ANNALES DE L'I. H. P., SECTION A

SCHÔICHI ÔTA

Unbounded representations of a $*$-algebra on indefinite metric space

Annales de l'I. H. P., section A, tome 48, nº 4 (1988), p. 333-353
<http://www.numdam.org/item?id=AIHPA_1988__48_4_333_0>

Ann. Inst. Henri Poincaré,
Vol. 48, n° 4, 1988, p. 333-353. *Physique théorique*

Unbounded representations of a $*$-algebra on indefinite metric space

by

Schôichi ÔTA
General Educations, Mathematics,
Kyushu Institute of Design,, Shiobaru, Fukuoka, 815 Japan

ABSTRACT. — We study unbounded representations of a $*$-algebra on an indefinite inner product space, the so-called unbounded J-representations and, at the same time, we also discuss the similarity between a non $*$-representation and a $*$-representation. In particular, we characterize the similarity of a J-representation to a self-adjoint $*$-representation in terms of invariant dual pairs.

RÉSUMÉ. — Nous étudions des représentations non bornées d'une algèbre involutive dans un espace muni d'un produit intérieur indéfini (J-représentations non bornées), et en même temps nous discutons de la similarité entre une représentation non involutive et une représentation involutive. En particulier, nous caractérisons en termes des « paires duales invariantes » une J-représentation similaire à une représentation auto-adjoint.

1. INTRODUCTION

Unbounded representations of a $*$-algebra on Hilbert space or algebras of unbounded linear operators on Hilbert space have recently been investigated as one of the most fascinating topics, in connection with quantum field theory and the representations of the enveloping algebra of a Lie algebra, in particular one associated with the Heisenberg commutation relations [*3*], [*4*], [*14*], [*20*], [*21*], [*23*] and [*24*].

If one feels it of interest, from a mathematical point of view, to develop the general theory of unbounded representations of a $*$-algebra, one cannot help facing up to a class of representations which are not $*$-hermitian, in other words, not $*$-representations. For example, the adjoint of a $*$-representation is in general not $*$-hermitian.

Most of our work will be done in the framework of an involution-preserving representation on an indefinite inner product space (a J-space), the so-called J-representation, where the involution means the adjoint operation with respect to the indefinite inner product. Given a non-$*$-hermitian representation of a $*$-algebra on a Hilbert space, one can easily construct a J-representation on a J-space induced by the representation and a J-representation may be considered, in a sense, as a typical example of a non-$*$-hermitian representation. Moreover, interesting results have appeared, concerning an approach to quantum theory of gauge fields with indefinite metric formulations, by making use of J-representations. For the physical aspects related to them, we refer to [*5*], [*11*]-[*13*], [*25*]-[26] and references stated in them. This motivates us to study representations of a $*$-algebra on J-space.

In Section 2, we state some definitions and facts concerning unbounded representations on a J-space. In Section 3, first we consider under what conditions the weak commutant of a (in general, non-$*$-hermitian) representation is an algebra and then study the relation between similarity of two representations and their weak commutants. These considerations yield some consequences related to self-adjointness of a $*$-representation and in particular we show in Theorem 3.9 that a $*$-representation similar to its adjoint representation is self-adjoint. Though, as we will see in Theorem 3.7, a $*$-representation similar to a self-adjoint $*$-representation is also self-adjoint, we construct a non-$*$-hermitian representation (Example 3.11) that is similar to a self-adjoint $*$-representation.

Section 4 is devoted to the study of a J-representation of a $*$-algebra similar to a $*$-representation. We show that one of the sufficient conditions under which a J-representation is similar to a $*$-representation is the existence of an invariant dual pair with some property. On the other hand, apart from bounded representations like C*-algebra-representations [*8*], there is a typical example of a cyclic J-representation of a $*$-algebra which is not similar to any $*$-representation (Example 3.13).

The concept of self-adjointness in the context of $*$-representations is very useful, as proposed in references cited above, to analyze $*$-representations and to classify them. We conclude Section 4, from this point of view, by giving our main theorem (Theorem 4.4) which provides a characterization for a J-representation of a $*$-algebra to be similar to a self-adjoint $*$-representation of the algebra, in terms of its invariant dual pair and J-self-adjointness, which is the J-analogue of ordinary self-adjointness.

2. PRELIMINARIES

We begin with introducing some notions and results on unbounded representations of $*$-algebras on Hilbert space, and on indefinite inner product spaces.

Throughout this paper, we will deal only with representations of a $*$-algebra which has a unit denoted by I.

Let A be a $*$-algebra. A (closable) *representation* π of A on a Hilbert space H is a mapping of A into all closable linear operators defined on a common subspace $D(\pi)$, which is dense in H, satisfying

(1) $\pi(I) = \mathbb{1}$ (the identity operator on H);

(2) $\pi(\alpha x + y)\xi = \alpha\pi(x)\xi + \pi(y)\xi$

for all $x, y \in A$, $\xi \in D(\pi)$ and complex numbers α; and

(3) $D(\pi)$ is globally invariant under each $\pi(x)$ and

$$\pi(x)\pi(y)\xi = \pi(xy)\xi$$

for all $x, y \in A$ and $\xi \in D(\pi)$.

$D(\pi)$ is called the *domain* of π.

Let π be a representation of A with domain $D(\pi)$. The induced topology on $D(\pi)$ is the locally convex topology generated by the family of semi-norms $\{ \| \pi(x)\xi \| : x \in A \}$. If $D(\pi)$ is complete with respect to the induced topology, then π is called *closed*.

For a closable operator T in H, we write $\overline{T}$ for the minimal closed extension of T.

We now define two representations derived from a representation π of A on a Hilbert space H with domain $D(\pi)$ as follows: Let $D(\tilde{\pi})$ be the completion of $D(\pi)$ with respect to the induced topology; that is, $D(\tilde{\pi})$ consists of the vectors ξ such that there is a net $\{ \xi_\alpha \}$ in $D(\pi)$ satisfying $\xi_\alpha \to \xi$ and $\pi(x)\xi_\alpha \to \overline{\pi(x)}\xi$ for all $x \in A$, and put

$$\tilde{\pi}(x)\xi = \overline{\pi(x)}\xi$$

for all $x \in A$ and $\xi \in D(\tilde{\pi})$, and

$$D(\pi^*) = \bigcap_{x \in A} D(\pi(x)^*)$$

$$\pi^*(x)\xi = \pi(x^*)^*\xi$$

for all $x \in A$ and $\xi \in D(\pi^*)$. Then $\tilde{\pi}$ is a closed representation of A on H, which is called the *closure* of π, and π^* is also a closed representation of A on Hilbert space $\overline{D(\pi^*)}$, which is called the $*$-*adjoint* of π.

A representation π of A with domain $D(\pi)$ is said to be (*)-*hermitian*, or sometimes to be a *-*representation* if it satisfies

$$(\pi(x)\xi, \eta) = (\xi, \pi(x^*)\eta)$$

for all $x \in A$ and $\xi, \eta \in D(\pi)$.

Let π be a *-representation of A on a Hilbert space H with domain $D(\pi)$. It follows that the closed representation $\tilde{\pi}$ is hermitian satisfying

$$D(\tilde{\pi}) = \bigcap_{x \in A} D(\overline{\pi(x)})$$

and is a minimal closed extension of π. Furthermore, π^{**} $(= (\pi^*)^*)$ is also a *-representation satisfying the relations

$$\pi \subset \tilde{\pi} \subset \pi^{**} \subset \pi^* .$$

Here, for two representations π and ρ, the relation $\pi \supset \rho$ means that $D(\pi) \supset D(\rho)$ and $\pi(x)\xi = \rho(x)\xi$ for all $x \in A$ and $\xi \in D(\rho)$. We remark that π^* may fail to be hermitian even if π is a *-representation [*21*]. In the above situation, $\pi(A)$ is an O_p^*-algebra on $D(\pi)$ in the sense of [*14*]. For more details and results on these subjects, we refer to [*7*], [*14*], [*20*], [*21*], [*24*] and references stated in them.

