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Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations

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ABSTRACT. — In this paper we shall study the scattering problem for the following Schrödinger equation:

$$(**)\quad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = V_1 u + (V_2 * |u|^2)u, & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = \phi(x), & x\in\mathbb{R}^n, \end{cases}$$

where $V_1 = V_1(x) = \lambda_1 |x|^{-\gamma_1}$, $(\lambda_1 \geqq 0,\ 1 < \gamma_1 < \min(2, n/2))$, $V_2 = V_2(x) = \sum_{k=2}^{3} \lambda_k |x|^{-\gamma_k}$, $(\lambda_k \geqq 0, 1 < \gamma_k < \min(2, n))$, $*$ denotes the convolution in $\mathbb{R}^n$. $H^{m,s} = \{ v \in L^2(\mathbb{R}^n); \| v \|_{m,s} = \| (1+|x|^2)^{s/2}(I-\Delta)^{m/2} v \|_{L^2} < \infty \}$, $m, s \in \mathbb{R}$.

We show that (1) if $\phi \in H^{0,1}$, all solutions of (**) are asymptotically free in $L^2(\mathbb{R}^n)$, (2) if $n \geqq 4$, $(3/2) \leqq \gamma_1, \gamma_2, \gamma_3 < 2$, $\phi \in H^{0,2}$, all solutions of (**) are asymptotically free in $H^{0,1}$, (3) if $\lambda_1 = 0$, $n \geqq 3$, $(4/3) < \gamma_2, \gamma_3 < 2$, $s \in \mathbb{N}$, $\phi \in H^{0,s}$, the wave operators and the scattering operator are well defined in $H^{0,s}$ and homeomorphisms from $H^{0,s}$ to $H^{0,s}$.

RÉSUMÉ. — Dans cet article nous étudions le problème de diffusion pour l'équation de Schrödinger :

$$(**)\quad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = V_1 u + (V_2 * |u|^2)u, & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = \phi(x), & x\in\mathbb{R}^n, \end{cases}$$

où $V_1 = V_1(x) = \lambda_1 |x|^{-\gamma_1}, \qquad (\lambda_1 \geqq 0,\ 1 < \gamma_1 < \min(2, n/2))$,

$$V_2 = V_2(x) = \sum_{k=2}^{3} \lambda_k |x|^{-\gamma_k}, \qquad (\lambda_k \geqq 0,\ 1 < \gamma_k < \min(2, n)),$$

et $*$ est la convolution dans $\mathbb{R}^n$.

$H^{m,s} = \{ v \in L^2(\mathbb{R}^n);\quad \| v \|_{m,s} = \| (1+|x|^2)^{s/2}(I-\Delta)^{m/2} v \|_{L^2} < \infty \}, m, s \in \mathbb{R}$.

Nous montrons que (1) si $\phi \in H^{0,1}$, toutes les solutions de (**) sont asymptotiquement libres dans $L^2(\mathbb{R}^n)$, (2) si $n \geqq 4$, $(3/2) \leqq \gamma_1, \gamma_2, \gamma_3 < 2$, $\phi \in H^{0,2}$, toutes les solutions de (**) sont asymptotiquement libres dans $H^{0,1}$, (3) si $\lambda_1 = 0$, $n \geqq 3$, $(4/3) < \gamma_2, \gamma_3 < 2$, $s \in \mathbb{N}$, $\phi \in H^{0,s}$, les opérateurs d'onde et l'opérateur de diffusion sont bien définis dans $H^{0,s}$ et sont des homéomorphismes de $H^{0,s}$.

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1. INTRODUCTION

In this paper we shall study the scattering problem for the following Schrödinger equation:

$$i\partial_t u + \frac{1}{2}\Delta u = V_1 u + (V_2 * |u|^2)u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \phi(x), \quad x\in\mathbb{R}^n, \tag{1.2}$$

where $V_1 = V_1(x) = \lambda_1 |x|^{-\gamma_1}$, $(\lambda_1 \geqq 0,\ 1 < \gamma_1 < \min(2, n/2))$,

$$V_2 = V_2(x) = \sum_{k=2}^{3} \lambda_k |x|^{-\gamma_k}, \qquad (\lambda_k \geqq 0,\ 1 < \gamma_k < \min(2, n)),$$

$*$ denotes the convolution in $\mathbb{R}^n$.

Throughout the paper we use the following notations and function spaces: $\partial_j = \partial/\partial x_j$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $\alpha \in (\mathbb{N} \cup \{0\})^n$; $U = U(t) = \exp(i(t/2)\Delta)$, $S = S(t) = \exp(i|x|^2/2t)$; $J_j = J_j(t) = (x_j + it\partial_j) = U(t)x_j U(-t)$; $J = (J_1, \ldots, J_n) = U(t)xU(-t)$, $|J|^\alpha = U(t)|x|^\alpha U(-t)$, $\alpha \in \mathbb{R}^+ = [0, \infty)$

L^p denotes the Lebesgue space $L^p(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n) \otimes \mathbb{C}^n$ with the norm $\| \cdot \|_p$, $1 \leqq p \leqq \infty$; $\| \cdot \| = \| \cdot \|_2$; $(.,.)$ denotes the L^2 scalar product; $H_p^{m,s}$ denotes the weighted Sobolev space with the norm $\| \psi \|_{m,s,p} = \| (1+|x|^2)^{s/2}(I-\Delta)^{m/2}\psi \|_p$, $m, s \in \mathbb{R}$, $1 \leqq p \leqq \infty$; $\| \cdot \|_{m,s} = \| \cdot \|_{m,s,2}$; $\dot{B}^s_{p,q}$ denotes the homogeneous Besov space with the semi-norm

$$\| \psi \|_{\dot{B}^s_{p,q}} = \left(\int_0^\infty t^{-\sigma q} \sup_{|k| \leq t} \sum_{|\alpha| \leqq [s]} \| \partial^\alpha(\psi_k - \psi) \|_p^q \frac{dt}{t} \right)^{1/q},$$

$s = [s] + \sigma$, $0 < \sigma < 1$, $\psi_k(x) = \psi(x + k)$; $[s]$ denotes the largest integer less than s; $C(I; E)$ denotes the space of continuous functions from an interval $I \subset \mathbb{R}$ to a Fréchet space E; $C^k(I; E)$ denotes the space of k-times continuously differentiable functions from I to E, $k \in \mathbb{N}$; $L^\theta(I; B)$ denotes the space of measurable functions u from I to a Banach space B such that $\| u(.) \|_B \in L^\theta(I)$, $1 \leqq \theta \leqq \infty$; $L^{q,\theta} = L^\theta(I; L^q)$ with the norm $||| \quad |||_{q,\theta}$; $X(a, \sigma) = \{ u \in C(I; L^2) \cap L^{\delta(\sigma), 8/\sigma}; ||| u |||_{X(a,\sigma)} = ||| u |||_{2,\infty} + ||| u |||_{\delta(\sigma), 8/\sigma} < \infty \}$, $\delta(\sigma) = 4n/(2n - \sigma)$, $\sigma < 2n$, $I = [-a, a]$, $a > 0$; the dilation operator $(D(t)\psi)(x) = (it)^{-n/2}\psi(x/t)$; the Fourier transform

$$(\mathscr{F}\psi)(\xi) = (2\pi)^{-n/2} \int \psi(x) \exp(-i\xi x) dx ;$$

different positive constants might be denoted by the same letter C. If necessary, by $C(*, \ldots, *)$ we denote constants depending on the quantities appearing in parentheses.

We note that

$$U(t) = S(t)D(t)\mathscr{F}S(t), \qquad D(t)^{-1} = i^n D\left(\frac{1}{t}\right),$$
$$U(-t) = S(-t)\mathscr{F}^{-1} D(t)^{-1} S(-t) = S(-t) i^n \mathscr{F}^{-1} D\left(\frac{1}{t}\right) S(-t)$$

imply the following relations:

$$\begin{aligned} J_j(t) &= U(t)x_j U(-t) = S(t)D(t)\mathscr{F}S(t)x_j S(-t)\mathscr{F}^{-1} i^n D\left(\frac{1}{t}\right)S(-t) \\ &= S(t)D(t)i^n(i\partial_j)D\left(\frac{1}{t}\right)S(-t) \\ &= S(t)D(t)i^n D\left(\frac{1}{t}\right)(it\partial_j)S(-t) = S(t)(it\partial_j)S(-t), \end{aligned}$$

and
$$|J|^\alpha(t) = S(t)(-t^2\Delta)^{\alpha/2}S(-t), \quad \alpha \in \mathbb{R}^+ .$$

We shall prove the following theorems.

THEOREM 1. — For any $\phi \in H^{0,1}$, there exist unique $u_\pm \in L^2$ such that

$$\| U(-t)u(t) - u_\pm \| \to 0 \quad \text{as} \quad t \to \pm\infty ,$$

where u is a unique solution of (1.1)-(1.2) satisfying $u \in C(\mathbb{R}\setminus\{0\}; H^{1,-1})$ and $U^{-1}u \in C(\mathbb{R}; H^{0,1})$.

