.13) in the form

$$Z(t_f, t_i) i\partial_t \Omega' = J(t) \cdot \frac{\delta_l Z(t_f, t_i)}{\delta \langle \Gamma^*(t) \rangle}. \tag{1.15}$$

Let now $| \Phi \rangle$ be a normalizable physical state from (1.3). By integrating (1.15) over $t$ within the interval $(t_i, t_f)$ and calculating the expectation value in the state $| \Phi \rangle$, one obtains

$$\int_{t_i}^{t_f} J(t) \cdot \frac{\delta_l W(J, \langle \Gamma^* \rangle)}{\delta \langle \Gamma^*(t) \rangle} dt = 0, \tag{1.16}$$

where the functional $W(J, \langle \Gamma^* \rangle)$ is defined as

$$W(J, \langle \Gamma^* \rangle) = -i \ln \left( \frac{\langle \Phi | Z(t_f, t_i) | \Phi \rangle}{\langle \Phi | \Phi \rangle} \right). \tag{1.17}$$

Let us, further, define the average values $\langle \Gamma(t) \rangle$ in a usual way as variational derivatives of the functional (1.17) with respect to external sources $J(t)$

$$\frac{\delta W(J, \langle \Gamma^* \rangle)}{\delta J(t)} = \langle \Gamma(t) \rangle \tag{1.18}$$

assume, as usual, that (at least within the perturbation theory) equation (1.18) is uniquely solvable with respect to the source J and enables one to express the latter as a functional of the quantities $\langle \Gamma \rangle, \langle \Gamma^* \rangle$

$$J(t) = J(t \,|\, \langle \Gamma \rangle, \langle \Gamma^* \rangle). \tag{1.19}$$

Then, let us define an effective action in the standard way

$$S(\langle \Gamma \rangle, \langle \Gamma^* \rangle) = W(J, \langle \Gamma^* \rangle) - \int_{t_i}^{t_f} J(t) \cdot \langle \Gamma(t) \rangle dt, \tag{1.20}$$

where the substitution (1.19) is understood in the r.-h. side.

The following relations hold true for the effective action (1.20)

$$\frac{\delta_r S}{\delta \langle \Gamma(t) \rangle} = -J(t), \qquad \frac{\delta_l S}{\delta \langle \Gamma^*(t) \rangle} = \frac{\delta_l W(J, \langle \Gamma^* \rangle)}{\delta \langle \Gamma^*(t) \rangle}, \tag{1.21}$$

where the derivative in the r.-h. side is calculated taking into account its explicit dependence on $\langle \Gamma^* \rangle$, the substitution (1.19) being performed after that.

Finally, after substituting the r.-h. sides from (1.21) into (1.16), one obtains

$$\int_{t_i}^{t_f} \frac{\delta_r S}{\delta \langle \Gamma(t) \rangle} \cdot \frac{\delta_l S}{\delta \langle \Gamma^*(t) \rangle} dt = 0. \tag{1.22}$$

Consider now the following formal definition. Let functions $\Phi^A(t)$ and $\Phi^*_A(t)$, $A = 1, \ldots, \mathcal{N}$, of opposite statistics

$$\varepsilon(\Phi^*_A) = \varepsilon(\Phi^A) + 1 \tag{1.23}$$

be given in the interval $t_i \leqslant t \leqslant t_f$, and let $X(\Phi, \Phi^*)$ and $Y(\Phi, \Phi^*)$ be any two functionals that depend on these functions, $\varepsilon_X$ and $\varepsilon_Y$ being Grassmanian partities of X and Y, respectively. Following ref. [*10-12*], let us define the binary combination

$$(X, Y) = \int_{t_i}^{t_f} \left( \frac{\delta_r X}{\delta\Phi(t)} \cdot \frac{\delta_l Y}{\delta\Phi^*(t)} - \frac{\delta_r Y}{\delta\Phi(t)} \cdot \frac{\delta_l X}{\delta\Phi^*(t)} (-1)^{(\varepsilon_X+1)(\varepsilon_Y+1)} \right) dt \tag{1.24}$$

called antibracket. With the identification

$$\Phi(t) = \langle \Gamma(t) \rangle, \qquad \Phi^*(t) = \langle \Gamma^*(t) \rangle, \tag{1.25}$$

equation (1.22) takes its most compact and symmetric form in terms of the antibracket (1.24)

$$(S, S) = 0. \tag{1.26}$$

Equation of the form of (1.26) is known as the master equation in the covariant (Lagrangian) quantization method of gauge systems [*10-12*]. In the covariant approach it appears at two levels. First, its proper solution determines (up to a local measure) the total action in the functional integrand and the Feynman rules in the Lagrangian formalism thereof. Second, the master equation holds true already for the effective Lagrangian action that results from the path integration and the Legendre transformation with respect to the external sources of the Lagrangian variables. It is in this second role, i. e. as an equation for the effective action, that the Lagrangian master equation is a direct counterpart to equation (1.26). It is, however, essential, that equation (1.26) has been obtained by us within the generalized canonical formalism as a consequence of exact operator equations of motion, whereas within the Lagrangian approach the master equation is introduced following special postulates of covariant quantization that are not, at first glance, *directly* connected with the requirement of physical unitarity. As a matter of fact, it turns out, nonetheless, that the equation that is a counterpart at the basic covariant quantization postulate immediately follows from the generalized canonical formalism.

Let us consider, indeed, the functional Fourier transformation for the functional W (1.17)

$$\exp\{ iW(J, \langle \Gamma^* \rangle) \} =$$
$$= (\text{const}) \int \exp\left\{ i\tilde{W}(\tilde{\Gamma}, \tilde{\Gamma}^*) + i \int_{t_i}^{t_f} J(t) \cdot \tilde{\Gamma}(t) dt \right\} \prod_t d\tilde{\Gamma}(t), \tag{1.27}$$

where we have set in the r.-h. side

$$\tilde{\Gamma}^* \equiv \langle \Gamma^* \rangle. \tag{1.28}$$

for the sake of symmetry in notations. Eq. (1.27) is an analog to the generating functional of covariant Green functions. In its turn, the functional $\tilde{W}(\tilde{\Gamma}, \tilde{\Gamma}^*)$ is nothing but an analog to the total covariant action that creates covariant Feynman rules. Now it follows directly from (1.16) that

$$\tilde{\Delta} \exp \{ i\tilde{W} \} = 0, \qquad \tilde{\Delta} \equiv \int_{t_i}^{t_j} \frac{\delta_r}{\delta\tilde{\Gamma}(t)} \cdot \frac{\delta_l}{\delta\tilde{\Gamma}^*(t)} dt, \tag{1.29}$$

or, what is the same, that

$$(\tilde{W}, \tilde{W}) = 2i\tilde{\Delta}\tilde{W}, \tag{1.30}$$

where the antibracket in the l.-h. side is defined relative to the variables

$$\Phi(t) \equiv \tilde{\Gamma}(t), \qquad \Phi^*(t) \equiv \tilde{\Gamma}^*(t). \tag{1.31}$$

Equation of the form of (1.30) expresses the basic quantization postulate in the covariant approach.

It is essential to note that in every equation of the present section, from (1.4) on, the set $\Gamma$ should not necessarily be the *complete* set of dynamical variables in the extended phase space, but may be its certain part, as well. It is only needed that the unique solvability of equation (1.18) in the sense of (1.19) take place for this part. In case $\Gamma$ is the complete set of dynamical variables indeed, equations (1.26) and (1.30) are more informative than their counterparts in the covariant approach. It would correspond to the covariant approach if one took for $\Gamma$ (after a certain canonical transformation) the part of the complete set of dynamical variables that makes the relativistic configuration space of the system. Simultaneously, the r.-h. side of equation (1.30) is proportional to $\hbar$ (this is apparent if the explicit way of writing is used) is the contribution of the configurational (local) integration measure. Therefore, disregarding the local measure, one can reduce, indeed, equation (1.30) to the master equation [*10-12*].

## 2. SOLUTION FOR THE UNITARIZING HAMILTONIAN. GAUGE FERMION. GAUGE INDEPENDENCE

Following [*1-6*] we shall seek for a solution to equation (1.2) for the unitarizing Hamiltonian in the form

$$\mathrm{H} = \mathscr{H} - i[\Psi, \Omega], \tag{2.1}$$

$$[\mathscr{H}, \Omega] = 0, \qquad \mathscr{H} = \mathscr{H}^\dagger, \qquad \varepsilon(\mathscr{H}) = 0, \qquad \mathrm{gh}(\mathscr{H}) = 0, \tag{2.2}$$

$$\Psi = -\Psi^\dagger, \qquad \varepsilon(\Psi) = 1, \qquad \mathrm{gh}(\Psi) = -1. \tag{2.3}$$

Representation (2.1) reflects the natural arbitrariness in solution of the equation (2.1), satisfied, in virtue of (1.1) and (2.2) for any $\Psi$ subject to (2.3). The fermion generator $\Psi$ is intended for generating necessary gauge conditions. This operator comprises the total gauge arbitrariness of the theory and is hence, called the gauge fermion. The gauge independence of a physical dynamics is its independence of any special choice of the gauge fermion $\Psi$. Let us consider this in more details.

Let $\Gamma$ be again the complete set of dynamical variables of the extended phase space. Denote as $\Gamma_\Psi(t)$ the solution of equations of motion (1.4) with the Hamiltonian H (2.1) and initial data $\Gamma(t_i)$. The subscript $\Psi$ in $\Gamma_\Psi(t)$ indicates the dependence of the dynamical evolution on the gauge fermion $\Psi$ involved explicitly in the Hamiltonian (2.1). The initial data $\Gamma(t_i)$ are meant to be independent of $\Psi$. Under an arbitrary form-variation $\Psi \to \Psi + \Delta\Psi$ of the gauge fermion the operators $\Gamma_\Psi$ transform according to the law

$$\Gamma_{\Psi+\Delta\Psi} = \mathfrak{A}^{-1}\Gamma_\Psi\mathfrak{A}, \tag{2.4}$$

where the transformation operator is determined by the equation

$$\partial_t\mathfrak{A} = -\ [\Omega, \Delta\Psi]\mathfrak{A}, \qquad \mathfrak{A}(t_i) = 1. \tag{2.5}$$

Define physical entities to be functions $\mathcal{O}(\Gamma)$ of the operators $\Gamma$ (but not of their time-derivatives) subject to the condition

$$[\mathcal{O}, \Omega] = 0. \tag{2.6}$$

Utilizing eqs. (2.4) and (2.5) in their infinitesimal limit $\Delta\Psi = \delta\Psi \to 0$ we can represent the gauge variation of the operator $\mathcal{O}(\Gamma_\Psi)$ subject to (2.6),

$$\delta_\Psi\mathcal{O} \equiv \mathcal{O}(\Gamma_{\Psi+\delta\Psi}) - \mathcal{O}(\Gamma_\Psi), \tag{2.7}$$

in the form

$$\delta_\Psi\mathcal{O} = \left[\Omega, \left[\int_{t_i}^{t_f} \delta\Psi(t')dt', \mathcal{O}\right]\right]. \tag{2.8}$$

Thus, the matrix element of the gauge variation (2.8) relative to any two states from (1.3) vanishes

$$\langle\, \Phi' \,|\, \delta_\Psi\mathcal{O}(\Gamma_\Psi) \,|\, \Phi'' \,\rangle \equiv 0, \tag{2.9}$$

and we come to the statement: physical matrix elements of physical operators do not depend on a specialization of the gauge fermion $\Psi$ in (2.1).

Let us present now the physical dynamics in an explicit gauge invariant form. After writing the equations of motion with the Hamiltonian (2.1) for physical operators and calculating their matrix element between any two physical states from (1.3) we have

$$i\partial_t \langle\, \Phi' \,|\, \mathcal{O} \,|\, \Phi'' \,\rangle = \langle\, \Phi' \,|\, [\mathcal{O}, \mathrm{H}] \,|\, \Phi'' \,\rangle = \langle\, \Phi' \,|\, [\mathcal{O}, \mathcal{H}] \,|\, \Phi'' \,\rangle, \tag{2.10}$$

where $\mathcal{H}$ is the gauge-independent part of H defined by (2.2) (i. e. the first term in (2.1) alone).

The second equality in (2.10) already confirms (2.9), but does not yet provide a closed description of the physical dynamics, since the r.-h. side has not been yet expressed exclusively in terms of the *physical* matrix elements of the physical operators $\mathcal{O}$ and $\mathcal{H}$. To make the next step we shall use the results of the important works [13], [14].

Let $\mathcal{V}$ be a vector space (its metrics is indefined) where our operators $\Gamma$ are to act. Denote as $\mathcal{V}_{\text{phys}} \subset \mathcal{V}$ the set of states $|\Phi\rangle$ from $\mathcal{V}$ obeying eq. (1.3). Denote, finally, as $\mathcal{V}_s \subset \mathcal{V}_{\text{phys}}$ the set of states $|\Phi_s\rangle$ from $\mathcal{V}_{\text{phys}}$ that satisfy the condition

$$|\Phi_s\rangle \neq \Omega\,|\,\text{anything}\,\rangle\,. \tag{2.11}$$

The states $|\Phi_s\rangle$ are called singlets.

Under the assumption that (1.1) holds true and that the ghost number operator has only integers as its eigenvalues it was proved in refs [13], [14] that the following key statement is valid for any $|\Phi'\rangle$ and $|\Phi''\rangle$ from $\mathcal{V}_{\text{phys}}$:

$$\langle \Phi' | \Phi'' \rangle = \langle \Phi' | \mathrm{P}(\mathcal{V}_s) | \Phi'' \rangle\,, \tag{2.12}$$

where $\mathrm{P}(\mathcal{V}_s)$ is the projection operator onto $\mathcal{V}_s$.

Due to a special choice of the metric structure (of the basis) in the space of states $\mathcal{V}$, the arbitrariness inherent in the definition (2.11) of the singlet states was lifted in ref. [14] in a natural way, the definition of the projection operator $\mathrm{P}(\mathcal{V}_s)$ being made singlevalued thereof.

Accepting the starting principles of the papers [13], [14] we shall employ relation (2.12) to represent equation (2.10) in the form

$$i\partial_t \langle \Phi' | \mathcal{O} | \Phi'' \rangle = \langle \Phi' | (\mathcal{O}\mathrm{P}(\mathcal{V}_s)\mathcal{H} - \mathcal{H}\mathrm{P}(\mathcal{V}_s)\mathcal{O}) | \Phi'' \rangle\,. \tag{2.13}$$

Equation (2.13) provides already a closed description of the physical dynamics. It shows that the restriction of the physical operator $\mathcal{H}$ onto the singlet subspace $\mathcal{V}_s$ of the physical space $\mathcal{V}_{\text{phys}}$ controls the time evolution of the physical operator $\mathcal{O}$ when restricted onto the same physical subspace $\mathcal{V}_s$.

Another very important consequence of the relation (2.12) is the unitarity of the evolution operator in the physical sector. Owing to the formal hermiticity of the Hamiltonian (2.1), the evolution operator corresponding to it

$$\mathrm{E}(t) = \exp\{-i\mathrm{H}t\} \tag{2.14}$$

is formally unitary

$$\mathrm{E}^\dagger\mathrm{E} = \mathrm{E}\mathrm{E}^\dagger = 1\,. \tag{2.15}$$

After calculating the matrix element between any two states $|\Phi'\rangle$ and $|\Phi''\rangle$ from $\mathcal{V}_{\text{phys}}$ and using (2.12), one has

$$\langle \Phi' | \mathrm{E}^\dagger\mathrm{P}(\mathcal{V}_s)\mathrm{E} | \Phi'' \rangle = \langle \Phi' | \mathrm{E}\mathrm{P}(\mathcal{V}_s)\mathrm{E}^\dagger | \Phi'' \rangle = \langle \Phi' | \Phi'' \rangle\,. \tag{2.16}$$

Therefore, the evolution operator (2.14), when restricted onto the physical subspace $\mathcal{V}_s$ of singlet states, remains unitary in this space.

To conclude this section, we concentrate again on expression (1.27). Since all the gauge arbitrariness of the theory is comprised, at the operator level, in the gauge fermion operator $\Psi$ in the Hamiltonian (2.1), as mentioned above, it is expected that the arbitrariness of the functional integral (1.27) is also gathered in the fermi-function $\tilde{\Psi}(\tilde{\Gamma})$ of the integration variables.

If the Hamiltonian (2.1) is substituted into the operator-valued equations of motion (1.12) in the representation determined by external sources, it is readily seen that, at the quasiclassical level, i. e. when the commutators become Poisson brackets, the gauge fermion $\Psi$ enters in the r.-h. side of (1.12) through

$$\langle \Gamma^* \rangle - \frac{\partial \Psi}{\partial \Gamma}. \tag{2.17}$$

It is therefore natural to expect that the functional integral (1.27) also contains, at least at the quasiclassical level, a combination analogous to (2.17), which corresponds to the replacement of $\tilde{\Gamma}^*$ (1.28) by

$$\tilde{\Gamma}^* = \langle \Gamma^* \rangle - \frac{\delta \tilde{\Psi}}{\delta \tilde{\Gamma}}. \tag{2.18}$$

Equation (1.30) retains its form under this replacement. As it is known [*10-12*], the covariant analogue of eq. (1.27) contains the gauge fermion just in the combination (2.18).

Employing the closed continual solution of the operator-valued equations of motion (1.12), one can show that, up to terms responsable for the ordering of equal-time operators in the total Hamiltonian in (2.12), the generating functional of the theory has indeed the structure corresponding to (2.18).

## 3. GENERATING THE GAUGE ALGEBRA. NORMAL-ORDERED EXPANSIONS OF THE GENERATING EQUATIONS IN POWERS OF THE GHOST SECTOR OPERATORS

In this section the process is considered through which the operator gauge algebra of the most general type is generated within the generating equations (1.1), (1.2). The solution of these equations for the generating operators will be found in the form of normal-ordered series in powers of the ghost operators. Coefficients in these series only depend on dynamical operator-valued variables of the original phase space of the system, and are the structural operators of the gauge algebra. The lowest terms in the expansions of the operators $\Omega$ and $\mathscr{H}$ contain as the structural coefficients the operators of the initial constraints and the Hamiltonian, respectively. The substitution of the normal-ordered expansions of the operators $\Omega$ and $\mathscr{H}$ into equations (1.1) and (2.2) gives, after their l.-h. sides are reduced to the normal form, recurrence equations for the structural operators, in each order with respect to the ghosts. These equations are nothing, but the structural relations of the operator gauge algebra. For instance, in the lowest order, one obtains the involution relations for the operators of the initial constraints and the Hamiltonian. In the next order one obtains relations that provide the fulfilment of the necessary conditions of compatibility in the system of involution relations, etc. Thus, a sequence of coupled structural relations arises, in which each solution provides the fulfilment of the necessary compatibility conditions for the previous ones.

A peculiarity of the operator approach is in that both the structural operators themselves and the form of the structural relations for them depend on a specialization in the normal ordering of ghosts. From the pure formal point of view this specialization is in no way restricted. Only special features of a given dynamical system may provide a preference for certain type of normal ordering. Throughout our previous works on the operatorial quantization we were using the normal form that places all ghost momenta to the left of all ghost coordinates. This type of normal form (as well as of the one dual to it) is most convenient for the purpose of general analysis of the structural relations, since only this normal form guarantees *linearity* to every structural relation with respect to the structural operators defined by it. Among other possibilities of specializing the normal ordering, the Weyl and Wick normal forms are of special interest. The Weyl form is distinguished by its property of being invariant under linear canonical transformations. On the other hand, the Wick form is most stable when one goes to an infinite number of degrees of freedom.

In this section we shall study in detail the expansions of the generating equations of the gauge algebra, accepting different types of normal ordering of the ghost operators including both the one used before and the Weyl and Wick normal forms.

## A. Canonical ghost sector.

Assume that the ghost sector of the theory is represented by the canonical operators pairs

$$(G^A, \overline{G}_A), \qquad \varepsilon(G^A) = \varepsilon(\overline{G}_A) \equiv \varepsilon_A, \qquad \mathrm{gh}\,(G^A) = -\,\mathrm{gh}\,(\overline{G}_A) \quad (3.1)$$

so that among the equal-time supercommutators only the following

$$[G^A, \overline{G}_B] = i\delta^A_B\,. \quad (3.2)$$

are nonvanishing. Assume further, that the operators (3.1) transform as follows under the Hermite conjugation

$$G^{\dagger A} = G^A, \qquad \overline{G}^\dagger_A = \overline{G}_A(-1)^{\varepsilon_A}. \quad (3.3)$$

The ghost sector (3.1)-(3.3) will be called canonical.

A1. $\overline{G}G$-*normal form.*

In this item we consider the expansion of the generating equations (1.1) and (1.2) in powers of the operators belonging to the canonical ghost sector (3.1)-(3.3), taking it in the $\bar{G}G$ normal form, i. e. with every $\bar{G}$ placed to the left of all G.

Let us start with equation (1.1), whose solution will be sought for using the Ansatz

$$\Omega = \sum_{m,n\geqslant 0} \bar{G}_{A_m} \dots \bar{G}_{A_1} X^{A_1\dots A_m}_{B_n\dots B_1} G^{B_1} \dots G^{B_n}. \tag{3.4}$$

Consider classical analogs to operators (3.1)

$$(\tilde{G}^A, \tilde{\bar{G}}_A), \qquad \varepsilon(\tilde{G}^A) = \varepsilon(\tilde{\bar{G}}_A) = \varepsilon_A \tag{3.5}$$

and define the symmetrizers

$$m\,!\,\bar{S}^{A_1\dots A_m}_{A'_m\dots A'_1} \equiv \left(\frac{\vec{\partial}}{\partial\tilde{\bar{G}}_{A_1}} \dots \frac{\vec{\partial}}{\partial\tilde{\bar{G}}_{A_m}} \tilde{\bar{G}}_{A'_m} \dots \tilde{\bar{G}}_{A'_1}\right) \tag{3.6}$$

$$n\,!\,S^{B'_1\dots B'_n}_{B_n\dots B_1} \equiv \left(\tilde{G}^{B'_1} \dots \tilde{G}^{B'_n} \frac{\overleftarrow{\partial}}{\partial\tilde{G}^{B_n}} \dots \frac{\partial}{\partial\tilde{G}^{B_1}}\right) \tag{3.7}$$

to be used for defining symmetrization of any quantity

$$\mathrm{Sym}\,(K^{A_1\dots A_m}_{B_n\dots B_1}) \equiv \bar{S}^{A_1\dots A_m}_{A'_m\dots A'_1} K^{A'_1\dots A'_m}_{B'_n\dots B'_1} S^{B'_1\dots B'_n}_{B_n\dots B_1}. \tag{3.8}$$

The structural operators X in expansion (3.4) are symmetric in the sense of the operation (3.8)

$$X = \mathrm{Sym}\,(X). \tag{3.9}$$

By substituting (3.4) into the l.-h. side of the first equation in (1.1) and reducing it to the normal form we obtain the following recurrence relations for the structural operators

$$\mathrm{Sym}\,(Q^{A_1\dots A_m}_{B_n\dots B_1}) = 0, \tag{3.10}$$

where the following notation is used:

$$Q^{A_1\dots A_m}_{B_n\dots B_1} \equiv \sum_{p=0}^{m}\sum_{q=0}^{n}\left\{\frac{1}{2}\left[X^{A_1\dots A_p}_{B_n\dots B_{q+1}}, X^{A_{p+1}\dots A_m}_{B_q\dots B_1}\right] + \right.$$

$$\left. + \sum_{s\geqslant 1} X^{A_1\dots A_p}_{B_n\dots B_{q+1}\mathscr{D}_s\dots\mathscr{D}_1} X^{\mathscr{D}_1\dots\mathscr{D}_s A_{p+1}\dots A_m}_{B_q\dots B_1} C^p_{qs}\right\}(-1)^{\varepsilon^p_q}, \tag{3.11}$$

$$C^p_{qs} \equiv \frac{(m-p+s)\,!\,i^s(n-q+s)\,!}{(m-p)\,!\;s\,!\;(n-q)\,!}, \tag{3.12}$$

$$\varepsilon^p_q \equiv \left(\sum_{i=p+1}^{m} \varepsilon_{A_i} + 1\right)\left(\sum_{j=q+1}^{n} \varepsilon_{B_j} + 1\right) + 1. \tag{3.13}$$

Analogously, the second equation (1.2) when taken together with (3.3), produces the transformation laws of the structural operators under the Hermite conjugation

$$\left(X^{A_1 \dots A_m}_{B_n \dots B_1}\right)^\dagger = \sum_{s \geq 0} C^0_{0s} X^{A_m \dots A_1 \mathscr{D}_1 \dots \mathscr{D}_s}_{\mathscr{D}_s \dots \mathscr{D}_1 B_1 \dots B_n} (-1)^{\varepsilon_A(\varepsilon_B + 1)}, \qquad (3.14)$$

where $C^0_{0s}$ is given by (3.12) at $p = 0$, $q = 0$ and, besides, the notation

$$\varepsilon_A \equiv \sum_{i=1}^{m} \varepsilon_{A_i}, \qquad \varepsilon_B \equiv \sum_{i=1}^{n} \varepsilon_{B_i}, \qquad (3.15)$$

is used.

The third equation in (1.2), coupled with the second equation from (3.1), determines the distribution of statistics of the structural operators

$$\varepsilon\left(X^{A_1 \dots A_m}_{B_n \dots B_1}\right) = \varepsilon_A + \varepsilon_B + 1, \qquad (3.16)$$

where the parities from (3.15) enter in the r.-h. side.