We next introduce the concept of indefinite inner product space and give a typical example of an unbounded representation of a *-algebra on such a space, which is naturally induced by a (non-hermitian) representation.

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and its norm $\|\cdot\|$, and let $[\cdot,\cdot]$ be a sesquilinear form on H.

DEFINITION 2.1. — A sesquilinear form $[\cdot,\cdot]$ is said to be a *Minkowsky form* on H if it satisfies

$$[\xi, \eta] = \overline{[\eta, \xi]}$$

for $\xi, \eta \in H$, and

$$\| \xi \| = \sup \{ | [\xi, \eta] | : \| \eta \| \leqq 1 \}$$

for all $\xi \in H$.

We can easily prove that a sesquilinear form $[\cdot,\cdot]$ is a Minkowsky form if and only if there exists uniquely a self-adjoint unitary (symmetry) J on H (i. e., $J = J^* = J^{-1}$) satisfying

$$[\xi, \eta] = (J\xi, \eta)$$

for all $\xi, \eta \in H$. In what follows we denote by $[\cdot,\cdot]_J$ this Minkowsky form defined by J, and such a Hilbert space H equipped with Minkowsky form $[\cdot,\cdot]_J$ is said to be a J-*space* or a *Krein space*, which is denoted by $\{ H, J \}$. Any topological concepts in J-spaces are always defined by the Hilbert space norm.

Let T be a densely defined linear operator in a J-space $\{ H, J \}$. We can define the adjoint operator (the J-adjoint) T^J of T with respect to the Minkowsky form $[.,.]_J$ in the obvious manner. Then, T^J is also a linear operator, determined by the equation $[T\xi, \eta]_J = [\xi, T^J\eta]_J$ for $\xi \in D(T)$ and $\eta \in D(T^J)$. In particular, $T^J = JT^*J$.

For further details on this notion we refer to [*1*], [*2*] and [*10*].

DEFINITION 2.2. — Let π be a representation of a $*$-algebra A on a Hilbert space H with domain $D(\pi)$. If there exists a Minkowsky form on H defined by some J such that

$$[\pi(x)\xi, \eta]_J = [\xi, \pi(x^*)\eta]_J$$

for all $x \in A$ and $\xi, \eta \in D(\pi)$, then π is said to be J-*hermitian*, or sometimes to be a J-*representation* on a J-space $\{H, J\}$.

Let π be a J-representation of a $*$-algebra A on a J-space $\{H, J\}$. By the analogous way to in the case of $*$-representations, we define the J-adjoint of π as follows:

$$D(\pi^J) = \bigcap_{x \in A} D(\pi(x)^J)$$

and

$$\pi^J(x) = \pi(x^*)^J|_{D(\pi^J)} \text{ (the restriction of } \pi(x^*)^J \text{ to } D(\pi^J))$$

for all $x \in A$. Then we can easily prove by the same way as in [*7*], Theorem 2, (4) and [*20*], Lemma 4.1 that π^J is a closed representation of A on H and $\tilde{\pi}$ and π^{JJ} are J-representations satisfying $\pi \subset \tilde{\pi} \subset \pi^{JJ} \subset \pi^J$. It is, moreover, easily seen that $\pi^{JJ} = \pi^{**}$.

DEFINITION 2.3. — A J-representation of a *-algebra A on a J-space $\{H, J\}$ is said to be J-*self-adjoint* if $\pi^J = \pi$; that is, $D(\pi^J) = D(\pi)$.

This notion is closely related to the commutant of π, which is discussed in Sections 3 and 4. We remark that, in the case that the Minkowsky form defined by J is definite ($J = \pm \mathbb{1}$), the concept of J-self-adjointness coincides with that of ($*$-) *self-adjointness* of $*$-representations in the sense of Powers [*20*].

Let π be a representation of a $*$-algebra A on a Hilbert space H with domain $D(\pi)$. Suppose the domain $D(\pi^*)$ of the $*$-adjoint of π is dense in H. We remark that this condition is equivalent that π is *adjointable* in the sense of [*7*], p. 372, and also that this condition holds for every J-representation by the relation $JD(\pi) \subset D(\pi^*)$. If this is the case, $\pi \subset \tilde{\pi} \subset \pi^{**}$ and $\pi^{***} = \pi^*$.

We now show that π induces a J-representation of A on some J-space. We take

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

on the Hilbert space $H \oplus H$ (two fold copies of H). Then $H \oplus H$ is a J-space with Minkowsky form $[\xi \oplus \eta, \xi' \oplus \eta']_J = (\xi, \eta') + (\eta, \xi')$ (ξ, ξ', η

and $\eta' \in H$). For each $x \in A$, we define a linear operator $\breve{\pi}(x)$ on dense domain $D(\pi) \oplus D(\pi^*)$ by

$$\breve{\pi}(x)(\xi \oplus \eta) = \begin{pmatrix} \pi(x) & 0 \\ 0 & \pi^*(x) \end{pmatrix}(\xi \oplus \eta)$$

for all $\xi \in D(\pi)$ and $\eta \in D(\pi^*)$. Since π^* is a representation of A on H, $\breve{\pi}$ is a representation of A satisfying

$$[\breve{\pi}(x)f, g]_J = [f, \breve{\pi}(x^*)g]_J$$

for all $x \in A$ and $f, g \in D(\pi) \oplus D(\pi^*)$. Thus $\breve{\pi}$ is a J-representation of A on the J-space $\left\{ H \oplus H, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\}$.

We call $\breve{\pi}$ the *J-representation induced by* π.

LEMMA 2.4. — *Let* π *be a representation of a* $*$*-algebra* A *on a Hilbert space* H *with domain* $D(\pi)$. *Suppose* $D(\pi^*)$ *is dense in* H. *Then the following statements hold:*

1. $\breve{\pi}$ *is closed if and only if so is* π;
2. $\breve{\pi}$ *is* J*-self-adjoint if and only if* $\pi = \pi^{**}$;
3. The adjoint $(\breve{\pi})^J$ *of* $\breve{\pi}$ *is* J*-hermitian with* $(\breve{\pi})^{JJ} = (\breve{\pi})^J$; *and*
4. The J*-representation* $\breve{\pi}^*$ *induced by* π^* *is always* J*-self-adjoint.*

Proof. — The first statement follows from the fact that π^* is always closed and π is unitarily equivalent to the restriction of $\breve{\pi}$ to $D(\pi) \oplus \{0\}$.

It is clear that

$$(\breve{\pi})^J = \begin{pmatrix} \pi^{**} & 0 \\ 0 & \pi^* \end{pmatrix},$$

so that the second statement follows. The rest follows from the relation $\pi^{***} = \pi^*$.

3. COMMUTANT AND SIMILARITY

We first recall the weak and strong commutants of the algebra of unbounded operators on Hilbert space.

Let A be a $*$-algebra and let π be a representation of A on a Hilbert space H with domain $D(\pi)$. We denote by B(H) the algebra of all bounded linear operators on H.

DEFINITION 3.1. — The *weak commutant* $C^w(\pi(A))$ of the algebra $\pi(A)$ is defined as

$$C^w(\pi(A)) = \{T \in B(H) : (T\pi(x)\xi, \eta) = (T\xi, \pi(x^*)\eta) \text{ for all } x \in A \text{ and } \xi, \eta \in D(\pi)\},$$

and the *strong commutant* $C^s(\pi(A))$ of $\pi(A)$ is defined as

$C^s(\pi(A)) = \{ T \in B(H) : T\pi(x) \subset \pi(x)T$ for all $x \in A$; that is, T leaves $D(\pi)$ globally invariant and $T\pi(x)\xi = \pi(x)T\xi$ for $x \in A$ and $\xi \in D(\pi) \}$.