THEOREM 2. — Let $n \geqq 4$, $(3/2) \leqq \gamma_1, \gamma_2, \gamma_3 < 2$. For any $\phi \in H^{0,2}$, there exist unique $u_\pm \in H^{0,1}$ such that

$$\| U(-t)u(t) - u_\pm \| \to 0 \quad \text{as} \quad t \to \pm\infty,$$

where u is a unique solution of (1.1)-(1.2) satisfying $u \in C(\mathbb{R}\setminus\{0\}; H^{2,-2})$ and $U^{-1}u \in C(\mathbb{R}; H^{0,2})$.

THEOREM 3. — Let $\lambda_1 = 0$, $n \geqq 3$, $(4/3) < \gamma_2, \gamma_3 < 2$, $s \in \mathbb{N}$. For any $\phi \in H^{0,s}$, the wave operators $W_\pm$ and the scattering operator $W_+^{-1}W_-$ are well defined in $H^{0,s}$ and homeomorphisms from $H^{0,s}$ to $H^{0,s}$.

REMARK 1. — (1) For any $\phi \in H^{0,l}$ $(l=1, 2)$, the existence and uniqueness of solutions for (1.1)-(1.2) have been proved by N. Hayashi-T. Ozawa [10] [11].

(2) When $\lambda = 0$ and $\phi \in H^{1,1}$, N. Hayashi-Y. Tsutsumi [7] showed Theorem 1 by using the pseudoconformal conservation law and the transform $u(t, x) = (Cv)(t, x) = (1/it)^{n/2} \exp(i|x|^2/2t)\overline{v(1/t, x/t)}$ (see also Y. Tsutsumi-K. Yajima [17]). In Section 3 we prove Theorems 1-2 by using a more direct method than that of [7] [17].

(3) When $\gamma_2 = \gamma_3$, $\phi \in H^{1,s}$, $n \geqq 2$, $s \in \mathbb{N}$, Theorem 3 was shown in [7]. In Section 4 we prove Theorem 3 by making use of the space-time estimates of the Schrödinger evolution group $\{U(t)\}$ with the operators J and $|J|^\alpha$.

2. PRELIMINARY ESTIMATES

LEMMA 1.1. — (The Gagliardo-Nirenberg inequality). Let q, r be any numbers satisfying $1 \leqq q, r \leqq \infty$, and let j, m be any integers satisfying $0 \leqq j < m$. If $u \in H_r^{m,0} \cap L^q$, then

$$\sum_{|\alpha|=j} \| \partial^\alpha u \|_p \leqq C \sum_{|\beta|=m} \| \partial^\beta u \|_r^a \| u \|_q^{1-a}, \tag{2.1}$$

where $(1/p) = (j/n) + a((1/r) - (m/n)) + (1-a)/q$ for all a in the interval $(j/m) \leqq a \leqq 1$, where C is a constant depending only on n, m, j, q, r, a, with the following exception: if $m - j - (n/r)$ is a nonnegative integer, then (2.1) holds for any $(j/m) \leqq a < 1$.

For Lemma 2.1 see, e. g., A. Friedman [3].

LEMMA 2.2. — Let $1 < p < q < \infty$, $0 < \gamma < n$ and $(1/q) = (1/p) - (n-\gamma)/n$. Then we have

$$\| I_\gamma(\psi) \|_q \leqq C \| \psi \|_p, \qquad \text{for} \quad \psi \in L^p, \tag{2.2}$$

where

$$I_\gamma(\psi) = \int_{\mathbb{R}^n} |x - y|^{-\gamma}\psi(y)dy.$$

If $n \geqq 3$, then we have

$$\int_{\mathbb{R}^n} |\psi(x)|^2/|x|^2 dx \leqq (2/(n-2))^2 \|\nabla\psi\|^2, \quad \text{for} \quad \psi \in H^{1,0}, \tag{2.3}$$

$$\int_{\mathbb{R}^n} |\psi(x)|^2/|x|^2 dx \leqq (2/(n-2))^2 t^{-2} \|J\psi\|^2, \text{ for } \psi \in H^{1,1} \text{ and } t \in \mathbb{R}\backslash\{0\}. \tag{2.4}$$

For (2.2) and (2.3) see, e. g., E. M. Stein [*13*], and for (2.4) see, e. g., N. Hayashi-T. Ozawa [*9*].

We put

$$(Gv)(t) = \int_0^t U(t - s)v(s)ds.$$

LEMMA 2.3. — Let $0 \leqq \sigma < 2$, $\delta(\sigma) = 4n/(2n-\sigma)$ and $1/\delta(\sigma) + 1/\delta'(\sigma) = 1$. Then there exist positive constants C independent of $I = [-a, a]$, $a \geqq 0$ such that

$$|||Gv|||_{\delta(\sigma),8/\sigma} \leqq C|||v|||_{\delta'(\theta),8/(8-\theta)}, \quad \text{for} \quad \sigma \geqq \theta, \quad v \in L^{\delta'(\theta),8/(8-\theta)}, \tag{2.5}$$

$$|||Gv|||_{2,\infty} \leqq C|||v|||_{\delta'(\sigma),8/(8-\sigma)}, \quad \text{for} \quad v \in L^{\delta'(\sigma),8/(8-\sigma)}, \tag{2.6}$$

$$|||U(\cdot)w|||_{\delta(\sigma),8/\sigma} \leqq C\|w\|, \quad \text{for} \quad w \in L^2, \tag{2.7}$$

(2.5) has been proved by T. Kato [*12*] and K. Yajima [*18*]. (2.5) plays an important role to prove Theorem 3. For Lemma 2.3, see, e. g., K. Yajima [*18*].

LEMMA 2.4. — Let $0 < \sigma < n$, $\delta(\sigma) = 4n/(2n - \sigma)$, $1/\delta(\sigma) + 1/\delta'(\sigma) = 1$ and $P(\psi) = ((|x|^{-\sigma} * |\psi|^2)\psi, \psi)$. Then we have for $\psi \in H^{1,1}_{\delta(\sigma)} \cap H^{1,1}$

$$\|(|x|^{-\sigma} * |\psi|^2)\psi\|_{\delta'(\sigma)} \leqq CP(\psi)^{1/2}\|\psi\|_{\delta(\sigma)}, \tag{2.8}$$

$$\sum_{|\alpha|=l} \|J^\alpha(|x|^{-\sigma} * |\psi|^2)\psi\|_{\delta'(\sigma)} \leqq C\|\psi\|^2_{\delta(\sigma)} \sum_{|\alpha|=l} \|J^\alpha\psi\|_{\delta(\sigma)}, \tag{2.9}$$

$$\|J(|x|^{-\sigma} * |\psi|^2)\psi\|_{\delta'(\sigma)} \leqq CP(\psi)^{1/2}\|J\psi\|_{\delta(\sigma)}, \quad \text{for} \quad 0 < \sigma \leqq n-2, \tag{2.10}$$

$$\begin{aligned} \||J|^\alpha(|x|^{-\sigma} * |\psi|^2)\psi\|_{\delta'(\sigma)} \leqq{}& CP(\psi)^{1/2}(|t|^{(\alpha-1)(1-b_1)}\||J|^\alpha\psi\|^{b_1}_{\delta(\sigma)}\|J\psi\|^{1-b_1} \\ &+ |t|^{\alpha(1-b_2)}\||J|^\alpha\psi\|^{b_2}_{\delta(\sigma)}\|\psi\|^{1-b_2}) \\ &+ C\||J|^\alpha\psi\|_{\delta(\sigma)}(P(\psi)^{b_3/2}\|\psi\|^{2(1-b_3)}_{2q_1} + P(\psi)^{b_4/2}\|\psi\|^{2(1-b_4)}_{2q_2}), \end{aligned} \tag{2.11}$$

for $0 < \alpha < \min\left\{\frac{4}{5}, \frac{1}{2}(n - \sigma) - (1 - b_3)\left(n\left(1 - \frac{1}{q_1}\right) - \frac{\sigma}{2}\right)\right\}$, $0 < b_1, b_2, b_3, b_4 < 1$, $2n/(2n - \sigma) < q_1 \leqq n/(n - 1)$, $1 < q_2 < 2n/(2n - \sigma)$.