Finally, the fourth equation in (1.2), joined with the third equation in (3.1), provides conditions on admissible values of $m$ and $n$ in the expansion (3.4):

$$-\sum_{i=1}^{m} \mathrm{gh}\,(G^{A_i}) + \sum_{i=1}^{n} \mathrm{gh}\,(G^{B_i}) = 1. \qquad (3.17)$$

Concentrate now on equations (2.2), whose solutions are to be sought for as a $\bar{G}G$-normal expansion, the same as (3.4):

$$\mathscr{H} = \sum_{m, n \geq 0} \bar{G}_{A_m} \dots \bar{G}_{A_1} Y^{A_1 \dots A_m}_{B_n \dots B_1} G^{B_1} \dots G^{B_n}, \qquad (3.18)$$

with the structural operators Y symmetric under the operation (3.8)

$$Y = \mathrm{Sym}\,(Y). \qquad (3.19)$$

After substituting the expansion (3.18) into the l.-h. side of the first equation in (2.2) and reducing it to the $\bar{G}G$-normal form, we obtain the following recurrence relations for the structural operators

$$\mathrm{Sym}\,\left(R^{A_1 \dots A_m}_{B_n \dots B_1}\right) = 0, \qquad (3.20)$$

where the notations are used

$$R^{A_1\dots A_m}_{B_n\dots B_1} \equiv \sum_{p=0}^{m}\sum_{q=0}^{n} \{[Y^{A_1\dots A_p}_{B_n\dots B_{q+1}}, X^{A_{p+1}\dots A_m}_{B_q\dots B_1}](-1)^{\varepsilon'^p_q} +$$

$$+ \sum_{s\geq 1} (Y^{A_1\dots A_p}_{B_n\dots B_{q+1}\mathscr{D}_s\dots\mathscr{D}_1} X^{\mathscr{D}_1\dots\mathscr{D}_s A_{p+1}\dots A_m}_{B_q\dots B_1}(-1)^{\varepsilon'^p_q} -$$

$$- X^{A_1\dots A_p}_{B_n\dots B_{q+1}\mathscr{D}_s\dots\mathscr{D}_1} Y^{\mathscr{D}_1\dots\mathscr{D}_s A_{p+1}\dots A_m}_{B_q\dots B_1}(-1)^{\varepsilon''^p_q})C^p_{qs}\} \tag{3.21}$$

$$\varepsilon'^p_q \equiv \varepsilon^p_q + \sum_{i=p+1}^{m} \varepsilon_{A_i}, \qquad \varepsilon''^p_q \equiv \varepsilon^p_q + \sum_{i=q+1}^{n} \varepsilon_{B_i}, \tag{3.22}$$

with $C^p_{qs}$ and $\varepsilon^p_q$ given by (3.12) and (3.13), respectively.

The second equation in (2.2), when coupled with (3.3) gives

$$(Y^{A_1\dots A_m}_{B_n\dots B_1})^\dagger = \sum_{s\geq 0} C^0_{0s} Y^{A_m\dots A_1\mathscr{D}_1\dots\mathscr{D}_s}_{\mathscr{D}_s\dots\mathscr{D}_1 B_1\dots B_n}(-1)^{[(\varepsilon_A+1)\varepsilon_B+\varepsilon_{\mathscr{D}}]} \tag{3.23}$$

where $C^p_{qs}$, $\varepsilon_A$, $\varepsilon_B$ are defined in the same way as in (3.14), and, besides, the notation

$$\varepsilon_{\mathscr{D}} \equiv \sum_{i=1}^{s} \varepsilon_{\mathscr{D}_i} \tag{3.24}$$

is used.

The third equation in (2.2), together with the second one in (3.1) gives

$$\varepsilon(Y^{A_1\dots A_m}_{B_n\dots B_1}) = \varepsilon_A + \varepsilon_B \tag{3.25}$$

where the parities from (3.15) appear in the r.-h. side.

The fourth equation in (2.2), together with the third equation from (3.1), produces the condition to be obeyed by the values $m$ and $n$ in the decomposition (3.18)

$$-\sum_{i=1}^{m} \mathrm{gh}(G^{A_i}) + \sum_{i=1}^{n} \mathrm{gh}(G^{B_i}) = 0. \tag{3.26}$$

The structural relations (3.10), (3.20), conjugation properties (3.14), (3.23), distributions of statistics (3.16), (3.25) and of the ghost number (3.17), (3.27) exhaust the conditions imposed by the equations (1.1) and (2.2) on the structural operators X and Y in the $\bar{G}G$-normal expansions (3.4), (3.18).

A2. *Weyl normal form.*

In this item we consider the expansion of the generating equations (1.1) and (1.2) in powers of the operators (3.1) of the canonical ghost sector in the Weyl normal form. Here we denote again as X and Y the structural coefficients in the expansions for $\Omega$ and $\mathscr{H}$, although now these are apparently the operators *other* than the structural coefficients in expansions (3.4) and (3.18) of the preceding item.

We start again with equations (1.1) and seek for its solution in the form of the Weyl-ordered expansion

$$\Omega = \sum_{l \geq 0} \frac{1}{l!} \left( \mathrm{G} \cdot \frac{\partial}{\partial \tilde{\mathrm{G}}} + \overline{\mathrm{G}} \cdot \frac{\partial}{\partial \tilde{\overline{\mathrm{G}}}} \right)^l \times \\ \times \sum_{m, n \geq 0} \tilde{\overline{\mathrm{G}}}_{\mathrm{A}_m} \dots \tilde{\overline{\mathrm{G}}}_{\mathrm{A}_1} \mathrm{X}^{\mathrm{A}_1 \dots \mathrm{A}_m}_{\mathrm{B}_n \dots \mathrm{B}_1} \tilde{\mathrm{G}}^{\mathrm{B}_1} \dots \tilde{\mathrm{G}}^{\mathrm{B}_n} |_{\substack{\tilde{\mathrm{G}} = 0 \\ \tilde{\overline{\mathrm{G}}} = 0}} \quad (3.27)$$

where $\tilde{\mathrm{G}}$ and $\tilde{\overline{\mathrm{G}}}$ are classical counterparts (3.5) of the operators (3.1). The structural operators X in expansion (3.27) are symmetric under (3.8), so that (3.9) holds true for them.

After substituting (3.27) into the l.-h. side of the first equation from (1.1) and reducing it to the Weyl-normal form, we obtain for the structural operators the recurrence relations of the form of (3.10), in which, this time the new operators

$$\mathrm{Q}^{\mathrm{A}_1 \dots \mathrm{A}_m}_{\mathrm{B}_n \dots \mathrm{B}_1} \equiv \sum_{p=0}^{m} \sum_{q=0}^{n} \left\{ \frac{1}{2} [\mathrm{X}^{\mathrm{A}_1 \dots \mathrm{A}_p}_{\mathrm{B}_n \dots \mathrm{B}_{q+1}}, \mathrm{X}^{\mathrm{A}_{p+1} \dots \mathrm{A}_m}_{\mathrm{B}_q \dots \mathrm{B}_1}] (-1)^{\varepsilon^{p0}_q} + \right. \\ \left. + \sum_{r,s}' \mathrm{X}^{\mathrm{A}_1 \dots \mathrm{A}_p \mathrm{C}_1 \dots \mathrm{C}_r}_{\mathrm{B}_n \dots \mathrm{B}_{q+1} \mathscr{D}_s \dots \mathscr{D}_1} \mathrm{X}^{\mathscr{D}_1 \dots \mathscr{D}_s \mathrm{A}_{p+1} \dots \mathrm{A}_m}_{\mathrm{C}_r \dots \mathrm{C}_1 \mathrm{B}_q \dots \mathrm{B}_1} (-1)^{\varepsilon^{pr}_q} \mathrm{C}^{pr}_{qs} \right\}, \quad (3.28)$$

are involved instead of (3.11). Here the notations are used:

$$\sum_{r,s}' \equiv \sum_{r,s>0} + \delta_{r0} \sum_{s>0} + \delta_{s0} \sum_{r>0}, \quad (3.29)$$

$$\varepsilon^{pr}_q \equiv \left( \sum_{i=p+1}^{m} \varepsilon_{\mathrm{A}_i} + 1 \right) \left( \sum_{j=q+1}^{n} \varepsilon_{\mathrm{B}_j} + 1 \right) + 1 + \\ + \left( \sum_{i=p+1}^{m} \varepsilon_{\mathrm{A}_i} + \sum_{i=q+1}^{n} \varepsilon_{\mathrm{B}_i} + 1 \right) \sum_{j=1}^{r} \varepsilon_{\mathrm{C}_j}, \quad (3.30)$$

$$\mathrm{C}^{pr}_{qs} \equiv \frac{(p+r)!\,(q+r)!\,(m-p+s)!\,(n-q+s)!}{p!\,r!\,q!\,(m-p)!\,s!\,(n-q)!} \left( -\frac{i}{2} \right)^r \left( \frac{i}{2} \right)^s. \quad (3.31)$$

The second equation in (1.1), together with (3.3), gives rise to the transformation law under the Hermite conjugation of the structural operators in the expansion (3.27)

$$(X^{A_1\dots A_m}_{B_n\dots B_1})^\dagger = X^{A_m\dots A_1}_{B_1\dots B_n}(-1)^{\varepsilon_A(\varepsilon_B+1)}, \tag{3.32}$$

with the parities from (3.15) appearing in the r.-h. side.

The distribution of statistics of the structural operators, and the condition for admissible values of $m$ and $n$ in the expansion (3.27) coincide with (3.16) and (3.17), respectively.

Consider now equations (2.2), solutions for which will be again sought for as Weyl-ordered expansions

$$\mathscr{H} = \sum_{l\geqslant 0}\frac{1}{l!}\left(G\cdot\frac{\partial}{\partial\tilde{G}} + \overline{G}\cdot\frac{\partial}{\partial\tilde{\tilde{G}}}\right)^l \times$$

$$\times \sum_{m,n\geqslant 0}\tilde{\tilde{G}}_{A_m}\dots\tilde{\tilde{G}}_{A_1}Y^{A_1\dots A_m}_{B_n\dots B_1}\tilde{G}^{B_1}\dots\tilde{G}^{B_n}|_{\tilde{G}=0,\tilde{\tilde{G}}=0}\,. \tag{3.33}$$

The structural operators Y are symmetric in the sense of the operation (3.8), so that eq. (3.19) holds true for them.

After using the expansion (3.33) in the l.-h. side of the first equation in (2.2) it may be reduced to the Weyl-normal form to give the recurrence relations of the form (3.20) for the structural operators, with the new operators

$$R^{A_1\dots A_m}_{B_n\dots B_1} \equiv \sum_{p=0}^{m}\sum_{q=0}^{n}\Big\{\,[Y^{A_1\dots A_p}_{B_n\dots B_{q+1}}, X^{A_{p+1}\dots A_m}_{B_q\dots B_1}](-1)^{\varepsilon'^{p0}_q} +$$

$$+ \sum_{r,s}{}'\,(Y^{A_1\dots A_pC_1\dots C_r}_{B_n\dots B_{q+1}\mathscr{D}_s\dots\mathscr{D}_1}X^{\mathscr{D}_1\dots\mathscr{D}_sA_{p+1}\dots A_m}_{C_r\dots C_1B_q\dots B_1}(-1)^{\varepsilon'^{pr}_q} -$$

$$-X^{A_1\dots A_pC_1\dots C_r}_{B_n\dots B_{q+1}\mathscr{D}_s\dots\mathscr{D}_1}Y^{\mathscr{D}_1\dots\mathscr{D}_sA_{p+1}\dots A_m}_{C_r\dots C_1B_q\dots B_1}(-1)^{\varepsilon''^{pr}_q})C^{pr}_{qs}\,\Big\}, \tag{3.34}$$

entering (3.20) this time, instead of (3.21). We refer to the designations

$$\varepsilon'^{pr}_q \equiv \varepsilon^{pr}_q + \sum_{i=p+1}^{m}\varepsilon_{A_i} + \sum_{i=1}^{r}\varepsilon_{C_i}, \tag{3.35}$$

$$\varepsilon''^{pr}_q \equiv \varepsilon^{pr}_q + \sum_{i=q+1}^{n}\varepsilon_{B_i} + \sum_{i=1}^{r}\varepsilon_{C_i}, \tag{3.36}$$

with $\varepsilon^{pr}_q$ and $C^{pr}_{qs}$ defined as (3.30) and (3.31), respectively.

The second one of equations (2.2), when taken together with (3.3), creates the Hermite conjugation law for the structural operators in the expansion (3.33)

$$\left(Y^{A_1 \dots A_m}_{B_n \dots B_1}\right)^{\dagger} = Y^{A_m \dots A_1}_{B_1 \dots B_n} (-1)^{(\varepsilon_A + 1)\varepsilon_B}, \tag{3.37}$$

where parities from (3.15) appear.

The distribution of statistics of the structural operators and the condition to be obeyed by the values of $m$ and $n$ in expansion (3.33) coincide with (3.25) and (3.26), respectively.

To conclude this item, we list formulas that relate the structural operators corresponding to the $\overline{\text{G}}$G-and Weyl-normal forms. There are

$$\sum_{s \geq 0} \left(\pm \frac{1}{2}\right)^s C^0_{0s} \left(X^{A_1 \dots A_m \mathscr{D}_s \dots \mathscr{D}_1}_{\mathscr{D}_1 \dots \mathscr{D}_s B_n \dots B_1}\right)_{\substack{\text{Weyl} \\ \overline{G}G}} = \left(X^{A_1 \dots A_m}_{B_n \dots B_1}\right)_{\substack{\overline{G}G \\ \text{Weyl}}}, \tag{3.38}$$

$$\sum_{s \geq 0} \left(\pm \frac{1}{2}\right)^s C^0_{0s} (-1)^{\varepsilon_{\mathscr{D}}} \left(Y^{A_1 \dots A_m \mathscr{D}_s \dots \mathscr{D}_1}_{\mathscr{D}_1 \dots \mathscr{D}_s B_n \dots B_1}\right)_{\substack{\text{Weyl} \\ \overline{G}G}} = \left(Y^{A_1 \dots A_m}_{B_n \dots B_1}\right)_{\substack{\overline{G}G \\ \text{Weyl}}} \tag{3.39}$$

Here $C^0_{0s}$ is given by (3.12) with $p = 0, q = 0$; $\varepsilon_{\mathscr{D}}$ is defined as (3.24).

## B. Wick ghost sector.

In the preceeding subsection A we proceeded from a unique principle of arranging the ghost sector of the theory formed as a set of canonical operator pairs (3.1). Now we go into more details. Assume that the ghost sector contains, first, the Wick operator pairs

$$(G^a, \overline{G}^{\dagger}_a), \qquad (\overline{G}_a, G^{\dagger a}), \tag{3.40}$$

$$\varepsilon(G^a) = \varepsilon(G^{\dagger a}) = \varepsilon(\overline{G}_a) = \varepsilon(\overline{G}^{\dagger}_a) \equiv \varepsilon_a, \tag{3.41}$$

$$\text{gh}(G^a) = \text{gh}(G^{\dagger a}) = -\text{gh}(\overline{G}_a) = -\text{gh}(\overline{G}^{\dagger}_a), \tag{3.42}$$

so that only the following equal-time supercommutators are different from zero for them

$$[G^a, \overline{G}^{\dagger}_b] = [\overline{G}_b, G^{\dagger a}] = \delta^a_b. \tag{3.43}$$

In prospect of considering the limiting case of infinite number of degrees of freedom, assume that the parameter which becomes infinite in this limit is just the dimension run by the index « $a$ » that numbers the Wick pairs (3.40).

Assume, second, that, besides (3.40), that ghost sector can also contain a *finite* number of ordinary canonical operator pairs $(G^{a_0}, \overline{G}_{a_0})$ like (3.1)-(3.3) labelled by the index « $a_0$ ». These operators are called, conventionally, null modes.

From the physical point of view, it is the Wick pairs that correspond to the gauge symmetry understood literally as a local symmetry, whereas the null modes correspond to an admixture of the global symmetry in the gauge transformation algebra. A typical example is provided by the Virasoro algebra for the boson string.

The complete set (3.1)-(3.3) of the ghost canonical pairs can be formed by the Wick pairs and null modes, following e. g. the relations

$$G^A = \left(\frac{1}{\sqrt{2}}(G^a + G^{\dagger a})\,;\, \frac{i}{\sqrt{2}}(G^a - G^{\dagger a})\,;\, G^{a_0}\right), \tag{3.44}$$

$$\overline{G}_A = \left(\frac{i}{\sqrt{2}}(\overline{G}^\dagger_a - \overline{G}_a(-1)^{\varepsilon_a})\,;\, \frac{1}{\sqrt{2}}(\overline{G}^\dagger_a + \overline{G}_a(-1)^{\varepsilon_a})\,;\, \overline{G}_{a_0}\right). \tag{3.45}$$

When studying the generating of the gauge algebra we shall only perform an explicit expansion of the generating operators $\Omega$ and $\mathscr{H}$ in powers the Wick pairs (3.40), keeping the dependence upon the null modes inside the operator-valued coefficients along with the original dynamical variables. Accordingly, we certainly shall not assume the ghost number of the operator-valued coefficients to be zero. Once the recurrence relations for the coefficient operators are obtained, they can readily be further expanded in powers of the null modes (if any) using the means analogous to those presented in subsection A. In the cases interesting for practice whe number of null modes is usually small, which is a source of further simplifications in cases when they are all fermions.

B1. *Wick normal form.*

In this item we perform the expansion of the generating equations (1.1) and (2.2) in powers of the Wick pairs (3.40) of the ghost operators taken in the Wick-normal form, i. e. placing all $\overline{G}^\dagger$, $G^\dagger$ to the left of all G, $\overline{G}$. The same as in items A1 and A2, we continue to designate the structural coefficients in expansions for $\Omega$ and $\mathscr{H}$ as X and Y. The operators X and Y now bear each four groups of small Roman indices enumerating the Wick pairs (3.40), and do not, evidently, coincide with the structural coefficients in the expansions (3.4), (3.18), nor in (3.27), (3.33).

We begin, the same as before, with equations (1.1) and seek their solution in the form of the Wick expansion

$$\Omega = \sum_{k,l,m,n\geqslant 0} \overline{G}^\dagger_{a_k} \dots \overline{G}^\dagger_{a_1} G^{\dagger b_l} \dots G^{\dagger b_1} \times \\ \times X^{a_1\dots a_k|c_m\dots c_1}_{b_1\dots b_l|d_n\dots d_1} G^{d_1} \dots G^{d_n} \overline{G}_{c_1} \dots \overline{G}_{c_m}. \tag{3.46}$$

Consider classical counterparts to the operators (3.40):

$$(\tilde{G}^a, \tilde{\overline{G}}{}^*_a), \qquad (\tilde{\overline{G}}_a, \tilde{G}^{*a}), \tag{3.47}$$

$$\varepsilon(\tilde{G}^a) = \varepsilon(\tilde{G}^{*a}) = \varepsilon(\tilde{\overline{G}}_a) = \varepsilon(\tilde{\overline{G}}{}^*_a) = \varepsilon_a, \tag{3.48}$$

and define the symmetrizers

$$k!\,l!\,\bar{S}^{a_1\ldots a_k|b'_l\ldots b'_1}_{b_1\ldots b_l|a'_k\ldots a'_1} \equiv \left( \frac{\vec{\partial}}{\partial \tilde{G}^*_{a_1}} \cdots \frac{\vec{\partial}}{\partial \tilde{G}^*_{a_k}} \tilde{G}^*_{a'_k} \cdots \tilde{G}^*_{a'_1} \right) \times$$

$$\times \left( \frac{\vec{\partial}}{\partial \tilde{G}^{*b_1}} \cdots \frac{\vec{\partial}}{\partial \tilde{G}^{*b_l}} \tilde{G}^{*b'_l} \cdots \tilde{G}^{*b'_1} \right), \quad (3.49)$$

$$m!\,n!\,S^{d'_1\ldots d'_n|c_m\ldots c_1}_{c'_1\ldots c'_m|d_n\ldots d_1} \equiv \left( \tilde{G}_{c'_1} \cdots \tilde{G}_{c'_m} \frac{\overleftarrow{\partial}}{\partial \tilde{G}_{c_m}} \cdots \frac{\overleftarrow{\partial}}{\partial \tilde{G}_{c_1}} \right) \times$$

$$\times \left( \tilde{G}^{d'_1} \cdots \tilde{G}^{d'_n} \frac{\overleftarrow{\partial}}{\partial \tilde{G}^{d_n}} \cdots \frac{\overleftarrow{\partial}}{\partial \tilde{G}^{d_1}} \right), \quad (3.50)$$

involved in the definition of the symmetrization operation of any quantity K bearing four groupes of indices

$$\mathrm{Sym}\left(K^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1}\right) \equiv \bar{S}^{a_1\ldots a_k|b'_l\ldots b'_1}_{b_1\ldots b_l|a'_k\ldots a'_1} K^{a'_1\ldots a'_k|c'_m\ldots c'_1}_{b'_1\ldots b'_l|d'_n\ldots d'_1} S^{d'_1\ldots d'_n|c_m\ldots c_1}_{c'_1\ldots c'_m|d_n\ldots d_1}. \quad (3.51)$$

Structural operators X in expansion (3.46) are symmetric under (3.51)

$$X = \mathrm{Sym}\,(X). \quad (3.52)$$

After substituting expansion (3.46) into the l.-h. side of the first equation in (1.1) and reducing it to the Wick normal form, we obtain the following recurrence relations for the structural operators

$$\mathrm{Sym}\left(Q^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1}\right) = 0, \quad (3.53)$$

where the symmetrization (3.51) appears in the l.-h. side, and the notations are used

$$Q^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1} \equiv$$

$$\equiv \sum_{p=0}^{k}\sum_{q=0}^{l}\sum_{r=0}^{m}\sum_{s=0}^{n}\left\{ \frac{1}{2}(-1)^{\varepsilon^{pr0}_{qs0}} \times \left[X^{a_1\ldots a_p|c_m\ldots c_{r+1}}_{b_1\ldots b_q|d_n\ldots d_{s+1}}, X^{a_{p+1}\ldots a_k|c_r\ldots c_1}_{b_{q+1}\ldots b_l|d_s\ldots d_1}\right] + \right.$$

$$\left. + \sum_{t,u}{}' X^{a_1\ldots a_p|c_m\ldots c_{r+1}e_t\ldots e_1}_{b_1\ldots b_q|d_n\ldots d_{s+1}f_u\ldots f_1} X^{f_1\ldots f_u a_{p+1}\ldots a_k|c_r\ldots c_1}_{e_1\ldots e_t b_{q+1}\ldots b_l|d_s\ldots d_1} \times C^{prt}_{qsu}(-1)^{\varepsilon^{prt}_{qsu}} \right\}, \quad (3.54)$$

$$\varepsilon_{qsu}^{prt} \equiv \left( \sum_{i=p+1}^{k} \varepsilon_{a_i} + \sum_{i=q+1}^{l} \varepsilon_{b_i} + 1 \right) \left( \sum_{j=r+1}^{m} \varepsilon_{c_j} + \sum_{j=s+1}^{n} \varepsilon_{d_j} + 1 \right) +$$
$$+ \left( \sum_{i=1}^{p} \varepsilon_{a_i} + \sum_{i=1}^{u} \varepsilon_{f_i} \right) \sum_{j=q+1}^{l} \varepsilon_{b_j} + \left( \sum_{i=1}^{r} \varepsilon_{c_i} + \sum_{i=1}^{t} \varepsilon_{e_i} \right) \sum_{j=s+1}^{n} \varepsilon_{d_j} +$$
$$+ \sum_{i=1}^{t} \varepsilon_{e_i} \sum_{j=1}^{u} \varepsilon_{f_j} + 1 , \tag{3.55}$$

$$\sum_{t,u}{}' \equiv \sum_{t,u>0} + \delta_{t0} \sum_{u>0} + \delta_{u0} \sum_{t>0} , \tag{3.56}$$

$$C_{qsu}^{prt} \equiv \frac{(u+k-p)!\,(u+n-s)!\,(t+l-q)!\,(t+m-r)!}{(k-p)!\,u!\,(n-s)!\,(l-q)!\,t!\,(m-r)!} . \tag{3.57}$$

The second equation in (1.1) induces the Hermite conjugation law

$$\left( X_{b_1 \dots b_l | d_n \dots d_1}^{a_1 \dots a_k | c_m \dots c_1} \right)^\dagger = X_{d_1 \dots d_n | b_l \dots b_1}^{c_1 \dots c_m | a_k \dots a_1} . \tag{3.58}$$

The third equation (1.1) gives rise to the distribution of statistics

$$\varepsilon\left( X_{b_1 \dots b_l | d_n \dots d_1}^{a_1 \dots a_k | c_m \dots c_1} \right) = \sum_{i=1}^{k} \varepsilon_{a_i} + \sum_{i=1}^{l} \varepsilon_{b_i} + \sum_{i=1}^{m} \varepsilon_{c_i} + \sum_{i=1}^{n} \varepsilon_{d_i} + 1 . \tag{3.59}$$

The fourth equation in (1.1) results in restricting the admissible values of $k$, $l$, $m$, $n$ in expansion (3.46):

$$1 = \mathrm{gh}\left( X_{b_1 \dots b_l | d_n \dots d_1}^{a_1 \dots a_k | c_m \dots c_1} \right) - \sum_{i=1}^{k} \mathrm{gh}(G^{a_i}) + \sum_{i=1}^{l} \mathrm{gh}(G^{b_i}) - \sum_{i=1}^{m} \mathrm{gh}(G^{c_i}) +$$
$$+ \sum_{i=1}^{n} \mathrm{gh}(G^{d_i}) . \tag{3.60}$$

Consider now equations (2.2), for solving which the Ansatz analogous to (3.46) will be used

$$\mathscr{H} = \sum_{k,l,m,n \geq 0} \overline{G}_{a_k}^\dagger \dots \overline{G}_{a_1}^\dagger G^{\dagger b_l} \dots G^{\dagger b_1} \times$$
$$\times Y_{b_1 \dots b_l | d_n \dots d_1}^{a_1 \dots a_k | c_m \dots c_1} G^{d_1} \dots G^{d_n} \overline{G}_{c_1} \dots \overline{G}_{c_m} \tag{3.61}$$

with the structural coefficients Y symmetric under (3.51)

$$\mathrm{Y} = \mathrm{Sym}\,(\mathrm{Y}). \tag{3.62}$$

Substituting (3.46), (3.61) into the first equation in (2.2) and reducing it to the Wick normal form results in the following recurrence relations for the structural operators

$$\mathrm{Sym}\,(\mathrm{R}^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1}) = 0\,, \tag{3.63}$$

where the symmetrization (3.51) appears in the l.-h. side, and the notations are referred to

$$\mathrm{R}^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1} \equiv \sum_{p=0}^{k}\sum_{q=0}^{l}\sum_{r=0}^{m}\sum_{s=0}^{n}\{(-1)^{\varepsilon'^{pr0}_{qs0}} \times$$

$$\times\,[\mathrm{X}^{a_1\ldots a_p|c_m\ldots c_{r+1}}_{b_1\ldots b_q|d_n\ldots d_{s+1}}, \mathrm{Y}^{a_{p+1}\ldots a_k|c_r\ldots c_1}_{b_{q+1}\ldots b_l|d_s\ldots d_1}] + \sum_{t,u}{}' (\mathrm{X}^{a_1\ldots a_p|c_m\ldots c_{r+1}e_t\ldots e_1}_{b_1\ldots b_q|d_n\ldots d_{s+1}f_u\ldots f_1} \times$$

$$\times\,\mathrm{Y}^{f_1\ldots f_u a_{p+1}\ldots a_k|c_r\ldots c_1}_{e_1\ldots e_t b_{q+1}\ldots b_l|d_s\ldots d_1}(-1)^{\varepsilon'^{prt}_{qsu}} - \mathrm{Y}^{a_1\ldots a_p|c_m\ldots c_{r+1}e_t\ldots e_1}_{b_1\ldots b_q|d_n\ldots d_{s+1}f_u\ldots f_1} \times$$

$$\times\,\mathrm{X}^{f_1\ldots f_u a_{p+1}\ldots a_k|c_r\ldots c_1}_{e_1\ldots e_t b_{q+1}\ldots b_l|d_s\ldots d_1}(-1)^{\varepsilon''^{prt}_{qsu}})\mathrm{C}^{prt}_{qsu}\}\,, \tag{3.64}$$

$$\varepsilon'^{prt}_{qsu} \equiv \varepsilon^{prt}_{qsu} + \sum_{j=r+1}^{m}\varepsilon_{c_j} + \sum_{j=s+1}^{n}\varepsilon_{d_j}\,, \tag{3.65}$$

$$\varepsilon''^{prt}_{qsu} \equiv \varepsilon^{prt}_{qsu} + \sum_{i=p+1}^{k}\varepsilon_{a_i} + \sum_{i=q+1}^{l}\varepsilon_{b_i}\,, \tag{3.66}$$

with $\varepsilon^{prt}_{qsu}$, $\sum_{t,u}{}'$, $\mathrm{C}^{prt}_{qsu}$ given by (3.55), (3.56), (3.57), respectively.