Let T be in B(H). Then it is easy to see, by the similar argument to [*20*], Lemma 4.5, that T belongs to $C^w(\pi(A))$ if and only if T maps $D(\pi)$ into $D(\pi^*)$ and satisfies

$$T\pi(x)\xi = \pi^*(x)T\xi$$

for all $x \in A$ and $\xi \in D(\pi)$.

The weak commutant $C^w(\pi(A))$ is a weakly closed set of B(H) and is closed under the adjoint operation with respect to the usual inner product, i. e., $T \to T^*$ ($T \in B(H)$), but nevertheless it need not be an algebra even if π is $*$-hermitian, [*20*], p. 91.

We remark that $C^w(\pi(A))$ contains $C^s(\pi(A))$ if π is hermitian i. e., a $*$-representation. We also note that the strong commutant $C^s(\pi(A))$ is a Banach subalgebra of B(H) if π is closed.

The sufficient conditions for the weak commutant of a $*$-representation to be an algebra (i. e., a von Neumann algebra) are well-known as, for example, the terms of self-adjointness [*20*], Lemma 4.6 (see for further details, [*4*], [*7*] and [*20*]). In the case of general representations we have the following.

PROPOSITION 3.2. — *Let π be a representation of a $*$-algebra* A *on a Hilbert space* H. *Suppose the adjoint representation π^* of π is hermitian i. e., a $*$-representation of* A *on* H. *Then the weak commutant* $C^w(\pi(A))$ *is a* W*-*algebra on* H *(a weakly closed, $*$-subalgebra of* B(H)*)*.

Proof. — Let T_1, T_2 be in $C^w(\pi(A))$. For all $x \in A$ and $\xi, \eta \in D(\pi)$, we have

$$\begin{aligned}(T_1T_2\pi(x)\xi, \eta) &= (T_2\pi(x)\xi, T_1^*\eta)\\ &= (\pi^*(x)T_2\xi, T_1^*\eta).\end{aligned}$$

Since T_1^* belongs to $C^w(\pi(A))$, both $T_1^*\eta$ and $T_2\xi$ belong to $D(\pi^*)$. Since π^* is hermitian, it follows that

$$\begin{aligned}(T_1T_2\pi(x)\xi, \eta) &= (T_2\xi, \pi^*(x^*)T_1^*\eta)\\ &= (T_2\xi, T_1^*\pi(x^*)\eta)\end{aligned}$$

for all $x \in A$ and $\xi, \eta \in D(\pi)$. Thus the product $T_1 \cdot T_2$ belongs to $C^w(\pi(A))$. This completes the proof.

COROLLARY 3.3. — *Let π be a $*$-representation of a $*$-algebra* A *on a Hilbert space* H. *If* $C^w(\pi^*(A))$ *is non-degenerate, then* $C^w(\pi(A))$ *is a von Neumann algebra acting on* H *with $\pi^* = \pi^{**}$.*

Proof. — Suppose $C^w(\pi^*(A))$ is non-degenerate. Since π^{**} is hermitian, $C^w(\pi^*(A))$ is a W*-algebra on H by proposition 3.2. It follows from the double commutation theorem of J. von Neumann that $C^w(\pi^*(A))$ contains the identity operator on H. This means that π^* is hermitian with $\pi^* = \pi^{**}$. The corollary follows from Proposition 3.2.

COROLLARY 3.4. — *Keep the same assumption as in Proposition 3.2. If the representation π is not $*$-hermitian, then there is no Minkowsky form (no symmetry J) on H which makes π J-hermitian.*

Proof. — Suppose there is a symmetry J on H such that π is a J-representation of A on the J-space $\{ H, J \}$. Since the equality in Definition 2.2 means that J belongs to $C^w(\pi(A))$, it follows from Proposition 3.2 that

$$\mathbb{1} = J \cdot J \in C^w(\pi(A)) .$$

This shows that π is a $*$-representation on H.

We now give an example of a non-$*$-hermitian representation satisfying the condition of Proposition 3.2, and see that it is not J-hermitian by Corollary 3.4.

Let A be the free commutative $*$-algebra on one hermitian generator S. Define the representation π of A on the Hilbert space $L^2[0, 1]$ in the obvious manner by

$$D(\pi) = \{ f \in C^\infty [0, 1] : f^{(n)}(0) = f^{(n)}(1) = 0 \quad n = 0, 1, 2, \dots \}$$

$$\pi(S) = -i\frac{d}{dt} \text{ (the usual differentiation)}\Big|_{D(\pi)} .$$

Then π is a $*$-representation of A on $L^2[0,1]$ and $\pi = \tilde{\pi} = \pi^{**} \subsetneqq \pi^*$ by [7] (see the comparable examples 3.11 and 3.13).

Take $\rho = \pi^*$ with domain $D(\rho) = C^\infty[0,1]$. It follows that the repretation ρ is not hermitian (even not J-hermitian) and ρ^* is a $*$-representation.

DEFINITION 3.5 (Similarity). — Let π and ρ be representations of a $*$-algebra A on Hilbert spaces H_π and H_ρ, respectively. If there exists a continuous linear transformation (an intertwining operator) T from H_π onto H_ρ with bounded inverse T^{-1} such that

$$TD(\pi) = D(\rho)$$

and

$$\rho(x)T\xi = T\pi(x)\xi$$

for all $x \in A$ and $\xi \in D(\pi)$, then we say that π is *similar* to ρ, which is denoted by $\pi \sim \rho$, and that T realizes the similarity $\pi \sim \rho$.

It is easily seen that the relation $\sim$ is an equivalence relation.

We give some elementary consequences of Definition 3.5.

LEMMA 3.6. — *Let π and ρ be representations of a $*$-algebra A on Hilbert spaces H_π and H_ρ, respectively. Assume that π is similar to ρ with intertwining operator T. Then the following statements hold:*

1. The adjoint T^* *of* T *realizes the similarity* $\rho^* \sim \pi^*$ *of the* $*$*-adjoint representations* ρ^* *and* π^**;*

2. The closure $\tilde{\pi}$ *is also similar to the closure* $\tilde{\rho}$ *with the intertwining operator* T*; and*

3. If π *is closed, then so is* ρ.

Proof. — We write $(\cdot,\cdot)_\pi$ for the inner product of H_π. To show the statement 1, take $f \in D(\rho^*)$. Since we have

$$(T^*f, \pi(x)\xi)_\pi = (f, T\pi(x)\xi)_\rho = (\rho(x)^*f, T\xi)_\rho$$

for all $x \in A$, it follows that T^*f belongs to $\bigcap_{x\in A} D(\pi(x)^*) = D(\pi^*)$ and $\pi(x)^*T^*f = T^*\rho(x)^*f$. Applying the above argument to T^{-1}, we get $(T^{-1})^*D(\pi^*) \subset D(\rho^*)$. Hence T^* realizes the similarity $\rho^* \sim \pi^*$.

Let ξ be in $D(\tilde{\pi})$. There exists a net $\{\xi_\alpha\}$ in $D(\pi)$ satisfying $\xi_\alpha \to \xi$ and $\pi(x)\xi_\alpha \to \overline{\pi(x)}\xi$, for all $x \in A$. Since $\rho(x)T\xi_n = T\pi(x)\xi_n \to T\overline{\pi(x)}\xi$, it follows that $T\xi \in D(\overline{\rho(x)})$ and $\overline{\rho(x)}T\xi = T\overline{\rho(x)}\xi$ for all $x \in A$. This shows that $T\xi \in D(\tilde{\rho})$ and $\tilde{\rho}(x)T\xi = T\tilde{\pi}(x)\xi$ for all $x \in A$, which implies the statement 2.

The statement 3 is clear from the statement 2. This completes the proof.