Proof. — For (2.8) and (2.9), see N. Hayashi-Y. Tsutsumi [7]. We only prove (2.10) and (2.11). We note that

$$\int_{\mathbb{R}^n} |x-y|^{-\sigma} g(y)dy = \pi^{n/2} 2^{n-\sigma} \Gamma((n-\sigma)/2)/\Gamma(\sigma/2)(-\Delta)^{-(n-\sigma)/2} g(x),$$

where Γ is the Gamma function. We have by using Hölder's inequality and Lemma 2.1

$$\begin{aligned} \| ((-\Delta)^{-(n-\sigma)/2} |\psi|^2) J\psi \|_{\delta'(\sigma)} &\leqq C \| (-\Delta)^{-(n-\sigma)/2} |\psi|^2 \|_{2n/\sigma} \| J\psi \|_{\delta(\sigma)} \\ &\leqq C \| (-\Delta)^{-(n-\sigma)/4} |\psi|^2 \| \, \| J\psi \|_{\delta(\sigma)} . \end{aligned} \quad (2.12)$$

We again use Hölder's inequality and Lemma 2.1 to obtain for $\sigma \leqq n-2$

$$\begin{aligned} \| (J(-\Delta)^{-(n-\sigma)/2} |\psi|^2)\psi \|_{\delta'(\sigma)} &= \| (S(it\nabla)S^{-1}(-\Delta)^{-(n-\sigma)/2} |\psi|^2)\psi \|_{\delta'(\sigma)} \\ &= |t| \, \| (\nabla(-\Delta)^{-(n-\sigma)/2} |\psi|^2) S^{-1}\psi \|_{\delta'(\sigma)} \\ &\leqq C \| (-\Delta)^{-(n-\sigma-1)/2} |\psi|^2 \|_{2n/(\sigma+2)} |t| \, \| S^{-1}\psi \|_{4n/(2n-\sigma-4)} \\ &\leqq C \| (-\Delta)^{-(n-\sigma)/4} |\psi|^2 \| \, \| J\psi \|_{\delta(\sigma)} . \end{aligned} \quad (2.13)$$

Since $\| (-\Delta)^{-(n-\sigma)/4} |\psi|^2 \| \leqq CP(\psi)^{1/2}$, (2.12) and (2.13) imply (2.10). By the relation $|J|^\alpha = S(-t^2\Delta)^{\alpha/2}S^{-1}$ we have

$$\| |J|^\alpha(|x|^{-\sigma} * |\psi|^2)\psi \|_{\delta'(\sigma)} = |t|^\alpha \| (-\Delta)^{\alpha/2}(|x|^{-\sigma} * |\phi|^2)\phi \|_{\delta'(\sigma)}, \quad (2.14)$$

where $\phi = S^{-1}\psi$. By Theorem 6.3.1 of [2], we have for $f = (-\Delta)^{-(n-\sigma)/2}|\phi|^2$,

$$\| (-\Delta)^{\alpha/2} f\phi \|_{\delta'(\sigma)} \leqq C \| f\phi \|_{\dot{B}^\alpha_{\delta'(\sigma),1}} = C\left(\int_0^\infty \tau^{-\alpha} \sup_{|k|\leqq\tau} \| f_k\phi_k - f\phi \|_{\delta'(\sigma)} \frac{d\tau}{\tau} \right). \quad (2.15)$$

Hölder's inequality gives for $\alpha < (2n-\sigma)/4$

$$\begin{aligned} \| f_k\phi_k - f\phi \|_{\delta'(\sigma)} &\leqq C \| f_k \|_{2n/\sigma} \| \phi_k - \phi \|_{4n/(2n-\sigma)} \\ &\quad + C \| f_k - f \|_{2n/(\sigma+2\alpha)} \| \phi \|_{4n/(2n-\sigma-4\alpha)} . \end{aligned}$$

From this, (2.15) and Lemma 2.1 we have

$$\begin{aligned} \| f\phi \|_{\dot{B}^\alpha_{\delta'(\sigma),1}} &\leqq C \| (-\Delta)^{(n-\sigma)/4} f \| \left(\int_0^\infty \tau^{-\alpha} \sup_{|k|\leqq\tau} \| \phi_k - \phi \|_{4n/(2n-\sigma)} \frac{d\tau}{\tau} \right) \\ &\quad + C \| (-\Delta)^{\alpha/2}\phi \|_{\delta(\sigma)} \left(\int_0^\infty \tau^{-\alpha} \sup_{|k|\leqq\tau} \| f_k - f \|_{2n/(\sigma+2\alpha)} \frac{d\tau}{\tau} \right). \end{aligned} \quad (2.16)$$

By Theorem 6.3.1 ([2]), $\| f \|_{\dot{B}^s_{p,\infty}} \leqq C \| (-\Delta)^{s/2} f \|_p$ for $s \in \mathbb{R}$, $1 \leqq p \leqq \infty$. Hence we have from (2.16)

$$\left(\int_0^\infty \tau^{-\alpha} \sup_{|k| \leqq \tau} \| \phi_k - \phi \|_{4n/(2n-\sigma)} \frac{d\tau}{\tau}\right)$$
$$\leqq \sup_{\tau \in \mathbb{R}^+} \sup_{|k| \leqq \tau} (\| \phi_k - \phi \|_{\delta(\sigma)} / \tau^{\alpha+\varepsilon_1}) \left(\int_0^1 \tau^{-1+\varepsilon_1} d\tau\right)$$
$$+ \sup_{\tau \in \mathbb{R}^+} \sup_{|k| \leqq \tau} (\| \phi_k - \phi \|_{\delta(\sigma)} / \tau^{\alpha-\varepsilon_2}) \left(\int_1^\infty \tau^{-1-\varepsilon_2} d\tau\right)$$
$$\leqq C(\| (-\Delta)^{(\alpha+\varepsilon_1)/2} \phi \|_{\delta(\sigma)} + \| (-\Delta)^{(\alpha-\varepsilon_2)/2} \phi \|_{\delta(\sigma)}), \quad \text{for} \quad 0 < \varepsilon_1, \varepsilon_2. \tag{2.17}$$

Similarly we obtain

$$\left(\int_0^\infty \tau^{-\alpha} \sup_{|k| \leqq \tau} \| f_k - f \|_{2n/(\sigma+2\alpha)} \frac{d\tau}{\tau}\right) \leqq C(\| (-\Delta)^{(\alpha+\varepsilon_3)/2} f \|_{2n/(\sigma+2\alpha)}$$
$$+ \| (-\Delta)^{(\alpha-\varepsilon_4)/2} f \|_{2n/(\sigma+2\alpha)}), \qquad \text{for} \quad 0 < \varepsilon_3, \varepsilon_4. \tag{2.18}$$

Let $\varepsilon_1 = (1-b_1)(1-5\alpha/4)$, $0 < b_1 < 1$, $\alpha < 4/5$. Then Lemma 2.1 yields

$$\| (-\Delta)^{(\alpha+\varepsilon_1)/2} \phi \|_{\delta(\sigma)} \leqq C \| (-\Delta)^{\alpha/2} \phi \|_{\delta(\sigma)}^{b_1} \| (-\Delta)^{1/2} \phi \|^{1-b_1}. \tag{2.19}$$

Similarly we have

$$\| (-\Delta)^{(\alpha-\varepsilon_2)} \phi \|_{\delta(\sigma)} \leqq C \| (-\Delta)^{\alpha/2} \phi \|_{\delta(\sigma)}^{b_2} \| \phi \|^{1-b_2},$$
$$\text{for} \quad \varepsilon_2 = (1-b_2)5\alpha/4, \quad 0 < b_2 < 1, \quad \alpha < 4/5. \tag{2.20}$$

From Lemma 2.1 and Hölder's inequality we have for

$$\varepsilon_3 = (1-b_3)\left(n\left(1 - \frac{1}{q_1}\right) - \frac{\sigma}{2}\right), \qquad 2n/(2n-\sigma) < q_1 \leqq n/(n-1),$$
$$0 < b_3 < 1, \quad 0 < \alpha < (n-\sigma)/2 - \varepsilon_3,$$
$$\| (-\Delta)^{(\alpha+\varepsilon_3)/2} f \|_{2n/(\sigma+2\alpha)} \leqq C \| (-\Delta)^{(\alpha+\varepsilon_3)/2-(n-\sigma)/2} | \phi |^2 \|_{2n/(\sigma+2\alpha)}$$
$$\leqq C \| (-\Delta)^{-(n-\sigma)/4} | \phi |^2 \|^{b_3} \| (-\Delta)^{\varepsilon_3/2(1-b_3)-(n-\sigma)/4} | \phi |^2 \|^{1-b_3}$$
$$\leqq C \| (-\Delta)^{-(n-\sigma)/4} | \phi |^2 \|^{b_3} \| | \phi |^2 \|_{q_1}^{1-b_3}$$
$$\leqq C \| (-\Delta)^{-(n-\sigma)/4} | \phi |^2 \|^{b_3} \| \phi \|_{2q_1}^{2(1-b_3)}. \tag{2.21}$$

In the same way as (2.21) we obtain

$$\| (-\Delta)^{(\alpha-\varepsilon_4)/2} f \|_{2n/(\sigma+2\alpha)} \leqq C \| (-\Delta)^{-(n-\sigma)/4} | \phi |^2 \|^{b_4} \| \phi \|_{2q_2}^{2(1-b_4)},$$
$$\text{for} \quad \varepsilon_4 = (1-b_4)\left(\frac{\sigma}{2} - n\left(1 - \frac{1}{q_2}\right)\right), \quad 1 < q_2 < 2n/(2n-\sigma),$$
$$0 < b_4 < 1, \quad \alpha < (n-\sigma)/2 + \varepsilon_4. \tag{2.22}$$

Collecting everything, we have

$$\| |J|^{\alpha}(|x|^{-\sigma} * |\psi|^2)\psi \|_{\delta'(\sigma)} \leqq C|t|^{\alpha} \|(-\Delta)^{-(n-\sigma)/4} |\phi|^2 \|$$
$$\times (\|(-\Delta)^{\alpha/2}\phi \|_{\delta(\sigma)}^{b_1} \|(-\Delta)^{1/2}\phi \|^{1-b_1} + \|(-\Delta)^{\alpha/2}\phi \|_{\delta(\sigma)}^{b_2} \|\phi\|^{1-b_2})$$
$$+ C|t|^{\alpha} \|(-\Delta)^{\alpha/2}\phi \|_{\delta(\sigma)}$$
$$\times (\|(-\Delta)^{-(n-\sigma)/4} |\phi|^2 \|^{b_3} \|\phi\|_{2q_1}^{2(1-b_3)} + \|(-\Delta)^{-(n-\sigma)/4} |\phi|^2 \|^{b_4} \|\phi\|_{2q_2}^{2(1-b_4)}). \qquad (2.23)$$

(2.23) gives (2.11). Q. E. D.