The second equation (2.2) creates the Hermite conjugation property

$$(\mathrm{Y}^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1})^{\dagger} = \mathrm{Y}^{c_1\ldots c_m|a_k\ldots a_1}_{d_1\ldots d_n|b_l\ldots b_1}\,. \tag{3.67}$$

The third equation in (2.2) leads to the distribution of statistics

$$\varepsilon(\mathrm{Y}^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1}) = \sum_{i=1}^{k}\varepsilon_{a_i} + \sum_{i=1}^{l}\varepsilon_{b_i} + \sum_{i=1}^{m}\varepsilon_{c_i} + \sum_{i=1}^{n}\varepsilon_{d_i}\,. \tag{3.68}$$

Finally, the fourth equation in (2.2) provides the condition on admissible values of $k, l, m, n$ in expansion (3.61):

$$0 = \mathrm{gh}\,(\mathrm{Y}^{a_1\ldots a_k|c_m\ldots c_1}_{b_1\ldots b_l|d_n\ldots d_1}) -$$

$$- \sum_{i=1}^{k}\mathrm{gh}\,(\mathrm{G}^{a_i}) + \sum_{i=1}^{l}\mathrm{gh}\,(\mathrm{G}^{b_i}) - \sum_{i=1}^{m}\mathrm{gh}\,(\mathrm{G}^{c_i}) + \sum_{i=1}^{n}\mathrm{gh}\,(\mathrm{G}^{d_i})\,. \tag{3.69}$$

We conclude this item by studying the connection between structural operators corresponding to the Wick and Weyl normal forms for ghosts. Unfortunately, the relations that connect directly the Wick and Weyl structural operators as explicitly as (3.38), (3.39), prove to be very cumbersome. By this reason we display here only a closed generating formula, whose expansion in powers of the *classical* variables induces the relations between the corresponding structural operators.

Let us write the division (3.44), (3.45) as

$$G^{A} = (G'^{a}\,;\,G''^{a}\,;\,G^{a_0}), \qquad \bar{G}_{A} = (\bar{G}'_{a}\,;\,\bar{G}''_{a}\,;\,\bar{G}_{a_0}), \tag{3.70}$$

and understand that the set of classical variables (3.5) corresponding to operators (3.1) has been divided in analogous way.

Let us denote as

$$\bar{\Omega}_{\mathrm{Weyl}}(\tilde{G}', \tilde{G}'', \tilde{\bar{G}}', \tilde{\bar{G}}'') \tag{3.71}$$

the result of the action of the operator

$$\exp\left\{ G^{a_0}\frac{\partial}{\partial \tilde{G}^{a_0}} + \bar{G}_{a_0}\frac{\partial}{\partial \tilde{\bar{G}}_{a_0}} \right\} \tag{3.72}$$

on the sum $\sum\limits_{m,n\geqslant 0} \ldots$ in eq. (3.27), taken at $\tilde{G}^{a_0} = 0$, $\tilde{\bar{G}}_{a_0} = 0$.

Then, let us denote as

$$\bar{\Omega}_{\mathrm{Wick}}(\tilde{G}', \tilde{G}'', \tilde{\bar{G}}', \tilde{\bar{G}}'') \tag{3.73}$$

the expansion to be obtained from (3.46) by the formal replacement of the operators (3.40) by the following values of their classical counterparts (3.47):

$$\sqrt{2}\tilde{G}^{a} = (\tilde{G}' - i\tilde{G}'')^{a}, \qquad \sqrt{2}\tilde{G}^{*a} = (\tilde{G}' + i\tilde{G}'')^{a}, \tag{3.74}$$

$$\sqrt{2}\tilde{\bar{G}}_{a} = (\tilde{\bar{G}}'' + i\tilde{\bar{G}}')_{a}(-1)^{\varepsilon_a}, \qquad \sqrt{2}\tilde{\bar{G}}^{*}_{a} = (\tilde{\bar{G}}'' - i\tilde{\bar{G}}')_{a}. \tag{3.75}$$

Operators (3.71) and (3.73) are related by the following generating formula

$$\bar{\Omega}_{\substack{\mathrm{Weyl}\\ \mathrm{Wick}}} = \exp\left\{ \mp\frac{1}{2}\left( \frac{\partial}{\partial \tilde{\bar{G}}''}\cdot\frac{\partial}{\partial \tilde{G}'} - \frac{\partial}{\partial \tilde{\bar{G}}'}\cdot\frac{\partial}{\partial \tilde{G}''} \right) \right\} \bar{\Omega}_{\substack{\mathrm{Wick}\\ \mathrm{Weyl}}} \tag{3.76}$$

By expanding (3.76) in powers of the classical variables present as arguments in (3.71), (3.73) we obtain the formulas connecting the Weyl and Wick structural operators.

Defining the operators

$$\bar{\mathscr{H}}_{\substack{\mathrm{Weyl}\\ \mathrm{Wick}}}(\tilde{G}', \tilde{G}'', \tilde{\bar{G}}', \tilde{\bar{G}}''), \tag{3.77}$$

with the help of expansions (3.33), (3.61), in full analogy with (3.71), (3.73), we shall obtain a generating formula analogous to (3.76) for them.

## C. Some remarks on structural relations.

Thus, we have obtained structural relations corresponding to the $\overline{\mathrm{G}}\mathrm{G}$ –, Weyl—and Wick—normal orderings of the ghost operators. Note to the point, that the case of $\mathrm{G}\overline{\mathrm{G}}$ – normal form has been as a matter of fact also included here, since, owing to (3.3) the $\mathrm{G}\overline{\mathrm{G}}$ – normal expansions of the generating operators $\Omega$ and $\mathscr{H}$ are obtained from (3.4) and (3.18) by means of the Hermite conjugation. Hence, the $\mathrm{G}\overline{\mathrm{G}}$ – and $\overline{\mathrm{G}}\mathrm{G}$ – structural operators are interrelated as follows

$$\left(\mathrm{X}^{\mathrm{A}_m \ldots \mathrm{A}_1}_{\mathrm{B}_1 \ldots \mathrm{B}_n}\right)_{\mathrm{G}\overline{\mathrm{G}}} = \left(\mathrm{X}^{\mathrm{A}_1 \ldots \mathrm{A}_m}_{\mathrm{B}_n \ldots \mathrm{B}_1}\right)^{\dagger}_{\overline{\mathrm{G}}\mathrm{G}} (-1)^{\varepsilon_{\mathrm{A}}}, \tag{3.78}$$

$$\left(\mathrm{Y}^{\mathrm{A}_m \ldots \mathrm{A}_1}_{\mathrm{B}_1 \ldots \mathrm{B}_n}\right)_{\mathrm{G}\overline{\mathrm{G}}} = \left(\mathrm{Y}^{\mathrm{A}_1 \ldots \mathrm{A}_m}_{\mathrm{B}_n \ldots \mathrm{B}_1}\right)^{\dagger}_{\overline{\mathrm{G}}\mathrm{G}} (-1)^{\varepsilon_{\mathrm{A}}}, \tag{3.79}$$

where $\varepsilon_{\mathrm{A}}$ is defined in (3.15). The operators in the l.-h. sides of eqs. (3.78) and (3.79) obey the structural relations to be obtained by the Hermite conjugation from (3.10)-(3.13) and (3.20)-(3.22), respectively. The operators in the r.-h. sides of eqs. (3.78) are given by (3.14) and (3.23).

Consider now the most characteristic general properties of the structural relations.

*First,* in accord with the above, we can observe from the formulas of section 3 that both the structural operators themselves, and the explicit form of the structural relations they are subject to, depend essentially on a special choice of the way of normal ordering in expansions of the generating operators $\Omega$ and $\mathscr{H}$. It is seen from the connection formulas (3.38), (3.39), (3.76), (3.78), (3.79) that the quantum correction acquired by each structural operator as the normal ordering of ghosts is changed is given by a linear superposition of correlated traces of all possible structural operators of a given type (i. e. X or Y) over their super- and subscripts.

Second, a peculiarity of the operator gauge algebra is in that the essential quantum terms are present in its structural relations, that are not removable by any alteration in the way of ordering the structural operators when multiplied among themselves. We call this phenomenon, first observed in our work [*1*], the quantum deformation. Consider, for example, the operator (3.11) in the l.-h. side of the structural relations (3.10). All the terms in the sum in (3.11) with $s > 1$ are the quantum deformation. Only the term with $s = 1$ has a classical analog. Exactly the same situation takes place for the sum over $s$ in (3.21). Next, in the Weyl structural relations the operators (3.28), (3.34) are present in the l.-h. sides instead of (3.11), (3.21). Here the quantum deformation is made by every term in the sum over the values $r + s > 1$. Finally, in the Wick structure relations (3.53), (3.63) every term in the sums (3.54), (3.64) with $t + u > 1$ corresponds to the quantum deformation. The phenomenon of quantum deformation proves to be a very peculiar feature of the operator approach since the corresponding terms in the structural relations, although unpredictable from the classical point of view, are, nevertheless, quite necessary for the algebraic compatibility to be fulfilled in the operator domain. As we shall see in what follows the quantum deformation, when taken into account in the evolution relations of constraints, leads immediately to the correct central extension of the Virasoro algebra in case of bosonic string in the critical dimension.

Third, we must emphasize the nontrivial nature of the operatorial existence problem for the generating equations (1.1), (2.2) and, hence, for the corresponding structural relations. The heart of this problem thought of as a problem of quantization lies in the following. Assume that there exists a solution to the classical analogs of the generating functionals (1.2), (2.2) corresponding, in the classical sense, to the gauge system under study. One may ask, whether it is possible to find in a constructive way a corresponding solution of the operator equations (1.1), (1.2) at least as formal operator expressions, given in the form of series in powers of the Planck constant.

Below we shall give affirmative answer to this question, at least in what concerns a system possessing a finite number of degrees of freedom and no profound dynamical pathology.

### D. Ghost number operator.

Limitations due to the fact that there exists such an internal dynamical characteristic as the ghost number are of extreme importance for quantization of gauge systems. These are discrete selection rules that should be respected by every admissible operator in the theory. They are given by the fourth equations of (1.1), (1.2) and (2.2) and the third equation of (2.3). As applied to the structural operators, these limitations are presented by conditions (3.17), (3.26), (3.60), (3.68) that separate admissible terms in the expansions (3.4), (3.18), (3.27), (3.33), (3.46), (3.61) of the generating functionals $\Omega$ and $\mathscr{H}$ in powers of ghosts.

The ghost number is a peculiar dynamical characteristic, only born by the ghost sector operators. Every entity of the theory that depends on ghost operators, is constructed so as to possess a definite value of the ghost number. As all our basic objects, including the generating operators $\Omega$ and $\mathscr{H}$ of the gauge algebra, the gauge fermion $\Psi$, the unitarizing Hamiltonian H, etc., are found in as series expansions in powers of ghosts, the requirement that any of these quantities should have a definite ghost number reduces to the mere uniformity condition, which reads that for all monomials present in the expansion of a given quantity in powers of ghosts the sum of the ghost numbers of the elementary operators included into them must be the same. This simple selection rule is used directly and suffices for treating ghost numbers of *operators*. If, however, we are going to attribute definite values of the ghost number also to *states* (at least to some of them), we should have at our disposal a special conserving operator that would correspond to the ghost number, *as itself.* This operator is, quite naturally, called the ghost number operator. It is denoted as $\mathcal{G}$ and is a hermitian boson

$$\varepsilon(\mathcal{G}) = 0, \qquad \mathcal{G} = \mathcal{G}^{\dagger}. \tag{3.80}$$

By definition, any operator A possessing the value gh (A) of the ghost number, obeys the condition

$$[i\mathcal{G}, \mathrm{A}] = \mathrm{gh}\,(\mathrm{A})\mathrm{A}, \tag{3.81}$$

that automatically guarantees the correct multiplication composition law for the ghost number.

Although, at first glance, it looks a paradox, the operator $\mathcal{G}$ itself obeys the relation

$$\mathrm{gh}\,(\mathcal{G}) = 0\,. \tag{3.82}$$

that follows from (3.81). In accordance with (3.81), the fourth equation in (1.1), (1.2) is equivalent to the conditions

$$[i\mathcal{G}, \Omega] = \Omega, \qquad [i\mathcal{G}, \mathrm{H}] = 0\,. \tag{3.83}$$

The second of them provides the conservation of the ghost number operator in time

$$i\partial_t \mathcal{G} = [\mathcal{G}, \mathrm{H}] = 0\,. \tag{3.84}$$

In full analogy with (3.83), the fourth equation in (2.2) and the third in (2.3) are equivalent to the conditions

$$[i\mathcal{G}, \Psi] = -\Psi, \qquad [i\mathcal{G}, \mathcal{H}] = 0\,. \tag{3.85}$$

We are now in a position to find an explicit expression for the ghost number operator in terms of elementary operators of the ghost sector. We shall study the two equivalent possibilities:

*i*) canonical ghost sector (3.1)-(3.3)

$$\mathcal{G} = \frac{1}{2}\sum_{\mathrm{A}} \mathrm{gh}\,(\mathrm{G}^{\mathrm{A}})(\overline{\mathrm{G}}_{\mathrm{A}}\mathrm{G}^{\mathrm{A}} + \mathrm{G}^{\mathrm{A}}\overline{\mathrm{G}}_{\mathrm{A}}(-1)^{\varepsilon_{\mathrm{A}}})\,; \tag{3.86}$$

*ii*) the Wick ghost sector (3.40)-(3.43)

$$\mathcal{G} = i\sum_{a} \mathrm{gh}\,(\mathrm{G}^{a})(\overline{\mathrm{G}}{}^{\dagger}_{a}\mathrm{G}^{a} - \mathrm{G}^{\dagger a}\overline{\mathrm{G}}_{a}) + \frac{1}{2}\sum_{a_0} \mathrm{gh}\,(\mathrm{G}^{a_0})(\overline{\mathrm{G}}_{a_0}\mathrm{G}^{a_0} + \mathrm{G}^{a_0}\overline{\mathrm{G}}_{a_0}(-1)^{\varepsilon_{a_0}})\,. \tag{3.87}$$

Here the Wick pairs enter in the first sum in (3.87), while the second one is comprised by canonical pairs of the null modes. The fulfilment of conditions (3.80) and (3.81) for elementary ghost operators is verified straightforwardly.

It remains to define the ghost number for states. By means of the operators (3.86) or (3.87), this is done in a quite obvious way. By definition, a state $|\Phi\rangle$ has its ghost number equal gh $(|\Phi\rangle)$ providing the condition

$$\mathcal{G}|\Phi\rangle = \mathrm{gh}(|\Phi\rangle)|\Phi\rangle, \tag{3.88}$$

is obeyed. A particularly important class of such states is made by singlets defined as (1.3) and (2.11) that have zero ghost number. These states are most closely related to physical degrees of freedom, singlet out, in the classical sense, by constraints and the unitary gauge.

## 4. OPERATORIAL EXISTENCE PROBLEM FOR THE GENERATING EQUATIONS OF THE GAUGE ALGEBRA VIEWED ON AS A PROBLEM OF QUANTIZATION

In this section we shall try to answer the following important question. Let a solution for the classical analogs of the generating equations (1.1), (2.2) be given, that correspond to a given gauge system. One may ask, if there can be found (at least in the form of series in powers of the Planck constant) a solution of the corresponding operator equations that would have this classical solutions as its $\hbar \to 0$ limit. We shall show that this correspondence can be indeed established for systems with a finite number of degrees of freedom at least locally, i. e. in every domain where an admissible gauge exists.

Since we are going to essentially exploit the technique of symbols, we find it appropriate to remember some elementary facts concerning the relations between operators and their symbols.

### A. Operators and Symbols.

Assume that a complete set $\Gamma$ of operator-valued dynamical variables is given as a set of canonical pairs

$$\Gamma = (P_M, Q^M), \ \varepsilon(P_M) = \varepsilon(Q^M) \equiv \varepsilon_M, \ \mathrm{gh}(P_M) = -\mathrm{gh}(Q^M), \tag{4.1}$$

so that the only nonzero equal-time supercommutators for them are

$$[Q^M, P_{M'}] = i\hbar\delta^M_{M'}. \tag{4.2}$$

We display the Planck constant $\hbar$ explicitly throughout this section.

Denote as

$$\tilde{\Gamma} = (\tilde{P}_M, \tilde{Q}^M), \qquad \varepsilon(\tilde{P}_M) = \varepsilon(\tilde{Q}^M) = \varepsilon_M, \qquad \mathrm{gh}\,(\tilde{P}_M) = -\,\mathrm{gh}\,(\tilde{Q}^M) \quad (4.3)$$

the classical counterparts of the operators (4.1).

Let (α) mark a certain way of the normal ordering for operators (4.1). Then any operator A(Γ) is defined providing a function of classical variables (4.3)

$$\tilde{A}_\alpha(\tilde{\Gamma}): \quad \varepsilon(\tilde{A}_\alpha) = \varepsilon(A), \qquad \mathrm{gh}\,(\tilde{A}_\alpha) = \mathrm{gh}\,(A), \quad (4.4)$$

is given according to the formula

$$A(\Gamma) = D_\alpha \tilde{A}_\alpha(\tilde{\Gamma})\,|_{\tilde{\Gamma}=0}\,, \quad (4.5)$$

where

$$D_\alpha \equiv \mathcal{N}_\alpha \exp\left\{ \Gamma \cdot \frac{\partial}{\partial \tilde{\Gamma}} \right\} \quad (4.6)$$

and $\mathcal{N}_\alpha$ designates normal ordering of the type (α) for operators Γ.

The function $\tilde{A}_\alpha(\tilde{\Gamma})$ is called the α-symbol of the operator A(Γ). Operator (4.6) replaces, effectively, classical arguments $\tilde{\Gamma}$ of the α-symbol by the corresponding operators Γ arranged in accord with the ordering rule $\mathcal{N}_\alpha$.

Let us display three examples of normal form, most commongly used in the canonical basis (4.1):

*i*) the PQ-normal form (α = PQ):

$$D_{PQ} = \exp\left\{ P \cdot \frac{\partial}{\partial \tilde{P}} \right\} \exp\left\{ Q \cdot \frac{\partial}{\partial \tilde{Q}} \right\}, \quad (4.7)$$

*ii*) the QP-normal form (α = QP):

$$D_{QP} = \exp\left\{ Q \cdot \frac{\partial}{\partial \tilde{Q}} \right\} \exp\left\{ P \cdot \frac{\partial}{\partial \tilde{P}} \right\}, \quad (4.8)$$

*iii*) the Weyl-normal form (α = Weyl):

$$D_{Weyl} = \exp\left\{ P \cdot \frac{\partial}{\partial \tilde{P}} + Q \cdot \frac{\partial}{\partial \tilde{Q}} \right\}, \quad (4.9)$$

Operators (4.7)-(4.9) are related as

$$D_{PQ} \exp\left\{ \frac{i\hbar}{2} \frac{\partial}{\partial \tilde{P}} \cdot \frac{\partial}{\partial \tilde{Q}} \right\} = D_{Weyl} = D_{QP} \exp\left\{ -\frac{i\hbar}{2} \frac{\partial}{\partial \tilde{P}} \cdot \frac{\partial}{\partial \tilde{Q}} \right\}. \quad (4.10)$$

This leads to the corresponding relation for the symbols

$$\exp\left\{ -\frac{i\hbar}{2} \frac{\partial}{\partial \tilde{P}} \cdot \frac{\partial}{\partial \tilde{Q}} \right\} \tilde{A}_{PQ} = \tilde{A}_{Weyl} = \exp\left\{ \frac{i\hbar}{2} \frac{\partial}{\partial \tilde{P}} \cdot \frac{\partial}{\partial \tilde{Q}} \right\} \tilde{A}_{PQ}. \quad (4.11)$$

Formally, the classical variables (4.3) are basic elements of the commutative superalgebra of functions on the phase space, while the operators (4.1) are basic elements of the noncommutative superalgebra of the corresponding operator functions. Eq. (4.5) gives for every ($\alpha$) a one-to-one correspondence

$$\mathrm{A} \leftrightarrow \tilde{\mathrm{A}}_\alpha \tag{4.12}$$

between elements of these two superalgebras, such that

$$\mathrm{AB} \leftrightarrow \tilde{\mathrm{A}}_\alpha * \tilde{\mathrm{B}}_\alpha \equiv \tilde{\mathrm{A}}_\alpha \exp \{ i\hbar \overleftrightarrow{\Delta}_\alpha \} \tilde{\mathrm{B}}_\alpha , \tag{4.13}$$

$$[\mathrm{A}, \mathrm{B}] \leftrightarrow [\tilde{\mathrm{A}}_\alpha, \tilde{\mathrm{B}}_\alpha]_* \equiv \tilde{\mathrm{A}}_\alpha * \tilde{\mathrm{B}}_\alpha - \tilde{\mathrm{B}}_\alpha * \tilde{\mathrm{A}}_\alpha (-1)^{\varepsilon(\mathrm{A})\varepsilon(\mathrm{B})} . \tag{4.14}$$

Eq. (4.13), where $\overleftrightarrow{\Delta}_\alpha$ is a differential operator determined by ($\alpha$) that acts both to the right and to the left, gives a non-commutative (but associative) $*$-multiplication law for $\alpha$-symbols. In its turn, eq. (4.14) defines the $*$-commutator of $\alpha$-symbols. For the normal forms (4.7)-(4.9) the operator is given by the relations

$$\overleftrightarrow{\Delta}_{\mathrm{PQ}} = \frac{\overleftarrow{\partial}}{\partial \tilde{\mathrm{Q}}} \cdot \frac{\overrightarrow{\partial}}{\partial \tilde{\mathrm{P}}}, \qquad \overleftrightarrow{\Delta}_{\mathrm{QP}} = - \frac{\overrightarrow{\partial}}{\partial \tilde{\mathrm{Q}}} \cdot \frac{\overleftarrow{\partial}}{\partial \tilde{\mathrm{P}}}, \tag{4.15}$$

$$2\overleftrightarrow{\Delta}_{\mathrm{Weyl}} = \overleftrightarrow{\Delta}_{\mathrm{PQ}} + \overleftrightarrow{\Delta}_{\mathrm{QP}} . \tag{4.16}$$

Equations (4.11) and expressions (4.15), (4.16) are particular cases of the following general relations

$$\tilde{\mathrm{A}}_\beta(\tilde{\Gamma}) = \exp \left\{ (\mathcal{N}_\alpha - \mathcal{N}_\beta) \frac{1}{2} \left( \Gamma \cdot \frac{\partial}{\partial \tilde{\Gamma}} \right)^2 \right\} \tilde{\mathrm{A}}_\alpha(\tilde{\Gamma}) , \tag{4.17}$$

$$i\hbar \overleftrightarrow{\Delta}_\alpha = (1 - \mathcal{N}_\alpha) \left( \frac{\overleftarrow{\partial}}{\partial \tilde{\Gamma}} \cdot \Gamma \right) \left( \Gamma \cdot \frac{\overrightarrow{\partial}}{\partial \tilde{\Gamma}} \right), \tag{4.18}$$

valid for any specialization of the normal ordering of the operators $\Gamma$. Moreover, it turns out that these formulas hold true in the general case of dynamical variables, when not only the canonical pairs (4.1) also any other set of operators whose all possible equal-time supercommutators make a $c$-numerical reversible matrix, proportional ($i\hbar$), can be used as operators $\Gamma$.

From now on, we shall omit the subscript indicating the type of a symbol. Unless indicated otherwise we shall mean that any normal ordering can be used in the general basis of the dynamical variables.

Now we shall give some general relations for expanding $*$-products and $*$-commutators of symbols needed in what follows. Assume that the symbols $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$ of operators A and B allow to be expanded in powers of

$$\tilde{\mathrm{A}} = \sum_{n=0}^{\infty} \hbar^n \tilde{\mathrm{A}}_n, \qquad \tilde{\mathrm{B}} = \sum_{n=0}^{\infty} \hbar^n \tilde{\mathrm{B}}_n . \tag{4.19}$$

Then, for the $*$-product (4.13) of these symbols one has the expansion

$$\tilde{A} * \tilde{B} = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^{n} \sum_{l=0}^{m} \frac{i^l}{l!} \tilde{A}_{m-l}(\overleftrightarrow{\Delta})^l \tilde{B}_{n-m}, \tag{4.20}$$

where $\overleftrightarrow{\Delta}$ is the concise notation (with the subscript $\alpha$ omitted) for the operator $\overleftrightarrow{\Delta}_\alpha$ from (4.18).