THEOREM 3.7. — *Let* π *and* ρ *be* $*$*-representations of a* $*$*-algebra* A *on Hilbert spaces* H_π *and* H_ρ*, respectively. Suppose* π *is similar to* ρ*. If* π *is self-adjoint, then* ρ *is also self-adjoint. In this case,* π *and* ρ *are mutually unitarily equivalent.*

Proof. — We first show that π and ρ are mutually unitarily equivalent. Suppose T realizes the similarity $\pi \sim \rho$. Since π is self-adjoint and T^* gives the similarity $\rho^* \sim \pi^* = \pi$ by Lemma 3.6, it follows that

$$\begin{aligned} T^*T\pi(x)\xi &= T^*\rho(x)T\xi \\ &= T^*\rho^*(x)T\xi \quad (\text{by } \rho^* \supset \rho) \\ &= \pi^*(x)T^*T\xi \\ &= \pi(x)T^*T\xi \quad (\text{by } \pi^* = \pi) \end{aligned}$$

for all $x \in A$ and $\xi \in D(\pi)$. Thus T^*T belongs to the strong commutant $C^s(\pi(A))$. Let $T = v\,|\,T\,|$ be the polar decomposition of T. It is clear that v is an isometry of H_π onto H_ρ by the boundedly invertibleness of T. Since π is self-adjoint, it follows from [*20*], Lemma 4.6 that $C^s(\pi(A), = C^w(\pi(A))$ is a von Neumann algebra on H_π. Hence $|\,T\,|$ and $|\,T\,|^{-1}$ belong to $C^s(\pi(A))$. It follows that $v = T\,|\,T\,|^{-1}$ maps $D(\pi)$ onto $D(\rho)$ and

$$\begin{aligned} v\pi(x)\xi &= T\pi(x)\,|\,T\,|^{-1}\xi \\ &= \rho(x)T\,|\,T\,|^{-1}\xi \\ &= \rho(x)v\xi \end{aligned}$$

for all $x \in A$ and $\xi \in D(\pi)$. Thus v gives the unitarily equivalence between π and ρ. It follows from Proposition 3.6 that v^* realizes the similarity $\rho^* \sim \pi^* = \pi$ and, in particular, $v^*D(\rho^*) = D(\pi)$. Noticing that v^* is the inverse of v, we get

$$vD(\pi) = D(\rho^*).$$

Since $vD(\pi) = D(\rho)$, it follows that $D(\rho) = D(\rho^*)$. This completes the proof of the theorem.

We next characterize a representation similar to some $*$-representation, in terms of the weak commutant.

LEMMA 3.8. — *Let π be a representation of a $*$-algebra A on a Hilbert space H_π. The following statements are equivalent:*

1. π is similar to some $*$-representation of A on a Hilbert space.

2. There exists a positive operator on H_π with bounded inverse, which belongs to the weak commutant of π.

Proof. — Suppose π is similar to a $*$-representation ρ of A on H_ρ. Let T be the intertwining operator between π and ρ. We show that T^*T satisfies the desired property. For $\xi, \eta \in D(\pi)$, we have

$$\begin{aligned}(T^*T\pi(x)\xi, \eta)_\pi &= (\rho(x)T\xi, T\eta)_\rho \\ &= (T\xi, \rho(x^*)T\eta)_\rho \\ &= (T\xi, T\pi(x^*)\eta)_\rho\end{aligned}$$

for all $x \in A$. Thus $T^*T \in C^w(\pi(A))$.

Conversely, let S be a boundedly invertible, positive operator on H_π which belongs to $C^w(\pi(A))$. Put

$$\rho(x)\xi = S^{\frac{1}{2}}\pi(x)S^{-\frac{1}{2}}\xi$$

for all $\xi \in S^{\frac{1}{2}}D(\pi)$. It then follows that ρ is a representation of A with domain $D(\rho) = S^{\frac{1}{2}}D(\pi)$.

To show that ρ is hermitian, take $\xi = S^{\frac{1}{2}}\xi_1$ and $\eta = S^{\frac{1}{2}}\eta_1(\xi_1, \eta_1 \in D(\pi))$. It follows that

$$\begin{aligned}(\xi, \rho(x^*)\eta)_\rho &= (S\xi_1, \pi(x^*)\eta_1)_\pi \\ &= (S\pi(x)\xi_1, \eta_1)_\pi \\ &= (\rho(x)\xi, \eta)_\rho\end{aligned}$$

for all $x \in A$. This completes the proof.

THEOREM 3.9. — *Let ρ be a $*$-representation of a $*$-algebra A on a Hilbert space H. If ρ is similar to its adjoint representation ρ^*, then ρ is self-adjoint.*

Proof. — Let T be an intertwining operator to realize $\rho \sim \rho^*$. By Lemma 3.6, T^* realizes the similarity $\rho^{**} \sim \rho^*$. By repeating this process, we see that $T = T^{**}$ realizes the similarity $\rho^{**} \sim \rho^*$ by using $\rho^{***} = \rho^*$. This means that $\rho = \rho^{**}$.

Since ρ^* is similar to ρ, it follows from Lemma 3.8 that there is a boundedly invertible, positive operator in $C^w(\rho^*(A))$. Hence $C^w(\rho^*(A))$ is non-degenerate on H. It follows from Corollary 3.3 that $\rho^* = \rho^{**}$. This completes the proof.

In general, even if a representation of a $*$-algebra is similar to a self-adjoint $*$-representation, it need not be hermitian. Before giving such an example, we will present a method to construct a J-representation induced by a pair of a $*$-representation and an operator, which is also useful in the next section for taking a look at the relation between J-representations and their invariant subspaces.

Let π be a $*$-representation of a $*$-algebra A on a Hilbert space H with domain $D(\pi)$ and let T be a bounded, skew-self-adjoint operator on H ($T^* = -T$). Assume T leaves $D(\pi)$ globally invariant. We define, for each $x \in A$, a linear operator with domain $D(\pi) \oplus D(\pi)$ in $H \oplus H$ by

$$\pi_T(x)(\xi \oplus \eta) = \begin{pmatrix} \pi(x) & T\pi(x) - \pi(x)T \\ 0 & \pi(x) \end{pmatrix}(\xi \oplus \eta)$$

for all $\xi, \eta \in D(\pi)$. Since each $\pi_T(x)$ leaves $D(\pi) \oplus D(\pi)$ globally invariant, it follows that

$$\pi_T(x)\pi_T(y)(\xi \oplus \eta) = \pi_T(xy)(\xi \oplus \eta)$$

for all $x \in A$ and $\xi, \eta \in D(\pi)$. Thus π_T gives a representation of A on $H \oplus H$ with domain $D(\pi_T) = D(\pi) \oplus D(\pi)$.

Take $J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Then we have

$$\begin{aligned} [\pi_T(x)(\xi \oplus \eta), \xi' \oplus \eta']_J &= (\pi(x)\eta, \xi') + (\pi(x)\xi, \eta') + (T\pi(x)\eta, \eta') - (\pi(x)T\eta, \eta') \\ &= (\eta, \pi(x^*)\xi') + (\xi, \pi(x^*)\eta') - (\eta, \pi(x^*)T\eta') + (\eta, T\pi(x^*)\eta') \\ &= [\xi \oplus \eta, \pi_T(x^*)(\xi' \oplus \eta')]_J \end{aligned}$$

for all $x \in A$ and $\xi, \xi', \eta, \eta' \in D(\pi)$. This means that π_T is a J-representation of A on the J-space $\left\{ H \oplus H, J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\}$.

We call π_T the *J-representation induced by* π and T.

PROPOSITION 3.10. — *Let π be a $*$-representation of a $*$-algebra* A *on a Hilbert space* H *with domain* $D(\pi)$ *and let* T *be a bounded, skew-self-adjoint operator on* H. *Assume* T *leaves* $D(\pi)$ *globally invariant. Then the* J*-representation* π_T *of* A *induced by* π *and* T *is similar to the $*$-representation* $\pi \oplus \pi$ *(the direct sum). In particular, if* π *is $*$-self-adjoint then* π_T *is* J*-self-adjoint.*

Furthermore, π_T *is $*$-hermitian if and only if* T *belongs to the strong commutant of* $\pi(A)$.