3. PROOF OF THEOREMS 1, 2

In [10] [11] we have obtained the following results.

PROPOSITION 1. — Let $\phi_j \in \mathscr{S}(\mathbb{R}^n)$, $j \in \mathbb{N}$. Then for each j there exists a unique u_j such that

$$i\partial_t u_j + \frac{1}{2}\Delta u_j = F_j(u_j) \quad \text{in} \quad C^1(\mathbb{R}; \mathscr{S}(\mathbb{R}^n)) \qquad (3.1)$$

with $u_j(0) = \phi_j$, where $F_j(u_j) = V_{1,j}u_j + (V_2 * |u_j|^2)u_j$, $V_{1,j} = \lambda_1/(|x| + 1/j)^{\gamma_1}$.

PROPOSITION 2. — Let $\phi \in H^{0,1}$. Then there exists a unique u such that

$$u = U\phi - iGF(u) \quad \text{in} \quad C(\mathbb{R}; L^2) \qquad (3.2)$$

with $U^{-1}u \in C(\mathbb{R}; H^{0,1})$,

where $GF(u) = \int_0^t U(t-s)F(u(s))ds, \qquad F(u) = V_1 u + (V_2 * |u|^2)u$.

PROPOSITION 3. — Let $\phi \in H^{0,2}$. Then there exists a unique u satisfying (3.2) with $U^{-1}u \in C(\mathbb{R}; H^{0,2})$.

PROPOSITION 4. — Let $\{\phi_j\}$ be a sequence in $\mathscr{S}(\mathbb{R}^n)$ such that $\phi_j \to \phi$ in $H^{0,1}$ as $j \to \infty$. Let u_j be the solution of (3.1) constructed in Proposition 1, and let u be the solution of (3.2) constructed in Proposition 2. Then we have

$$U^{-1}u_j \to U^{-1}u \quad \text{in} \quad C(\mathbb{R}; H^{0,1}) \quad \text{as} \quad j \to \infty, \qquad (3.3)$$
$$\sup_{j \in \mathbb{N}} \alpha_j(t) \leqq C(\sup_{j \in \mathbb{N}} \|\phi_j\|_{0,1})(1+|t|)^{2-\gamma_0}, \quad t \in \mathbb{R}, \qquad (3.4)$$
$$\alpha(t) \leqq C(\|\phi\|_{0,1})(1+|t|)^{2-\gamma_0}, \quad t \in \mathbb{R}, \qquad (3.5)$$

where $\gamma_0 = \min(\gamma_1, \gamma_2, \gamma_3)$,

$$\alpha_j(t) = \|Ju_j(t)\|^2 + 2t^2(V_{1,j}u_j(t), u_j(t)) + t^2((V_2 * |u_j|^2)u_j(t), u_j(t)),$$
$$\alpha(t) = \|Ju(t)\|^2 + 2t^2(V_1 u(t), u(t)) + t^2((V_2 * |u|^2)u(t), u(t)).$$

PROPOSITION 5. — Let $\{ \phi_j \}$ be a sequence in $\mathscr{S}(\mathbb{R}^n)$ such that $\phi_j \to \phi$ in $H^{0,2}$ as $j \to \infty$. Let u_j be the solution of (3.1) constructed in Proposition 1, and let u be the solution of (3.2) constructed in proposition 3. Let $n \geqq 4$, $(3/2) \leqq \gamma_1, \gamma_2, \gamma_3 < 2$. Then we have

$$U^{-1}u_j \to U^{-1}u \quad \text{in} \quad C(\mathbb{R}; H^{0,2}) \quad \text{as} \quad j \to \infty, \tag{3.6}$$

$$\sup_{j\in\mathbb{N}} \beta_j(t) \leqq C(\sup_{j\in\mathbb{N}} \| \phi_j \|_{0,2})(1 + |t|)^{2(2-\gamma_0)}, \quad t \in \mathbb{R}, \tag{3.7}$$

$$\beta(t) \leqq C(\| \phi \|_{0,2})(1 + |t|)^{2(2-\gamma_0)}, \quad t \in \mathbb{R}, \tag{3.8}$$

$$\sup_{j\in\mathbb{N}} \int_{\mathbb{R}\setminus[-1,1]} |t|^{2\gamma_0-5}(\| Ju_j(t) \|^2 + t^2 \| V_{1,j}^{1/2} Ju_j(t) \|^2)dt \leqq C(\sup_{j\in\mathbb{N}} \| \phi_j \|_{0,2}). \tag{3.9}$$

$$\int_{\mathbb{R}\setminus[-1,1]} |t|^{2\gamma_0-5}(\| Ju(t) \|^2 + t^2 \| V_1^{1/2} Ju(t) \|^2)dt \leqq C(\| \phi \|_{0,2}), \tag{3.10}$$

$$\sup_{j\in\mathbb{N}} \| Ju_j(t) \| \leqq C(\sup_{j\in\mathbb{N}} \| \phi_j \|_{0,2}), \quad t \in \mathbb{R}, \tag{3.11}$$

$$\| Ju(t) \| \leqq C(\| \phi \|_{0,2}), \quad t \in \mathbb{R}, \tag{3.12}$$

where

$$\beta_j(t) = \| J^2 u_j(t) + 2t^2 F_j(u_j(t)) \|^2 - 4t^2(V_2 * \operatorname{Im} \overline{u_j} J u_j(t), \ \operatorname{Im} \overline{u_j} J u_j(t)),$$
$$\beta(t) = \| J^2 u(t) + 2t^2 F(u(t)) \|^2 - 4t^2(V_2 * \operatorname{Im} \bar{u} J u(t), \ \operatorname{Im} \bar{u} J u(t)).$$

Proof of Theorem 1. — Let u_j be the solution of (3.1) constructed in Proposition 1, and let u be the solution of (3.2) constructed in Proposition 2. Let $w(t) = S(t)U(-t)u(t)$, $w_j(t) = S(t)U(-t)u_j(t)$, $t \neq 0$. We restrict our attention to the case $t > 0$, since the other case can be treated analogously. We first prove that there exists $u_+ \in L^2$ satisfying $w(t) \to u_+$ in L^2 as $t \to \infty$. It suffices to show that $\{ w(t); t > 1 \}$ is Cauchy in L^2. Let $t > \tau > 1$. Since $\| w(t) \| = \| \phi \|$, $t \neq 0$, we have

$$\| w(t) - w(\tau) \|^2 = -2\operatorname{Re}(w(t) - w(\tau), \ w(\tau)).$$

We estimate the R. H. S. of the above equality.
From (3.1) we have

$$\begin{aligned} \frac{d}{ds} w_j(s) &= S(s)\left(-i \frac{|x|^2}{2s^2} U(-s)u_j(s) - iU(-s)\left(\frac{1}{2}\Delta u_j(s) + i\frac{d}{ds} u_j(s)\right)\right) \\ &= -iS(s)\left(\frac{|x|^2}{2s^2} U(-s)u_j(s) + U(-s)F_j(u_j(s))\right) \\ &= iS(s)\frac{|x|^2}{2s^2} U(-s)u_j(s) - i\mathscr{F}^{-1}D(s)^{-1}S(-s)F_j(u_j(s)). \end{aligned}$$

A direct calculation shows

$$D(s)^{-1}S(-s)V_{1,j}u_j(s) = s^{-\gamma_1}V_{1,js}D(s)^{-1}S(-s)u_j(s),$$

$$D(s)^{-1}S(-s)(V_2 * |u_j|^2)u_j(s)$$

$$= \sum_{k=2}^{3} s^{-\gamma_k}(V_2^{(k)} * |D(s)^{-1}S(-s)u_j(s)|^2)D(s)^{-1}S(-s)u_j(s),$$

where $V_2^{(2)} = \lambda_2 |x|^{-\gamma_2}$, $V_2^{(3)} = \lambda_3 |x|^{-\gamma_3}$. Thus we have the identity

$$(w_j(t)-w_j(\tau), \quad w_j(\tau)) = \left(\int_\tau^t \frac{d}{ds}(w_j(s))ds, \quad w_j(\tau)\right) = I_1 + I_2 + I_3, \tag{3.13}$$

where

$$I_1 = -i\left(\int_\tau^t S(s)\frac{|x|^2}{2s^2}U(-s)u_j(s)ds, \quad w_j(\tau)\right)$$

$$= -i\left(\int_\tau^t S(s)\frac{1}{2s^2}U(-s)J(s)u_j(s)ds, \quad S(\tau)U(-\tau)J(\tau)u_j(\tau)\right),$$

$$I_2 = -i\left(\int_\tau^t \mathscr{F}^{-1}D(s)^{-1}S(-s)V_{1,j}u_j(s)ds, \quad \mathscr{F}^{-1}D(\tau)^{-1}S(-\tau)u_j(\tau)\right)$$

$$= -i\left(\int_\tau^t s^{-\gamma_1}V_{1,js}D(s)^{-1}S(-s)u_j(s)ds, \quad D(\tau)^{-1}S(-\tau)u_j(\tau)\right),$$

$$I_3 = -i\left(\int_\tau^t \mathscr{F}^{-1}D(s)^{-1}S(-s)(V_2 * |u_j|^2)u_j(s)ds, \quad \mathscr{F}^{-1}D(\tau)^{-1}S(-\tau)u_j(\tau)\right),$$

$$= -i\sum_{k=2}^{3}\left(\int_\tau^t s^{-\gamma_k}(V_2^{(k)} * |D(s)^{-1}S(-s)u_j(s)|^2)D(s)^{-1}S(-s)u_j(s)ds, \right.$$
$$\left. D(\tau)^{-1}S(-\tau)u_j(\tau)\right), \quad t > \tau > 1.$$