In its turn, for the $*$-commutator (4.14) of the symbols (4.19) one has the expansion

$$[\tilde{A}, \tilde{B}]_* = \sum_{n=1}^{\infty} \hbar^n \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{i^l}{l\,!} \{ \tilde{A}_{m-l}, \tilde{B}_{n-m} \}_l, \tag{4.21}$$

where we have denoted

$$\{ \tilde{A}, \tilde{B} \}_l \equiv \tilde{A}(\overleftrightarrow{\Delta})^l \tilde{B} - \tilde{B}(\overleftrightarrow{\Delta})^l \tilde{A} (-1)^{\varepsilon(\tilde{A})\varepsilon(\tilde{B})}. \tag{4.22}$$

For $l = 1$, the binary operation (4.22) coincides with the Poisson superbracket

$$\{ \tilde{A}, \tilde{B} \}_1 = \{ \tilde{A}, \tilde{B} \}, \tag{4.23}$$

which takes its standart from in the canonical basis

$$\{ \tilde{A}, \tilde{B} \} = \frac{\partial_r \tilde{A}}{\partial \tilde{Q}} \cdot \frac{\partial_l \tilde{B}}{\partial \tilde{P}} - \frac{\partial_r \tilde{B}}{\partial \tilde{Q}} (-1)^{\varepsilon(\tilde{A})\varepsilon(\tilde{B})}. \tag{4.24}$$

In the classical limit $\hbar \to 0$ eqs. (4.20), (4.21) result in

$$\lim_{\hbar \to 0} \tilde{A} * \tilde{B} = \tilde{A}_0 \tilde{B}_0, \tag{4.25}$$

$$\lim_{\hbar \to 0} (i\hbar)^{-1} [\tilde{A}, \tilde{B}]_* = \{ \tilde{A}_0, \tilde{B}_0 \}. \tag{4.26}$$

The last remark of this subsection will concern behaviour of the symbol of a given operator under the Hermite conjugation. Generally, the Hermite conjugation of an operator maps its $\alpha$-symbol into the complex conjugate $(\alpha^\dagger)$-symbol, where $(\alpha^\dagger)$ is the normal form, Hermite-conjugate relative to $(\alpha)$. In the general basis of dynamical variables it is natural to require that the complete set of elementary operators $\Gamma$ be closed with respect to Hermite conjugation:

$$\Gamma^\dagger = \Gamma \eta, \tag{4.27}$$

where $\eta$ is a reversible $c$-numerical matrix. If, besides, the normal form $(\alpha)$ chosen is Hermite-symmetric

$$\alpha^\dagger = \alpha, \tag{4.28}$$

then under Hermite conjugation of an operator its $\alpha$-symbol only undergoes the complex conjugation, while the symbol of a hermitian operator is a real function.

In the canonical basis (4.1) condition (4.27) is fulfilled as a consequence of the usual hermiticity requirement for operators Q and P:

$$Q^{\dagger M} = Q^{M}, \qquad P^{\dagger}_{M} = P_{M}(-1)^{\varepsilon_M}. \tag{4.29}$$

Then, property (4.28) holds true for the Weyl-normal form (4.9), so that the Weyl symbol of any hermitian operator is real:

$$\alpha = \text{Weyl}: \quad A = A^{\dagger} \leftrightarrow \tilde{A} = \tilde{A}^{*}. \tag{4.30}$$

Finally, we shall need two important properties of the binary operation (4.22) corresponding to the Weyl-normal form (i. e. for $\overleftrightarrow{\Delta}$ in (4.22) coinciding with $\overleftrightarrow{\Delta}$ Weyl from (4.15), (4.16)):

$$\alpha = \text{Weyl}: \quad \{\tilde{A}, \tilde{B}\}_{2m} \equiv 0, \qquad m = 0, 1, \ldots \tag{4.31 a}$$

$$\alpha = \text{Weyl}: \quad (\{\tilde{A}, \tilde{B}\}_{l})^{*} = \{\tilde{A}^{*}, \tilde{B}^{*}\}_{l}(-1)^{\varepsilon(\tilde{A})\varepsilon(\tilde{B})}. \tag{4.31 b}$$

## B. Expansion of generating equations of the gauge algebra in powers of the Planck constant.

Henceforward in this section we shall only refer to the Weyl symbols characterized by eqs. (4.9), (4.16), (4.30), (4.31). Denote as $\tilde{\Omega}$ and $\tilde{\mathscr{H}}$ the Weyl symbols of generating operators $\Omega$ and $\mathscr{H}$, respectively. In virtue of the equations (1.1), (2.2), of the correspondences (4.13), (4.14) and also of the properties (4.4), (4.30) ,we have the following equations for the symbols

$$[\tilde{\Omega}, \tilde{\Omega}]_{*} = 0, \qquad \tilde{\Omega} = \tilde{\Omega}^{*}, \qquad \varepsilon(\tilde{\Omega}) = 1, \qquad \text{gh}\,(\tilde{\Omega}) = 1\,, \tag{4.32}$$

$$[\tilde{\mathscr{H}}, \tilde{\Omega}]_{*} = 0, \qquad \tilde{\mathscr{H}} = \tilde{\mathscr{H}}^{*}, \qquad \varepsilon(\tilde{\mathscr{H}}) = 0, \qquad \text{gh}\,(\tilde{\mathscr{H}}) = 0\,, \tag{4.33}$$

where the $*$-commutators are defined by eqs. (4.13), (4.14) with $\alpha$=Weyl, as well as by (4.16) together with (4.15). For more convenience, we reproduce this definition here again in a more explicit form

$$[\tilde{A}, \tilde{B}]_{*} \equiv \tilde{A} \exp\{ i\hbar\overleftrightarrow{\Delta} \} \tilde{B} - \tilde{B} \exp\{ i\hbar\overleftrightarrow{\Delta} \} \tilde{A}(-1)^{\varepsilon(\tilde{A})\varepsilon(\tilde{B})}, \tag{4.34}$$

where

$$\overleftrightarrow{\Delta} = \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial\tilde{Q}} \cdot \frac{\overrightarrow{\partial}}{\partial\tilde{P}} - \frac{\overrightarrow{\partial}}{\partial\tilde{Q}} \cdot \frac{\overleftarrow{\partial}}{\partial\tilde{P}}\right). \tag{4.35}$$

We shall solve equations (4.32), (4.33) using the expansion in powers of $(i\hbar)$:

$$\tilde{\Omega} = \sum_{n=0}^{\infty} (i\hbar)^{n}\tilde{\Omega}_{n}, \qquad \tilde{\mathscr{H}} = \sum_{n=0}^{\infty} (i\hbar)^{n}\tilde{\mathscr{H}}_{n}. \tag{4.36}$$

With the help of the formulas (4.21), (4.22), (4.35) for expansions of $*$-commutators, one obtains the following equations for the coefficient functions $\tilde{\Omega}_n$ and $\tilde{\mathscr{H}}_n$:

$$\{ \tilde{\Omega}_0, \tilde{\Omega}_0 \} = 0, \tag{4.37}$$

$$\{ \tilde{\Omega}_0, \tilde{\Omega}_n \} = \tilde{\Lambda}_n(\tilde{\Omega}_{m<n}), \tag{4.38}$$

$$\{ \tilde{\Omega}_0, \tilde{\mathscr{H}}_0 \} = 0, \tag{4.39}$$

$$\{ \tilde{\Omega}_0, \tilde{\mathscr{H}}_n \} = \tilde{K}_n(\tilde{\mathscr{H}}_{m<n}, \tilde{\Omega}_{l\leqslant n}), \tag{4.40}$$

where $n = 1, \ldots,$ and the r.-h. sides are given as

$$- \tilde{\Lambda}_n \equiv \frac{1}{2(n+1)!} \{ \tilde{\Omega}_0, \tilde{\Omega}_0 \}_{n+1} + \sum_{m=2}^{n} \frac{1}{m!} \{ \tilde{\Omega}_0, \tilde{\Omega}_{n+1-m} \}_m + + \sum_{m=2}^{n} \sum_{l=1}^{m-1} \frac{1}{2l!} \{ \tilde{\Omega}_{m-l}, \tilde{\Omega}_{n+1-m} \}_l, \tag{4.41}$$

$$\tilde{K}_n \equiv \frac{1}{(n+1)!} \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_0 \}_{n+1} + \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_n \} + + \sum_{m=2}^{n} \frac{1}{m!} ( \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_{n+1-m} \}_m + \{ \tilde{\mathscr{H}}_{n+1-m}, \tilde{\Omega}_0 \}_m ) + + \sum_{m=2}^{n} \sum_{l=1}^{m-1} \frac{1}{l!} \{ \tilde{\mathscr{H}}_{m-l}, \tilde{\Omega}_{n+1-m} \}_l . \tag{4.42}$$

The braces without subscripts $\{ , \}$ in eqs. (4.37)-(4.40) and eqs. (4.41), (4.42) stand for the Poisson superbracket (4.24), while the braces with subscripts $\{ , \}_l$ denote the binary operation (4.22) with the operators $\overleftrightarrow{\Delta}$ from (4.35). In the r.-h. sides of equations (4.38), (4.40) the coefficient functions on which eqs. (4.41), (4.42) in fact depend are indicated in the brackets.

The lowest equations (4.37), (4.39) are classical, whereas the higher equations (4.38), (4.40) determine quantum corrections in a recurrent way. One sees that, e. g. the $n$-th equation (4.38) contains the quantity $\tilde{\Omega}_n$ to be fixed by it, only in its l.-h. side, while its r.-h. side (4.41) contains only the preceding quantities $\tilde{\Omega}_m$ with $m < n$. The same situation occurs in equations (4.40) for the quantities $\tilde{\mathscr{H}}_n$. Here, however, the r.-h. sides contain also all the quantities $\tilde{\Omega}_l$ with $l \leqslant n$ found from (4.37), (4.38). In virtue of the lowest equation (4.37), every $n$-th equation out of the higher equations (4.38) or (4.40) requires for its solvability that the necessary compatibility conditions

$$\{ \tilde{\Omega}_0, \tilde{\Lambda}_n \} = 0, \qquad \{ \tilde{\Omega}_0, \tilde{K}_n \} = 0, \tag{4.43}$$

be fulfilled as consequences of the preceding equations for $\tilde{\Omega}_m$ with $m < n$, or $\tilde{\mathscr{H}}_m$ with $m < n$.

The set of equations (4.37)-(4.40) is equivalent to the first equations in (4.32), (4.33). The second equations are the reality conditions for the symbols $\tilde{\Omega}$ and $\tilde{\mathscr{H}}$. They give

$$\tilde{\Omega}_n = (-1)^n \tilde{\Omega}_n^*, \qquad \tilde{\mathscr{H}}_n = (-1)^n \tilde{\mathscr{H}}_n^* . \tag{4.44}$$

The third and fourth conditions from (4.32), (4.33) are straightforwardly extended to the coefficient functions

$$\varepsilon(\tilde{\Omega}_n) = 1, \qquad \mathrm{gh}\,(\tilde{\Omega}_n) = 1, \qquad \varepsilon(\tilde{\mathscr{H}}_n) = 0, \qquad \mathrm{gh}\,(\tilde{\mathscr{H}}_n) = 0 . \tag{4.45}$$

## C. **The basic theorem.**

Assume that we have at our disposal a regular solution $\tilde{\Omega}_0$, $\tilde{\mathscr{H}}_0$ of the classical equations (4.37), (4.39), obeying the conditions (4.44), (4.45) for $n = 0$, such that any regular solution of the equation

$$\{ \tilde{\Omega}_0, \tilde{X} \} = 0, \qquad \mathrm{gh}\,(\tilde{X}) \neq 0 \tag{4.46}$$

can be represented, at least locally, as

$$\tilde{X} = \{ \tilde{\Omega}_0, \tilde{Y} \}, \qquad \varepsilon(\tilde{Y}) = \varepsilon(\tilde{X}) + 1, \qquad \mathrm{gh}\,(\tilde{Y}) = \mathrm{gh}\,(\tilde{X}) - 1 , \tag{4.47}$$

where $\tilde{Y}$ is also a regular function.

Then, we state that, at least locally, there exist and are regular all functions $\tilde{\Omega}_n$, $\tilde{\mathscr{H}}_n$, $n > 0$ satisfying the equations (4.38), (4.40) and the conditions (4.44), (4.45). Thereby, the quantities $\tilde{\Omega}$ and $\tilde{\mathscr{H}}$, that satisfy equations (4.32), (4.33) exist as formal series (4.36) in powers of the Planck constant $\hbar$ and hence, in virtue of the correspondence (4.12), there exist formal expansions in powers of $\hbar$ for the operators $\Omega$ and $\mathscr{H}$, that satisfy equations (1.1), (2.2).

*The proof* will be given following the induction method.

*i)* For $n = 0$ and $n = 1$ the statement of the theorem is evident.

*ii)* Suppose, there exist functions $\tilde{\Omega}_n$, $\tilde{\mathscr{H}}_n$, $n \leqslant \mathscr{N}$, for which the statement of the theorem holds true (the induction assumption).

*iii)* Let us prove then that there exist functions $\Omega_{\mathscr{N}+1}$, $\tilde{\mathscr{H}}_{\mathscr{N}+1}$ for which the statement of the theorem also holds true.

Consider first equations (4.38) for the functions $\tilde{\Omega}_n$.

Let us form the partial sum

$$\tilde{\Omega}^{(\mathcal{N})} = \sum_{n=0}^{\mathcal{N}} (i\hbar)^n \tilde{\Omega}_n \tag{4.48}$$

of the functions $\tilde{\Omega}_n$ with $n < \mathcal{N}$, whose existence is provided by the induction assumption. Then

$$\frac{1}{2}(i\hbar)^{-1}[\tilde{\Omega}^{(\mathcal{N})}, \tilde{\Omega}^{(\mathcal{N})}]_* = \sum_{n=0}^{\mathcal{N}} (i\hbar)^n (\{\tilde{\Omega}_0, \tilde{\Omega}_n\} - \Lambda_n(\tilde{\Omega}_{m<n})) - \\ - (i\hbar)^{(\mathcal{N}+1)} \tilde{\Lambda}_{\mathcal{N}+1}(\tilde{\Omega}_{m<\mathcal{N}+1}) + O((i\hbar)^{(\mathcal{N}+2)}). \tag{4.49}$$

Since the induction assumption implies that the function $\tilde{\Omega}_n$, $n \leqslant \mathcal{N}$ satisfy equations (4.38), every term of the sum $\Sigma_{n=0}^{\mathcal{N}}$ the r.-h. side of (4.49) disappears for each $n$.

Let us employ now the identities valid for symbols

$$[\tilde{A}, [\tilde{A}, \tilde{A}]_*]_* \equiv 0. \tag{4.50}$$

By setting $\tilde{A} = \tilde{\Omega}^{(\mathcal{N})}$ one obtains from (4.49), (4.50) in the $(\mathcal{N} + 1)$-th order in $(i\hbar)$ that

$$\{\tilde{\Omega}_0, \tilde{\Lambda}_{\mathcal{N}+1}(\tilde{\Omega}_{m<\mathcal{N}+1})\} = 0, \tag{4.51}$$

as a consequence of equations (4.38) for $n \leqslant \mathcal{N}$.

Further on, in accord with the induction assumption, the functions $\tilde{\Omega}_n$ with $n \leqslant \mathcal{N}$ are regular and satisfy the conditions

$$\varepsilon(\tilde{\Omega}_{n\leqslant\mathcal{N}}) = 1, \qquad \text{gh}\,(\tilde{\Omega}_{n\leqslant\mathcal{N}}) = 1. \tag{4.52}$$

Then, it follows from the explicit formulas (4.41) that the functions $\tilde{\Lambda}_n$ with $n \leqslant \mathcal{N} + 1$ are also regular and possess the properties

$$\varepsilon(\tilde{\Lambda}_{n\leqslant\mathcal{N}+1}) = 0, \qquad \text{gh}\,(\tilde{\Lambda}_{n\leqslant\mathcal{N}+1}) = 2. \tag{4.53}$$

Then, in virtue of (4.53) and of the implication (4.46) $\Rightarrow$ (4.47) the general regular solution of equation (4.51) with respect to $\tilde{\Lambda}_{\mathcal{N}+1}$ has the form

$$\tilde{\Lambda}_{\mathcal{N}+1}(\tilde{\Omega}_{m<\mathcal{N}+1}) = \{\tilde{\Omega}_0, \tilde{\Omega}_{\mathcal{N}+1}\}, \tag{4.54}$$

$$\varepsilon(\tilde{\Omega}_{\mathcal{N}+1}) = 1, \qquad \text{gh}\,(\tilde{\Omega}_{\mathcal{N}+1}) = 1, \tag{4.55}$$

where $\tilde{\Omega}_{\mathcal{N}+1}$ is a regular function.

It remains to prove that the function $\tilde{\Omega}_{\mathcal{N}+1}$ can be subjected to the condition (4.44) for $n = \mathcal{N} + 1$. In accord with the induction assumption the function $\tilde{\Omega}_n$, with $n \leqslant \mathcal{N}$ satisfies eq. (4.44). Then it follows from (4.31), (4.41) that

$$\tilde{\Lambda}_n^* = \tilde{\Lambda}_n(-1)^{n+1}, \qquad n \leqslant \mathcal{N} + 1. \tag{4.56}$$

From (4.54) and (4.56) for $n = \mathcal{N} + 1$ we conclude that

$$\left\{ \tilde{\Omega}_0, \frac{1}{2}(\tilde{\Omega}_{\mathcal{N}+1} - \tilde{\Omega}^*_{\mathcal{N}+1}(-1)^{\mathcal{N}+1}) \right\} = 0. \tag{4.57}$$

The homogeneity of equation (4.57) shows that the new function

$$\tilde{\Omega}'_{\mathcal{N}+1} = \tilde{\Omega}_{\mathcal{N}+1} - \frac{1}{2}(\tilde{\Omega}_{\mathcal{N}+1} - \tilde{\Omega}^*_{\mathcal{N}+1}(-1)^{\mathcal{N}+1}) \tag{4.58}$$

obeying the condition

$$\tilde{\Omega}'_{\mathcal{N}+1} = (\tilde{\Omega}'_{\mathcal{N}+1})^*(-1)^{\mathcal{N}+1}, \tag{4.59}$$

obeys also equations (4.54), (4.55) with all the functions $\tilde{\Omega}_n$, $n \leqslant \mathcal{N}$ being same, including the same $\tilde{\Omega}_0$:

$$\tilde{\Lambda}_{\mathcal{N}+1}(\tilde{\Omega}_{m<\mathcal{N}+1}) = \{ \tilde{\Omega}_0, \tilde{\Omega}'_{\mathcal{N}+1} \}, \tag{4.60}$$

$$\varepsilon(\tilde{\Omega}'_{\mathcal{N}+1}) = 1, \qquad \text{gh}\,(\tilde{\Omega}'_{\mathcal{N}+1}) = 1. \tag{4.61}$$

The key point here is the circumstance that the transformation (4.58) only alters the subsequent functions ($\tilde{\Omega}_n$ with $n > \mathcal{N}$), and not the preceding ones ($\tilde{\Omega}_n$ with $n \leqslant \mathcal{N}$: this fact guarantees that for any classical function $\tilde{\Omega}_0$ chosen in accord with the theorem premise we are always able to step-by-step guarantee the existence of all functions $\tilde{\Omega}_n$, $n \leqslant \mathcal{N}$, $\tilde{\Omega}'_{\mathcal{N}+1}, \ldots$, for which the statement of the theorem would hold true. The quintessence of these considerations is that we can without loss of generality, in the sense just mentioned, to merely subject the function $\tilde{\Omega}_{\mathcal{N}+1}$ to condition (4.44) with $n = \mathcal{N} + 1$

$$\tilde{\Omega}_{\mathcal{N}+1} = \tilde{\Omega}^*_{\mathcal{N}+1}(-1)^{\mathcal{N}+1}. \tag{4.62}$$

Thus, as far as equations (4.38) are concerned the proof of the theorem is completed.

Consider now equations (4.40). Here our treatment is quite close to the above. We, hence, describe it in less details, at least in some points.

Let all the functions $\tilde{\Omega}_n$, $n > 0$ have been found. Let us make up the partial sum

$$\tilde{\mathcal{H}}^{(\mathcal{N})} = \sum_{n=0}^{\mathcal{N}} (i\hbar)^n \tilde{\mathcal{H}}_n \tag{4.63}$$

of the functions $\mathcal{H}_n$, $n \leqslant \mathcal{N}$ whose existence is provided by the induction assumption concerning the equation (4.40). Then we have

$$(i\hbar)^{-1}[\tilde{\Omega}^{(\mathcal{N}+1)}, \tilde{\mathcal{H}}^{(\mathcal{N})}]_* =$$
$$= \sum_{n=0}^{\mathcal{N}} (i\hbar)^n(\{ \tilde{\Omega}_0, \tilde{\mathcal{H}}_n \} - \tilde{K}_n(\tilde{\mathcal{H}}_{m<n}, \tilde{\Omega}_{l\leqslant n})) -$$
$$- (i\hbar)^{\mathcal{N}+1}\tilde{K}_{\mathcal{N}+1}(\tilde{\mathcal{H}}_{m<\mathcal{N}+1}, \tilde{\Omega}_{l\leqslant\mathcal{N}+1}) + O((i\hbar)^{\mathcal{N}+2}). \tag{4.64}$$

Since, in accord with the induction assumption, the functions $\tilde{\mathcal{H}}_n$, $n \leqslant \mathcal{N}$ satisfy equations (4.40), each term in the summand in (4.40) vanishes.

Using, further, (4.46), with $\mathcal{N}$ replaced by $\mathcal{N} + 1$ and taking into account the fact that all the functions $\tilde{\Omega}_n$ satisfy equations (4.37), (4.38), we have

$$(i\hbar)^{-1}[\tilde{\Omega}^{(\mathcal{N}+1)}, (i\hbar)^{-1}[\tilde{\Omega}^{(\mathcal{N}+1)}, \tilde{\mathcal{H}}^{(\mathcal{N})}]_*]_* \equiv$$
$$\equiv \frac{1}{2}(i\hbar)^{-1}[(i\hbar)^{-1}[\tilde{\Omega}^{(\mathcal{N}+1)}, \tilde{\Omega}^{(\mathcal{N}+1)}]_*, \tilde{\mathcal{H}}^{(\mathcal{N})}]_* =$$
$$= \frac{1}{2}(i\hbar)^{-1}[-(i\hbar)^{\mathcal{N}+2}\tilde{\Lambda}_{\mathcal{N}+2} + 0((i\hbar)^{\mathcal{N}+3}), \tilde{\mathcal{H}}^{(\mathcal{N})}]_* = 0((i\hbar)^{\mathcal{N}+2}). \quad (4.65)$$

From (4.64), (4.65) we obtain in the $(\mathcal{N} + 1)$-th order in $(i\hbar)$

$$\{\tilde{\Omega}_0, \tilde{K}_{\mathcal{N}+1}(\tilde{\mathcal{H}}_{m<\mathcal{N}+1}, \tilde{\Omega}_{l\leqslant\mathcal{N}+1})\} = 0,$$

as a consequence of equations (4.40) for $n \leqslant \mathcal{N}$.

By the induction assumption, the functions $\tilde{\mathcal{H}}_n$, with $n \leqslant \mathcal{N}$ are regular and obey the conditions

$$\varepsilon(\tilde{\mathcal{H}}_{n\leqslant\mathcal{N}}) = 0, \qquad \mathrm{gh}\,(\tilde{\mathcal{H}}_{n\leqslant\mathcal{N}}) = 0. \quad (4.67)$$

On the other hand, all functions $\tilde{\Omega}_n$ with $n > 0$ have been already proved to be regular and satisfy conditions (4.45). Then it follows from the explicit formulas (4.42) that the functions $\tilde{K}_n$, $n \leqslant \mathcal{N} + 1$, are also regular and possess the properties

$$\varepsilon(\tilde{K}_{n\leqslant\mathcal{N}+1}) = 1, \qquad \mathrm{gh}\,(K_{n\leqslant\mathcal{N}+1}) = 1. \quad (4.68)$$

Then, in virtue of (4.68) and of the implication (4.46) $\Rightarrow$ (4.47) the general regular solution of equation (4.66) for $\tilde{K}_{\mathcal{N}+1}$ has the form

$$\tilde{K}_{\mathcal{N}+1}(\tilde{\mathcal{H}}_{m<\mathcal{N}+1}, \tilde{\Omega}_{l\leqslant\mathcal{N}+1}) = \{\tilde{\Omega}_0, \tilde{\mathcal{H}}_{\mathcal{N}+1}\}, \quad (4.69)$$
$$\varepsilon(\tilde{\mathcal{H}}_{\mathcal{N}+1}) = 0, \qquad \mathrm{gh}\,(\tilde{\mathcal{H}}_{\mathcal{N}+1}) = 0, \quad (4.70)$$

where $\tilde{\mathcal{H}}_{\mathcal{N}+1}$ is a regular function.

The induction assumption implies further that the functions $\tilde{\mathcal{H}}_n$ with $n \leqslant \mathcal{N}$ satisfy conditions (4.44), whereas all the functions $\tilde{\Omega}_n$ satisfy these conditions in accord with what has been proved. Then it follows from the explicit formulas (4.42), owing to the property (4.31), that the functions $\tilde{K}_n$ with $n \leqslant \mathcal{N} + 1$ possess the property

$$\tilde{K}_n^* = \tilde{K}_n(-1)^n, \qquad n \leqslant \mathcal{N} + 1. \quad (4.71)$$

From (4.69), (4.71) with $n = \mathcal{N} + 1$ we conclude that

$$\left\{\tilde{\Omega}_0, \frac{1}{2}(\tilde{\mathcal{H}}_{\mathcal{N}+1} - \tilde{\mathcal{H}}^*_{\mathcal{N}+1}(-1)^{\mathcal{N}+1})\right\} = 0. \quad (4.72)$$

Following the same arguments that have lead us to (4.62), it may be concluded from (4.72) that the function $\tilde{\mathscr{H}}_{\mathcal{N}+1}$ can be subjected to the condition

$$\tilde{\mathscr{H}}_{\mathcal{N}+1} = \tilde{\mathscr{H}}^{*}_{\mathcal{N}+1}(-1)^{\mathcal{N}+1}. \tag{4.73}$$

Thus, in what concerns equations (4.40), the theorem has been completely proved, as well.

## D. Some further consequences.

Note, first of all, that all conditions of the theorem proved above are certainly fulfilled for any gauge system with a finite number of degrees of freedom and with constraints of finite stage of reducibility in every neighbourhood of initial data, where admissible gauges exist. This assertion includes a number of statements of pure classical nature, whose proof is to be performed using the method of ref. [*15*], adjusted to the Hamiltonian formalism. We shall not prove here all these statements, but confine ourselves to a brief explanation concerning the implication (4.46) $\Rightarrow$ (4.47) that took a crucial part in the proof of the basic theorem.

The key point here is the property of Abelian factorizability, which is a possible version of the known general property of the local Abelizability of a gauge algebra.