Proof. — Let us define an operator on $H \oplus H$ by

$$\tilde{T} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.$$

It then follows that $\tilde{T}$ is boundedly invertible with

$$\tilde{T}^{-1} = \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix},$$

and it is easy to see that $\tilde{T}$ maps $D(\pi_T)$ onto itself. Furthermore we have

$$\pi_T(x)(\xi \oplus \eta) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi(x) & 0 \\ 0 & \pi(x) \end{pmatrix}\begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}(\xi \oplus \eta)$$

for all $x \in A$ and $\xi, \eta \in D(\pi)$. Hence $\tilde{T}$ is the intertwining operator to realize the similarity $\pi_T \sim \pi \oplus \pi$.

Now $\tilde{T}^*$ realizes the similarity $(\pi \oplus \pi)^* \sim (\pi_T)^*$ by Lemma 3.6. Hence we have

$$\begin{aligned} D((\pi_T)^J) &= JD((\pi_T)^*) \\ &= J(\tilde{T})^* D((\pi \oplus \pi)^*) \\ &= \begin{pmatrix} -T & 1 \\ 1 & 0 \end{pmatrix} D((\pi \oplus \pi)^*) \quad (T^* = -T). \end{aligned}$$

Suppose π is $*$-self-adjoint. Then $\pi \oplus \pi$ is also $*$-self-adjoint. Since $\begin{pmatrix} -T & 1 \\ 1 & 0 \end{pmatrix}$ maps $D(\pi \oplus \pi)$ onto $D(\pi \oplus \pi)$, it follows that π_T is J-self-adjoint.

It is clear from the definition of π_T that π_T is $*$-hermitian if and only if $T \in C^s(\pi(A))$.

Remark. — In the case that π is the bounded representation of a C*-algebra, the above representation π_T is useful for studying derivations in C*-algebras. For further informations on this subject, we refer to [*16*]-[*18*].

EXAMPLE 3.11. — Let A_0 be the free commutative $*$-algebra on one-hermitian generator K_0. Consider the representation π_0 of A_0 on the Hilbert space $L^2[0, 1]$ defined by

$$D(\pi_0) = \{ f \in C^\infty[0, 1]; f^{(n)}(0) = f^{(n)}(1) \quad (n = 0, 1, 2, \dots) \},$$

$$\pi_0(K_0) \doteq -i \frac{d}{dt}\bigg|_{D(\pi_0)}.$$

It is then well-known that π_0 is a $*$-self-adjoint representation of A_0 on $L^2[0, 1]$ ([*7*] [*9*] and [*20*]).

For any fixed non-constant $\xi \in D(\pi_0)$, we write e_ξ for the operator of rank one defined by ξ; $e_\xi(\eta) = (\eta, \xi)\xi$ for all $\eta \in L^2[0, 1]$. Put $T = ie_\xi$. It is easily verified that T does not belong to $C^s(\pi_0(A_0))$. It follows from

Proposition 3.10 that the J-representation $(\pi_0)_T$ induced by π_0 and T is a non-$*$-hermitian which is similar to the $*$-self-adjoint representation $\pi_0 \oplus \pi_0$ of A_0.

We observed that $(\pi_0)_T$ in the above example is J-self-adjoint. In general, we will show (in the following section) that a J-representation of a $*$-algebra on a J-space which is similar to a $*$-self-adjoint representation is always J-self-adjoint. There is, however, a J-self-adjoint representation which is not similar to any $*$-representation. We conclude this section with such an example. For our convenience, we present an immediate consequence of Definition 3.1, which is easily verified.

LEMMA 3.12. — *Let π be a J-representation of a $*$-algebra A on a J-space. It follows that*

$$JC^w(\pi(A)) \supset C^s(\pi(A)).$$

In particular, the equality in this relations holds whenever π is J-self-adjoint.

EXAMPLE 3.13. — Let A_0, K_0 be the same as in Example 3.11. Consider the representation π_1 of A_0 on $L^2[0,1]$ defined by

$$D(\pi_1) = D(\pi_0)$$

$$\pi_1(K_0) = \frac{d}{dt}\Big|_{D(\pi_1)}.$$

Now let us define J_0 by

$$(J_0 f)(t) = f(1-t) \quad (0 \leqq t \leqq 1)$$

for all $f \in L^2[0,1]$. Clearly, J_0 is a symmetry on $L^2[0,1]$. For all $f, g \in D(\pi_1)$ and any complex number α, we have

$$[\pi_1(\alpha K_0)f, g]_{J_0} - [f, \pi_1(\bar{\alpha} K_0)g]_{J_0} = \alpha[-f(1-t)\overline{g(t)}]_0^1 = 0.$$

By repeating this calculations, we get

$$[\pi_1(P(K_0))f, g]_{J_0} = [f, \pi_1(\bar{P}(K_0))g]_{J_0}$$

for all polynomials P and $f, g \in D(\pi_1)$. Hence π_1 is a J-representation of A_0 on the J-space $\{ L^2[0,1], J = J_0 \}$. It is easy to see, by the analogous arguments to [9], Theorem 2.4 and Example 2.5, that π_1 is J-self-adjoint. We finally show that π_1 is not similar to any $*$-representations of A_0.

Suppose π_1 is similar to a $*$-representation of A_0 with intertwining operator T. Since π_1 is J-self-adjoint, it follows from Lemma 3.8 and Lemma 3.12 that

$$J_0 T^* T \in C^s(\pi_1(A_0)).$$

Let us take $f(t) = e^{2\pi i t}$ in $D(\pi_1)$. It follows that

$$\frac{d}{dt}(J_0 T^* T f) = J_0 T^* T\left(\frac{d}{dt} f\right)$$
$$= 2\pi i \cdot J_0 T^* T f.$$

Hence there is a constant γ such that

$$J_0 T^* T f = \gamma f.$$

Since $(J_0 f)(t) = e^{-2\pi i t}$, it follows that

$$(T^* T f, f) = \gamma(f, J_0 f) = 0.$$

Since T is invertible, it follows that $f = 0$. This is a contradiction. Thus π_1 is not similar to any $*$-representation.

4. INVARIANT SUBSPACES IN J-SPACES

In this section we will be concerned with characterizing a J-representation which is similar to some $*$-representation (especially, a $*$-self-adjoint representation), in terms of its invariant subspaces. Let us recall some concepts in J-spaces.

Let $\{ H, J \}$ be a J-space with the Minkowsky form $[\cdot, \cdot]_J = (J\cdot, \cdot)$. A subspace M is said to be J-*non-negative*, J-*positive definite* and J-*uniformly positive* if it satisfies

$$[\xi, \xi]_J \geqq 0$$

for all $\xi \in M$,

$$[\xi, \xi]_J > 0$$

for all $\xi \neq 0 \in M$ and there exists a constant $\lambda > 0$ such that

$$[\xi, \xi]_J \geqq \lambda \| \xi \|^2$$

for all $\xi \in M$, respectively. Similarly J-*non-positive*, J-*negative definite* and J-*uniformly negative* subspaces are defined. A J-non-negative subspace is said to be *maximal* if it is not a proper part of any other J-non-negative subspace of H, and we can analogously define *maximal* J-*positive* and *maximal* J-*uniformly positive* subspaces, etc.

Two subsets M and N of H are called J-*orthogonal* if $[\xi, \eta]_J = 0$ for all $\xi \in M$ and $\eta \in N$, which is denoted by $M[\perp]N$. If H is a direct sum of closed subspaces M and N with $M[\perp]N$, then H is called the J-*orthogonal direct sum* of M and N, which is denoted by $H = M[+]N$. For each subspace M, we define the J-*orthogonal complement* $M^{[\perp]}$ by

$$M^{[\perp]} = \{ \xi \in H ; [\xi, \eta]_J = 0 \text{ for all } \eta \in M \}.$$

If M is a maximal J-uniformly positive subspace, then $M^{[\perp]}$ is maximal J-uniformly negative and it satisfies $H = M[+]M^{[\perp]}$. For further informations on these notions in J-space we refer to [*2*] and [*10*].