I_1 is estimated by

$$|I_1| \leqq \int_\tau^t s^{-2}\|J(s)u_j(s)\|ds\|J(\tau)u_j(\tau)\|.$$

I_2 is estimated by

$$|I_2| \leqq \int_\tau^t s^{-\gamma_1}\|V_1^{1/2}D(s)^{-1}S(-s)u_j(s)\|ds\|V_1^{1/2}D(\tau)^{-1}S(-\tau)u_j(\tau)\|$$

$$= \int_\tau^t s^{-\gamma_1/2}\|V_1^{1/2}u_j(s)\|ds \cdot \tau^{\gamma_1/2}\|V_1^{1/2}u_j(\tau)\|.$$

I_3 is estimated by

$$|I_3| \leqq \frac{1}{2}\sum_{k=2}^{3}\int_\tau^t s^{-\gamma_k}((V_2^{(k)} * |D(s)^{-1}S(-s)u_j(s)|^2, |D(s)^{-1}S(-s)u_j(s)|^2)$$

$$+ \| I_{(\gamma_k+n)/2} | D(s)^{-1} S(-s) u_j(s) |^2 \| \, \| I_{(\gamma_k+n)/2} | D(\tau)^{-1} S(-\tau) u_j(\tau) |^2 \|) ds$$
$$= \frac{1}{2} \int_\tau^t (V_2 * | u_j(s) |^2, | u_j(s) |^2) ds$$
$$= \frac{1}{2} \sum_{k=2}^{3} \int_\tau^t s^{-\gamma_k/2} \| I_{(\gamma_k+n)/2} | u_j(s) |^2 \| ds \cdot \tau^{\gamma_k/2} \| I_{(\gamma_k+n)/2} | u_j(\tau) |^2 \| .$$

Since $w_j \to w$ in $C(\mathbb{R} \setminus \{ 0 \}; L^2)$ as $j \to \infty$, the L. H. S. of (3.13) tends to

$$(w(t) - w(\tau), w(\tau)).$$

In view of (3.4), the R. H. S. of (3.13) is bounded uniformly in $j \in \mathbb{N}$ by

$$C \cdot (\tau^{-\gamma_0/2} + t^{-\gamma_0/2}) \cdot \tau^{1-\gamma_0/2} + C \cdot \sum_{k=1}^{3} (\tau^{1-\gamma_k/2-\gamma_0/2} + t^{1-\gamma_k/2-\gamma_0/2}) \cdot \tau^{\gamma_k/2-\gamma_0/2}$$
$$+ C \cdot (\tau^{1-\gamma_0} + t^{1-\gamma_0}), \quad t > \tau > 1 .$$

This proves that $w(t)$ converges in L^2 as $t \to \infty$. Now

$$\| U(-t)u(t) - u_+ \| \leqq \| w(t) - u_+ \| + \| S(t)u_+ - u_+ \|$$

so that $U(-t)u(t) \to u_+$ in L^2 as $t \to \infty$. Q. E. D.

Proof of Theorem 2. — Let u_j be the solution of (3.1) constructed in Proposition 1, and let u be the solution of (3.2) constructed in Proposition 3. We already know that there exist $u_\pm \in L^2$ such that

$$U(-t)u(t) \to u_\pm \quad \text{in} \quad L^2 \quad \text{as} \quad t \to \pm \infty . \tag{3.14}$$

From now on we consider only the case $t > 0$. We first claim that for any $\psi \in L^2$, $\{ (xU(-t)u(t), \psi); t > 0 \}$ is Cauchy in $\mathbb{C}$. Indeed, we have, for $\psi_\varepsilon \in \mathcal{S}$ ($\varepsilon > 0$) such that $\psi_\varepsilon \to \psi$ in L^2 as $\varepsilon \to +0$.

$$| (xU(-t)u(t) - xU(-\tau)u(\tau), \psi) | \leqq \| Ju(t) - Ju(\tau) \| \; \| \psi - \psi_\varepsilon \|$$
$$+ \| U(-t)u(t) - U(-\tau)u(\tau) \| \; \| x\psi_\varepsilon \| ,$$

so that our claim follows from (3.12) and (3.14). Thus $u_+ \in H^{0,1}$ and $xU(-t)u(t) \to xu_+$ weakly in L^2 as $t \to \infty$. This gives $S(t)xU(-t)u(t) \to xu_+$ weakly in L^2 as $t \to \infty$, since the operator $S(t)$ tends to I strongly in L^2 as $t \to \pm\infty$. We now prove that $S(t)xU(-t)u(t) \to xu_+$ in L^2 as $t \to \infty$. For this purpose we compute

$$(S(t)xU(-t)u_j(t) - S(\tau)xU(-\tau)u_j(\tau), S(\tau)xU(-\tau)u_j(\tau)) = I_4 + I_5 + I_6 ,$$

where

$$I_4 = -i\left(\int_\tau^t S(s)x\frac{|x|^2}{2s^2}U(-s)u_j(s)ds,\ S(\tau)xU(-\tau)u_j(\tau)\right)$$
$$= -i\left(\int_\tau^t S(s)\ \frac{1}{2s^2}U(-s)J^2u_j(s)ds,\ S(\tau)U(-\tau)J^2u_j(\tau)\right),$$
$$I_5 = -i\left(\int_\tau^t S(s)xU(-s)V_{1,j}u_j(s)ds,\ S(\tau)xU(-\tau)u_j(\tau)\right),$$
$$I_6 = -i\left(\int_\tau^t S(s)xU(-s)(V_2*|u_j|^2)u_j(s)ds,\ S(\tau)xU(-\tau)u_j(\tau)\right).$$

I_4 is estimated by

$$|I_4| \leqq \int_\tau^t s^{-2}\|J^2u_j(s)\|ds\|J^2u_j(\tau)\|$$
$$\leqq \left(\int_\tau^t s^{1-2\gamma_0}ds\right)^{1/2}\left(\int_\tau^t s^{2\gamma_0-5}\|J^2u_j(s)\|^2ds\right)^{1/2}\|J^2u_j(\tau)\|\ .$$

In order to estimate I_5, we write

$$\begin{aligned}(S(s)xU(-s)V_{1,j}u_j(s),\ S(\tau)xU(-\tau)u_j(\tau))&\\ &= (S(s)U(-s)JV_{1,j}u_j(s),\ S(\tau)U(-\tau)Ju_j(\tau))\\ &= (S(s)U(-s)V_{1,j}Ju_j(s),\ S(\tau)U(-\tau)Ju_j(\tau))\\ &\quad + is(S(s)U(-s)(\nabla V_{1,j})u_j(s),\ S(\tau)U(-\tau)Ju_j(\tau))\\ &= s^{-\gamma_1}(V_{1,js}D(s)^{-1}S(-s)Ju_j(s),\ D(\tau)^{-1}S(-\tau)Ju_j(\tau))\\ &\quad + is^{-\gamma_1}((\nabla V_{1,js})D(s)^{-1}S(-s)u_j(s),\ D(\tau)^{-1}S(-\tau)Ju_j(\tau))\end{aligned}$$

and therefore

$$\begin{aligned}(S(s)xU(-s)V_{1,j}u_j(s),\ S(\tau)xU(-\tau)u_j(\tau))&\\ &\leqq s^{-\gamma_1}\|V_1^{1/2}D(s)^{-1}S(-s)Ju_j(s)\|\ \|V_1^{1/2}D(\tau)^{-1}S(-\tau)Ju_j(\tau)\|\\ &\quad + s^{-\gamma_1}\||x|^{-1}V_1^{1/2}D(s)^{-1}S(-s)u_j(s)\|\ \|V_1^{1/2}D(\tau)^{-1}S(-\tau)Ju_j(\tau)\|\\ &\leqq s^{-\gamma_1/2}\|V_1^{1/2}Ju_j(s)\|\tau^{\gamma_1/2}\|V_1^{1/2}Ju_j(\tau)\|\\ &\quad + s^{-(\gamma_1/2)+1}\||x|^{-1}V_1^{1/2}u_j(s)\|\tau^{\gamma_1/2}\|V_1^{1/2}Ju_j(\tau)\|\\ &\leqq Cs^{-\gamma_1/2}\|V_1^{1/2}Ju_j(s)\|\tau^{\gamma_1/2}\|V_1^{1/2}Ju_j(\tau)\|,\end{aligned}$$

since (see [*10*] [*11*].)