Let the index « $\mu$ » label all independent constraints in the theory, contained among the full set of constraints, both initial, and those arising at every stage of reduction for the ghosts of the previous stage (in the irreducible case all is exhausted by the initial constraints, that are then independent). Let $\varepsilon_\mu$ and $(-g_\mu)$ be the distributions of statistics and of ghost numbers of independent constraints, respectively (in the irreducible case the ghost number of every irreducible constraints is zero). The Abelian factorization is expressed by the following representation for

$$\tilde{\Omega}_0 = \tilde{\mathscr{T}}_\mu \tilde{\Theta}^\mu, \tag{4.74}$$

$$\{\tilde{\mathscr{T}}_\mu, \tilde{\mathscr{T}}_\nu\} = 0, \ \{\tilde{\mathscr{T}}_\mu, \tilde{\Theta}^\nu\} = 0, \ \{\tilde{\Theta}^\mu, \tilde{\Theta}^\nu\} = 0, \tag{4.75}$$

$$\varepsilon(\tilde{\mathscr{T}}_\mu) = \varepsilon_\mu, \ \mathrm{gh}\,(\tilde{\mathscr{T}}_\mu) = -g_\mu, \ \varepsilon(\tilde{\Theta}^\mu) = \varepsilon_\mu + 1, \ \mathrm{gh}\,(\tilde{\Theta}^\mu) = g_\mu + 1, \tag{4.76}$$

where the functions $\tilde{\mathscr{T}}_\mu$, $\tilde{\Theta}^\mu$ are all independent and canonically conjugated to independent functions $\tilde{\mathscr{K}}^\mu$, $\tilde{\Xi}_\mu$ :

$$\{\tilde{\mathscr{K}}^\mu, \tilde{\mathscr{K}}^\nu\} = 0, \quad \{\tilde{\mathscr{K}}^\mu, \tilde{\mathscr{T}}_\nu\} = \delta^\mu_\nu, \quad \{\tilde{\mathscr{K}}^\mu, \tilde{\Theta}^\nu\} = 0, \tag{4.77}$$

$$\{\tilde{\Xi}_\mu, \tilde{\Xi}_\nu\} = 0, \quad \{\tilde{\Theta}^\mu, \tilde{\Xi}_\nu\} = \delta^\mu_\nu, \tag{4.78}$$

$$\{\tilde{\mathscr{K}}^\mu, \tilde{\Xi}\} = 0, \quad \{\tilde{\mathscr{T}}_\mu, \tilde{\Xi}_\nu\} = 0, \tag{4.79}$$

$$\varepsilon(\tilde{\mathscr{K}}^\mu) = \varepsilon_\mu, \ \mathrm{gh}\,(\tilde{\mathscr{K}}^\mu) = g_\mu, \ \varepsilon(\tilde{\Xi}_\mu) = \varepsilon_\mu + 1, \ \mathrm{gh}\,(\tilde{\Xi}_\mu) = -(g_\mu + 1). \tag{4.80}$$

So that a local vicinity inside the phase space $\{ \tilde{\Gamma} \}$ allows the regular canonical reparametrization

$$\{ \tilde{\Gamma} \} \to \{ \tilde{\Gamma}_{\text{phys}} \} \cup \{ \tilde{\Gamma}_{\text{aux}} \}, \tag{4.81}$$

$$\text{gh}\,(\tilde{\Gamma}_{\text{phys}}) = 0, \qquad \{ \tilde{\Gamma}_{\text{aux}} \} = \{ (\tilde{\mathscr{K}}, \tilde{\mathscr{T}}) \} \cup \{ (\tilde{\Theta}, \tilde{\Xi}) \} \tag{4.82}$$

with $\{ \tilde{\Gamma}_{\text{phys}} \}$ taken as the phase subspace of physical degrees of freedom.

Representation (4.74)-(4.76) makes the fulfilment of the classical equation (4.37) and of the first and second conditions out of (4.45) evident for $n = 0$. The validity of the first condition in (4.44) with $n = 0$ is guaranteed by the fact that one can subject the functions $(\tilde{\mathscr{T}}, \tilde{\mathscr{K}})$, $(\tilde{\Xi}, \tilde{\Theta})$ to the conditions

$$\tilde{\mathscr{T}}_{\mu}^{*} = \tilde{\mathscr{T}}_{\mu}, \qquad (\tilde{\Theta}^{\mu})^{*} = \tilde{\Theta}^{\mu}, \tag{4.83}$$

$$(\tilde{\mathscr{K}}^{\mu})^{*} = \tilde{\mathscr{K}}^{\mu}(-1)^{\varepsilon_{\mu}}, \; \tilde{\Xi}_{\mu}^{*} = \tilde{\Xi}_{\mu}(-1)^{\varepsilon_{\mu}+1}. \tag{4.84}$$

The nontrivial aspect of the Abelian factorization is that the representation (4.74) satisfies also boundary conditions that correspond to any gauge system belonging to the class considered.

Let, now, the following equation be given

$$\{ \tilde{\Omega}_{0}, \tilde{\text{X}} \} = 0. \tag{4.85}$$

The reparametrization (4.81), when taken together with eq. (4.74) puts equation (4.85) to the following standard form

$$\tilde{\text{J}} \cdot \frac{\partial}{\partial \tilde{\text{Z}}} \tilde{\mathscr{X}} = 0, \qquad \varepsilon(\tilde{\text{J}}) = \varepsilon(\tilde{\text{Z}}) + 1, \tag{4.86}$$

where we have denoted

$$\tilde{\text{J}} = (\tilde{\mathscr{T}}_{\mu}, \tilde{\Theta}^{\mu}(-1)^{\varepsilon_{\mu}+1}), \qquad \tilde{\text{Z}} = (\tilde{\Xi}_{\mu}, \tilde{\mathscr{K}}^{\mu}), \tag{4.87}$$

$$\tilde{\text{X}}(\tilde{\Gamma}) = \tilde{\mathscr{X}}(\tilde{\Gamma}_{\text{phys}}, \tilde{\Gamma}_{\text{aux}}). \tag{4.88}$$

In what concerns the standard equation (4.86) with the independent variables $\tilde{\text{J}}$ and $\tilde{\text{Z}}$ possessing opposite statistics, a Lemma was proved in ref. [*15*], that states that any regular solution allows the representation

$$\tilde{\mathscr{X}}(\tilde{\text{J}}, \tilde{\text{Z}}) = \tilde{\mathscr{X}}(0,0) + \tilde{\text{J}} \cdot \frac{\partial}{\partial \tilde{\text{Z}}} \tilde{\mathscr{Y}}, \tag{4.89}$$

$$\tilde{\mathscr{Y}} = \tilde{\mathscr{Y}}_{\tilde{\mathscr{X}}} + \tilde{\mathscr{X}}^{(1)}, \tag{4.90}$$

$$\tilde{\mathscr{Y}}_{\tilde{\mathscr{X}}} \equiv \tilde{\text{Z}} \cdot \frac{\partial}{\partial \tilde{\text{J}}} \int_{0}^{1} (\tilde{\mathscr{X}}(\alpha \tilde{\text{J}}, \alpha \tilde{\text{Z}}) - \tilde{\mathscr{X}}(0,0)) \frac{d\alpha}{\alpha}, \tag{4.90 a}$$

$$\tilde{\mathscr{X}}^{(s)}(\tilde{\text{J}}, \tilde{\text{Z}}) = \tilde{\mathscr{X}}^{(s)}(0,0) + \tilde{\text{J}} \cdot \frac{\partial}{\partial \tilde{\text{Z}}} \tilde{\mathscr{Y}}^{(s)}, \tag{4.90 b}$$

$$\tilde{\mathscr{Y}}^{(s)} = \tilde{\mathscr{Y}}_{\tilde{\mathscr{X}}^{(s)}} + \tilde{\mathscr{X}}^{(s+1)}, \tag{4.90 c}$$

where $s = 1, \ldots$.

To avoid a possible misunderstanding, we should emphasize that we have pointed the overall dependence of solutions of equation (4.86) on the variables $\tilde{J}$, $\tilde{Z}$ in (4.90). Referring to the reparametrized (in the sense of (4.81)) form (4.88) of writing we note that the components of (4.87) enter in $\tilde{\mathscr{X}}$ through $\tilde{\Gamma}_{aux}$ defined by (4.82). As for the dependence on $\tilde{\Gamma}_{phys}$ in (4.89), (4.90), the latter is meant, but not shown explicitly for the sake of brevity.

Coming back to the $\tilde{\Gamma}$-parametrization in (4.89), we obtain the following representation for the solution of equation (4.85)

$$\tilde{X} = \tilde{X}_{phys} + \{ \tilde{\Omega}_0, \tilde{Y} \}, \tag{4.91}$$

where we have denoted

$$\tilde{X}_{phys} \equiv \tilde{\mathscr{X}}(\tilde{\Gamma}_{phys}, \tilde{\Gamma}_{aux} = 0) = \tilde{X}(\tilde{\Gamma}) |_{\substack{\tilde{\mathscr{J}}=0,\, \tilde{\mathscr{X}}=0 \\ \tilde{\Xi}=0,\, \tilde{\Theta}=0}} . \tag{4.92}$$

$$\tilde{Y}(\tilde{\Gamma}) = \tilde{\mathscr{Y}}(\tilde{\Gamma}_{phys}, \tilde{\Gamma}_{aux}) . \tag{4.93}$$

The first term in the r.-h. side of the expansion (4.91) given by eq. (4.92) is called a singlet component of the classical observable $\tilde{X}$ subject to equation (4.85), while the second one is referred to as a doublet component.

From (4.92) we conclude that

$$\text{gh}\,(\tilde{X}) \neq 0 \;\Rightarrow\; \tilde{X}_{phys} = 0 . \tag{4.94}$$

So that the implication (4.46) $\Rightarrow$ (4.47) follows immediately from (4.91), (4.94). If, however, $\text{gh}\,(\tilde{X}) = 0$ the representation (4.91)-(4.93) provides an essential additional information.

Once the implication (4.46) $\Rightarrow$ (4.47) is established we can derive the principal consequence of the theorem proved above. This is: to any finite-dimensional gauge system given classically by a solution $\tilde{\Omega}_0$, $\tilde{\mathscr{H}}_0$ of equations (4.37), (4.39) supplemented by corresponding initial data, one really can put into correspondence formal expressions for the operators $\Omega$, $\mathscr{H}$, satisfying the generating equations (1.1), (1.2).

Clearly enough, this correspondence is far from being one-to-one. Let, indeed, $\tilde{\Omega}'_n$, $\tilde{\mathscr{H}}'_n$ and $\tilde{\Omega}''_n$, $\tilde{\mathscr{H}}''_n$ simultaneously satisfy equations (4.38), (4.40) for some $n > 0$. Then

$$\{ \tilde{\Omega}_0, (\tilde{\Omega}'_n - \tilde{\Omega}''_n) \} = 0, \qquad \{ \tilde{\Omega}_0, (\tilde{\mathscr{H}}'_n - \tilde{\mathscr{H}}''_n) \} = 0 . \tag{4.95}$$

Hence we derive, referring to (4.91), (4.94), the second consequence

$$\tilde{\Omega}'_n - \tilde{\Omega}''_n = \{ \tilde{\Omega}_0, \tilde{\mathfrak{M}} \}, \quad \tilde{\mathscr{H}}'_n - \tilde{\mathscr{H}}''_n = (\tilde{\mathscr{H}}'_n - \tilde{\mathscr{H}}''_n)_{phys} + \{ \tilde{\Omega}_0, \tilde{\mathfrak{H}}_n \} , \tag{4.96}$$

$$\begin{aligned} \tilde{\mathfrak{M}}^*_n &= \tilde{\mathfrak{M}}_n(-1)^n, \quad \varepsilon(\tilde{\mathfrak{M}}_n) = 0, \quad \text{gh}\,(\tilde{\mathfrak{M}}_n) = 0 , \\ \tilde{\mathfrak{H}}^*_n &= \tilde{\mathfrak{H}}_n(-1)^{n+1}, \quad \varepsilon(\tilde{\mathfrak{H}}_n) = 1, \quad \text{gh}\,(\tilde{\mathfrak{H}}_n) = -1 , \end{aligned} \tag{4.97}$$

where eqs. (4.97) provide the fulfilment of (4.44), (4.45) for the given $n$. The arbitrariness (4.96) any affects the higher functions $\tilde{\Omega}_m$, $\tilde{\mathscr{H}}_m$ with $m > n$.

Consider, finally, quasiclassical expansion of the Weyl symbol at the total unitarizing Hamiltonian (2.1)

$$\tilde{\mathrm{H}} = \tilde{\mathscr{H}} + (i\hbar)^{-1}[\tilde{\Psi}, \tilde{\Omega}]_* , \tag{4.98}$$

where $\tilde{\mathscr{H}}$ and $\tilde{\Omega}$ are represented by the series (4.36), the $*$-commutator is defined in (4.34), (4.35), and $\tilde{\Psi}$ is the Weyl symbol of the gauge fermion operator

$$\tilde{\Psi}^* = -\tilde{\Psi}, \quad \varepsilon(\tilde{\Psi}) = 1, \quad \mathrm{gh}\,(\tilde{\Psi}) = -1 . \tag{4.99}$$

By writing the expansions

$$\tilde{\Psi} = \sum_{n=0}^{\infty} (i\hbar)^n \tilde{\Psi}_n, \qquad \tilde{\mathrm{H}} = \sum_{n=0}^{\infty} (i\hbar)^n \tilde{\mathrm{H}}_n , \tag{4.100}$$

where

$$\tilde{\Psi}_n^* = \tilde{\Psi}_n(-1)^{n+1}, \quad \varepsilon(\tilde{\Psi}_n) = 1, \quad \mathrm{gh}\,(\tilde{\Psi}_n) = -1 , \tag{4.101}$$

and employing expansion (4.2) for the $*$-commutator, one obtains for $\tilde{\mathrm{H}}_n$:

$$\tilde{\mathrm{H}}_n = \tilde{\mathscr{H}}_n + \sum_{m=1}^{n+1} \sum_{l=1}^{m} \frac{1}{l!} \{ \tilde{\Psi}_{m-l}, \tilde{\Omega}_{n+1-m} \}_l . \tag{4.102}$$

From this we have, for instance

$$\tilde{\mathrm{H}}_0 = \tilde{\mathscr{H}}_0 + \{ \tilde{\Psi}_0, \tilde{\Omega}_0 \} , \tag{4.103}$$

$$\tilde{\mathrm{H}}_1 = \tilde{\mathscr{H}}_1 + \{ \tilde{\Psi}_0, \tilde{\Omega}_1 \} + \{ \tilde{\Psi}_1, \tilde{\Omega}_0 \} , \tag{4.104}$$

$$\tilde{\mathrm{H}}_2 = \tilde{\mathscr{H}}_2 + \{ \tilde{\Psi}_0, \tilde{\Omega}_2 \} + \{ \tilde{\Psi}_1, \tilde{\Omega}_1 \} + \{ \tilde{\Psi}_2, \tilde{\Omega}_0 \} + \frac{1}{6} \{ \tilde{\Psi}_0, \tilde{\Omega}_0 \}_3 . \tag{4.105}$$

Let us also display the equations for the functions $\tilde{\Omega}_n$, $\tilde{\mathscr{H}}_n$, $n = 1, 2$ in a more explicit form

$$\{ \tilde{\Omega}_0, \tilde{\Omega}_1 \} = 0 , \tag{4.106}$$

$$\{ \tilde{\Omega}_0, \tilde{\Omega}_2 \} = -\frac{1}{12} \{ \tilde{\Omega}_0, \tilde{\Omega}_0 \}_3 - \frac{1}{2} \{ \tilde{\Omega}_1, \tilde{\Omega}_1 \} \tag{4.107}$$

$$\{ \tilde{\Omega}_0, \tilde{\mathscr{H}}_1 \} = \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_1 \} , \tag{4.108}$$

$$\{ \tilde{\Omega}_0, \tilde{\mathscr{H}}_2 \} = \frac{1}{6} \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_0 \}_3 + \{ \tilde{\mathscr{H}}_0, \tilde{\Omega}_2 \} + \{ \tilde{\mathscr{H}}_1, \tilde{\Omega}_1 \} . \tag{4.109}$$

Solution for $\tilde{\Omega}_1$ has the form

$$\tilde{\Omega}_1 = \{ \tilde{\Omega}_0, \tilde{\mathfrak{M}}_1 \}. \tag{4.110}$$

After substituting this into (4.108) and taking (4.39) into account we find

$$\{ \tilde{\Omega}_0, (\tilde{\mathscr{H}}_1 - \{ \tilde{\mathscr{H}}_0, \tilde{\mathfrak{M}}_1 \}) \} = 0, \tag{4.111}$$

from where the solution for $\tilde{\mathscr{H}}_1$ follows to be

$$\tilde{\mathscr{H}}_1 = \{ \tilde{\mathscr{H}}_0, \tilde{\mathfrak{M}}_1 \} + \tilde{\mathscr{F}}_1(\tilde{\Gamma}_{\text{phys}}) + \{ \tilde{\Omega}_0, \tilde{\mathfrak{H}}_1 \}. \tag{4.112}$$

With the help of the symbols (4.98) of the total Hamiltonian one can represent the Weyl symbol of the evolution operator (2.14) as a path integral in the phase space (4.3)

$$\tilde{\mathrm{E}}(\tilde{\Gamma}_0, t_f, t_i) = \int \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} \left( \frac{1}{2} \tilde{\Gamma} \cdot \xi \cdot \dot{\tilde{\Gamma}} - \tilde{\mathrm{H}}(\tilde{\Gamma}) \right) dt + \right.$$

$$\left. + \frac{i}{2\hbar} \tilde{\Gamma}(t_f) \cdot \xi \cdot \tilde{\Gamma}(t_i) \right\} \mathscr{D}\tilde{\Gamma}, \ \mathscr{D}\tilde{\Gamma} \equiv \prod_{t_i \leqslant t \leqslant t_f} \frac{d\tilde{\Gamma}(t)}{2\pi\hbar}. \tag{4.113}$$

Here the external phase argument is denoted as $\tilde{\Gamma}_0$, while the virtual phase trajectory is called $\tilde{\Gamma}(t)$. The integration includes all the trajectories subject to the condition

$$\tilde{\Gamma}(t_i) + \tilde{\Gamma}(t_f) = 2\tilde{\Gamma}_0. \tag{4.114}$$

The matrix $\xi$ in the « action » in (4.113) is inverse to the Poisson bracket matrix of the variables (4.3).

Condition (4.114) can be explicitly satisfied provided that one sets

$$\tilde{\Gamma}(t) = \tilde{\Gamma}_0 + \frac{1}{2} \int_{t_i}^{t} \tilde{\mathrm{V}}(t')dt' - \frac{1}{2} \int_{t}^{t_f} \tilde{\mathrm{V}}(t')dt' \tag{4.115}$$

and takes specifically the phase space velocities

$$\tilde{\mathrm{V}}(t) = \dot{\tilde{\Gamma}}(t) \tag{4.116}$$

for the functional integration variables. Then

$$\mathscr{D}\tilde{\Gamma} = \prod_{t_i \leqslant t \leqslant t_f} \frac{d\tilde{\mathrm{V}}(t)\Delta t}{2\pi\hbar}, \ \theta(0) \equiv \frac{1}{2}. \tag{4.117}$$

The first equality here defines the volume element in the phase space velocities while the second one fixes the step function $\theta(t - t')$, discontinuous for equal times, in a fashion coordinated with the symmetrical character of the Weyl-normal form. This additional fixation is necessary for making the formal calculation of the functional integral (4.113) based on the stationary phase method single-valued.

Using the expansion (4.102), one can write eq. (4.113) in the following instructive form

$$\tilde{\mathrm{E}}(\tilde{\Gamma}_0, t_f, t_i) = \int \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f}\left(\frac{1}{2}\tilde{\Gamma}\cdot\xi\cdot\dot{\tilde{\Gamma}} - \tilde{\mathrm{H}}_0(\tilde{\Gamma})\right)dt + \right.$$
$$\left. + \frac{i}{2\hbar}\tilde{\Gamma}(t_f)\cdot\xi\cdot\tilde{\Gamma}(t_i)\right\}d\mu(\tilde{\Gamma}), \quad (4.118)$$

$$d\mu(\tilde{\Gamma}) \equiv \exp\left\{\sum_{n=0}^{\infty}(i\hbar)^n\int_{t_i}^{t_f}\tilde{\mathrm{H}}_{n+1}(\tilde{\Gamma})dt\right\}\mathscr{D}\tilde{\Gamma}. \quad (4.119)$$

The exponent in (4.118) contains the explicitly displayed classical action that only includes the classical Hamiltonian (4.103). It is this action that determines the extremum serving the calculations within the stationary phase method. The admissibility conditions when imposed on the gauge fermion $\tilde{\Psi}_0$ make the classical action nondegenerate. In (4.118) the quantum terms of the expansion (4.102) are gathered into the integration measure (4.119). It is of interest to study in detail the lowest term in the exponential in (4.119) that includes $\tilde{\mathrm{H}}_1$ (4.104). This term is specialized by the fact that its contribution into (4.119) is of the zeroth order in $(i\hbar)$ and hence, in the course of calculations following the stationary phase method its extremum solution remains in the exponent. With the use of solutions (4.110) (4.112) eq. (4.104) can be written as

$$\tilde{\mathrm{H}}_1 = \{\tilde{\mathrm{H}}_0, \tilde{\mathfrak{M}}_1\} + \tilde{\mathscr{F}}_1(\tilde{\Gamma}_{\text{phys}}) + \{\tilde{\Omega}_0, (\tilde{\Psi}_1 - \{\tilde{\Psi}_0, \tilde{\mathfrak{M}}_1\} + \tilde{\mathfrak{H}}_1)\}. \quad (4.120)$$

The meaning of the individual terms in the r.-h. side of (4.120) is quite transparent. The first term is (up to $0((i\hbar)^2)$) the result of a canonical transformation applied to $\tilde{\mathrm{H}}_0$. The second term is the physical contribution itself, while the third one is of pure gauge origin and can be removed by the following choice of $\tilde{\Psi}_1$:

$$\tilde{\Psi}_1 = \{\tilde{\Psi}_0, \tilde{\mathfrak{M}}_1\} - \tilde{\mathfrak{H}}_1. \quad (4.121)$$

At this point we finish our study of the solvability problem for the generating equations of the gauge operator algebra, which is the quantization problem for gauge systems. To conclude this section the following two remarks are in order.

First, to use specifically Weyl symbols was in our analysis technically convenient, but by no means necessary. It was already mentioned, though, that with the number of degrees of freedom finite, all the types of normal orderings are quite equivalent and any priority in their choice should be only a matter of technical convenience. In the limit of infinite number of degree of freedom, as it was also pointed above the Wick symbol is most stable, which fact distinguishes it in reality. Certainly, wemight well exploit the Wick symbols in our analysis, instead of the Weyl ones. Unfortunately, this does not imply any possibility of the direct extension of our results to the situation with infinite number of degrees of freedom. The unitary nonequivalent representations of the commutation relations present in the latter case may give rise to new restrictions that must be fulfilled for the basic implication (4.46) $\Rightarrow$ (4.47) to remain valid. An example of these restrictions is provided by the necessity of critical dimension in string theories.

Second, the requirement that symbols of the generating operators should allow to be expanded in powers of $\hbar$ is a source of an essential limitation in our analysis. The correspondence principle cannot rule out nonanalytical solutions of the generating equations at the point $\hbar = 0$. The study of such situation would require different methods and lies beyond the scope of the present considerations.

## 5. RETURN TO OPERATORS. DYNAMICAL SUPERALGEBRA OF GAUGE SYSTEMS

Let us turn again to our basic operator equations. Let us gather together the generating equations (1.1), (2.2) and equations (3.83), (3.85) for the hermitian fermion $\Omega$ and bosons $\mathcal{G}$, $\mathcal{H}$:

$$[\Omega, \Omega] = 0, \tag{5.1}$$

$$[i\mathcal{G}, \Omega] = \Omega, \tag{5.2}$$

$$[\mathcal{H}, \Omega] = 0, \tag{5.3}$$

$$[i\mathcal{G}, \mathcal{H}] = 0. \tag{5.4}$$

These equations may be thought of as defining the universal dynamical superalgebra of gauge systems.

Let $\mathfrak{A}$ be a unitary boson operator with its ghost number zero

$$\mathfrak{A}^{\dagger} = \mathfrak{A}^{-1},\ [i\mathcal{G}, \mathfrak{A}] = 0, \tag{5.5}$$

and let $\mathfrak{R}$ be an antihermitian fermion operator with the ghost number equal to $(-1)$:

$$\mathfrak{R}^{\dagger} = -\mathfrak{R},\ [i\mathcal{G}, \mathfrak{R}] = -\mathfrak{R}. \tag{5.6}$$

Then a natural automorphism of the superalgebra (5.1)-(5.4) is given as

$$\Omega \to \mathfrak{A}^{-1}\Omega\mathfrak{A},\ \mathcal{G} \to \mathcal{G}, \tag{5.7}$$

$$\mathcal{H} \to \mathfrak{A}^{-1}(\mathcal{H} - i[\mathfrak{R}, \Omega])\mathfrak{A}. \tag{5.8}$$

Denoting the total unitarizing Hamiltonian (2.1), depending on the gauge fermion $\Psi$ as

$$\mathrm{H}_{\Psi} = \mathcal{H} - i[\Psi, \Omega], \tag{5.9}$$

we see that it transforms under (5.7), (5.8) as

$$H_\Psi \to \mathfrak{A}^{-1} H_{\mathfrak{A}\Psi\mathfrak{A}^{-1}+\mathfrak{R}} \mathfrak{A} . \tag{5.10}$$

In the classical domain ($\hbar = 0$) a classical analog of the superalgebra (5.1)-(5.4) occurs in terms of the Poisson superbrackets. As for this classical dynamical superalgebra it was pointed above that it can be locally Abelized by means of the classical analog of the automorphism (5.7), (5.8). This possibility is expressed by the classical factorization (4.74) and, as its consequence, by eq. (4.91), presenting the structure of a classical quantity that is physical in the sense of equation (4.85). Our goal now is to establish operator counterparts of these facts in quantum domain.

The basic Abelizability property of the operator superalgebra (5.1)-(5.4) is expected to be as follows: there exists a unitary operator $\mathfrak{A}$ involved in (5.5), such that the relation holds true

$$\mathfrak{A}^{-1}\Omega\mathfrak{A} = \Omega_{\text{Abelian}} , \tag{5.11}$$

$$\Omega_{\text{Abelian}} \equiv \mathfrak{T}_A G^A, \ [\mathfrak{T}_A, \mathfrak{T}_B] = 0, \tag{5.12}$$

where $\mathfrak{T}_A$ are hermitian operators of Abelian constraints, and $G^A$ are ghost coordinates operators from (3.1)-(3.3).