A pair $\{ P, N \}$ of subspaces is said to be a *dual pair* if P is J-non-negative and N is J-non-positive with $P[\perp]N$. A dual pair $\{ P, N \}$ is said to be *maximal* if $\overline{P}$ and $\overline{N}$ are maximal among subspaces of the same kind, respectively, where the bar denotes the closure of each subspace. If $\{P, N\}$ forms a dual pair of subspaces such that $P + N$ is dense in H, then P and N are J-positive and J-negative definite, respectively, in particular $P \cap N = \{0\}$ [*19*], Lemma 6.1.

DEFINITION 4.1. —Let π be a representation of a $*$-algebra A on a J-space $\{H, J\}$ with domain $D(\pi)$. A dual pair $\{P, N\}$ of subspaces of $\{H, J\}$ is said to be a *π-invariant dual pair* if it satisfies the following conditions:

1. P and N are contained in $D(\pi)$ with $P + N = D(\pi)$; that is, the domain $D(\pi)$ is the algebraic direct sum of P and N;

and

2. Both P and N are globally invariant under each $\pi(x)$ $(x \in A)$.

If this is the case, we say that π *has an invariant dual pair* $\{P, N\}$.

If the projection $2^{-1}(J + \mathbb{1})$ has a finite rank, then a J-space or a Krein space is said to be a *Pontryagin space*. We note [*2*], Chapter IX that, if a closed subspace of such a space is J-positive definite (resp. J-negative definite), it is J-uniformly positive (resp. J-uniformly negative). Hence, we should remark that, if the representation space is a Pontryagin space, its invariant dual pair consists of J-uniformly positive and J-uniformly negative subspaces. In this point of view, we will be mainly interested in J-representations with invariant dual pairs of such subspaces.

Let $\{P_i, N_i\}$ $(i = 1, 2)$ be dual pairs. If $P_1 \subset P_2$ and $N_1 \subset N_2$, then we write $\{P_1, N_1\} \subset \{P_2, N_2\}$.

PROPOSITION 4.2. — *Let π be a J-representation of a $*$-algebra A on a J-space $\{H, J\}$. Suppose π has an invariant dual pair $\{M, N\}$ of J-uniformly positive and J-uniformly negative subspaces. Then each of $\tilde{\pi}, \pi^J$ and π^{JJ} has an invariant dual pair of J-uniformly positive and J-uniformly negative subspaces and, moreover, each dual pair can be chosen to be a maximal dual pair satisfying the following properties: Let us write $\{M_\sim, N_\sim\}$ ($\{M_J, N_J\}$ and $\{M_{JJ}, N_{JJ}\}$, resp.) for its $\tilde{\pi}$ (π^J and π^{JJ})-invariant dual pair. Then*

1. $\{M, N\} \subset \{M_\sim, N_\sim\} \subset \{M_{JJ}, N_{JJ}\} \subset \{M_J, N_J\}$;

2. $\overline{M}_\sim$ *and* $\overline{N}_\sim$ *are maximal J-uniformly positive and maximal J-uniformly negative subspaces, respectively, such that*

$$\overline{M}_\sim = \overline{M_{JJ}} = \overline{M_J}, \quad \overline{N_\sim} = \overline{N_{JJ}} = \overline{N_J} = M_\sim^{[\perp]}$$

and

$$\overline{M}_\sim [+] \overline{N}_\sim = H;$$

and moreover

3. If E is the J-projection onto $\overline{M}_\sim$, then E belongs to the intersection of $C^s(\pi(A))$, $C^s(\pi^J(A))$ and $C^s(\pi^{JJ}(A))$.

Proof. — By [*1*], Theorem 1.2 or [*19*], Theorem 2.1, there exists a maximal J-uniformly positive (closed) subspace M_0 which contains M with $M_0[\perp]N$. In particular, $H = M_0[+]M_0^{[\perp]}$. Let E be the projection onto M_0. It follows from the closed graph theorem that E is continuous and, moreover, that E is a J-projection which means that $E^J = E = E^2$. It is clear that $\mathbb{1} - E$ is the J-projection onto $M_0^{[\perp]}$. Since $D(\pi)$ is uniquely decomposed into the sum of M and N, both of which are contained in M_0 and $M_0^{[\perp]}$ respectively, it follows that $ED(\pi) = M$ and $(\mathbb{1} - E)D(\pi) = N$. Since M and N are invariant under each $\pi(x)$, it follows that, for $\xi = \xi_1 + \xi_2$ ($\xi_1 \in M$, $\xi_2 \in N$), $E\pi(x)\xi = E\pi(x)\xi_1 = \pi(x)E\xi$. Hence E belongs to $C^s(\pi(A))$, and so to $C^s(\tilde{\pi}(A))$.

Put

$$M_\sim = ED(\tilde{\pi}) \quad \text{and} \quad N_\sim = (\mathbb{1} - E)D(\tilde{\pi}).$$

Since E belongs to $C^s(\tilde{\pi}(A))$, it follows that a pair $\{ M_\sim, N_\sim \}$ is a $\tilde{\pi}$-invariant dual pair of J-uniformly positive and J-uniformly negative subspaces. It is obvious that $\overline{M}_\sim$ coincides with the maximal J-uniformly positive subspace M_0, and so that $\overline{N}_\sim = M_0^{[\perp]}$.

Let ξ be in $D(\pi^J)$. Since $E \in C^s(\pi(A))$, it follows that

$$\begin{aligned}[\pi(x)\eta, E\xi]_J &= [\pi(x)E\eta, \xi]_J \\ &= [E\eta, \pi^J(x^*)\xi]_J\end{aligned}$$

for all $x \in A$ and $\eta \in D(\pi)$. This implies that $E\xi \in D(\pi^J)$ and

$$\pi^J(x)E\xi = E\pi^J(x)\xi$$

for all $x \in A$. Hence E belongs to $C^s(\pi^J(A))$.

It is easily seen, in the analogous manner, that E also belongs to $C^s(\pi^{JJ}(A))$. Put

$$\begin{aligned} M_J &= ED(\pi^J) \quad \text{and} \quad N_J = (\mathbb{1} - E)D(\pi^J); \\ M_{JJ} &= ED(\pi^{JJ}) \quad \text{and} \quad N_{JJ} = (\mathbb{1} - E)D(\pi^{JJ}). \end{aligned}$$

Then we can prove, by the same arguments as above, that the dual pairs $\{ M_J, N_J \}$ and $\{ M_{JJ}, N_{JJ} \}$ possess the desired properties. This completes the proof.

In order to illustrate clearly the relationship between the similarity of representations and the existence of an invariant dual pair, let us take a look at the J-representation given in Proposition 3.9.

Let π be a $*$-representation of a $*$-algebra A on a Hilbert space H and let $T = -T^*$ be a bounded operator on H which leaves $D(\pi)$ globally invariant. The J-representation π_T induced by π and T on the J-space $\left\{ H \oplus H, J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\}$ is, as is seen in Proposition 3.10, similar to the $*$-representation $\pi \oplus \pi$.