$$s^{-(\gamma_1/2)+1}\||x|^{-1}V_1^{1/2}u_j(s)\| \leqq Cs^{-\gamma_1/2}\|JV_1^{1/2}u_j(s)\|$$
$$\leqq Cs^{-\gamma_1/2}\|V_1^{1/2}Ju_j(s)\|.$$

Thus I_5 is estimated by

$$|I_5| \leqq \left(\int_\tau^t s^{3-2\gamma_0-\gamma_1}ds\right)^{1/2}\left(\int_\tau^t s^{2\gamma_0-3}\|V_1^{1/2}Ju_j(s)\|^2ds\right)^{1/2}$$
$$\times\ \tau^{\gamma_1/2}\|V_1^{1/2}Ju_j(\tau)\|.$$

For I_6, we write

$$\begin{aligned}
&(S(s)xU(-s)(V_2 * |u_j|^2)u_j(s), S(\tau)xU(-\tau)u_j(\tau)) \\
&\quad = (S(s)U(-s)(V_2 * |u_j|^2)Ju_j(s), S(\tau)U(-\tau)Ju_j(\tau)) \\
&\qquad + is(S(s)U(-s)((\nabla V_2) * |u_j|^2)u_j(s), S(\tau)U(-\tau)Ju_j(\tau)) \\
&\quad = \sum_{k=2}^{3} s^{-\gamma_k}((V_2^{(k)} * |D(s)^{-1}S(-s)u_j(s)|^2)D(s)^{-1}S(-s)Ju_j(s), D(\tau)^{-1}S(-\tau)Ju_j(\tau)) \\
&\quad + \sum_{k=2}^{3} is^{-\gamma_k}(((\nabla V_2^{(k)}) * |D(s)^{-1}S(-s)u_j(s)|^2)D(s)^{-1}S(-s)u_j(s), \\
&\qquad\qquad D(\tau)^{-1}S(-\tau)Ju_j(\tau)),
\end{aligned}$$

where $V_2^{(2)} = \lambda_2 |x|^{-\gamma_2}$, $V_2^{(3)} = \lambda_3 |x|^{-\gamma_3}$. Since

$$\| \nabla D(s)^{-1}S(-s)u_j(s) \| = \| Ju_j(s) \|,$$

the first two terms are estimated by

$$\begin{aligned}
&\sum_{k=2}^{3} s^{-\gamma_k} \| V_2^{(k)} * |D(s)^{-1}S(-s)u_j(s)|^2 \|_\infty \| Ju_j(s) \| \| Ju_j(\tau) \| \\
&\leqq C \sum_{k=2}^{3} s^{-\gamma_k} \| Ju_j(s) \|^{\gamma_k} \| u_j(s) \|^{2-\gamma_k} \| Ju_j(s) \| \| Ju_j(\tau) \|,
\end{aligned}$$

and the last two terms are estimated by

$$\begin{aligned}
&\sum_{k=2}^{3} s^{-\gamma_k} \| (\nabla V_2^{(k)}) * |D(-s)S(-s)u_j(s)|^2 \|_n \\
&\qquad \times \| D(-s)S(-s)u_j(s) \|_{2n/(n-2)} \| Ju_j(\tau) \| \\
&\qquad \leqq C \sum_{k=2}^{3} s^{-\gamma_k} \| Ju_j(s) \|^{\gamma_k} \| u_j(s) \|^{2-\gamma_k} \| Ju_j(s) \| \| Ju_j(\tau) \|.
\end{aligned}$$

Combining these estimates with (3.6)-(3.12), we conclude that

$$|(xu_+ - S(\tau)xU(-\tau)u(\tau), S(\tau)xU(-\tau)u(\tau))| \to 0 \quad \text{as} \quad \tau \to \infty .$$

This yields

$$\begin{aligned}
\| xu_+ - S(\tau)xU(-\tau)u(\tau) \|^2 &= (xu_+ - S(\tau)xU(-\tau)u(\tau), xu_+) \\
&\quad - (xu_+ - S(\tau)xU(-\tau)u(\tau), S(\tau)xU(-\tau)u(\tau)) \to 0 \quad \text{as} \quad \tau \to \infty .
\end{aligned}$$

Thus

$$\| xU(\tau)u(\tau) - xu_+ \| \leqq \| S(\tau)xU(-\tau)u(\tau) - xu_+ \| + \| xu_+ - S(\tau)xu_+ \| \to 0 \quad \text{as} \quad \tau \to \infty,$$

as desired. Q. E. D.

REMARK 2. — In the case of nonlinear Schrödinger equation (NLS eq.) such that $i\partial_t u + \frac{1}{2}\Delta u = |u|^{p_1-1}u + |u|^{p_2-1}u$ with $u(0) = \phi \in H^{1,1}$, Y. Tsutsumi-K. Yajima [17] showed Theorem 1 if $1+(2/n) < p_1 = p_2 < \alpha(n)$, where $\alpha(n) = \infty$ for $n = 1, 2$, $\alpha(n) = (n+2)/(n-2)$ for $n \geqq 3$. For any $\phi \in H^{0,1}$, we can apply our method of Theorem 1 to the NLS eq. if $1+(2/n) < p_1 \leqq p_2 < 1+(4/n)$. Indeed, we can prove Theorem 1 in the case of the NLS eq. as follows: we put

$$X(a, p) = \{ u \in C(I; L^2) \cap L^{\frac{4}{n}\left(\frac{p+1}{p-1}\right)}(I; L^{p+1}); \ |||u|||_{X(a,p)} = |||u|||_{2,\infty} + |||u|||_{p+1,\frac{4}{n}\left(\frac{p+1}{p-1}\right)} < \infty \},$$

where $I = [-a, a]$, $a > 0$. By the existence theorem of solutions for the NLS eq. obtained by T. Kato [12], the NLS eq. has a unique solution such that u, $Ju \in X(a, p_2)$ for any $a > 0$.

Also we have from the pseudoconformal conservation law and u, $Ju \in X(a, p_2)$

$$|t|^{-2} \| Ju(t) \|^2 + \| u(t) \|_{p_1+1}^{p_1+1} + \| u(t) \|_{p_2+1}^{p_2+1} \leqq C |t|^{-n(p_1+1)/2}$$

for $t \neq 0$, where C is a positive constant depending only on $\phi \in H^{0,1}$ (see also [1] [3] [15] [16]). From this and the same argument as Theorem 1 we have the desired result.

4. PROOF OF THEOREM 3

Proof of Theorem 3. — For simplicity we let $\gamma_2 \geqq \gamma_3$ and we suppress the subscript j of u_j in (3.1). By (3.1) we have

$$u(t) = U(t)\phi_j - i\int_0^t U(t-s)(V_2 * |u|^2)u(s)ds, \tag{4.1}$$

$$Ju(t) = U(t)x\phi_j - i\int_0^t U(t-s)J(V_2 * |u|^2)u(s)ds. \tag{4.2}$$

We first prove that the solutions of (4.1) form a bounded sequence in $X(\infty, \gamma_2)$.

We apply Lemma 2.3 and Lemma 2.4 to (4.1) to obtain

$$\begin{aligned}
|||\, u\,|||_{X(a,\gamma_2)} &\leqq C\,\|\phi_j\| + C\sum_{k=2}^{3} |||\,(\,|x|^{-\gamma_k} * |u|^2)u\,|||_{\delta'(\gamma_k),8/(8-\gamma_k)} \\
&\leqq C\,\|\phi_j\| + C\sum_{k=2}^{3}\left(\int_{-a}^{a}\|u(t)\|_{\delta(\gamma_k)}^{24/(8-\gamma_k)}dt\right)^{(8-\gamma_k)/8} \\
&\leqq C\,\|\phi_j\| + C\sum_{k=2}^{3}\left(\int_{-a}^{a}dt\right)^{2/(2-\gamma_k)}\left(\int_{-a}^{a}\|u(t)\|_{\delta(\gamma_k)}^{8/\gamma_k}dt\right)^{3\gamma_k/8} \\
&\leqq C\,\|\phi_j\| + C\sum_{k=2}^{3} a^{2/(2-\gamma_k)}\,|||\,u\,|||_{\delta(\gamma_k),8/\gamma_k}^{3} \\
&\leqq C\,\|\phi_j\| + C\sum_{k=2}^{3} a^{2/(2-\gamma_k)}\,|||\,u\,|||_{X(a,\gamma_2)}^{3}\,. \qquad (4.3)
\end{aligned}$$

We put $y(a) = |||\,u\,|||_{X(a,\gamma_2)}$. Then we have

$$y(a) \leqq C_1\,\|\phi_j\| + C_2 a^{2/(2-\gamma_3)}y(a)^3 \qquad \text{for} \qquad 0 < a < 1\,. \qquad (4.4)$$

Let $a_1 = \min\{\,1, (50C_2C_1^2\,\|\phi_j\|^2)^{-2(2-\gamma_3)}\}$. Then we have by (4.4) and Lemma 3.7 of [*14*]

$$y(a) \leqq 2C_1\,\|\phi_j\| \qquad \text{for} \qquad 0 < a < a_1\,. \qquad (4.5)$$