In case of irreducibility, the Abelian constraints $\mathfrak{T}_A$ do not depend on the ghost operators (3.1) and, hence, are functions of the original operator-valued dynamical variables alone, whose ghost numbers are zero by definition. In case of reducibility the Abelian constraints $\mathfrak{T}_A$ can contain, apart from dependence on the original operator-valued dynamical variables, only linear dependence on the ghost momenta operators $\overline{G}_A$ from (3.1), but not on the ghost coordinates operators. Thus, for every value of subscript « A » the following relations are true for the general Abelian constraint

$$[\mathfrak{T}_A, G^A] = 0, \qquad \mathfrak{T}_B [G^B, \mathfrak{T}_A] = 0 . \tag{5.13}$$

We are going now to perform expansion in powers of $\hbar$ in the equation for Abelization (5.11), like what we did in the preceding section when we dealt with the generating equations (1.1), (2.2). There is, however, a subtle point here. The matter is that the operator $\mathfrak{A}$ in (5.11) is, clearly, essentially nonanalytic in the point $\hbar = 0$. It cannot therefore be expanded in powers of $\hbar$, whereas the l.-h. side of equation (5.11), as a whole, can. To avoid this difficulty we shall use the exponential Ansatz for the unitary operator $\mathfrak{A}$ (the Planck constant $\hbar$ will be kept explicitly up to eq. (5.29))

$$\mathfrak{A} = \exp\left\{ -\frac{i}{\hbar}\mathfrak{S} \right\}, \tag{5.14}$$

where $\mathfrak{S}$ is a hermitian boson operator with zero ghost number. We claim that if the Abelization equation (5.11) is written for symbols, then perturbation theory in powers of $\hbar$ exists for the symbol of the generator $\mathfrak{S}$.

To make this evident, let us introduce a numerical parameter $\alpha$ into (5.14), so that the operator family

$$\mathfrak{A}(\alpha) = \exp\left\{ -\frac{i}{\hbar}\alpha\mathfrak{S} \right\}, \qquad 0 \leqslant \alpha \leqslant 1\,, \tag{5.15}$$

$$\mathfrak{A} = \mathfrak{A}(\alpha = 1)\,. \tag{5.16}$$

appear. Define the $\alpha$-dependent operator

$$\Omega(\alpha) \equiv \mathfrak{A}^{-1}(\alpha)\Omega\mathfrak{A}(\alpha)\,, \tag{5.17}$$

where $\Omega$ is the generating operator from the l.-h. side of eq. (5.11). After differentiating the definition (5.17) with respect to $\alpha$ and taking eq. (5.16) into account we find that the Abelization equation (5.11) can be equivalently presented as the set of equations

$$i\hbar\partial_\alpha\Omega(\alpha) = [\Omega(\alpha), \mathfrak{S}]\,, \tag{5.18}$$

$$\Omega(\alpha = 0) = \Omega, \quad \Omega(\alpha = 1) = \Omega_{\text{Abelian}}\,. \tag{5.19}$$

After changing here to the Weyl symbols according to (4.5), (4.9) we have

$$i\hbar\partial_\alpha\tilde{\Omega}(\alpha) = [\tilde{\Omega}(\alpha), \tilde{\mathfrak{S}}]_*\,, \tag{5.20}$$

$$\tilde{\Omega}(\alpha = 0) = \tilde{\Omega}, \quad \tilde{\Omega}(\alpha = 1) = \tilde{\Omega}_{\text{Abelian}}\,, \tag{5.21}$$

where the $*$-commutator is most explicitly defined by (4.34), (4.35).

By using the expansions

$$\tilde{\Omega}(\alpha) = \sum_{n=0}^{\infty}(i\hbar)^n\tilde{\Omega}_n(\alpha), \quad \tilde{\mathfrak{S}} = \sum_{n=0}^{\infty}(i\hbar)^n\tilde{\mathfrak{S}}_n\,, \tag{5.22}$$

$$\tilde{\Omega}_{\text{Abelian}} = \sum_{n=0}^{\infty}(i\hbar)^n(\tilde{\Omega}_n)_{\text{Abelian}}\,, \tag{5.23}$$

together with the first expansion in (4.36) and also eq. (4.21) for the expansion of the $*$-commutator we obtain from (5.20), (5.21) the following equations for the coefficient functions

$$(\partial_\alpha + \hat{\mathfrak{S}}_0)\tilde{\Omega}_0(\alpha) = 0\,, \tag{5.24 a}$$

$$(\partial_\alpha + \hat{\mathfrak{S}}_0)\tilde{\Omega}_n(\alpha) = \{\tilde{\Omega}_0(\alpha), \tilde{\mathfrak{S}}_n\} + \tilde{\pi}_n(\alpha)\,, \tag{5.24 b}$$

$$\begin{aligned}\tilde{\pi}_n(\alpha) \equiv \frac{1}{(n+1)!}\{\tilde{\Omega}_0(\alpha), \tilde{\mathfrak{S}}_0\}_{n+1} + \sum_{m=2}^{n}\frac{1}{m!}(\{\tilde{\Omega}_0(\alpha), \tilde{\mathfrak{S}}_{n+1-m}\}_m + \\ + \{\tilde{\Omega}_{n+1-m}(\alpha), \tilde{\mathfrak{S}}_0\}_m) + \sum_{m=2}^{n}\sum_{l=1}^{m-1}\frac{1}{l!}\{\tilde{\Omega}_{m-l}(\alpha), \tilde{\mathfrak{S}}_{n+1-m}\}_l\,,\end{aligned} \tag{5.25}$$

$$\tilde{\Omega}_n(\alpha = 0) = \tilde{\Omega}_n, \quad \tilde{\Omega}_n(\alpha = 1) = (\tilde{\Omega}_n)_{\text{Abelian}}\,. \tag{5.26}$$

Here $\hat{\mathfrak{S}}_0$ is an operator defined for any $\tilde{\mathrm{A}}$ by

$$\hat{\mathfrak{S}}_0\tilde{\mathrm{A}} \equiv \{ \tilde{\mathfrak{S}}_0, \tilde{\mathrm{A}} \}, \tag{5.27}$$

the functions $\tilde{\Omega}_n$ are coefficient functions for the first expansion (4.36); the r.-h. side of the second equality in (5.26) can be more explicitly written as

$$(\tilde{\Omega}_n)_{\text{Abelian}} = \tilde{\mathfrak{T}}_{n\mathrm{A}}\tilde{\mathrm{G}}^{\mathrm{A}}, \tag{5.28}$$

where $\tilde{\mathfrak{T}}_{n\mathrm{A}}$ are coefficient functions in the expansion in powers of $(i\hbar)$ of the Weyl symbols of Abelian constraints

$$\tilde{\mathfrak{T}}_{\mathrm{A}} = \sum_{n=0}^{\infty} (i\hbar)^n \tilde{\mathfrak{T}}_{n\mathrm{A}}. \tag{5.29}$$

In virtue of the second equations in (5.12) and (5.13), we have for the symbols (5.29)

$$[\tilde{\mathfrak{T}}_{\mathrm{A}}, \tilde{\mathfrak{T}}_{\mathrm{B}}]_* = 0, \qquad \tilde{\mathfrak{T}}_{\mathrm{B}} * \{ \tilde{\mathrm{G}}^{\mathrm{B}}, \tilde{\mathfrak{T}}_{\mathrm{A}} \} = 0, \tag{5.30}$$

while the first equation in (5.13) implies that the symbol $\tilde{\mathfrak{T}}_{\mathrm{A}}$ does not depend on $\tilde{\tilde{\mathrm{G}}}_{\mathrm{A}}$ for any value of the subscript. From (5.29), (5.30) we have

$$\sum_{m=1}^{n} \sum_{l=1}^{m} \frac{1}{l!} \{ \tilde{\mathfrak{T}}_{(m-l)\mathrm{A}}, \tilde{\mathfrak{T}}_{(n-m)\mathrm{B}} \}_l = 0, \tag{5.31}$$

$$\sum_{m=0}^{n} \sum_{l=0}^{m} \tilde{\mathfrak{T}}_{(m-l)\mathrm{B}} \frac{(\overleftrightarrow{\Delta})^l}{l!} \{ \tilde{\mathrm{G}}^{\mathrm{B}}, \tilde{\mathfrak{T}}_{(n-m)\mathrm{A}} \} = 0. \tag{5.32}$$

By solving (5.24)-(5.26) as linear differential equations for $\tilde{\Omega}_n(\alpha)$ we find the following equations for the coefficient functions $\tilde{\mathfrak{S}}_n$ of the second one of the expansions (5.22) serving the symbol $\tilde{\mathfrak{S}}$ of the generator $\mathfrak{S}$ from (5.14), (5.11):

$$\exp \{ \hat{\mathfrak{S}}_0 \} (\tilde{\Omega}_0)_{\text{Abelian}} = \tilde{\Omega}_0, \tag{5.33}$$

$$\left\{ \tilde{\Omega}_0, \int_0^1 \exp \{ \alpha\hat{\mathfrak{S}}_0 \} d\alpha \tilde{\mathfrak{S}}_n \right\} = \mathscr{W}_n, \tag{5.34}$$

$$\mathscr{W}_n = \exp \{ \hat{\mathfrak{S}}_0 \} (\tilde{\Omega}_n)_{\text{Abelian}} - \tilde{\Omega}_n - \int_0^1 \exp \{ \alpha\hat{\mathfrak{S}}_0 \} \tilde{\pi}_n(\alpha) d\alpha, \tag{5.35}$$

$$\tilde{\Omega}_0(\alpha) = \exp \{ -\alpha\hat{\mathfrak{S}}_0 \} \tilde{\Omega}_0, \tag{5.36 a}$$

$$\tilde{\Omega}_p(\alpha) = \exp \{ -\alpha\hat{\mathfrak{S}}_0 \} \left[ \tilde{\Omega}_p + \int_0^\alpha \exp \{ \beta\hat{\mathfrak{S}}_0 \} (\{ \tilde{\Omega}_0(\beta), \tilde{\mathfrak{S}}_p \} + \tilde{\pi}_p(\beta)) d\beta \right]. \tag{5.36 b}$$

Equations (5.31)-(5.36) should be solved according to the following scheme. Once the classical gauge system is Abelizable, we have at our disposal the Abelian classical constraints $\tilde{\mathfrak{T}}_{0A}$

$$\{\tilde{\mathfrak{T}}_{0A}, \tilde{\mathfrak{T}}_{0B}\} = 0, \qquad \tilde{\mathfrak{T}}_{0B}\{\tilde{G}^B, \tilde{\mathfrak{T}}_{0A}\} = 0, \tag{5.37}$$

so that in fact we know the classical function

$$(\Omega_0)_{\text{Abelian}} = \tilde{\mathfrak{T}}_{0A}\tilde{G}^A, \tag{5.38}$$

as well as the classical generators $\tilde{\mathfrak{S}}_0$ from (5.27), (5.33). Then, equations (5.31), (5.32) should determine the coefficient functions in the expansion (5.29) of the symbols of Abelian constraints for $n > 0$, while equations (5.34)-(5.36) should determine those in the second expansion in (5.22) of the symbol of the generator for $n > 0$.

Clearly, solvability of equations (5.31)-(5.36) is not evident beforehand, so that the existence problem is actual for them. Nonetheless, one can prove that, given classical Abelian constraints subject to (5.37) and a classical generator from (5.33) there exist all coefficient functions in the (second of) expansions (5.22) and (5.29) possessing the needed special properties:

$$\tilde{\mathfrak{T}}^*_{nA} = \tilde{\mathfrak{T}}_{nA}(-1)^n, \tag{5.39}$$

$$\varepsilon(\tilde{\mathfrak{T}}_{nA}) = \varepsilon(\tilde{G}^A) + 1, \quad \text{gh}(\tilde{\mathfrak{T}}_{nA}) = \text{gh}(\tilde{G}^A) - 1, \tag{5.40}$$

$$\tilde{\mathfrak{S}}^*_n = \tilde{\mathfrak{S}}_n(-1)^n, \quad \varepsilon(\tilde{\mathfrak{S}}_n) = 0, \quad \text{gh}(\tilde{\mathfrak{S}}_n) = 0, \tag{5.41}$$

and satisfying equations (5.31)-(5.36). The proof of this fact by induction, although technically more complicated than that of the basic existence theorem given above is very close to it in its structure. We shall not therefore display here the corresponding considerations.

Let, now, the symbols $\tilde{\mathfrak{S}}$ and $\tilde{\mathfrak{T}}_A$ of the generator and the Abelian constraints, respectively, be at our disposal, as well as the operators $\mathfrak{A}$ and $\mathfrak{T}_A$ from (5.11), (5.12). Let us then write the Abelization equation for operators (5.11) in a more explicit form

$$\Omega = \mathfrak{A}\mathfrak{T}_A G^A \mathfrak{A}^{-1}. \tag{5.42}$$

It can be shown that operators of the reducible Abelian constraints $\mathfrak{T}_A$ allow separation of independent Abelian constraints $\tau_\mu$

$$\mathfrak{T}_A = \tau_\mu P^\mu_A, \qquad \tau_\mu = \mathfrak{T}_A \bar{P}^A_\mu, \tag{5.43}$$

$$[\tau_\mu, \tau_\nu] = 0, \quad \varepsilon(\tau_\mu) \equiv \varepsilon_\mu, \quad \text{gh}(\tau_\mu) \equiv -g_\mu. \tag{5.44}$$

Here the filtering operators $P^\mu_A$ and $\bar{P}^A_\mu$ do not depend on the ghost operators in (3.1), and obey the conditions

$$[P^\mu_A G^A, \tau_\nu] = 0, \qquad [\bar{G}_A \bar{P}^A_\mu, \tau_\nu] = 0, \tag{5.45}$$

$$\left.\begin{aligned} [P^\mu_A G^A, P^\nu_B G^B] = 0, \quad [\bar{G}_A \bar{P}^A_\mu, \bar{G}_B \bar{P}^B_\nu] = 0, \\ [P^\mu_A G^A, \bar{G}_B \bar{P}^B_\nu] = i\delta^\mu_\nu, \end{aligned}\right\} \tag{5.46}$$

$$(P^\mu_A)^\dagger = P^\mu_A(-1)^{\varepsilon_\mu \varepsilon_A}, \quad (\bar{P}^A_\mu)^\dagger = \bar{P}^A_\mu(-1)^{(\varepsilon_\mu + 1)(\varepsilon_A + 1)}, \tag{5.47}$$

where $\varepsilon_\mu$ and $\varepsilon_A$ are defined in (5.44) and (3.1), respectively. Then we have from (5.42)

$$\Omega = \mathcal{T}_\mu \Theta^\mu , \tag{5.48}$$

$$\mathcal{T}_\mu \equiv \mathfrak{A} \tau_\mu \mathfrak{A}^{-1}, \qquad \Theta^\mu \equiv \mathfrak{A} P^\mu_A G^A \mathfrak{A}^{-1} , \tag{5.49}$$

and

$$[\mathcal{T}_\mu, \mathcal{T}_\nu] = 0, \qquad [\mathcal{T}_\mu, \Theta^\nu] = 0, \qquad [\Theta^\mu, \Theta^\nu] = 0 , \tag{5.50}$$

$$\mathcal{T}^\dagger_\mu = \mathcal{T}_\mu, \qquad \varepsilon(\mathcal{T}_\mu) \equiv \varepsilon_\mu, \qquad \mathrm{gh}\,(\mathcal{T}_\mu) \equiv - g_\mu , \tag{5.51}$$

$$(\Theta^\mu)^\dagger = \Theta^\mu, \qquad \varepsilon(\Theta^\mu) = \varepsilon_\mu + 1, \qquad \mathrm{gh}\,(\Theta^\mu) = g_\mu + 1 . \tag{5.52}$$

The independent commuting operators $\mathcal{T}_\mu$, $\Theta^\mu$ allow an introduction of the canonically conjugate operators $\mathcal{K}^\mu$, $\Xi_\mu \equiv \mathfrak{A} \overline{G}_A \overline{P}^A_\mu \mathfrak{A}^{-1}$ :

$$[\mathcal{K}^\mu, \mathcal{K}^\nu] = 0, \qquad [\mathcal{K}^\mu, \mathcal{T}_\nu] = i\delta^\mu_\nu, \qquad [\mathcal{K}^\mu, \Theta^\nu] = 0 , \tag{5.53}$$

$$[\Xi_\mu, \Xi_\nu] = 0, \qquad [\Theta^\mu, \Xi_\nu] = i\delta^\mu_\nu , \tag{5.54}$$

$$[\mathcal{K}^\mu, \Xi_\nu] = 0, \qquad [\mathcal{T}_\mu, \Xi_\nu] = 0 . \tag{5.55}$$

$$(\mathcal{K}^\mu)^\dagger = \mathcal{K}^\mu(-1)^{\varepsilon_\mu}, \qquad \varepsilon(\mathcal{K}^\mu) = \varepsilon_\mu, \qquad \mathrm{gh}\,(\mathcal{K}^\mu) = g_\mu , \tag{5.56}$$

$$\Xi^\dagger_\mu = \Xi_\mu(-1)^{\varepsilon_\mu + 1}, \qquad \varepsilon(\Xi_\mu) = \varepsilon_\mu + 1, \qquad \mathrm{gh}\,(\Xi_\mu) = -(g_\mu + 1) . \tag{5.57}$$

Representation (5.48) is nothing but the operator analog of the classical Abelian factorization (4.74).

In case of irreducibility, i.e. when the Abelian constraints (5.43) are independent and do not contain ghost operators, every linguistic difference between the subscripts « $\mu$ » and « A » in the r.-h. side of (5.43) disappears, since now they have the same nature and dimensionality. One may set

$$P^\mu_A = \delta^\mu_A, \qquad \overline{P}^A_\mu = \delta^A_\mu , \tag{5.58}$$

$$\mathcal{T}_\mu = \mathfrak{A} \mathfrak{T}_\mu \mathfrak{A}^{-1}, \qquad \Theta^\mu = \mathfrak{A} G^\mu \mathfrak{A}^{-1} . \tag{5.59}$$

Similarly to (4.81), (4.82), the operatorial Abelian factorization defines in a natural way the canonical reparametrization for the complete set $\{ \Gamma \}$ of canonical pairs of operator-valued dynamical variables that make the extended phase space of the system

$$\{ \Gamma \} \rightarrow \{ \Gamma_{\mathrm{phys}} \} \cup \{ \Gamma_{\mathrm{aux}} \} , \tag{5.60}$$

$$\mathrm{gh}\,(\Gamma_{\mathrm{phys}}) = 0, \qquad \{ \Gamma_{\mathrm{aux}} \} = \{ (\mathcal{K}, \mathcal{T}) \} \cup \{ (\Theta, \Xi) \} . \tag{5.61}$$

Here $\Gamma_{\mathrm{phys}}$ is the set of canonical operator pairs corresponding to physical degrees of freedom. The canonicity of the reparametrization (5.60) implies, in particular, that

$$[\Gamma_{\mathrm{phys}}, \Gamma_{\mathrm{aux}}] = 0 . \tag{5.62}$$

Let us now consider the operator equation of the form

$$[\Omega, X] = 0 , \tag{5.63}$$

carry out the reparametrization (5.60) in it

$$X(\Gamma) = \mathscr{X}(\Gamma_{\text{phys}}, \Gamma_{\text{aux}}), \tag{5.64}$$

and pass to the Weyl symbols relative to the new operators

$$\mathscr{X}(\Gamma_{\text{phys}}, \Gamma_{\text{aux}}) = \exp\left\{ \Gamma_{\text{phys}} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{\text{phys}}} + \Gamma_{\text{aux}} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{\text{aux}}} \right\} \tilde{\mathscr{X}}(\tilde{\Gamma}_{\text{phys}}, \tilde{\Gamma}_{\text{aux}}) \,|_0 \tag{5.65}$$

Here the sign of substitution $|_0$ indicates that the zero values of the classical variables $\tilde{\Gamma}_{\text{phys}}$ and $\tilde{\Gamma}_{\text{aux}}$ are to be taken.

Using the Abelian factorization (5.48), we obtain from (5.63) the following equation for symbols

$$\left(\tilde{\mathscr{T}}_\mu - \frac{1}{2} i \frac{\vec{\partial}}{\partial \tilde{\mathscr{K}}^\mu} (-1)^{\varepsilon_\mu}\right)\left(\tilde{\Theta}^\mu + \frac{1}{2} i \frac{\vec{\partial}}{\partial \tilde{\Xi}_\mu}\right)\tilde{\mathscr{X}} -$$

$$- \tilde{\mathscr{X}}\left(\tilde{\mathscr{T}}_\mu + \frac{1}{2} i \frac{\overleftarrow{\partial}}{\partial \tilde{\mathscr{K}}^\mu}\right)\left(\tilde{\Theta}^\mu - \frac{1}{2} i \frac{\overleftarrow{\partial}}{\partial \tilde{\Xi}_\mu} (-1)^{\varepsilon_\mu + 1}\right)(-1)^{\varepsilon_{\tilde{\mathscr{X}}}} = 0 . \tag{5.66}$$

The terms free of derivatives, as well as the ones containing the second derivatives, all cancel, and we are left with

$$\left(\tilde{\mathscr{T}}_\mu \frac{\partial}{\partial \tilde{\Xi}_\mu} + \tilde{\Theta}^\mu \frac{\partial}{\partial \tilde{\mathscr{K}}^\mu} (-1)^{\varepsilon_\mu + 1}\right)\tilde{\mathscr{X}} = 0 . \tag{5.67}$$

We are facing again the standard equation (4.86)-(4.88) whose solution allows the standard representation (4.89), (4.90). Basing on it, we draw immediately the conclusion that the general solution of equation (5.63) can be written as

$$X = X_{\text{phys}} - i\,[\Omega, Y] , \tag{5.68}$$

where

$$X_{\text{phys}} \equiv \exp\left\{ \Gamma_{\text{phys}} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{\text{phys}}} \right\} \tilde{\mathscr{X}}(\tilde{\Gamma}_{\text{phys}}, \tilde{\Gamma}_{\text{aux}}) \,|_0 , \tag{5.69}$$

$$Y(\Gamma) = \mathscr{Y}(\Gamma_{\text{phys}}, \Gamma_{\text{aux}}) \equiv \exp\left\{ \Gamma_{\text{phys}} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{\text{phys}}} + \Gamma_{\text{aux}} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{\text{aux}}} \right\} \tilde{\mathscr{Y}}(\tilde{\Gamma}_{\text{phys}}, \tilde{\Gamma}_{\text{aux}}) \,|_0 \tag{5.70}$$

Here $\tilde{\mathscr{X}}$ in the r.-h. side of (5.69) is a solution of equation (5.67), while $\tilde{\mathscr{Y}}$ in the r.-h. side of (5.70) is related to $\tilde{\mathscr{X}}$ from (5.67) through the standard formula (4.90) with the notations (4.87) used for the variables involved in (5.67).

In full similarity with the classical expansion (4.91), the first term in the r.-h. side of expansion (5.68), defined by eq. (5.69), is called the singlet component of the operator-valued observable X from equation (5.63), whereas the second term of the r.-h. side of (5.68) is called the doublet component. Consider the basic properties of singlet operators (5.69).

First, the same as in the classical case there holds

$$\mathrm{gh}(X) \neq 0 \Rightarrow X_{\text{phys}} = 0, \tag{5.71}$$

so that there is an operator analog of the classical implication (4.46) ⇒ (4.47), i. e. each regular solution of the equation

$$[\Omega, X] = 0, \qquad \mathrm{gh}(X) \neq 0, \tag{5.72}$$

can be presented in the form

$$X = -i[\Omega, Y], \qquad \varepsilon(Y) = \varepsilon(X) + 1, \qquad \mathrm{gh}(Y) = \mathrm{gh}(X) - 1. \tag{5.73}$$

Second, it appear that for any operators X subject to (5.63), the restriction

$$X \to X_{\text{phys}} \tag{5.74}$$

is associative relative to the multiplication, in full analogy with the classical situation.

Indeed, let X′ and X″ be two operators subjected to (5.63). Then, according to (5.68), we have

$$X' = X'_{\text{phys}} - i[\Omega, Y'], \qquad X'' = X''_{\text{phys}} - i[\Omega, Y'']. \tag{5.75}$$

By multiplying these equalities we get

$$X'X'' = X'_{\text{phys}}X''_{\text{phys}} - i[\Omega, 'Y'']. \tag{5.76}$$

On the other hand, by applying the expansion (5.68) directly to the product X′X″ we find

$$X'X'' = (X'X'')_{\text{phys}} - i[\Omega, Y''']. \tag{5.77}$$

The subtraction of (5.76) from (5.77) results in the equation

$$(X'X'')_{\text{phys}} - X'_{\text{phys}}X''_{\text{phys}} = i[\Omega, Y''' - 'Y''] \tag{5.78}$$

whence

$$(X'X'')_{\text{phys}} = X'_{\text{phys}}X''_{\text{phys}}, \tag{5.79}$$

as well as

$$[\Omega, Y''' - 'Y''] = 0. \tag{5.80}$$

Equations like (5.78) are worth being now considered from a somewhat more general point of view. Let us consider, instead of (5.63), the equation of the form that look, at first glance, more general

$$-i[\Omega, X] = F. \tag{5.81}$$

Here the r.-h. side F is an operator that does not depend on the operators

$\Gamma_{aux}$ from (5.61) after reparametrization. Going over to the Weyl symbols for the operators (5.61) we have

$$\tilde{J} \cdot \frac{\partial}{\partial \tilde{Z}} \tilde{\mathscr{X}} = \tilde{\mathscr{F}}(\tilde{\Gamma}_{phys}), \tag{5.82}$$

$$F(\Gamma) = \mathscr{F}(\Gamma_{phys}) = \exp\left\{ \Gamma_{phys} \cdot \frac{\partial}{\partial \tilde{\Gamma}_{phys}} \right\} \tilde{\mathscr{F}}(\tilde{\Gamma}_{phys}) \,|_0 , \tag{5.83}$$

where the notations (4.87) are used for the variables corresponding to the operators $\Gamma_{aux}$ involved in (5.61), while is defined as (5.64), (5.65).

Following [*15*], let us apply to (5.82) the Fourier-transformed operator

$$\left(\tilde{Z} \cdot \frac{\partial}{\partial \tilde{J}}\right)\left(\tilde{J} \cdot \frac{\partial}{\partial \tilde{Z}}\right) \tilde{\mathscr{X}} = 0 . \tag{5.84}$$

The zero in the r.-h. side is just due to the fact that the r.-h. side of (5.82) does not depend on the variables $\tilde{\Gamma}_{aux}$. Equation (5.84) has exactly the same form as it would have if F in the r.-h. side of equation (5.81) were zero, so that the latter coincided with (5.63).

The method of ref. [*15*], when applied to equation (5.84), leads directly to representation (4.89), (4.90) for the regular solution of this equation. Hence we have ome to representation (5.68) for solution of equation (5.81). Then, it is obvious, however, that

$$F = 0 . \tag{5.85}$$

Thus, if the r.-h. side F of equation (5.81) does not depend on the operators $\Gamma_{aux}$ involved in (5.61), a regular solution to equation (5.81) only exists provided that (5.85) is satisfied. In this case it coincides with the solution (5.68) of the homogeneous equation (5.63). This result has a clear physical sense: by its nature, the doublet component (the l.-h. side of (5.81)) cannot reproduce the « extracted » contribution of physical degrees of freedom $\Gamma_{phys}$ alone for any regular X.