Take $\varepsilon > 0$ and put $T_\varepsilon = (T + \varepsilon\mathbb{1})^{-1}$. It then follows from [16], Theorem 2.2 that the graph $G(T_\varepsilon)$ of T_ε in $H \oplus H$ (i. e., $G(T_\varepsilon) = \{ \xi \oplus T_\varepsilon\xi : \xi \in H \}$)

is a maximal J-uniformly positive subspace of $\left\{ H \oplus H, J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.
Define

$$P = G(T_\varepsilon) \cap D(\pi_T) \quad \text{and} \quad N = G(T_\varepsilon)^{[\perp]} \cap D(\pi_T).$$

We shall show that $\{ P, N \}$ is a π_T-invariant maximal dual pair of J-uniformly positive and J-uniformly negative subspaces satisfying

$$\overline{P} = G(T_\varepsilon) \quad \text{and} \quad \overline{N} = G(T_\varepsilon)^{[\perp]}.$$

Observe that $G(T_\varepsilon)^{[\perp]} = G(-T_\varepsilon^*)$. Let $\xi_1 \oplus \xi_2$ be in $D(\pi_T)$. Since $G(T_\varepsilon)[+]G(T_\varepsilon)^{[\perp]} = H \oplus H$, there are vectors η_1 and η_2 in H such that

$$\xi_1 = \eta_1 + \eta_2$$
$$\xi_2 = T_\varepsilon \eta_1 - T_\varepsilon^* \eta_2.$$

It follows that

$$\begin{aligned} (T - \varepsilon 1)\xi_2 &= -(T_\varepsilon^*)^{-1}(T_\varepsilon \eta_1 - T_\varepsilon^* \eta_2) \\ &= (T + \varepsilon - 2\varepsilon)T_\varepsilon \eta_1 + \eta_2 \\ &= \xi_1 - 2\varepsilon T_\varepsilon \eta_1 . \end{aligned}$$

Since T maps $D(\pi)$ into $D(\pi)$, it follows that $T_\varepsilon \eta_1 \in D(\pi)$, so that $\eta_1 \in D(\pi)$. Hence $\eta_1 \oplus T_\varepsilon \eta_1 \in P$ and $\eta_2 + (-T_\varepsilon^*)\eta_2 \in N$. Thus $P + N = D(\pi_T)$. Since $(T\pi(x) - \pi(x)T)T_\varepsilon \xi = T_\varepsilon^{-1}\pi(x)T_\varepsilon \xi - \pi(x)\xi$ for all $\xi \oplus T_\varepsilon \xi \in P$ and $x \in A$, P is globally invariant under each $\pi(x)$. It follows that $\{ P, N \}$ is a π_T-invariant dual pair of J-uniformly positive and J-uniformly negative subspaces.

To show that its dual pair is maximal, it suffices to prove that $\overline{P} = G(T_\varepsilon)$.

For any $\xi \in H$, there are sequences $\{ \xi_n \oplus T_\varepsilon \xi_n \}$ in P and $\{ \eta_n \oplus T_\varepsilon \eta_n \}$ in N such that

$$\xi_n + \eta_n \to \xi \quad \text{and} \quad T_\varepsilon \xi_n - T_\varepsilon^* \eta_n \to T_\varepsilon \xi$$

as $n \to \infty$, since $P + N$ is dense in $H \oplus H$. Hence $T_\varepsilon \eta_n + T_\varepsilon^* \eta_n \to 0$ as $n \to \infty$. Noticing that $T_\varepsilon + T_\varepsilon^* = -2\varepsilon T_\varepsilon T_\varepsilon^*$, we have $\eta_n \to 0$ as $n \to \infty$. It follows that $\{ \xi_n \oplus T_\varepsilon \xi_n \}$ in P converges to $\xi \oplus T_\varepsilon \xi$.

PROPOSITION 4.3. — *If a* J*-representation* π *of a* *-*algebra* A *on a* J*-space* $\{ H, J \}$ *has an invariant dual pair of* J*-uniformly positive and* J*-uniformly negative subspaces, then* π *is similar to a* *-*representation.*

Proof. — Let $\{ M, N \}$ be a π-invariant dual pair of J-uniformly positive and J-uniformly negative subspaces. Keeping the same notations as in Proposition 4.2, we should note that the J-projection E on a maximal J-uniformly positive subspace $\overline{M}_\sim$ belongs to $C^s(\pi(A))$. Put

$$Q = 2E - 1.$$

It is clear that Q is a J-unitary, J-self-adjoint operator on H.

We now introduce a new inner product on H as follows:

$$\begin{aligned}\langle \xi, \eta \rangle &= [Q\xi, \eta]_J \\ &= [E\xi, E\eta]_J - [(\mathbb{1} - E)\xi, (\mathbb{1} - E)\eta]_J\end{aligned}$$

for all $\xi, \eta \in H$. It is easy to check that the form $\langle \cdot,\cdot \rangle$ is an (usual) inner product, which makes H a pre-Hilbert space.

We next show that the new norm defined by $\langle \cdot,\cdot \rangle$, which is denoted by $|\cdot|_0$, is equivalent to the original norm $\|\cdot\|$. We have, for $\xi \in H$,

$$|\xi|_0^2 = [Q\xi, \xi]_J \leqq \|Q\| \, \|\xi\|^2,$$

so that $|\xi|_0 \leqq \sqrt{\|Q\|} \, \|\xi\|$. Since Q is an isometry with respect to $| \quad |_0$, it follows that

$$\|\xi\|^2 = [J\xi, \xi]_J = \langle QJ\xi, \xi \rangle \leqq |J\xi|_0 |\xi|_0 \leqq \sqrt{\|Q\|} \, |J\xi|_0 \, \|\xi\|$$

for all $\xi \in H$. Replacing ξ with $J\xi$, we obtain

$$\sqrt{\|Q\|} \, |\xi|_0 \geqq \|\xi\|$$

for all $\xi \in H$. Hence H equipped with $\langle \cdot,\cdot \rangle$ turns out to be a Hilbert space. Since we have to distinguish two Hilbert spaces, we write $\tilde{H}$ for H equipped with $\langle \cdot,\cdot \rangle$.

Let ι be the identity transformation from H onto $\tilde{H}$. We now define a representation ρ of A on $\tilde{H}$ as follows:

$$\begin{aligned}D(\rho) &= \iota D(\pi), \\ \rho(x)\xi &= \iota\pi(x)\iota^{-1}\xi \quad \text{for} \quad \xi \in D(\rho).\end{aligned}$$

Since Q belongs to $C^s(\pi(A))$, it follows that

$$\begin{aligned}\langle \rho(x)\xi, \eta \rangle &= [Q\pi(x)\iota^{-1}\xi, \iota^{-1}\eta]_J \\ &= [Q\iota^{-1}\xi, \pi(x^*)\iota^{-1}\eta]_J \\ &= \langle \xi, \iota\pi(x^*)\iota^{-1}\eta \rangle\end{aligned}$$

for all $x \in A$ and $\xi, \eta \in D(\rho)$. Thus ρ is $*$-hermitian, which implies the proposition.

Remark. — Even in case that the assumption of the J-uniformly definiteness on the dual pair $\{M, N\}$ in Proposition 4.3 is dropped, we can show, by the slight modification of the proof, that there are a $*$-representation ρ of A on a Hilbert space H_ρ and a closed (in general, unbounded), densely defined injective linear transformation S from H to H_ρ with dense range such that S maps $D(\pi)$ onto $D(\rho)$ and $S\pi(x)\xi = \rho(x)S\xi$ for all $x \in A$ and $\xi \in D(\pi)$.

In that case, we say that π is Naimark-related to ρ with intertwining operator S ([*15*] and [*27*], p. 232). In general Naimark-relatedness is not an equivalence relation, since it need not be transitive. One may obtain the analogous results to Lemmas 3.6 and 3.8 by introducing a certain unbounded commutant, though the relation between Naimark-relatedness and an invariant dual pair is more complicated.

THEOREM 4.4. — *Let π be a* J*-representation of a $*$-algebra* A *on a* J*-space* $\{H, J\}$. *For π to be similar to a $*$-self-adjoint representation of* A *on a Hilbert space, it is necessary and sufficient that*

1. π is J*-self-adjoint; and*

2. π has an invariant dual pair of J*-uniformly positive and* J*-uniformly negative subspaces.*

Proof. — Suppose the conditions 1 and 2 are satisfied. Then π is similar to a $*$-representation ρ of A on a Hilbert space by Proposition 4.3. It is clear, by Lemma 3.6, that π^J is similar to ρ^*. Since π is J-self-adjoint, it follows from Theorem 3.9 that ρ is $*$-self-adjoint.