We have by (4.5) and the fact that $|||\,u\,|||_{2,\infty} = \|\phi_j\|$

$$y(T) \leqq C(\,\|\phi_j\|, T) \qquad \text{for any} \quad T \in \mathbb{R}^+\,. \qquad (4.6)$$

In the same way as in the proof of (4.3) we obtain

$$\begin{aligned}
|||\, Ju\,|||_{X(a,\gamma_2)} &\leqq C\,\|x\phi_j\| + C\sum_{k=2}^{3} a^{2/(2-\gamma_k)}\,|||\,u\,|||_{X(a,\gamma_2)}^{2}\;|||\,Ju\,|||_{X(a,\gamma_2)} \\
&\leqq C\,\|x\phi_j\| + C(\,\|\phi_j\|, T)\sum_{k=2}^{3} a^{2/(2-\gamma_k)}\,|||\,Ju\,|||_{X(a,\gamma_2)}\,, \qquad (4.7)
\end{aligned}$$

from which we get $|||\,Ju\,|||_{X(a,\gamma_2)} \leqq C(\,\|\phi_j\|_{0,1})$ for a sufficiently small. By using (3.4), we iterate this process to get $|||\,Ju\,|||_{X(la,\gamma_2)} \leqq C(\|\phi_j\|_{0,1})$, $l \in \mathbb{N}$, inductively. Thus,

$$|||\,Ju\,|||_{X(T,\gamma_2)} \leqq C(\,\|\phi_j\|_{0,1}, T) \qquad \text{for any} \quad T \in \mathbb{R}^+\,. \qquad (4.8)$$

We let $T = \infty$. By virtue of Lemma 2.3, Lemma 2.4 (2.8), (4.3) and Proposition 4 we have

$$\begin{aligned} \||u\||_{X(\infty,\gamma_2)} &\leqq C\|\phi_j\| + C\sum_{k=2}^{3} \||(|x|^{-\gamma_k} * |u|^2)u\||_{\delta'(\gamma_k),8/(8-\gamma_k)} \\ &\leqq C\|\phi_j\| + C\sum_{k=2}^{3}\left(\int_{-b}^{b} \|u(s)\|_{\delta(\gamma_k)}^{24/(8-\gamma_k)} ds\right)^{(8-\gamma_k)/8} \\ &\quad + C\sum_{k=2}^{3}\left(\int_{\mathbb{R}\setminus[-b,b]} (((|x|^{-\gamma_k} * |u|^2)u(s), u(s))^{1/2} \times \|u(s)\|_{\delta(\gamma_k)})^{8/(8-\gamma_k)} ds\right)^{(8-\gamma_k)/8} \end{aligned} \tag{4.9}$$

Proposition 4, (4.6) and (4.7) imply

$$\||u\||_{X(\infty,\gamma_2)} \leqq C(\|\phi_j\|, b) + C(\|\phi_j\|_{0,1}) \times \sum_{k=2}^{3}\left(\left(\int_{\mathbb{R}\setminus[-b,b]} (|s|^{-\gamma_3/2}\|u(s)\|_{\delta(\gamma_k)}\right)^{8/(8-\gamma_k)} ds\right)^{(8-\gamma_k)/8}, \quad \text{for} \quad b > 1. \tag{4.10}$$

Since $4/3 < \gamma_3, \gamma_3$, a simple calculation gives

$$\begin{aligned} &\sum_{k=2}^{3}\left(\int_{\mathbb{R}\setminus[-b,b]} |s|^{-4\gamma_3/(8-\gamma_k)} \|u(s)\|_{\delta(\gamma_k)}^{8/(8-\gamma_k)} ds\right)^{(8-\gamma_k)/8} \\ &\leqq \sum_{k=2}^{3}\left(\int_{\mathbb{R}\setminus[-b,b]} |s|^{-2\gamma_3/(4-\gamma_k)} ds\right)^{(4-\gamma_k)/4}\left(\int_{\mathbb{R}\setminus[-b,b]} \|u(s)\|_{\delta(\gamma_k)}^{8/\gamma_k} ds\right)^{\gamma_k/8} \\ &\leqq C\sum_{k=2}^{3} b^{(1-(2\gamma_3/(4-\gamma_k)))(4-\gamma_k)/4} \||u\||_{\delta(\gamma_k),8/\gamma_k} \\ &\leqq Cb^{(4-3\gamma_3)/4}\sum_{k=2}^{3} \||u\||_{\delta(\gamma_k),8/\gamma_k}, \qquad \text{for} \quad b > 1. \end{aligned} \tag{4.11}$$

Thus we have from (4.10) and (4.11)

$$\||u\||_{X(\infty,\gamma_2)} \leqq C(\|\phi_j\|, b) + C(\|\phi_j\|_{0,1}) b^{(4-3\gamma_3)/4} \||u\||_{X(\infty,\gamma_2)}, \quad \text{for} \quad b > 1. \tag{4.12}$$

We choose b large enough to ensure that $C(\|\phi_j\|_{0,1}) b^{(4-3\gamma_3)/4} < 1/2$. Finally we get

$$\||u\||_{X(\infty,\gamma_2)} \leqq C(\|\phi_j\|_{0,1}), \tag{4.13}$$

as required. We continue the proof of the theorem and treat the cases $n \geqq 4$ and $n = 3$ separately. We first consider the case $n \geqq 4$. In the same way as in the proof of (4.13) we obtain by Lemma 2.4 (2.10) if $4/3 < \gamma_2, \gamma_3 < n - 2$

$$||| Ju |||_{X(\infty,\gamma_2)} \leqq C(\| \phi_j \|_{0,1}) . \qquad (4.14)$$

Let $j \to \infty$ in (4.13) and (4.14). We have

$$||| u |||_{X(\infty,\gamma_2)} + ||| Ju |||_{X(\infty,\gamma_2)} \leqq C(\| \phi \|_{0,1}) . \qquad (4.15)$$

We now consider the following integral equation for any $u_+ \in H^{0,1}$

$$u(t) = U(t)u_+ - i \int_t^{\infty} U(t-s)(V_2 * | u |^2)u(s)ds . \qquad (4.16)$$

(4.16) is the integral version of the initial value problem (1.1) with the initial data given at $+\infty$ and $\lambda_1 = 0$. In the same way as in the proof of Theorem 5 of [*10*], we can prove that there exists a unique solution u of (4.16) such that $u, Ju \in C(\mathbb{R}; L^2) \cap L^{8/\gamma_2}(\mathbb{R}; L^{\delta(\gamma_2)})$ for any $u_+ \in H^{0,1}$. Let u be the solution of (4.16) mentioned above. In the same way as in the proof of (4.8) we have for sufficiently large T

$$\left(\int_T^{\infty} \| u(t) \|_{\delta(\gamma_2)}^{8/\gamma_2} dt \right)^{\gamma_2/8} + \sup_{t \in [T,\infty)} \| Ju(t) \| + \left(\int_T^{\infty} \| Ju(t) \|_{\delta(\gamma_2)}^{8/\gamma_2} dt \right)^{\gamma_2/8} \leqq C(\| u_+ \|_{0,1}) . \qquad (4.17)$$

By (4.15) we can take $T = -\infty$ in(4.17). We put

$$\phi = u(0) = u_+ - i \int_0^{\infty} U(-s)(V_2 * | u |^2)u(s)ds .$$

This and (4.17) with $T = 0$ imply that there exists the wave operator $W_+ : u_+ \mapsto \phi$ in $H^{0,1}$. In the same way for any $\phi \in H^{0,1}$ there exists a unique $u_- \in H^{0,1}$ such that $\| U(-t)u(t) - u_- \|_{0,1} \to 0$ as $t \to -\infty$. This implies that there exists the inverse wave operator $W_-^{-1} : \phi \mapsto u_-$. Therefore the inverse of the scattering operator $W_-^{-1}W^+$ exists in $H^{0,1}$. In the case $s = 1$, Theorem 3 follows from the same argument as in the proof of Corollary 5.1 in [*7*]. We prove the case $s \geqq 2$. In the same way as in the proof of (4.8) we have by Lemma 2.4 (2.9)

$$\sum_{|\beta| \leqq s} ||| J^{\beta} u |||_{X(T,\gamma_2)} \leqq C(\| \phi_j \|_{0,s}, T) , \qquad \text{for any} \quad T \in \mathbb{R}^+ . \qquad (4.18)$$

From (4.15) and Lemma 2.1 we get

$$\| u(t) \|_{\delta(\gamma_2)} \leqq C | t |^{-\gamma_2/4} \| u(t) \|^{(4-\gamma_2)/4} \| Ju(t) \|^{\gamma_2/4} \leqq C(\| \phi \|_{0,1}) | t |^{-\gamma_2/4}, \qquad \text{for} \quad t \neq 0. \qquad (4.19)$$

In the similar way as in the proof of (4.15) we obtain by using (4.18), (4.19) and Lemma 2.4 (2.9) in place of Lemma 2.4 (2.8)

$$\sum_{|\beta| \leqq s} |||\, J^{\beta} u \,|||_{X(\infty,\gamma_2)} \leqq C(\| \phi_j \|_{0,s}) . \tag{4.20}$$