If, now, we come back to equation (5.78) it will be completely clear that its consequence (5.79) is nothing but a particular case of (5.85) while (5.80) is an equation of the form (5.63).

Representation (5.68) for the solution of equation (5.63) is the main result of the study performed in this section. We have established the general structure of any physical (in the sense of the definition (2.6)) operator

$$\mathscr{O} = \mathscr{O}_{phys} - i\,[\Omega, \mathfrak{B}] . \tag{5.86}$$

Let us apply this result to the two principal physical operators involved in our theory, i. e. to the unitarizing Hamiltonian (2.1) and the corresponding evolution operator (2.14).

First of all, in accord with (5.86) we have the following representation for the operator $\mathscr{H}$ in (2.2)

$$\mathscr{H} = \mathscr{H}_{\text{phys}} - i[\Omega, \mathfrak{D}]. \tag{5.87}$$

Then the hermiticity of $\mathscr{H}$ implies

$$\mathscr{H}_{\text{phys}} - \mathscr{H}^{\dagger}_{\text{phys}} = i[\Omega, \mathfrak{D} + \mathfrak{D}^{\dagger}]. \tag{5.88}$$

This in an equation of the form of (5.81). Due to the implication (5.81) $\Rightarrow$ (5.85) we get from (5.88)

$$\mathscr{H}_{\text{phys}} = \mathscr{H}^{\dagger}_{\text{phys}}, \qquad [\Omega, \mathfrak{D} + \mathfrak{D}^{\dagger}] = 0. \tag{5.89}$$

Then we find for the unitarizing Hamiltonian (2.1)

$$\mathrm{H} = \mathscr{H}_{\text{phys}} - i\left[\Omega, \Psi + \frac{1}{2}(\mathfrak{D} - \mathfrak{D}^{\dagger})\right]. \tag{5.90}$$

so that in the class of gauges

$$\Psi = -\frac{1}{2}(\mathfrak{D} - \mathfrak{D}^{\dagger}) - i[\Omega, \mathfrak{F}] \tag{5.91}$$

we obtain

$$\mathrm{H} = \mathrm{H}_{\text{phys}} = \mathscr{H}_{\text{phys}}. \tag{5.92}$$

Let us turn now to the evolution operator (2.14) corresponding to the unitarizing Hamiltonian (2.1). In this case we also have

$$\mathrm{E} = \mathrm{E}_{\text{phys}} - i[\Omega, \mathfrak{H}], \qquad \mathrm{E}^{\dagger} = \mathrm{E}^{\dagger}_{\text{phys}} + i[\Omega, \mathfrak{H}^{\dagger}]. \tag{5.93}$$

The multiplication of these equalities, and the use of the unitarity property (2.15) leads to

$$1 = \mathrm{E}^{\dagger}_{\text{phys}}\mathrm{E}_{\text{phys}} - i[\Omega, \mathfrak{M}], \tag{5.94}$$

$$1 = \mathrm{E}_{\text{phys}}\mathrm{E}^{\dagger}_{\text{phys}} - i[\Omega, \mathfrak{M}']. \tag{5.95}$$

These are again equations of the form of (5.81). Hence we obtain from them

$$\mathrm{E}^{\dagger}_{\text{phys}}\mathrm{E}_{\text{phys}} = \mathrm{E}_{\text{phys}}\mathrm{E}^{\dagger}_{\text{phys}} = 1, \tag{5.96}$$

$$[\Omega, \mathfrak{M}] = [\Omega, \mathfrak{M}'] = 0. \tag{5.97}$$

Equation (5.96) shows that the extracted singlet component of $\mathrm{E}_{\text{phys}}$ is also a unitary operator.

To conclude this section we shall study the consequences of the Abelian factorization (5.48) regarding physical states.

The use of (5.84) in the definition (1.3) implies that

$$\Omega\,|\,\Phi\,\rangle = \mathscr{T}_{\mu}\Theta^{\mu}\,|\,\Phi\,\rangle = 0. \tag{5.98}$$

From this we readily conclude that

$$(\mathscr{T}_{\mu} + i\Omega\Xi_{\mu})\,|\,\Phi\,\rangle = 0, \qquad (\Theta^{\mu} - i\Omega\mathscr{K}^{\mu}(-1)^{\varepsilon_{\mu}})\,|\,\Phi\,\rangle = 0 \tag{5.99}$$

so that the physical matrix elements of operators $\mathscr{T}_\mu$, $\Theta^\mu$ vanish:

$$\langle \Phi' | \mathscr{T}_\mu | \Phi \rangle = 0, \qquad \langle \Phi' | \Theta^\mu | \Phi \rangle = 0. \tag{5.100}$$

Now we shall obtain the analog of expansion (5.86) for physical states. For this to be possible it is important that the set of operators $\mathscr{T}_\mu$ as well as the set $\Theta^\mu$ be divisible in a symmetric way in two sets of equal dimensionality with the same distributions of statistics and ghost number.

$$\mathscr{T} = (\mathscr{T}', \mathscr{T}''), \qquad \Theta = (\Theta', \Theta''). \tag{5.101}$$

This division puts operator $\Omega$ to the form

$$\Omega = \mathscr{T}' \cdot \Theta' + \mathscr{T}'' \cdot \Theta''. \tag{5.102}$$

Let us also make the explicit division of the set of operators $\Gamma_{phys}$ from (5.60) into pairs of canonically conjugated momenta $P_{phys}$ and coordinates $Q_{phys}$

$$\Gamma_{phys} = (P_{phys}, Q_{phys}). \tag{5.103}$$

Consider the realization of the operator (5.102) in the representation where the operators

$$Q_{phys}, \mathscr{K}', \mathscr{T}'', \Theta', \Xi'' \tag{5.104}$$

are diagonal and have the eigenvalues

$$\tilde{Q}_{phys}, \tilde{\mathscr{K}}', \tilde{\mathscr{T}}'', \tilde{\Theta}', \tilde{\Xi}''. \tag{5.105}$$

In this representation operators $\mathscr{T}'$ and $\Theta''$ can be realized in the form

$$\mathscr{T}' = i \frac{\partial}{\partial \tilde{\mathscr{K}}'} (-1)^{\varepsilon' + 1}, \qquad \Theta'' = i \frac{\partial}{\partial \tilde{\Xi}''}, \tag{5.106}$$

$$\varepsilon' \equiv \varepsilon(\mathscr{K}') = \varepsilon(\mathscr{T}'). \tag{5.107}$$

Denoting the components of the vector $| \Phi \rangle$ in the basis of the representation (5.104)-(5.107) by

$$\tilde{\Phi} = \tilde{\Phi}(\tilde{Q}_{phys}, \tilde{\mathscr{K}}', \tilde{\mathscr{T}}'', \tilde{\Theta}', \tilde{\Xi}''), \tag{5.108}$$

we can write equation (5.98) with the operator (5.102) as

$$\left( \tilde{\mathscr{T}}'' \cdot \frac{\partial}{\partial \tilde{\Xi}''} + \tilde{\Theta}' \cdot \frac{\partial}{\partial \tilde{\mathscr{K}}'} (-1)^{\varepsilon' + 1} \right) \tilde{\Phi} = 0, \tag{5.109}$$

which only differs from equation (5.67) considered above in the dimensionality of the variables involved.

Using the notations

$$\tilde{J} = (\tilde{\mathscr{T}}'', \tilde{\Theta}'(-1)^{\varepsilon' + 1}), \qquad \tilde{Z} = (\tilde{\Xi}'', \tilde{\mathscr{K}}'), \tag{5.110}$$

in equation (5.109), we come again to the standard equation (4.86), whence we conclude, with the help of (4.89) that

$$\tilde{\Phi} = \tilde{\Phi}_{\text{phys}} + \Omega\tilde{\mathfrak{P}}, \tag{5.111}$$

where the notation

$$\tilde{\Phi}_{\text{phys}} \equiv \tilde{\Phi}(\tilde{Q}_{\text{phys}}, 0, 0, 0, 0), \tag{5.112}$$

is used, and the function $\tilde{\mathfrak{P}}$ is given by the representation (4.90) in the variables (5.110) and with the components (5.108) for $\tilde{\mathscr{X}}(\tilde{J}, \tilde{Z})$.

Eq. (5.111) gives the needed expansion that serves for the physical state as an analog of the operatorial expansion (5.86). Restriction (5.112) defines the singlet component of the state, while the second term in (5.111) gives the doublet one.

Expansions (5.86), (5.111) show that the contribution from nonphysical degrees of freedom $\Gamma_{\text{aux}}$ can be completely included into the corresponding doublet components $-i[\Omega, \mathfrak{B}]$ in the case of physical operators from (2.6), and into $\Omega\tilde{\mathfrak{P}}$ in the case of physical states from (1.3), while the corresponding singlet components $\mathscr{O}_{\text{phys}}$ and $\tilde{\Phi}_{\text{phys}}$ only contain the extracted contribution of physical degrees of freedom $\Gamma_{\text{phys}}$.

One should, however, take some care, when expansions like (5.86), (5.111) are used for finding explicitly the norms of physical states or physical matrix elements of physical operators. A sharp « switch off » of nonphysical degrees of freedom, that separates pure singlet components in expansions (5.86), (5.111) can, generally, lead to uncertainties in expressions for the norm or the matrix elements. To avoid this, one should be using the « contraction » formulas (4.89), (4.90) in the form to be obtained by the replacement

$$\tilde{\mathscr{X}}(0,0) \to \tilde{\mathscr{X}}(\alpha_0\tilde{J}, \alpha_0\tilde{Z}), \qquad \tilde{\mathscr{X}}^{(s)}(0,0) \to \tilde{\mathscr{X}}^{(s)}(\alpha_0\tilde{J}, \alpha_0\tilde{Z}). \tag{5.113}$$

made in the r.-h. sides of (4.89), (4.90 *b*), supplemented by the substitution of $\alpha_0$ in place of the zero in the lower integration limit over $\alpha$ in (4.90 *a*). In the final expressions for the norm and the matrix elements, and no sooner, one should go to the limit $\alpha_0 \to 0$.

## 6. QUANTUM DEFORMATION IN THE INVOLUTION RELATIONS

In this section we shall study in more detail the phenomenon of quantum deformation of structural relations, already outlined in Section 3. We confine ourselves to the practically most interesting case of irreducible rank-1 theories and shall as a matter of fact only analyse the involution relations for constraints and the Hamiltonian. The same as in Section 3 we consider here both the canonical and Wick realizations of the ghost sector. Here, however, we refrain from explicitly using the standartized way of writing accepted in Section 3 for expansions of the generating operators $\Omega$ and $\mathscr{H}$ in powers of ghosts but use, instead, more conventional normalizations for writing these expansions. It goes without saying that, nevertheless, the same basic types of normal orderings for ghost operators will be covered, as those dealt with in Section 3.

### A. **Canonical ghost sector.**

Let us start with consideration of the canonical ghost sector (3.1)-(3.3) in the case of irreducibility, when

$$\mathrm{gh}\,(\mathrm{G}^{\mathrm{A}}) = -\,\mathrm{gh}\,(\overline{\mathrm{G}}_{\mathrm{A}}) = 1\,. \tag{6.1}$$

Consider also the notation

$$\varepsilon'_{\mathrm{A}} \equiv \varepsilon_{\mathrm{A}} + 1\,, \tag{6.2}$$

where $\varepsilon_{\mathrm{A}}$ in the r.-h. side in the Grassmanian parity of ghosts from (3.1).

A1. $\overline{\mathrm{G}}\mathrm{G}$-*normal form*

In this normal form we have the following expression for generating operators in the irreducible rank-1 theory

$$\Omega = \mathrm{T}_{\mathrm{A}}\mathrm{G}^{\mathrm{A}} + \frac{1}{2}\overline{\mathrm{G}}_{\mathrm{A}}\mathrm{U}^{\mathrm{A}}_{\mathrm{BC}}\mathrm{G}^{\mathrm{C}}\mathrm{G}^{\mathrm{B}}(-1)^{\varepsilon'_{\mathrm{B}}}, \tag{6.3}$$

$$\mathscr{H} = \mathrm{H}_0 + \overline{\mathrm{G}}_{\mathrm{A}}\mathrm{V}^{\mathrm{A}}_{\mathrm{B}}\mathrm{G}^{\mathrm{B}}\,. \tag{6.4}$$

Equations (1.1), (2.2) give rise in the $\mathrm{G}^2$-order to the following involution relations for operator-valued constraints $\mathrm{T}_{\mathrm{A}}$ and the Hamiltonian $\mathrm{H}_0$

$$[\mathrm{T}_{\mathrm{A}}, \mathrm{T}_{\mathrm{B}}] = i\mathrm{T}_{\mathrm{C}}\mathrm{U}^{\mathrm{C}}_{\mathrm{AB}}\,, \tag{6.5}$$

$$[\mathrm{H}_0, \mathrm{T}_{\mathrm{B}}] = i\mathrm{T}_{\mathrm{C}}\mathrm{V}^{\mathrm{C}}_{\mathrm{B}}\,. \tag{6.6}$$

These equations differ from their classical counterpart to a small extent. The operatorial character of the involution (6.5), (6.6) shows itself formally in two points. The first is that, instead of the Poisson superbracket, the supercommutator enters in the l.-h. sides and the second, that the order of multipliers in the r.-h. sides is fixed so that the operator-valued constraints are located to the *left* of the structural operators U and V.

Relations (6.5), (6.6) are apparently linear in the structural operators. The terms of quantum deformation are absent from here.

A2. *Weyl-normal form.*

In this normal form the generating operators are written as

$$\Omega = T_A G^A + \frac{1}{6}\overline{G}_A U^A_{BC} G^C G^B (-1)^{\varepsilon'_B} +$$

$$+ \frac{1}{6} G^C G^B U^A_{BC} \overline{G}_A (-1)^{[\varepsilon'_A \varepsilon'_B + (\varepsilon'_A + 1)\varepsilon'_C]} + \frac{1}{6} G^B \overline{G}_A U^A_{BC} G^C (-1)^{\varepsilon'_B}, \quad (6.7)$$

$$\mathscr{H} = H_0 + \frac{1}{2}\overline{G}_A V^A_B G^B - \frac{1}{2} G^B V^A_B \overline{G}_A (-1)^{\varepsilon'_A \varepsilon'_B}. \quad (6.8)$$

To avoid a misunderstanding we should note that although the constraints, the Hamiltonian and the structural operators are designated in (6.7), (6.8) in the same way as in (6.3), (6.4), they are certainly different operators now.

Equations (1.1), (1.2) give rise in the $G^2$-approximation to the following involution relations for the operators involved in (6.7)

$$[T_A, T_B] = \frac{i}{2}\left(T_C U^C_{AB} + U^C_{AB} T_C (-1)^{(\varepsilon'_A + \varepsilon'_B + 1)\varepsilon'_C}\right) -$$

$$- \frac{1}{4}[U^C_{A\mathscr{D}}, U^{\mathscr{D}}_{CB}](-1)^{(\varepsilon'_A + 1)\varepsilon'_C}, \quad (6.9)$$

$$[H_0, T_B] = \frac{i}{2}\left(T_C V^C_B + V^C_B T_C (-1)^{(\varepsilon'_B + 1)\varepsilon'_C}\right) - \frac{1}{4}[V^C_{\mathscr{D}}, U^{\mathscr{D}}_{CB}](-1)^{\varepsilon'_C}. \quad (6.10)$$

Contrary to (6.5), (6.6), symmetrized products of the operator-valued constraints and structural operators are present in the r.-h. sides of (6.10). Besides, the r.-h. sides of (6.9), (6.10) contain terms quadratic with respect to structural operators (these are [U, U] in (6.9) and [V, U] in (6.10)). It is these terms that correspond to the quantum deformation of the Weyl involution.

## B. Wick ghost sector.

Let now the ghost sector be realized according to (3.40)-(3.43) together with the sector of null modes $(G^{a_0}, \overline{G}_{a_0})$, satisfying (3.1)-(3.3) for $A = a_0$. In case of rank-1 irreducible theories one has

$$\mathrm{gh}(G^\alpha) = \mathrm{gh}(G^{\dagger\alpha}) = -\mathrm{gh}(\overline{G}_\alpha) = -\mathrm{gh}(\overline{G}{}^\dagger_\alpha) = 1, \quad (6.11)$$

where the collective notations are referred to

$$G^\alpha \equiv (G^a, G^{a_0}), \qquad \overline{G}_\alpha \equiv (\overline{G}_a, \overline{G}_{a_0}), \quad (6.12)$$

as well as

$$\varepsilon_\alpha \equiv (\varepsilon_a, \varepsilon_{a_0}), \qquad \varepsilon'_\alpha \equiv (\varepsilon'_a, \varepsilon'_{a_0}) \equiv \varepsilon_\alpha + 1, \quad (6.13)$$

and $\varepsilon_a$ and $\varepsilon_{a_0}$ are the Grassmann parities, respectively, from (3.41) and (3.1) for $A = a_0$.

B1. *Wick-normal form.*

In this normal form we have the following expressions for the generating operators

$$\Omega = T^\dagger_\alpha G^\alpha + G^{\dagger\alpha} T_\alpha + \left(\frac{1}{2}\overline{G}^\dagger_\gamma U^{\dagger\gamma}_{\alpha\beta} G^\alpha G^\beta + \frac{1}{2} G^{\dagger\beta} G^{\dagger\alpha} U^\gamma_{\alpha\beta} \overline{G}_\gamma + \right.$$

$$+ G^{\dagger\alpha} \overline{U}^\gamma_{\alpha\beta} \overline{G}_\gamma G^\beta + G^{\dagger\beta} \overline{G}^\dagger_\gamma \overline{U}^{\dagger\gamma}_{\alpha\beta} G^\alpha + \frac{1}{2} G^{\dagger\alpha} G^{\dagger\beta} \overline{G}^\dagger_\gamma \overline{\overline{U}}{}^{\dagger\gamma}_{\alpha\beta} +$$

$$\left. + \frac{1}{2} \overline{\overline{U}}{}^\gamma_{\alpha\beta} \overline{G}_\gamma G^\beta G^\alpha \right) (-1)^{\varepsilon'_\beta}, \tag{6.14}$$

$$\mathscr{H} = H_0 + \overline{G}^\dagger_\gamma V^{\dagger\gamma}_\beta G^\beta + G^{\dagger\beta} V^\gamma_\beta \overline{G}_\gamma +$$

$$+ (G^{\dagger\beta} \overline{G}^\dagger_\gamma \overline{V}^{\dagger\gamma}_\beta + \overline{V}^\gamma_\beta \overline{G}_\gamma G^\beta)(-1)^{\varepsilon'_\beta}. \tag{6.15}$$

These expressions contain both Wick ghost pairs and canonical pairs of ghost null modes in a unique way. In writing (6.15), (6.14) it was assumed that the components of the structural operators in the $a_0$-sector of null modes obey the relations that provided the Weyl-normal form for the corresponding canonical pairs $(G^{a_0}, \overline{G}_{a_0})$.

Since the Weyl-normal form in the canonical ghost basis has been already considered above, we do not display here the explicit form of the complete set of involution relations that follow from (1.1), (2.2), (6.14), (6.15) in every ghost sector, but confine ourselves, instead, to consideration of the only case when the null mode sector is absent. Then only the pure Wick sector is left, with the following involution relations

$$[T_a, T_b] = U^c_{ab} T_c + T^\dagger_c \overline{\overline{U}}{}^{\dagger c}_{ba} + (\overline{U}^c_{ad} \overline{\overline{U}}{}^{\dagger d}_{bc} - \overline{U}^c_{bd} \overline{\overline{U}}{}^{\dagger d}_{ac}(-1)^{\varepsilon_b \varepsilon'_a})(-1)^{\varepsilon'_c \varepsilon'_a}, \tag{6.16}$$

$$[T_a, T^\dagger_b] = \overline{U}^c_{ab} T_c + T^\dagger_c \overline{U}^{\dagger c}_{ba} + (\overline{U}^c_{ad} \overline{U}^{\dagger d}_{bc} - \overline{\overline{U}}{}^c_{bd} \overline{\overline{U}}{}^{\dagger d}_{ac}(-1)^{\varepsilon'_a \varepsilon_b})(-1)^{\varepsilon'_c \varepsilon'_a}, \tag{6.17}$$

$$[T_b, H_0] = V^c_b T_c + T^\dagger_c V^{\dagger c}_b + (\overline{U}^c_{bd} \overline{V}^{\dagger d}_c - \overline{V}^c_d \overline{U}^{\dagger d}_{bc})(-1)^{\varepsilon'_c \varepsilon'_a}. \tag{6.18}$$

All the terms quadratic in the structural operators correspond here to the quantum deformation of the Wick involution.

B2. *Central extension of the Virasoro algebra viewed upon as quantum deformation effect.*

A particular example of a gauge system with rank-1 — irreducible gauge algebra is given by the bosonic string. The quantum gauge algebra of this system is the central-extended Virasoro algebra. The central extension of this algebra arises naturally as a consequence of the quantum deformation described by the third term in the r.-h. side of the involution relation (6.17). The explicit expression for the generating operator (6.14) for the case of the Virasoro algebra was obtained in ref. [*16*]. We shall not reproduce it here, but consider directly the involution relation that is in this case an analog to relation (6.17) distinguished by the presence of the null mode

$$[L_n, L_m^\dagger] = \sum_{p=0}^{\infty} (\overline{U}_{nm}^p L_p + L_p^\dagger \overline{U}_{mn}^{\dagger p}) + \sum_{p,q=0}^{\infty} \overline{U}_{nq}^p \overline{U}_{mp}^{\dagger q}, \quad n, m \geqslant 0, \qquad (6.19)$$

here

$$L_n = L_{-n}^\dagger \qquad (6.20)$$

are generators of the operator Virasoro algebra, and

$$\overline{U}_{nm}^p = \overline{U}_{nm}^{\dagger p} \equiv (n+m)\theta(n-m)\delta_{n-m}^p, \qquad (6.21)$$

are structural coefficients that are in this case $c$-numbers. In (6.21) the fixation

$$\theta(0) \equiv \frac{1}{2} \qquad (6.22)$$

is meant.

The third term in the r.-h. side of (6.19) is an obvious analog of the similar term in (6.17), while the counterpart of the fourth term in the r.-h. side of (6.17) in the case of the Virasoro algebra is absent, since every $\overline{\overline{U}}_{nm}^p$ vanishes in this case.

Using (6.21), (6.22) one readily finds

$$\sum_{p,q=0}^{\infty} \overline{U}_{nq}^p \overline{U}_{mp}^{\dagger q} = \delta_{n-m}^0 \frac{13}{6} n(n^2-1) + \delta_{n-m}^0 2n. \qquad (6.23)$$

Here the first term in the r.-h. side exactly corresponds to the correct central extension of the Virasoro algebra in the critical dimension $\alpha = 26$, while the second term can be removed from (6.19) by the change in the definition of

$$L_n \to L_n - \delta_n^0. \qquad (6.24)$$

It is remarkable that the correct central extension (6.23) is obtained without exploiting any regularization, in a purely formal way. As for the contribution of infinite $p$, $q$ into the summation in (6.26), it is eliminated by the $\theta$-function present in the expression (6.21) for the structural coefficients. A similar mechanism leads to correct central extensions of algebra for every type of strings known.

## CONCLUSION

Here we point out two problems that need further study.

The first is to formulate closed operatorial postulates concerning the initial constraints and the Hamiltonian, that would guarantee directly the existence of solutions for the generating equations of the operator gauge algebra, with the needed properties in the classical limit. In the present paper we built the solution of generating equations step by step, in powers of the Planck constant, starting straight from the classical level. The non-closed character of this construction is evident: all the structural operators, including the operator-valued initial constraints and the Hamiltonian operator, are built in an iterative way, whereas a correctly posed closed postulate should be an exact assertion concerning the original operators of the theory.

The second problem, though not independent of the first one, is the need of a general and technically efficient description for the structure of solution of an equation for physical states.

# APPENDIX 1

## FORMAL OPERATORIAL QUANTIZATION OF ANTISYMMETRIC SECOND-RANK TENSOR FIELD IN INTERACTION

Classically this system is given in terms of Lagrange variables $A_\mu^a(x)$ and $B_{\mu\nu}^a(x) = -B_{\nu\mu}^a(x)$ by the action (*)

$$\mathcal{S} = \int \left( \frac{i}{4} F_{\mu\nu}^a B_{\rho\sigma}^a \varepsilon_{\mu\nu\rho\sigma} - \frac{1}{2}(A_\mu^a)^2 \right) d^4x \,, \tag{A.1}$$

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \,, \tag{A.2}$$

or by an equivalent Hamilton action

$$\mathcal{S} = \int \left( A_i^a \dot{B}_i^a - \frac{1}{2}(A_i^a)^2 - \frac{1}{2}(\nabla_i^{ab} B_i^b)^2 + B_{0i}^a \frac{1}{2} \varepsilon_{ijk} F_{jk}^a \right) d^4x \,, \tag{A.3}$$

where $A_i^a$ and $B_i^a \equiv \frac{1}{2}\varepsilon_{ijk} B_{jk}^a$ are canonical momenta and coordinates, respectively, $\nabla_i^{ab} \equiv \partial_i \delta^{ab} + f^{acb} A_i^c$, and $B_{0i}^a$ are Lagrange multipliers to first-class constraints.

Action (A.3) generates a Hamiltonian gauge algebra of the first stage of reducibility. Following ref. [*4*], we shall construct a formal operator expression for the complete unitarizing Hamiltonian that corresponds to (A.3). To this end consider the following set of operator-valued canonical momenta $P_M$ and coordinates $Q^M$ of the extended phase space (4.1) in the case of first stage reducibility

$$P_M = (A_i^a, \pi_{0i}^a, \overline{\mathcal{P}}_{0i}^a, \overline{C}_{0i}^a, \pi_1^a, \overline{\mathcal{P}}_1^a, \overline{C}_1^a, \pi_1^{1a}, \overline{C}_1^{1a}) \,, \tag{A.4}$$

$$Q^M \equiv (B_i^a, \lambda_{0i}^a, C_{0i}^a, \mathcal{P}_{0i}^a, \lambda_1^a, C_1^a, \mathcal{P}_1^a, \lambda_1^{1a}, \mathcal{P}_1^{1a}) \,, \tag{A.5}$$

with the following distribution of statistics

$$\varepsilon(P_M) = \varepsilon(Q^M) = (0, 0, 1, 1, 1, 0, 0, 0, 1) \,, \tag{A.6}$$

and the ghost number

$$\mathrm{gh}(P_M) = -\mathrm{gh}(Q^M) = -(0, 0, 1, 1, 1, 2, 2, 0, 1) \,. \tag{A.7}$$

The only nonzero equal-time supercommutators for the operators (A.4) and (A.5) evidently are those of the type of (4.2).