Conversely, suppose π is similar to a $*$-self-adjoint representation, which is also dented by ρ, of A on a Hilbert space H_ρ. Let T be the intertwining operator to realize the similarity $\pi \sim \rho$. It then follows from Lemma 3.6 and the self-adjointness of ρ that JT^* realizes the similarity $\rho \sim \pi^J$.

Noting that

$$TJT^*D(\rho) = TJD(\pi^*) = TD(\pi^J)$$
$$TD(\pi) = D(\rho),$$

we have

$$\begin{aligned}(TJT^*)^{-1}\rho(x)\xi &= (JT^*)^{-1}\pi(x)T^{-1}\xi \\ &= (JT^*)^{-1}\pi^J(x)T^{-1}\xi \\ &= \rho(x)(JT^*)^{-1}T^{-1}\xi\end{aligned}$$

for all $x \in A$ and $\xi \in D(\rho)$. Hence $(TJT^*)^{-1}$ belongs to $C^s(\rho(A))$. On the other hand, $C^s(\rho(A))$ is a von Neumann algebra by the self-adjointness of ρ, and so TJT^* also belongs to $C^s(\rho(A))$. In particular,

$$D(\pi^J) = JT^*D(\rho) = T^{-1}D(\rho) = D(\pi).$$

Hence, π is J-self-adjoint.

Let $\{e(\lambda)\}$ be the spectral projections of unit corresponding to TJT^*. Then $TJT^* = \int_{-\infty}^{-\varepsilon} \lambda de(\lambda) + \int_{\varepsilon}^{+\infty} \lambda de(\lambda)$ with some $\varepsilon > 0$. Put $E = \int_{\varepsilon}^{+\infty} de(\lambda)$. Since $C^s(\rho(A))$ is a von Neumann algebra, it follows that E belongs to $C^s(\rho(A))$. This means that $ED(\rho)$ is contained in $D(\rho)$ and is globally invariant under each $\rho(x)$ $(x \in A)$.

Let us define subspaces in $D(\pi)$ by

$$M = JT^*ED(\rho) \quad \text{and} \quad N = JT^*(\mathbb{1} - E)D(\rho).$$

It then follows that

$$\begin{aligned}\pi(x)M &= T^{-1}(T\pi(x)JT^*ED(\rho)) \\ &= T^{-1}(\rho(x)TJT^*ED(\rho)) \\ &= JT^*\rho(x)ED(\rho) \\ &\subset JT^*ED(\rho)\end{aligned}$$

for each $x \in A$. Thus M is globally invariant under each $\pi(x)$ and, it is analogously verified that N is also globally invariant under each $\pi(x)$. It is clear that M and N are J-orthogonal and $M + N = D(\pi)$. Now we obtain

$$\begin{aligned}[JT^*\xi, JT^*\xi]_J &= (TJT^*\xi, \xi)_\rho \\ &\geqq \varepsilon \| \xi \|^2 \\ &\geqq \varepsilon' \| JT^*\xi \|^2\end{aligned}$$

for all $\xi \in ED(\rho)$, where $\varepsilon' = \varepsilon \| T^* \|^{-2}$. Hence $\overline{M} = JT^*\overline{ED(\rho)}$ is J-uniformly positive and it is seen, by the similar argument, that $\overline{N}$ is J-uniformly negative. Moreover we have

$$\begin{aligned}\overline{M} + \overline{N} &= JT^*(\overline{ED(\rho)} + \overline{(1\!\!1 - E)D(\rho)}) \\ &= JT^*H_\rho = H_\pi .\end{aligned}$$

Thus $H_\pi = \overline{M}\,[+]\,\overline{N}$, which implies by [2], Theorem 7.1 that $\overline{M}$ and $\overline{N}$ are maximal J-uniformly positive and maximal J-uniformly negative, respectively. This completes the proof of the theorem.

Remark. — In the proof of the above theorem, the following result, in connection with Theorem 3.7, has been shown:

Suppose a J-representation π of a $*$-algebra A on a J-space is similar to a $*$-representation ρ of A on a Hilbert space. Then π is J-self-adjoint if and only if ρ is $*$-self-adjoint.

We should note that, as we have observed in Example 3.13 and the example cited before Proposition 4.3, the conditions 1 and 2 are mutually independent.

ACKNOWLEDGEMENTS

The author would like to express his hearty thanks to Professor P. Jorgensen for the warm hospitality at the University of Iowa in autumn of 1984, where this work was begun, and to Professor S. Sakai for his encouragements. He also would like to thank Professor A. Inoue for reading carefully the manuscript with many comments and suggestions.

REFERENCES

[1] T. ANDO, *Linear operators on Krein spaces*. Lectures Note in Hokkaido Univ., 1979.
[2] J. BOGNAR, *Indefinite inner product spaces*. Berlin, Springer Verlag, 1974.
[3] H. J. BORCHERS, *Algebraic aspects of Wighttman field theory in Statistical Mechanics and Field Theory Lectures*, 1971, New York, Jerusalem, London, 1972.
[4] H. J. BORCHERS and J. YUNGVASON, *Commun. Math. Phys.*, t. **42**, 1975, p. 231-252.
[5] Yu. DADASHYAN and S. S. KHORUZHII, *Theor. Math. Phys.*, t. **54**, 1983, p. 35-48.
[6] O. BRATTELI and D. W. ROBINSON, *Operator algebras and quantum statistical mechanics*, Berlin, Springer Verlag, 1979.
[7] S. GUDDER and W. SCRUGGS, *Pacific J. Math.*, t. **70**, 1977, p. 369-382.
[8] U. HAAGERUP, *Ann. Math.*, t. **118**, 1983, p. 215-224.
[9] A. INOUE and K. TAKESUE, *Trans. Amer. Math.*, t. **280**, 1983, p. 393-400.
[10] I. S. IOHIVIDOV, M. G. KREIN and H. LANGER, *Introduction to the spectral theory of operators in spaces with indefinite metric*, Berlin, Akademie Verlag, 1982.
[11] A. Z. JADCZYK, *Rep. Math. Phys.*, t. **2**, 1971, p. 263-276.
[12] L. JAKOBCZYK, *J. Math. Phys.*, t. **25**, 1984, p. 617-622.
[13] L. JAKOBCZYK, *Ann. Phys.*, t. **161**, 1985, p. 314-336.
[14] G. LASSNER, *Rep. Math. Phys.*, t. **3**, 1972, p. 270-293.
[15] N. A. NAIMARK, *Linear representations of the Lorentz groups*, London, Pergamon, 1964.
[16] S. ÔTA, *Acta Sci. Math. (Szeged)*, t. **39**, 1977, p. 129-133.
[17] S. ÔTA, *J. Funct. Anal.*, t. **30**, 1978, p. 238-244.
[18] S. ÔTA, *J. Reine Angew. Math.*, t. **347**, 1984, p. 21-32.
[19] R. S. PHILLIPS, *The extension of dual subspaces invariant under algebra*. Proc. Inter. Symp. On Linear Spaces. Jerusalem, 1960, p. 366-398.
[20] R. T. POWERS, *Commun. Math. Phys.*, t. **21**, 1971, p. 85-124.
[21] R. T. POWERS, *Trans. Amer. Math. Soc.*, t. **187**, 1974, p. 261-293.
[22] S. SAKAI, *C*-algebras and W*-algebras*, Berlin, Springer Verlag, 1971.
[23] K. SCHMÜDGEN, *Publ. R. I. M. S. (Kyoto)*, t. **19**, 1983, p. 601-671.
[24] A. N. VASL'EV, *Theor. Math. Phys.*, t. **2**, 1970, p. 113-123.
[25] A. H. VÖLKEL, *J. Math. Phys.*, t. **26**, 1985, p. 2956-2973.
[26] A. H. VÖLKEL, *J. Math. Phys.*, t. **27**, 1986, p. 1113-1127.
[27] G. WARNER, *Harmonic analysis on semi-simple Lie groups I*, Berlin, Springer Verlag, 1972.

(Manuscrit reçu le 22 août 1987)