In the case $n \geqq 4$, Theorem 3 follows from (4.20) and the same argument as in the case $s = 1$. We next consider the case $n = 3$. In the same way as in the proof of (4.9) we have

$$|||\, |J|^{\alpha} u \,|||_{X(T,\gamma_2)} \leqq C \| \,|x|^{\alpha} \phi_j \| + C \sum_{k=2}^{3} \left(\int_{-T}^{T} \| \,|J|^{\alpha}(|x|^{-\gamma_k} * |u|^2) u(s) \|_{\delta'(\gamma_k)}^{8/(8-\gamma_k)} ds \right)^{(8-\gamma_k)/8} . \tag{4.21}$$

By Lemma 2.4 (2.11)

$$\begin{aligned} \| \,|J|^{\alpha}(|x|^{-\gamma_k} * |u|^2) u \|_{\delta'(\gamma_k)} &\leqq C P(u)^{1/2} (|s|^{(\alpha-1)(1-b_1)} \| Ju \|^{1-b_1} \| \,|J|^{\alpha} u \|_{\delta(\gamma_k)}^{b_1} \\ &\quad + |s|^{\alpha(1-b_1)} \| u \|^{1-b_1} \| \,|J|^{\alpha} u \|_{\delta(\gamma_k)}^{b_2}) \\ &\quad + C \sum_{l=1}^{2} P(u)^{b_{2+l}/2} \| u \|_{2q_l}^{2(1-b_{2+l})} \| \,|J|^{\alpha} u \|_{\delta(\gamma_k)} , \quad s \neq 0 . \end{aligned} \tag{4.22}$$

Let $\varepsilon_l = 1 - b_l > 0$, $(1 \leqq l \leqq 4)$, $\varepsilon_5 = 2q_1 - \delta(\gamma_k) > 0$, $\varepsilon_6 = \delta(\gamma_k) - 2q_2 > 0$ be sufficiently small and $b_1 = b_2$. Since $P(u) \leqq C \| u \|_{\delta(\gamma_k)}^4$ by Lemma 2.2, we have from (4.22) and (4.8)

$$\begin{aligned} \| \,|J|^{\alpha}(|x|^{-\gamma_k} * |u|^2) u \|_{\delta'(\gamma_k)} &\leqq C(\| \phi_j \|_{0,1}, T) \| u \|_{\delta(\gamma_k)}^2 (|s|^{(\alpha-1)\varepsilon_1} + |s|^{\alpha\varepsilon_1}) \| \,|J|^{\alpha} u \|_{\delta(\gamma_k)}^{1-\varepsilon} \\ &\quad + C \sum_{l=1}^{2} \| u \|_{\delta(\gamma_k)}^{2(1-\varepsilon_{l+2})} \| u \|_{2q_l}^{2\varepsilon_{l+2}} \| \,|J|^{\alpha} u \|_{\delta(\gamma_k)} . \end{aligned} \tag{4.23}$$

Hölder's inequality and Lemma 2.1 give

$$\| u \|_{2q_1} \leqq C |s|^{-\varepsilon_7} \| u \|_{\delta(\gamma_k)}^{1-\varepsilon_7} \| Ju \|^{\varepsilon_7}, \qquad s \neq 0 , \tag{4.24}$$

$$\| u \|_{2q_2} \leqq C \| u \|_{\delta(\gamma_k)}^{1-\varepsilon_8} \| u \|^{\varepsilon_8} , \tag{4.25}$$

where

$$\varepsilon_7 = 3(2q_1 - \delta(\gamma_k)/q_1(6 - \delta(\gamma_k)) = 3\varepsilon_5/q_1(6 - \delta(\gamma_k)) ,$$

$$\varepsilon_8 = (\delta(\gamma_k) - 2q_2)/q_2(\delta(\gamma_k) - 2) = \varepsilon_6/q_2(\delta(\gamma_k) - 2) .$$

(4.23)-(4.25) and (4.8) imply

$$\begin{aligned} &\| |J|^\alpha(|x|^{-\gamma_k} * |u|^2)u \|_{\delta'(\gamma_k)} \\ &\leqq C(\|\phi_j\|_{0,1}, T)\big(\|u\|^2_{\delta(\gamma_k)}(|s|^{(\alpha-1)\varepsilon_1}+|s|^{\alpha\varepsilon_1})\,\||J|^\alpha u\|^{1-\varepsilon_1}_{\delta(\gamma_k)} \\ &+(\|u\|^{2(1-\varepsilon_3)+2\varepsilon_3(1-\varepsilon_7)}_{\delta(\gamma_k)}|s|^{-\varepsilon_7}+\|u\|^{2(1-\varepsilon_4)+2\varepsilon_4(1-\varepsilon_8)}_{\delta(\gamma_k)})\,\||J|^\alpha u\|_{\delta(\gamma_k)}, \quad s\neq 0. \end{aligned} \tag{4.26}$$

In the same way as in the proof of (4.8) we obtain by (4.21) and (4.26)

$$|||\, |J|^\alpha u\, |||_{X(T,\gamma_2)} \leqq C(\|\phi_j\|_{0,1}, T) \qquad \text{for any} \quad T\in\mathbb{R}^+. \tag{4.27}$$

We have by (4.22) and Proposition 4

$$\begin{aligned} \| |J|^\alpha(|x|^{-\gamma_k} * |u|^2)u \|_{\delta'(\gamma_k)} &\leqq C(\|\phi_j\|_{0,1}) \\ &\times \big(|s|^{-\gamma_3/2}(|s|^{(\alpha-1)\varepsilon_1}+|s|^{\alpha\varepsilon_1})\,\||J|^\alpha u\|^{1-\varepsilon_1}_{\delta(\gamma_k)}\big) \\ &+ \big(|s|^{-\varepsilon_3(1-\varepsilon_3)/2-\varepsilon_7}\,\|u\|^{2\varepsilon_3(1-\varepsilon_7)}_{\delta(\gamma_k)} \\ &+ |s|^{-\varepsilon_3(1-\varepsilon_4)/2}\,\|u\|^{2\varepsilon_4(1-\varepsilon_8)}_{\delta(\gamma_k)}\big)\,\||J|^\alpha u\|_{\delta(\gamma_k)}\big), \qquad |s|>1. \end{aligned} \tag{4.28}$$

In the same way as in the proof of (4.14), we have by (4.27), (4.28) and (4.21) with $T=\infty$

$$|||\, |J|^\alpha u\, |||_{X(\infty,\gamma_2)} \leqq C(\|\phi_j\|_{0,1}), \tag{4.29}$$

since $\gamma_2 \geqq \gamma_3 > 4/3$. By the conditions of Theorem 3, we can see that (4.29) holds valid for any α such that $0<\alpha<1/2$. We get by Lemma 2.1 and (4.29)

$$\begin{aligned} \|u(t)\|_{\delta(\gamma_2)} &\leqq C|t|^{-\gamma_2/4}\,\|u(t)\|^{1-(\gamma_2/4\alpha)} \\ &\times \||J|^\alpha u(t)\|^{\gamma_2/4\alpha}, \qquad \text{for} \quad \gamma_2<4\alpha, \qquad t\neq 0. \end{aligned} \tag{4.30}$$

(4.30) is the same estimate as (4.19). The proof for $n=3$ now proceeds from (4.30) in the same way as that for $n\geqq 4$ from (4.19). This completes the proof of Theorem 3. Q. E. D.

REMARK 3. — When $V_2(x)=|x|^{-\gamma_2}$, Theorem 3 holds valid for $n=2$. Indeed, by Lemma 2.4 (2.8) and Proposition 1 we have (4.19), from which we get (4.20). This yeilds Theorem 3.

REMARK 4. — In the case of the NLS eq. (see Remark 2), Y. Tsutsumi [*16*], N. Hayashi-Y. Tsutsumi [*7*] showed that Theorem 3 holds valid in the $H^{1,1}$ space if $\gamma(n)<p_1=p_2<\alpha(n)$, where $\gamma(n)=(n+2+\sqrt{n^2+12n+4})/2n$, $\alpha(n)$ is the same one as that in Remark 2. We can prove Theorem 3 in the $H^{0,1}$ space in the case of the NLS eq. if $\gamma(n)<p_1\leqq p_2<1+(4/n)$. Indeed, the pseudoconformal conservation law (see Remark 2), Lemma 2.3 and the fact that u, $Ju\in X(a,p_2)$ yield u, $Ju\in X(\infty,p_2)$ if $\gamma(n)<p_1\leqq p_2<1+(4/n)$. From this we obtain the desired result (see [*16*] [*7*]).

REMARK 5. — J. Ginibre-G. Velo [*6*] have proved Theorem 3 in the energy space $H^{1,0}$ if $2<\gamma_2\leqq\gamma_3<\min(4,n)$, and in [*5*] they also proved Theorem 3 in the energy space in the case of the NLS eq. if $1+(4/n)<p_1\leqq p_2<\alpha(n)$.

Added Remark. — The proof of Theorem 2 relies heavily on the estimates (3.7)-(3.12) in Proposition 5, which can be derived from a new identity for $\beta(t)$ (see (3.87) of [*10*] and (2.27) of [*11*]). Recently J. Ginibre gave a simple derivation of the identity for $\beta(t)$ (see [*19*]).

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