In terms of the complete canonical set (A.4), (A.5) we have the following expressions for the fermion and boson generating operators of the gauge algebra:

$$\Omega = \int d^3x \left( -\frac{1}{2} F_{ij}^a \varepsilon_{ijk} C_{0k}^a + \overline{\mathcal{P}}_{0i}^a \nabla_i^{ab} C_1^b + \pi_{0i}^a \mathcal{P}_{0i}^a + \pi_1^a \mathcal{P}_1^a + \pi_1^{1a} \mathcal{P}_1^{1a} \right) , \tag{A.8}$$

$$\mathcal{H} = \int d^3x \left( \frac{1}{2}(A_i^a)^2 + \frac{1}{2}(D^a)^2 \right) , \tag{A.9}$$

(*) Throughout this Appendix the Greek subscripts $\mu$, $\nu$, $\rho$, $\sigma$ refer to the 4-dimensional Minkowski space, the Roman subscripts $i$, $j$, $k$ refer to its 3-dimensional subspace, while the Roman superscripts $a$, $b$, $c$ are the internal indices labelling the adjoint representation of the semisimple compact group whose structural constants are $f^{abc}$.

where the designation is used

$$D^a \equiv \nabla_i^{ab} B_i^b + f^{abc}(\mathcal{P}_{0i}^b C_{0i}^c + \mathcal{P}_1^b C_1^c). \tag{A.10}$$

The complete unitarizing Hamiltonian of the theory has the general form (2.1):

$$H = \mathcal{H} - i[\Psi, \Omega], \tag{A.11}$$

where the operators $\Omega$ and $\mathcal{H}$ are given, respectively, by (A.8) and (A.9), (A.10), while the operator-valued gauge fermion $\Psi$ has the general structure

$$\Psi = \int d^3x(\bar{\mathcal{P}}_{0i}^a \lambda_{0i}^a + \bar{C}_{0i}^a \chi_{0i}^a + \bar{\mathcal{P}}_1^a \lambda_1^a + \bar{C}_1^a \chi_1^a + \bar{C}_1^{1a} \chi_1^{1a} + \bar{\chi}_1^{1a} \lambda_1^{1a}). \tag{A.12}$$

Here $\chi_{0i}^a$, $\chi_1^a$, $\chi_1^{1a}$, $\bar{\chi}_1^{1a}$ are the gauges for $B_i^a$, $C_{0i}^a$, $\lambda_{0i}^a$, $\bar{C}_{0i}^a$, respectively. The simplest is the choice of covariant nonsingular linear gauges

$$\chi_{0i}^a = -\partial_j B_{ji}^a + \frac{1}{2\alpha}\pi_{0i}^a, \tag{A.13}$$

$$\chi_1^a = -\partial_i C_{0i}^a + \frac{1}{2\beta}\mathcal{P}_1^{1a}, \tag{A.14}$$

$$\chi_1^{1a} = -\partial_i \lambda_{0i}^a + \frac{1}{2\alpha}\pi_1^{1\alpha}, \tag{A.15}$$

$$\bar{\chi}_1^{1a} = \partial_i \bar{C}_{0i}^a + \frac{1}{2\beta}\pi_1^a. \tag{A.16}$$

The unitarizing Hamiltonian given by eqs. (A.8-16) leads, as far as a formal functional integral is concerned, to the following covariant effective action in the configuration space

$$\begin{aligned} S = \int d^4x\Big(&\frac{i}{4}F_{\mu\nu}^a B_{\rho\sigma}^a \varepsilon_{\mu\nu\rho\sigma} - \frac{1}{2}(A_\mu^a)^2 - \frac{\alpha}{2}(\partial_\mu B_{\mu\nu}^a + \partial_\nu C_1'^a)^2 - \\ &- \frac{1}{2}(\partial_\mu \bar{C}_{0\nu}^a - \partial_\nu \bar{C}_{0\mu}^a)(\nabla_\mu^{ab} C_{0\nu}^b - \nabla_\nu^{ab} C_{0\mu}^b) - \beta(\partial_\mu \bar{C}_{0\mu}^a)(\partial_\nu C_{0\nu}^a) - (\partial_\mu \bar{C}_1^a)(\nabla_\mu^{ab} C_1^b) + \\ &+ \frac{1}{2i}(\partial_\mu \bar{C}_{0\nu}^a)(\partial_\rho \bar{C}_{0\sigma}^b) f^{abc} \varepsilon_{\mu\nu\rho\sigma} C_1^c\Big). \end{aligned} \tag{A.17}$$

The time components $B_{0i}^a$, $C_{00}^a$, $\bar{C}_{00}^a$ of the relativistic fields $B_{\mu\nu}^a$, $C_{0\mu}^a$, $\bar{C}_{0\mu}^a$ are identified as: $B_{0i}^a = \lambda_{0i}^a$, $C_{00}^a = \lambda_1^a$, $\bar{C}_{00}^a = \bar{C}_1^{1a}$. We have designated, besides, $C_1'^a = \lambda_1^{1a}$. The time component $A_0^a$ of the field $A_\mu^a$ is the integration variable serving the Gaussian parametrization, which reproduces the second-class constraint $A_0^a = -D^a$, see (A.9), (A.10).

# APPENDIX 2

## PHYSICAL UNITARITY WITHIN THE FUNCTIONAL INTEGRAL FORMALISM IN THE THEORIES WITH DEPENDENT FIRST-CLASS CONSTRAINTS

As it has been said in Section 2 above the proof of physical unitarity for general gauge systems was as a matter of fact given within the operatorial approach in refs. [*13*], [*14*]. Besides, it was shown in Section 5 of the present paper, independently of the result of ref. [*13*], [*14*], that the extracted singlet component of the evolution operator is itself a unitary operators. These results hold true both for the systems with independent first-class constraints and for a more general case when the constraints may be linearly dependent. Therefore, these general results make already a basis for the assertion that e. g. in the model of interacting antisymmetric tensor field, studied briefly in the preceding Appendix, the quartet mechanism of Kugo and Ojima is indeed realized and the physical unitarity holds true.

On the other hand it is not at present conventional, as a rule, to handle gauge field theories directly in the operator formalism; instead, the Hamiltonian functional integral, heuristically constructed at the formal level, is used as a basis for the quantization procedure. Within this approach, the formal reduction of the Hamiltonian path integral to the phase space of independent physical degrees of freedom is referred to as a « proof » of the physical unitarity. Just to illustrate the matter, we shall trace here this procedure for the case of linearly dependent first-class constraints, confining ourselves for simplicity to theories of first-stage reducibility (i. e. to the ones where the null-vectors of the original first-class constraints are linearly independent) without second-class constraints. We start by remembering, once again, the general algorithm for the formal construction of the Hamiltonian path integral in the extended phase space for such theories.

At the classical level the theory is fixed in the original phase space of canonical variables

$$(p_i, q^i), \qquad i = 1, \ldots, n, \tag{A.18}$$

by the Hamiltonian $H_0(p, q)$ and the first-class constraints

$$T_{0\alpha_0}(p, q), \qquad \alpha_0 = 1, \ldots, m_0, \qquad \varepsilon(T_{0\alpha_0}) \equiv \varepsilon_{0\alpha_0}, \tag{A.19}$$

that possess, generally, the linearly-independent null-vectors

$$Z^{\alpha_0}_{1\alpha_1}(p, q), \qquad \alpha_1 = 1, \ldots, m_1 < m_0, \qquad \varepsilon\big(Z^{\alpha_0}_{1\alpha_1}\big) = \varepsilon_{0\alpha_0} + \varepsilon_{1\alpha_1} \tag{A.20}$$

that is

$$T_{0\alpha_0} Z^{\alpha_0}_{1\alpha_1} = 0, \qquad \operatorname{rank} \| Z^{\alpha_0}_{1\alpha_1} \| = m_1, \tag{A.21}$$

the genuine number of independent physical degrees of freedom on the constraint hypersurface $T_0 = 0$ being

$$n^* \equiv n - \operatorname{rank} \left\| \frac{\partial T_{0\alpha_0}}{\partial (p_i, q^i)} \right\| = n - (m_0 - m_1) > 0. \tag{A.22}$$

The extended phase space of the system

$$\Gamma = (P_M, Q^M) \tag{A.23}$$

is spanned by the complete set of canonical momenta

$$P_M \equiv \left(p_i, \pi_{0\alpha_0}, \overline{\mathscr{P}}_{0\alpha_0}, \overline{C}_{0\alpha_0}, \pi_{1\alpha_1}, \overline{\mathscr{P}}_{1\alpha_1}, \overline{C}_{1\alpha_1}, \pi^1_{1\alpha_1}, \overline{C}^1_{1\alpha_1}\right), \tag{A.24}$$

and coordinates

$$Q^M \equiv \left(q^i, \lambda_0^{\alpha_0}, C_0^{\alpha_0}, \mathscr{P}_0^{\alpha_0}, \lambda_1^{\alpha_1}, C_1^{\alpha_1}, \mathscr{P}_1^{\alpha_1}, \lambda_1^{1\alpha_1}, \mathscr{P}_1^{1\alpha_1}\right), \tag{A.25}$$

with the following distribution of statistics

$$\varepsilon(P_M) = \varepsilon(Q^M) = (\varepsilon_i, \varepsilon_{0\alpha_0}, \varepsilon_{0\alpha_0} + 1, \varepsilon_{0\alpha_0} + 1, \varepsilon_{1\alpha_1} + 1, \varepsilon_{1\alpha_1}, \varepsilon_{1\alpha_1}, \varepsilon_{1\alpha_1}, \varepsilon_{1\alpha_1} + 1), \tag{A.26}$$

and the ghost number

$$\mathrm{gh}(P_M) = -\,\mathrm{gh}(Q^M) = -\,(0, 0, 1, 1, 1, 2, 2, 0, 1). \tag{A.27}$$

The unitarizing Hamiltonian of the system is written as

$$H = \mathscr{H} + \{\Psi, \Omega\}. \tag{A.28}$$

Here the functions $\Omega$ and $\mathscr{H}$ are a solution of our standard equations

$$\{\Omega, \Omega\} = 0, \qquad \Omega = \Omega^*, \qquad \varepsilon(\Omega) = 1, \qquad \mathrm{gh}(\Omega) = 1, \tag{A.29}$$

$$\{\mathscr{H}, \Omega\} = 0, \qquad \mathscr{H} = \mathscr{H}^*, \qquad \varepsilon(\mathscr{H}) = 0, \qquad \mathrm{gh}(\mathscr{H}) = 0, \tag{A.30}$$

with the special structure

$$\Omega = \Omega_{\min}(\Gamma_{\min}) + \pi_{0\alpha_0}\mathscr{P}_0^{\alpha_0} + \pi_{1\alpha_1}\mathscr{P}_1^{\alpha_1} + \pi^1_{1\alpha_1}\mathscr{P}_1^{1\alpha_1}, \tag{A.31}$$

$$\mathscr{H} = \mathscr{H}_{\min}(\Gamma_{\min}). \tag{A.32}$$

The functions $\Omega_{\min}$ and $\mathscr{H}_{\min}$ are a solution of the same equations (A.29), (A.30), but only depend on the variables of the so-called minimum sector

$$\Gamma_{\min} \equiv (P_{\min}, Q_{\min}), \tag{A.33}$$

$$P_{\min} \equiv (p_i, \overline{\mathscr{P}}_{0\alpha_0}, \overline{\mathscr{P}}_{1\alpha_1}), \qquad Q_{\min} \equiv (q^i, C_0^{\alpha_0}, C_1^{\alpha_1}). \tag{A.34}$$

Besides, these functions satisfy the conditions

$$\left.\frac{\partial \Omega_{\min}}{\partial C_0^{\alpha_0}}\right|_0 = T_{0\alpha_0}, \qquad \left.\frac{\partial_l \partial_r \Omega_{\min}}{\partial \overline{\mathscr{P}}_{0\alpha_0} \partial C_1^{\alpha_1}}\right|_0 = Z^{\alpha_0}_{1\alpha_1}, \tag{A.35}$$

$$\mathscr{H}_{\min}|_0 = H_0, \tag{A.36}$$

where $|_0$ denotes the substitution of zero values for all the variables from (A.34) that have their ghost number nonzero.

The function $\Psi$ in (A.28), obeying the conditions

$$\varepsilon(\Psi) = 1, \qquad \mathrm{gh}(\Psi) = -1, \qquad \Psi^* = -\Psi, \tag{A.37}$$

is written as

$$\Psi = \overline{\mathscr{P}}_{0\alpha_0}\lambda_0^{\alpha_0} + \overline{C}_{0\alpha_0}\chi_0^{\alpha_0} + \overline{\mathscr{P}}_{1\alpha_1}\lambda_1^{\alpha_1} + \overline{C}_{1\alpha_1}\chi_1^{\alpha_1} + \overline{C}^1_{1\alpha_1}\chi_1^{1\alpha_1} + \overline{\chi}^1_{1\alpha_1}\lambda_1^{1\alpha_1}, \tag{A.38}$$

where $\chi_0$, $\chi_1$, $\chi_1^1$, $\overline{\chi}_1^1$ are gauges imposed on $(p_i, q^i)$, $C_0$, $\lambda_0$, $\overline{C}_0$ respectively:

$$\mathrm{rank}\left\|\frac{\partial_r \chi_0^{\alpha_0}}{\partial(p_i, q^i)}\right\| = \mathrm{rank}\,\|\{\chi_0^{\alpha_0}, T_{0\beta_0}\}\| = m_0 - m_1, \tag{A.39}$$

$$\mathrm{rank}\left\|\frac{\partial_r \chi_1^{\alpha_1}}{\partial C_0^{\alpha_0}} Z^{\alpha_0}_{1\beta_1}\right\| = \mathrm{rank}\left\|\frac{\partial_r \chi_1^{1\alpha_1}}{\partial \lambda_0^{\alpha_0}} Z^{\alpha_0}_{1\beta_1}\right\| = \mathrm{rank}\left\|\overline{Z}^{\alpha_1}_{1\alpha_0}\frac{\partial_l \overline{\chi}^1_{1\beta_1}}{\partial \overline{C}_{0\alpha_0}}\right\| = \mathrm{rank}\,\|\overline{Z}^{\alpha_1}_{1\alpha_0}\| = m_1, \tag{A.40}$$

$$\overline{Z}^{\alpha_1}_{1\alpha_0}\chi_0^{\alpha_0} = 0, \qquad \varepsilon(\overline{Z}^{\alpha_1}_{1\alpha_0}) = \varepsilon_{0\alpha_0} + \varepsilon_{1\alpha_1}. \tag{A.40 a}$$

These conditions of gauge admissibility must be fulfilled on the hypersurface of the complete set of constraints and unitary gauges (see below).

The formal expression for the transition amplitude (the S-matrix) is given as the following Hamiltonian functional integral over paths in the extended phase space

$$Z = \int \exp\left\{\frac{i}{\hbar}\int (P_M\dot{Q}^M - H)dt\right\} d\Gamma, \tag{A.41}$$

$$d\Gamma \equiv \prod_t \prod_M \frac{dP_M dQ^M}{2\pi\hbar}, \tag{A.42}$$

where the unitarizing Hamiltonian H is given by eq. (A.28).

Our next goal is to reduce the path integral (A.41) to the phase space of independent physical degrees of freedom. This object will be, in fact, attained provided that we reduce eq. (A.41) to the standard form, corresponding to independent constraints and unitary gauges, since the latter form is already known to be equivalent to canonical quantization of physical degrees of freedom alone.

Note, first of all, that eq. (A.41) does not depend on the gauge fermion $\Psi$ involved in the r.-h. side of eq. (A.28). To see this, let us make the following transformation in eq. (A.41)

$$\delta P_M = \{ P_M, \Omega \} \mu, \qquad \delta Q^M = \{ Q^M, \Omega \} \mu ; \tag{A.43}$$

$$\mu = \frac{i}{\hbar}\int \delta\Psi dt . \tag{A.44}$$

Then, denoting the r.-h. side of (A.41) by $Z_\Psi$, we have in virtue of (A.29), (A.30):

$$Z_{\Psi+\delta\Psi} = Z_\Psi , \tag{A.45}$$

which means that the integral (A.41) does not indeed depend on $\Psi$, and hence on the choice of any gauge function in the r.-h. side of (A.38).

We may use the fact of the gauge independence (i. e. $\Psi$-independence) of eq. (A.41) to introduce parameters $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_1^1$ into it by the following formal change of the gauge functions in the r.-h. side of (A.38):

$$\chi_0 \to \frac{1}{\varepsilon_0}\chi_0, \qquad \chi_1 \to \frac{1}{\varepsilon_1}\chi_1 , \tag{A.46}$$

$$\chi_1^1 \to \frac{1}{\varepsilon_1^1}\chi_1^1, \qquad \bar{\chi}_1^1 \to \frac{1}{\varepsilon_0}\bar{\chi}_1^1 . \tag{A.47}$$

Owing to (A.45), expression (A.41) remains independent of the parameters $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_1^1$ after the formal change (A.46), (A.47) is made.

Let us fulfil now the unimodular transformation of the integration variables in (A.41)

$$\pi_0 \to \varepsilon_0\pi_0, \qquad \bar{C}_0 \to \varepsilon_0\bar{C}_0 , \tag{A.48}$$

$$\pi_1 \to \varepsilon_1\pi_1, \qquad \bar{C}_1 \to \varepsilon_1\bar{C}_1 , \tag{A.49}$$

$$\pi_1^1 \to \varepsilon_1^1\pi_1^1, \qquad \bar{C}_1^1 \to \varepsilon_1^1\bar{C}_1^1 . \tag{A.50}$$

After the change (A.46), (A.47) and the transformation (A.48-50) the limiting transition

$$\varepsilon_0 \to 0, \qquad \varepsilon_1 \to 0, \qquad \varepsilon_1^1 \to 0, \tag{A.51}$$

leads to the following form of eq. (A.41), corresponding to unitary gauges

$$Z = \int \exp \left\{ \frac{i}{\hbar} \int dt \left( p_i \dot{q}^i - H_0 - T_{0\alpha_0} \lambda_0^{\alpha_0} - \pi_{0\alpha_0} \chi_0^{\alpha_0} - \overline{C}_{0\alpha_0} \{ \chi_0^{\alpha_0}, T_{0\beta_0} \} C_0^{\beta_0} - \right. \right.$$

$$- \pi_{1\alpha_1}^1 \chi_1^{1\alpha_1} + \overline{C}_{1\alpha_1}^1 \frac{\partial_r \chi_1^{1\alpha_1}}{\partial \lambda_0^{\alpha_0}} Z_{1\beta_1}^{\alpha_0} \lambda_1^{\beta_1} - \pi_{1\alpha_1} \chi_1^{\alpha_1} - \overline{C}_{1\alpha_1} \frac{\partial_r \chi_1^{\alpha_1}}{\partial C_0^{\alpha_0}} Z_{1\beta_1}^{\alpha_0} C_1^{\beta_1} -$$

$$\left. \left. - \overline{\chi}_{1\alpha_1}^1 \mathscr{P}_1^{1\alpha_1} - \pi_{0\alpha_0} \frac{\partial_l \overline{\chi}_{1\alpha_1}^1}{\partial \overline{C}_{0\alpha_0}} \lambda_1^{1\alpha_1} \right) \right\} d\Gamma' , \tag{A.52}$$

$$d\Gamma' \equiv (\text{const}) \prod_t dp dq d\pi_0 d\lambda_0 dC_0 d\overline{C}_0 d\pi_1 d\lambda_1 \times dC_1 d\overline{C}_1 d\pi_1^1 d\overline{C}_1^1 d\mathscr{P}_1^1 . \tag{A.53}$$

Here we restrict ourselves to the following class of gauges, which will be sufficient for our purposes

$$\chi_0 = \chi_0(p, q), \qquad \chi_1 = \chi_1(C_0), \tag{A.54}$$

$$\chi_1^1 = \chi_1^1(\lambda_0), \qquad \overline{\chi}_1^1 = \overline{\chi}_1^1(\overline{C}_0). \tag{A.55}$$

This class is a minimum in the sense of fulfilling the conditions of gauge admissibility (A.39), (A.40). In this class of gauges functions $\chi_1(C_0)$ and $\overline{\chi}_1^1(\overline{C}_0)$ prove to be automatically linear, due to (A.37). We choose functions $\chi_1^1(\lambda_0)$ to be also linear and, moreover, to obey the conditions

$$\frac{\partial_r \chi_1^{\alpha_1}}{\partial C_0^{\alpha_0}} = \frac{\partial_r \chi_1^{1\alpha_1}}{\partial \lambda_0^{\alpha_0}} . \tag{A.56}$$

The action in (A.52) only contains kinetic terms for canonical variables of the original phase space. All the other variables in (A.52) are dynamically passive. In this sense, the r.-h. side of (A.52) corresponds to effective linearly dependent second-class constraints $(T_{0\alpha_0}, \chi_0^{\alpha_0})$. It remains to reduce eq. (A.52) to the form which corresponds to linearly independent constraints and gauges. In this way we shall come to the independent effective second-class constraints.

Denoting the linear-independent constraints contained in the complete set of original constraints (A.19) as

$$T_\alpha(p, q), \qquad \alpha = 1, \ldots, (m_0 - m_1) \tag{A.57}$$

we have

$$T_{0\alpha_0} = T_\alpha P_{\alpha_0}^\alpha, \qquad T_\alpha = T_{0\alpha_0} \overline{P}_\alpha^{\alpha_0}, \tag{A.58}$$

$$P_{\alpha_0}^\alpha \overline{P}_\beta^{\alpha_0} = \delta_\beta^\alpha, \qquad P_{\alpha_0}^\alpha Z_{1\alpha_1}^{\alpha_0} = 0 . \tag{A.59}$$

In the same way, denote the linearly independent gauges contained in the complete set of functions $\chi_0^{\alpha_0}$ as

$$\chi^\alpha(p, q), \qquad \alpha = 1, \ldots, (m_0 - m_1) . \tag{A.60}$$

They are subject to the property

$$\text{rank} \| \{ \chi^\alpha, T_\beta \} \| = m_0 - m_1 . \tag{A.61}$$

Then we have

$$\chi_0^{\alpha_0} = \Pi_\alpha^{\alpha_0} \chi^\alpha, \qquad \chi^\alpha = \overline{\Pi}_{\alpha_0}^\alpha \chi_0^{\alpha_0} , \tag{A.62}$$

$$\overline{\Pi}_{\alpha_0}^\alpha \Pi_\beta^{\alpha_0} = \delta_\beta^\alpha, \qquad \overline{Z}_{1\alpha_0}^{\alpha_1} \Pi_\alpha^{\alpha_0} = 0 , \tag{A.63}$$

The natural arbitrariness contained in the definition of the functions $\overline{P}_\alpha^{\alpha_0}$, $\overline{\Pi}_{\alpha_0}^\alpha$ allows one to impose the conditions

$$\frac{\partial_r \chi_1^{\alpha_1}}{\partial \lambda_0^{\alpha_0}} \overline{P}_\alpha^{\alpha_0} = 0, \qquad \overline{\Pi}_{\alpha_0}^\alpha \frac{\partial_l \overline{\chi}_{1\alpha_1}^1}{\partial \overline{C}_{0\alpha_0}} = 0 . \tag{A.64}$$

Let us introduce now the new integration variables in

$$\lambda_0^{\alpha_0} = Z_{1\alpha_1}^{\alpha_0}\varphi^{\alpha_1} + \overline{P}_\alpha^{\alpha_0}\lambda^\alpha, \qquad (A.65)$$

$$C_0^{\alpha_0} = Z_{1\alpha_1}^{\alpha_0}F^{\alpha_1} + \overline{P}_\alpha^{\alpha_0}C^\alpha, \qquad (A.66)$$

$$\pi_{0\alpha_0} = \psi_{\alpha_1}\overline{Z}_{1\alpha_1}^{\alpha_1} + \pi_\alpha\overline{\Pi}_{\alpha_0}^\alpha, \qquad (A.67)$$

$$\overline{C}_{0\alpha_0} = \overline{F}_{\alpha_1}\overline{Z}_{1\alpha_0}^{\alpha_1} + \overline{C}_\alpha\overline{\Pi}_{\alpha_0}^\alpha, \qquad (A.68)$$

with the property

$$d\lambda_0 dC_0 = d\varphi d\lambda dF dC, \qquad d\pi_0 d\overline{C}_0 = d\psi d\pi d\overline{F} dC. \qquad (A.70)$$

Then we get

$$Z = (\text{const}) \int \exp\left\{\frac{i}{\hbar}\int dt(p_i\dot{q}^i - H_0 - T_\alpha\lambda^\alpha - \pi_\alpha\chi^\alpha - \overline{C}_\alpha\{\chi^\alpha, T_\beta\}C^\beta)\right\} \prod_t dp dq d\pi d\lambda d\overline{C} dC \qquad (A.70)$$

The expression in the r.-h. side of (A.70) just corresponds to the case of linearly independent constraints $T_\alpha$ and unitary gauges $\chi^\alpha$. Using the collective notations

$$\mathcal{T}_A \equiv (T_\alpha, \chi^\alpha) \qquad (A.71)$$

for the $2(m_0 - m_1)$ effective linearly independent second-class constraints, we can write the path integral (A.70) in a more conventional form

$$Z = (\text{const}) \int \exp\left\{\frac{i}{\hbar}\int dt(p_i\dot{q}^i - H_0)\right\} \times \prod_t (\delta(\mathcal{T})(\text{Ber} \,\|\, \{\mathcal{T}, \mathcal{T}\} \,\|)^{1/2} dp dq). \qquad (A.72)$$

As for the latter expression, it is well known to allow an equivalent representation in the form

$$Z = (\text{const}) \int \exp\left\{\frac{i}{\hbar}\int dt(p^*\dot{q}^* - H^*)\right\} \prod_t dp^* dq^*, \qquad (A.73)$$

where $(p^*, q^*)$ are $2n^* = 2(n - (m_0 - m_1))$ canonical variables that correspond to $n^*$ independent physical degrees of freedom on the hypersurface of constraints $\mathcal{T} = 0$, while $H^*$ designates the physical Hamiltonian

$$H^* \equiv H^*(p^*, q^*) = H_0(p, q)|_{\mathcal{T}=0}. \qquad (A.74)$$

We summarize. Having started with the Hamiltonian functional integral (A.41) over paths in the extended phase space (A.23-25), we came finally to the Hamiltonian integral (A.72) or (A.73) over paths in the physical phase space as if the physical degrees of freedom $(p^*, q^*)$ governed by the physical Hamiltonian (A.74) were alone subjected canonical quantization. This situation is usually claimed to contain a proof, or at least a strong evidence in favour of the physical unitarity. A more consistent treatment requires, certainly, an appeal to the operator formalism, as it was clarified above.

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